EXISTENCE AND APPROXIMATION FOR VARIATIONAL PROBLEMS UNDER UNIFORM CONSTRAINTS ON THE GRADIENT BY POWER PENALTY

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Abstract. Variational problems under uniform quasiconvex constraints on the gradient are studied. Our technique consists in approximating the original problem by a one−parameter family of smooth unconstrained optimization problems. Existence of solutions to the problems under consideration is proved as well as existence of lagrange multipliers associated to the uniform constraint; no constraint qualification condition is required. The solution-multiplier pairs are shown to satisfy an Euler-Lagrange equation and a complementarity property. Numerical experiments confirm the ability of our method to accurately compute solutions and Lagrange multipliers.

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1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 1$ and $T : \Omega \times \mathbb{R}^{m\times N} \to [0,\infty[$ a Carathéodory function. Let $s \geq 1$ and consider a functional $J : W^{1,s}(\Omega;\mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\}$, which is supposed to be bounded from below and sequentially lower semicontinuous in the weak topology of $W^{1,s}(\Omega;\mathbb{R}^m)$. We study a class of constrained Dirichlet problems from the calculus of variations of the type

$$\inf\{J(v) : |T(x,\nabla v(x))| \leq 1 \ a.e\ x\ in\ \Omega,\ v = g\ on\ \partial\Omega\}. \tag{1.1}$$

In particular, we prove existence and approximability of solutions and Lagrange multipliers associated to the uniform constraint on the gradient. We approximate the problem by a sequence of unconstrained problems penalizing the uniform constraint by a power term.

The model case of (1.1) is the problem of the elastoplastic torsion of a cilindrical bar of section Ω:

$$\min_{v\in K_0} \frac{1}{2}\int_\Omega (|\nabla v(x)|^2 - h(x)v(x))dx \tag{1.2}$$

for $K_0 = \{v \in H_0^1(\Omega) \mid |\nabla v(x)| \leq 1\ a.e\ x \in \Omega\}$. Problem (1.2) has been extensively studied by Ting (1969); Brézis (1972); Caffarelli and Friedman (1979) and in the numerical aspects by Glowinski et al. (1981). Brézis (1972) proves the existence and uniqueness of a multiplier $\lambda \in L^\infty$ satisfying the system

$$\lambda \geq 0 \qquad a.e\ on\ \Omega \tag{1.3a}$$

$$\lambda(1 - |\nabla u|) = 0 \qquad a.e\ on\ \Omega \tag{1.3b}$$

$$-\Delta u - \sum_{i=1}^N \frac{\partial}{\partial x_i}(\lambda \frac{\partial u}{\partial x_i}) = h \quad in\ \mathcal{D}' \tag{1.3c}$$

when the right hand side h is constant. Chiadò Piat and Percivale (1994) reconsider the problem for a general elliptic operator and nonconstant right hand side h, obtaining a measure multiplier satisfying a system analogous to (1.3b)-(1.3c). Brézis

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(1972) uses the characteristics method to solve (1.3c) for λ, obtaining a semi-explicit formula for the multiplier. Chiadò Piat and Percivale (1994) approximate the problem by a sequence of nonsmooth problems penalizing the violation of the constraint $|\nabla u| \leq 1$ *a.e.*

Whether similar results could be obtained in the framework of a general duality theory standed as an open question for a long time. Ekeland and Temam (1976) attempted to apply the classical duality theory to this problem with unsatisfactory results. The question was solved positively by Daniele et al. (2007) using a new infinite dimensional duality theory (see also Donato, 2011; Maugeri and Puglisi, 2014). Daniele et al. (2007) show, for a class of problems including Problem (1.2), that if the problem is solvable and the solution satisfies a constraint qualification condition, then there exists a Lagrange multiplier $\lambda \in L^\infty_+(\Omega)$ satisfying (1.3b), which is indeed the solution of a dual problem.

Concerning existence of solutions for the general Problem (1.1), we can cite the results of Ball (1977b), showing existence of solutions in Orlicz-Sobolev spaces for variational problems under constraints of the type $F(\nabla v(x)) \in C(x)$ for almost every $x \in \Omega$, by assuming some convexity. Nonetheless, there is still a lack for practical ways to compute numerical approximations of the Lagrange multipliers associated with the uniform constraint on the gradient, and solutions in standard Sobolev spaces for vectorial, nonconvex problems under general boundary conditions.

We address these issues by providing an approximation scheme for Problem (1.1) by simpler problems that can be solved using existing mature numerical methods. The original problem is approximated by a sequence of unconstrained smooth problems whose solution converges to a solution of the constrained problem. Moreover, by analyzing the optimality conditions we identify a term which is then showed to converge to a Lagrange multiplier associated to the uniform constraint on the gradient. In this way, we recover and in many cases improve the existence results and provide a practical approximation scheme. Our approach is illustrated through numerical simulations.

1.1. Statement of the problem and main results. We are interested in the minimization problem

$$\inf\{J(v) \mid \|T(\cdot, \nabla v)\|_{\infty,\Omega} \leq 1,\ v \in g + W^{1,s}_0(\Omega; \mathbb{R}^m)\}, \tag{1.4}$$

where

$$\|T(\cdot, \nabla v)\|_{\infty,\Omega} = \text{ess-}\sup\{T(x, \nabla v(x)) \mid x \in \Omega\},$$

and $g \in W^{1,\infty}(\Omega; \mathbb{R}^m) \cap C(\overline{\Omega}; \mathbb{R}^m)$ is a given function satisfying

$$J(g) < +\infty \text{ and } T(x, \nabla g(x)) \leq 1 \quad \text{for a.e. } x \in \overline{\Omega}. \tag{1.5}$$

Define $J_\infty : W^{1,s}(\Omega; \mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\}$ by

$$J_\infty(v) = \begin{cases} J(v) & \text{if } \|T(\cdot, \nabla v)\|_{\infty,\Omega} \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Then (1.4) may be rewritten as

$$\inf\left\{J_\infty(v) \mid v \in g + W^{1,s}_0(\Omega; \mathbb{R}^m)\right\}. \tag{1.6}$$

By (1.5), we have that $J_\infty(g) < +\infty$.

From now on, we assume that T is *quasiconvex* in the sense of Morrey, *i.e.* for almost for every $x_0 \in \Omega$ and any $\xi_0 \in \mathbb{R}^{m\times N}$

$$T(x_0, \xi_0) \leq \frac{1}{\mathcal{L}(D)} \int_D T(x_0, \xi_0 + \nabla\phi(x))dx, \tag{1.7}$$

where D is an arbitrary bounded domain in $\mathbb{R}^N$ and ϕ is any function in $W_0^{1,\infty}(D;\mathbb{R}^m)$. Here, $\mathcal{L}$ stands for the Lebesgue measure in $\mathbb{R}^N$. Suppose also that

$$\alpha_1|\xi|^r \leq T(x,\xi) \leq \beta_1(1+|\xi|^r) \tag{1.8}$$

where $0 < \alpha_1 \leq \beta_1$ and $1 \leq r < \infty$. Concerning the functional J, in most interesting applications it will take the integral form

$$J(u) = \int_\Omega f(x, u(x), \nabla u(x))dx \tag{1.9}$$

where $f : \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{m\times N}$ is a Carathéodory integrand satisfying, for almost every $x \in \Omega$ and for every $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{m\times N}$

$$\xi \mapsto f(x,u,\xi) \text{ is quasiconvex} \tag{1.10a}$$
$$\gamma_1(x) \leq f(x,u,\xi) \leq \beta_2(|\xi|^s + |u|^t) + \gamma_2(x) \tag{1.10b}$$

where $\beta_2 \geq 0$, $\gamma_1, \gamma_2 \in L^1(\Omega)$ and $1 \leq t < \infty$.

Define the p-power penalty functional $J_p : W^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}\cup\{+\infty\}$ by

$$J_p(v) = J(v) + \frac{r}{p}\int_\Omega Tv(x)^{p/r}dx,$$

where $p \in \,]\max(r,s), \infty[$ and

$$Tv(x) = T(x, \nabla v(x))$$

and consider the penalized problems

$$\inf\{J_p(v) \mid v \in g + W_0^{1,p}(\Omega;\mathbb{R}^m)\}. \tag{1.11}$$

Under the above conditions, the existence of solutions u_p to (1.11) follows from a standard application of the direct method of the calculus of variations (Dacorogna, 2007, Theorem 8.29). In this direction, notice that the quasiconvexity of T yields the quasiconvexity of T^p for every $1 < p < \infty$. Hypothesis (1.10) can be replaced by any alternative set of hypothesis ensuring sequential lower semicontinuity of J, such as those related to polyconvexity (Ball, 1977b,a).

Any selection of solutions to Problems (1.11) uniformly converges to a solution of Problem (1.1), whose existence is not supposed a priori. Indeed, we have the following:

Theorem 1.1. *Under the previous assumptions, we have that:*

(i) *For every $q \geq \max\{N+1, r, s\}$, the net $\{u_p \mid p \geq q,\, p \to \infty\}$ is bounded in $W^{1,q}(\Omega;\mathbb{R}^m)$ and relatively compact in $C^\alpha(\overline{\Omega};\mathbb{R}^m)$, for every $0 \leq \alpha \leq 1-N/q$.*

(ii) *If u_∞ is a cluster point of $\{u_p \mid p \to \infty\}$ in $C(\overline{\Omega}; \mathbb{R}^m)$, then u_∞ is an optimal solution to* (1.4) *and, moreover,*

$$\lim_{p\to\infty} \min J_p = \lim_{p\to\infty} J_p(u_p) = \lim_{p\to\infty} J(u_p) = J(u_\infty) = \min J_\infty.$$

Remark 1. *The convergence*

$$\lim_{p\to\infty} J(u_p) = J(u_\infty) \tag{1.12}$$

will have important consequences if the functional J satisfies certain conditions, as it reinforces weak convergence of u_p in $W^{1,s}(\Omega; \mathbb{R}^m)$ to strong convergence.

The method of power penalties was applied by Attouch and Cominetti (1999) to problems with L^∞ constraints $|u| \le 1$ *a.e.* They also announced the results of Theorem 1.1 for scalar problems with zero boundary condition under the constraint $|\nabla u| \le 1$. Ishii and Loreti (2005) address the uniform convergence as $p \to \infty$ of scalar critical points of J_p for functionals J in the form (1.9) with $f(x, u(x)) = h(x)u(x)$, showing the convergence to solutions of the constrained problem in some particular cases, such as dimension one and radial solutions. In problems with non-unique solutions, variational solutions obtained by p-Laplacian approximation can be shown to minimize the L^∞ norm among solutions. This is notably the case in finite dimension (Attouch and Cominetti, 1999) and in infinite dimension for $J \equiv 0$, where variational solutions converge to absolute minimizers which are indeed unique by a celebrated result of Jensen (1993).

Next we address the existence and approximation of Lagrange multipliers for the uniform constraint on the gradient. The underlying rationale bears some resemblances to certain methods for showing existence of Lagrange multipliers without recourse to separation theorems, such as the Fritz John optimality conditions in nonlinear programming (cf. Bertsekas, 1999, Sec. 3.3.5). Let us consider the Lagrange functional $L : H^1(\Omega) \times L^\infty_+(\Omega) \to \overline{\mathbb{R}}$

$$L(u, \lambda) = J(u) + \int_\Omega \lambda(Tu - 1) \tag{1.13}$$

If a solution u to Problem (1.4) satisfies a constraint qualification condition, then there exists $\lambda \in L^\infty_+(\Omega)$ such that (u, λ) is a saddle point of L (Daniele et al., 2007). Let (u, λ) be a saddle point of L, and suppose that the derivative T_ξ of T with respect to its second argument exists. The minimality condition on u reads

$$J'(u)[v] + \int_\Omega \lambda \; T_\xi u \cdot \nabla v = 0 \quad \forall v \in C_0^\infty(\Omega).$$

On the other hand, the optimality conditions for the penalized problem (1.11) yields

$$J'(u_p)[v] + \int_\Omega (Tu_p)^{p-1} \; T_\xi u_p \cdot \nabla v = 0 \quad \forall v \in C_0^\infty(\Omega).$$

Suppose that $J'(u_p) \to J'(u)$ as $p \to \infty$, then

$$\int_\Omega (Tu_p)^{p-1} \; T_\xi u_p \cdot \nabla v \to \int_\Omega \lambda \; T_\xi u \cdot \nabla v \qquad \forall v \in C_0^\infty(\Omega)$$

This formal derivation strongly suggests that cluster points of $\lambda_p := (Tu_p)^{p-1}$, if any, must play the role of a Lagrange multiplier. Under very general conditions we are able to obtain a uniform bound on the $L^1(\Omega)$ norm of λ_p, which in general is not enough to have a weakly$-*$ convergent subsequence. Using parameterized measures we prove the existence of a measure-valued Lagrange multiplier, which satisfies an Euler-Lagrange equation and a complementarity property. We also prove that the sequence of approximating multipliers converges in the bitting sense (see further Definition 3.7), providing a clue for computing numerically the Lagrange multiplier in problems exhibiting concentration phenomena. The main result is the following;

THEOREM 1.2. *Let $T(x,\xi) = |\xi|$, and $g \in C^2(\bar{\Omega})$ be such that $\|\nabla g\|_{\infty,\Omega} < 1/2$. Let u_∞ be a cluster point of $\{u_p\}_{p\geq p_1}$ for the topology of $C(\overline{\Omega})$. Suppose that f satisfies* (3.13), (3.14) *and* (3.23). *There exists a nonnegative Radon measure multiplier μ such that:*

(i) *For a nonnegative Radon measure σ, and measurable non-negative functions λ and η,*

$$\mu = \lambda\mathcal{L} + \eta\sigma$$

Moreover, $\lambda \in L^1(\Omega)$.

(ii) *The primal-dual pair (u_∞, μ) satisfies the system*

$$\begin{aligned} -\mathrm{div}(f_\xi(x,u_\infty,\nabla u_\infty) + \nabla u_\infty \mu) + f_s(x,u_\infty,\nabla u_\infty) = 0 \quad &\textit{in } \mathcal{D}'(\Omega). \\ \lambda(x) \geq 0 \ \mathcal{L} - a.e \textit{ in } \Omega, \qquad \eta(x) \geq 0 \ \sigma - a.e \textit{ in } \Omega&. \\ \lambda(x)(|\nabla u_\infty(x)| - 1) = 0 \ \mathcal{L} - a.e \textit{ in } \Omega, \quad \eta(x)(|\nabla u_\infty(x)| - 1) = 0 \ \sigma - a.e \textit{ in } \Omega&. \end{aligned}$$

(iii) *The sequence $\{|\nabla u_p|^{p-1}\}_{p\geq p_1}$ converges to λ in the bitting sense.*

Notice that combining the complementarity property and the Euler-Lagrange equation we can formally show that u_∞ solves the boundary value problem

$$\begin{aligned} (|\nabla u| - 1)(-\mathrm{div}(f_\xi(\cdot,u,\nabla u)) + f_s(\cdot,u,\nabla u)) &= 0 \text{ in } \Omega, \\ |\nabla u| - 1 &\leq 0 \text{ in } \Omega, \\ u &= g \text{ in } \partial\Omega. \end{aligned}$$

An interesting question left open is whether u_∞ is a solution of this problem in the viscosity sense (Lions, 1982).

Then we focus on a subclass of simpler problems which includes the elastoplastic torsion problem (1.2). We use differential equations methods to prove the convergence of the approximating multipliers in $L^\infty(\Omega)$. We consider scalar problems of the following form

$$\min\left\{ J(v) := \int_\Omega \frac{1}{2}W(|\nabla v|^2) - \phi(v) \ : \ |\nabla v| \leq 1, \ v \in g + H_0^1(\Omega) \right\},$$

where g is a real constant, Ω is a convex domain and additionally

$$t \mapsto W(t^2) \textit{ and } \phi \text{ are convex and of class } C^2(\mathbb{R}) \tag{1.14}$$

$$G(s) := W'(s) + 2sW''(s) > 0, \quad \text{for } s > 0. \tag{1.15}$$

Under the previous hypothesis, we prove the following.

THEOREM 1.3. *Let u_∞ be a cluster point of $\{u_p\}_{p\geq p_1}$ for the topology of $C(\overline{\Omega})$. There exists $\lambda \in L^\infty(\Omega)$ such that*

(i) *The sequence* $\{|\nabla u_p|^{p-2}\nabla u_p\}_{p\geq p_1}$ *weakly*$-*$ *converges to* $\lambda\nabla u_\infty$, *up to subsequence.*
(ii) *The primal-dual pair* (u_∞, λ) *satisfy the system*

$$-\mathrm{div}(W'(|\nabla u_\infty|^2)\nabla u_\infty) - \mathrm{div}(\lambda\nabla u_\infty) = \phi'(u_\infty) \text{ in } \mathcal{D}'. \tag{1.16}$$

$$\lambda(x) \geq 0 \text{ a.e in } \Omega. \tag{1.17}$$

$$\lambda(x)(|\nabla u_\infty(x)| - 1) = 0 \text{ a.e in } \Omega \tag{1.18}$$

For the elastoplastic torsion problem (1.2), Brézis (1972) proved the uniqueness of $\lambda \in L^\infty_+(\Omega)$ verifying (1.16)–(1.18), therefore the whole net $\{|\nabla u_p|^{p-2}\}_{p\geq p_1}$ is convergent.

1.2. Organization of the paper. Section 2 is devoted to the proofs of primal convergence results. The proof of Theorem 1.1 is decomposed into a series of lemmas of independent interest. In Section 3 we prove Theorems 1.2 and 1.3. The numerical aspects of our method are presented in Section 4. We present an algorithm for computing solutions and Lagrange multipliers for Problem (1.4). Our algorithm is validated by computing numerical approximations to solutions and Lagrange multipliers of Problem (1.2), for which there exists explicit solutions on the 2D disk to compare. We also obtain an explicit formula for the Lagrange multiplier, in this way we are able to evaluate the algorithm at computing both solutions and multipliers. We close the paper with a summary presented in Section 5

2. Primal convergence results. In this section we provide the proof of Theorem 1.1. The proof is divided into a series of lemmas. For clarity of the exposition we put $r = 1$, the general case being completely analogous.

Lemma 2.1 (Compactness). *we have that:*
(i) $\sup_{p\geq s} \frac{1}{p}\|Tu_p\|^p_{p,\Omega} < +\infty$, *where*

$$\|Tu_p\|^p_{p,\Omega} = \int_\Omega T(x, \nabla u_p(x))^p dx$$

(ii) *Let* $p_1 = \max\{N+1, s\}$. *For every* $q > 1$, $\{u_p\}_{p\geq p_1}$ *is bounded in* $W^{1,q}(\Omega;\mathbb{R}^m)$
(iii) $\{u_p\}_{p\geq p_1}$ *is relatively compact in* $C(\overline{\Omega};\mathbb{R}^m)$.
(iv) *For every uniform cluster point* u_∞ *of* $\{u_p\}_{p\geq p_1}$, *we have that*

$$u_\infty \in g + W^{1,\infty}_0(\Omega;\mathbb{R}^m).$$

(v) *If* $u_{p_j} \to u_\infty$ *in* $C(\overline{\Omega};\mathbb{R}^m)$ *then* $u_{p_j} \rightharpoonup u_\infty$ *weakly in* $W^{1,q}(\Omega;\mathbb{R}^m)$ *for every* $q \in [p_1, \infty[$.

Proof. From the optimality of u_p it follows that

$$\alpha + \frac{1}{p}\|Tu_p\|^p_{p,\Omega} \leq J(g) + \frac{1}{p}\|Tg\|^p_{\infty,\Omega}\mathcal{L}(\Omega), \tag{2.1}$$

where $\alpha = \inf\{J(v) \mid v \in W^{1,s}(\Omega;\mathbb{R}^m)\} \in \mathbb{R}$ (recall that J is supposed to be bounded from below). Using (1.5) we deduce that

$$\sup_{p\geq s} \frac{1}{p}\|Tu_p\|^p_{p,\Omega} < +\infty,$$

hence

$$C_1 := \sup_{p \geq s} \|Tu_p\|_{p,\Omega} < +\infty.$$

In particular,

$$\|\nabla u_p\|_{p,\Omega} \leq \alpha_1 C_1.$$

On the other hand, the Poincaré inequality yields

$$\|u\|_{p,\Omega} \leq C(\Omega, p) \left(\|\nabla u\|_{p,\Omega} + \|\nabla g\|_{p,\Omega}\right) + \|g\|_{p,\Omega},$$

for every $u \in g + W_0^{1,p}(\Omega; \mathbb{R}^m)$ and a suitable constant $C(\Omega, p) > 0$. Combining these estimates, and recalling that the constant $C(\Omega, p)$ may be chosen such that (Adams, 1975)

$$\sup_{p \in [N+1, \infty[} C(\Omega, p) < +\infty,$$

we deduce that there exists a constant $C_2 > 0$ such that

$$\forall p \in [p_1, +\infty[,\ \|u_p\|_{1,p,\Omega} = \|u_p\|_{p,\Omega} + \|\nabla u_p\|_{p,\Omega} \leq C_2,$$

where $p_1 = \max\{N+1, s\}$. In particular, $\{w_p := u_p - g\}_{p \geq p_1}$ is bounded in $W_0^{1,q}(\Omega; \mathbb{R}^m)$ for each $q \geq p_1$, hence for every $q > 1$ by Hölder inequality. Since $p_1 > N$, we deduce that $\{w_p\}_{p \geq p_1}$ is relatively compact in $C(\overline{\Omega}; \mathbb{R}^m)$ by the Rellich-Kondrachov theorem (since we deal with W_0^{1,p_1} we do not require any regularity condition on $\partial\Omega$). Thus, we deduce that $\{u_p\}_{p \geq p_1}$ is relatively compact in $C(\overline{\Omega}; \mathbb{R}^m)$.

Let u_∞ be a cluster point of $\{u_p\}_{p \geq p_1}$ in $C(\overline{\Omega}; \mathbb{R}^m)$. First, we prove that $u_\infty \in W^{1,\infty}(\Omega; \mathbb{R}^m)$. By Morrey's theorem there exists a constant $C'(\Omega, p) > 0$ such that

$$|w_p(x) - w_p(y)| \leq C'(\Omega, p) ||w_p||_{1,p,\Omega} |x - y|^{1 - N/p}$$

for every $x, y \in \Omega$. In fact, the constant can be chosen in such a way that

$$\sup_{p \in [q, \infty[} C'(\Omega, p) < +\infty$$

for every $q > N$ (Adams, 1975). Therefore, we conclude that for a suitable constant $C_3 > 0$, $|u_p(x) - u_p(y)| \leq C_3 |x - y|^{1 - N/p}$, for every $x, y \in \Omega$ and $p \in [p_1, \infty[$. We deduce that

$$|u_\infty(x) - u_\infty(y)| \leq C_3 |x - y|,$$

then $u_\infty \in W^{1,\infty}(\Omega; \mathbb{R}^m)$. Of course, $u_\infty = g$ on $\partial\Omega$.

Next, fix $q \in]1, \infty[$. From our previous analysis it follows that $\{u_p\}_{p \in [p_1, \infty[}$ is bounded in $W^{1,q}(\Omega; \mathbb{R}^m)$ and therefore relatively compact for the weak topology of $W^{1,q}(\Omega; \mathbb{R}^m)$. Consequently, if $p_j \to \infty$ is a sequence such that $u_{p_j} \to u_\infty$ uniformly on $\overline{\Omega}$, then $u_{p_j} \rightharpoonup u_\infty$ weakly in $W^{1,q}(\Omega; \mathbb{R}^m)$. □

Lemma 2.2. *If u_∞ is a cluster point of $\{u_p \mid p \to \infty\}$ in $C(\overline{\Omega}; \mathbb{R}^m)$ then $\|Tu_\infty\|_{\infty,\Omega} \leq 1$. Moreover, u_∞ is an optimal solution to* (1.6)*, and we have that*

$$\lim_{p \to \infty} J_p(u_p) = \lim_{p \to \infty} J(u_p) = J(u_\infty) = \min J_\infty.$$

Proof. Let $u_{p_j} \to u_\infty$ in $C(\overline{\Omega};\mathbb{R}^m)$ and fix $q \in]1,\infty[$. By Lemma 2.1, $u_{p_j} \rightharpoonup u_\infty$ weakly in $W^{1,q}(\Omega;\mathbb{R}^m)$. It follows from the weak lower semicontinuity in $W^{1,q}(\Omega;\mathbb{R}^m)$ of $v \mapsto \|Tv\|_{q,\Omega}$, that

$$\|Tu_\infty\|_{q,\Omega} \leq \liminf_{j\to\infty} \|Tu_{p_j}\|_{q,\Omega}.$$

For every $p \in [q,\infty[$, the Hölder inequality yields

$$\|Tu_p\|_{q,\Omega} \leq \|Tu_p\|_{p,\Omega}\mathcal{L}(\Omega)^{\frac{1}{q}-\frac{1}{p}}.$$

Then, Lemma 2.1 ensures that

$$\|Tu_p\|_{q,\Omega} \leq (pC)^{\frac{1}{p}}\mathcal{L}(\Omega)^{\frac{1}{q}-\frac{1}{p}}$$

for some constant $C > 0$. Hence

$$\|Tu_\infty\|_{q,\Omega} \leq \mathcal{L}(\Omega)^{\frac{1}{q}}$$

Letting $q \to \infty$, we get the desired inequality.

Let $v \in g + W^{1,\infty}(\Omega;\mathbb{R}^m)$ with $\|Tv\|_{\infty,\Omega} \leq 1$. By optimality of u_p we have that

$$J(u_p) \leq J_p(u_p) \leq J_p(v) = J(v) + \frac{1}{p}\|Tv\|_{p,\Omega}^p.$$

Since $\|Tv\|_{\infty,\Omega} \leq 1$, we have that

$$\limsup_{p\to\infty} J(u_p) \leq \limsup_{p\to\infty} J_p(u_p) \leq \limsup_{p\to\infty} J_p(v) = J(v).$$

As v is arbitrary, we obtain that

$$\limsup_{p\to\infty} J(u_p) \leq \limsup_{p\to\infty} J_p(u_p) \leq \inf J_\infty.$$

Now, let $u_{p_j} \to u_\infty$ in $C(\overline{\Omega};\mathbb{R}^m)$. By the weak lower semicontinuity of J, we have that

$$J(u_\infty) \leq \liminf_{j\to\infty} J(u_{p_j}),$$

and due to the previous lemmas, we know that $J(u_\infty) = J_\infty(u_\infty)$. This proves the optimality of u_∞ and moreover

$$\lim_{j\to\infty} J_{p_j}(u_{p_j}) = \lim_{j\to\infty} J(u_{p_j}) = \min J_\infty.$$

Finally, note that, up to a subsequence, the same is valid for an arbitrary sequence $\{p_k\}_{k\in\mathbb{N}}$ with $p_k \to \infty$. This fact together with a compactness argument proves indeed the result. ∎

3. Dual convergence results. In this section we are concerned with the existence and approximation of Lagrange multipliers for the constrained problem (1.4). We treat separately a class of simpler instances of the problem which can be tackled by differential equations methods, and a more general case with few additional assumptions with respect to Section 2. For the former class of problems, we prove a strong existence result of Lagrange multipliers in $L^\infty(\Omega)$. In the more general case, we prove the existence of a Radon measure multiplier, and show how to obtain information on the uniformly integrable part of the multiplier.

3.1. Regular L^∞ multipliers. In this part, we consider the following instances of (1.4)

$$\min\left\{ J(v) := \int_\Omega (\frac{1}{2}W(|\nabla v|^2) - \phi(v)) \ : \ |\nabla v| \le 1,\ v \in g + H_0^1(\Omega) \right\}. \tag{3.1}$$

Let us consider the penalized problem

$$\min\left\{ \frac{1}{2}\int_\Omega W(|\nabla v|^2) + \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega \phi(v) \ : v \in g + H_0^1(\Omega) \right\}. \tag{3.2}$$

By the convexity assumptions on the functions W and ϕ, that problem has a unique solution u_p which is a weak solution of the Euler-Lagrange equation:

$$-\text{div}((W'(|\nabla u_p|^2) + |\nabla u_p|^{p-2})\nabla u_p) = \phi'(u_p). \tag{3.3}$$

Let us define:

$$\Psi(x) = \int_0^{|\nabla u_p|^2} G(s)ds + 2\frac{p-1}{p}|\nabla u_p|^p + 2\phi(u_p)$$

Maximum principles of Payne and Philippin (1977, 1979) state that under mild conditions the maximum of $\Psi(\cdot)$ is attained at a critical point of u_p. The application of maximum principle techniques require to work with classical $C^2(\Omega)$ solutions. Results of Uhlenbeck (1977), Tolksdorf (1984) and Lieberman (1988) show that bounded solutions to equations of the type (3.3) are $C^{1,\alpha}(\overline{\Omega})$-regular, provided that hypothesis (1.15) holds. Higher regularity can be obtained by a bootstrap argument at points where $\nabla u_p \neq 0$. However, if the function G defined in (1.15) is degenerate, *i.e.* $G(0) = 0$, a further regularization is necessary (Kawohl, 1990). Following a classic procedure (see eg. Evans and Gangbo, 1999; Bhattacharya et al., 1989; Sakaguchi, 1987; DiBenedetto, 1983) the term $|\nabla u_p|^p$ is regularized by $(\varepsilon^2 + |\nabla u_p|^2)^{p/2}$ to obtain a sequence of regular functions u_p^ε converging to u_p pointwise and in $W^{1,p}(\Omega)$ norm as $\varepsilon \to 0$. In this way degenerate problems can be handled by approximation.

THEOREM 3.1. *Under hypothesis* (1.14)- (1.15)*, if Ω is convex and $\partial\Omega \in C^2$, then the sequence $\{|\nabla u_p|^p\}_{p \ge p_1}$ is uniformly bounded in $L^\infty(\Omega)$.*

Proof. Note that by (1.15), $|\nabla u_p|^p + 2\phi(u_p) \le \Psi(x)$. By Payne and Philippin (1979, Corollary 1), the function $\Psi(x)$ attains its maximum at a critical point of u_p. In such a point $\Psi(x) = 2\phi(u_p(x))$, therefore

$$|\nabla u_p|^p + 2\phi(u_p) \le \Psi(x) \le \max_{\bar\Omega} \Psi(x) \le 2 \max_{\bar\Omega} \phi(u_p(x)),$$

whence

$$|\nabla u_p|^p \le 4 \max_{\bar\Omega} \phi(u_p) < +\infty$$

and conclude by Theorem 1.1 and the continuity of ϕ. □

REMARK 2. *For a non-convex domain Ω, let $\kappa(y)$ denote the mean curvature of $\partial\Omega$ at y, and define $K = (n-1)\max_{y\in\partial\Omega}[-\kappa(y)]_+$. Let also $F_p = \max_{y\in\bar\Omega}(\phi'(g) - \phi'(u_p))$, then if $\phi'(g) > 0$ and*

$$\limsup_{p\to\infty} F_p < \frac{2\phi'(g)}{K},$$

Theorem 3.1 remains valid.

COROLLARY 3.2. *Let u_∞ be a cluster point of $\{u_p\}_{p\geq p_1}$. Then, passing if neccesary to a further subsequence,*

(i) *$\nabla u_p(x) \to \nabla u_\infty(x)$ for a.e $x \in \Omega$.*

(ii) *$\nabla u_p \overset{*}{\rightharpoonup} \nabla u_\infty$ in the weak$-*$ topology of $L^\infty(\Omega)$.*

(iii) *There exists $\Lambda \in L^\infty(\Omega)^N$ such that the sequence $\{|\nabla u_p|^{p-2}\nabla u_p\}_{p\geq p_1}$ converges to Λ in the weak$-*$ topology.*

Proof. From Visintin (1984), using hypothesis (1.14) and the convergence of the functional (1.12), $u_p \to u_\infty$ strongly, proving assertion (i). Points (ii) and (iii) are consequences of the Banach–Alaoglu Theorem. □

We are now in position to state our existence and approximation result for both primal and dual solutions of Problem (1.4).

THEOREM 3.3. *Let u be a cluster point of $\{u_p\}_{p\geq p_1}$ in $C(\overline{\Omega})$ achieving the convergences of Corollary 3.2. There exists $\lambda \in L^\infty(\Omega)$ such that*

(i) *The sequence $\{|\nabla u_p|^{p-2}\nabla u_p\}_{p\geq p_1}$ weakly$-*$ converges to $\lambda\nabla u$.*

(ii) *The primal-dual pair (u, λ) satisfies the system*

$$-\mathrm{div}(W'(|\nabla u|^2)\nabla u) - \mathrm{div}(\lambda\nabla u) = \phi'(u) \text{ in } \mathcal{D}'. \tag{3.4}$$

$$\lambda(x) \geq 0 \text{ a.e in } \Omega. \tag{3.5}$$

$$\lambda(x)(|\nabla u(x)| - 1) = 0 \text{ a.e in } \Omega \tag{3.6}$$

Proof. We shall show that the limit field Λ in Corollary 3.2 (iii) verifies

$$|\Lambda| = \Lambda \cdot \nabla u \quad \text{a.e in } \Omega. \tag{3.7}$$

Using $u - g$ as test function in (3.3) we have

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \nabla u = -\int_\Omega W'(|\nabla u_p|^2)\nabla u_p \nabla u - \phi'(u_p)(u-g) \tag{3.8}$$

Then by Corollary 3.2(iii), Corollary 3.2(i) and (1.10b)

$$\int_\Omega \Lambda\nabla u = -\int_\Omega W'(|\nabla u|^2)|\nabla u|^2 - \phi'(u)(u-g). \tag{3.9}$$

The same procedure using $u_p - g \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function (recall the growth condition (1.10b)) shows that

$$\int_\Omega |\nabla u_p|^p \longrightarrow -\int_\Omega W'(|\nabla u|^2)|\nabla u|^2 - \phi'(u)(u-g). \tag{3.10}$$

and therefore

$$\int_\Omega |\nabla u_p|^p \longrightarrow \int_\Omega \Lambda\nabla u. \tag{3.11}$$

Then by lower semicontinuity and Hölder inequality

$$\int_\Omega |\Lambda| \leq \liminf_{p\to\infty} \int_\Omega |\nabla u_p|^{p-1} \leq \lim_{p\to\infty} \left(\int_\Omega |\nabla u_p|^p \right)^{1-1/p} (\mathcal{L}(\Omega))^{1/p},$$

which combined with (3.11) yields

$$\int_\Omega |\Lambda| \le \int_\Omega \Lambda \cdot \nabla u, \tag{3.12}$$

and (3.7) follows using $|\nabla u| \le 1$ a.e (Theorem 1.1). The existence of $\lambda \in L^\infty(\Omega)$ satisfying (3.5) & (3.6) follows from (3.7). Taking the limit in (3.3) using Theorem 1.1, Corollary 3.2 and the representation (3.7) gives (3.6). □

REMARK 3. *Theorem 3.3 generalizes a result by Evans and Gangbo (1999).*

The results of Theorem 3.3 can be slightly extended, for instance, to convex domains with piecewise smooth border or with small deviations from convexity. However, any condition promoting concentrations on the gradients such as interior corners is to be excluded. We have not reasons to doubt that the results of this section remain valid for smooth explicit dependencies of the functions W, ϕ and g on the variable x. However, the proof of this results for more general problems would require new, substantially different techniques. Though, such problems are covered by the theory developed in the following section. In this regard, notice that the regularity results discussed above apply to a class of problems much wider than those covered by Theorem 3.3.

3.2. Generalized multipliers. In this part we prove the more general dual convergence result. We attempt to keep the hypothesis a close as possible to those of the primal analysis. Hereafter, we take the following hypothesis on the integrand f

$$f(x,\cdot,\cdot) \text{ is differentiable for almost all } x \in \Omega. \tag{3.13}$$

$$\langle f_s(x,s,\xi), s\rangle \le \Gamma(1+|s|^t) \tag{3.14}$$

for some constant $\Gamma \in \mathbb{R}$ and $1 \le t \le \infty$. Also, we suppose the differentibility of T for almost all ξ, and a 'relative coercivity' hypothesis on the derivative:

$$\gamma T(x,\xi) \le \langle T_\xi(x,\xi), \xi\rangle \qquad a.e\ x \in \Omega. \tag{3.15}$$

for some $\gamma > 0$.

REMARK 4. *Note that the quasiconvexity and the controlled growth hypothesis* (1.10b) *implies the controlled growth of the derivative (Marcellini, 1985, pp. 6–7)*

$$|f_\xi(x,s,\xi)| \le \Gamma(1+|\xi|^{s-1}) \tag{3.16}$$

similarly, the quasiconvexity of T and the growth condition (1.8) *implies*

$$|T_\xi(x,\xi)| \le \tilde{\Gamma}(1+|\xi|^{r-1}) \le \Gamma(1+T(x,\xi)). \tag{3.17}$$

The first result is an uniform bound on the L^1 norm of the candidates to multiplier.

PROPOSITION 3.4. *Consider problem* (1.6) *and any asociated sequence $\{u_p\}$ obtained by the penalization process described in section 2. Let the general hypothesis stated in Section 1 be in force. Then, under hypothesis* (3.14) *and* (3.15), *there exists a constant C independent of p such that*

$$\int_\Omega (Tu_p)^p dx \le C, \tag{3.18}$$

provided that $\|\nabla g\|_{\infty,\Omega} < \frac{\gamma}{2\Gamma}$, *where* γ *and* Γ *are the constants of* (3.15) *and* (3.17) *respectively.*

REMARK 5. *If* $T(x,\xi) = |\xi|^r$ *we can take* $\gamma = \Gamma = r$.

Proof. From the optimality of u_p we obtain :

$$\int_\Omega f_s(x,u_p,\nabla u_p)v dx + \int_\Omega \langle f_\xi(x,u_p,\nabla u_p),\nabla v\rangle dx + \int_\Omega (Tu_p)^{p-1}\langle T_\xi u_p,\nabla v\rangle dx = 0 \tag{3.19}$$

for all $v \in W_0^{1,p}(\Omega)$. Hence using (3.19) we obtain, by choosing $v = u_p - g$

$$\int_\Omega \langle f_s(x,u_p,\nabla u_p),u_p-g\rangle dx + \int_\Omega \langle f_\xi(x,u_p,\nabla u_p),\nabla(u_p-g)\rangle dx + \int_\Omega (Tu_p)^{p-1}\langle T_\xi u_p,\nabla(u_p-g)\rangle dx = 0 \tag{3.20}$$

the coercivity condition (3.15) imply

$$\int_\Omega (Tu_p)^p dx \le \frac{1}{\gamma}\int_\Omega (Tu_p)^{p-1}\langle \nabla_\xi Tu_p,\nabla u_p\rangle dx$$

and combining the growth conditions (3.14) and (3.16) we have

$$\int_\Omega |\langle f_s(x,u_p,\nabla u_p),u_p-g\rangle| dx + |\langle f_\xi(x,u_p,\nabla u_p),\nabla(u_p-g)\rangle| dx \le C(\|u_p\|,g)$$

from Lemma 2.1 we know that the constant appearing in the previous equation can be chosen to be independent of p, and we will note it simply by C. Using (3.17) we summarize (3.20) as

$$\begin{aligned}\int_\Omega (Tu_p)^p dx &\le \frac{1}{\gamma}\left(C + \int_\Omega (Tu_p)^{p-1}\langle T_\xi u_p,\nabla g\rangle dx\right)\\ &\le \frac{1}{\gamma}\left(C + \Gamma\|\nabla g\|_{\infty,\Omega}\int_\Omega (Tu_p)^p + (Tu_p)^{p-1} dx\right)\end{aligned}$$

from the classical inequality $a^\nu b^{1-\nu} \le \nu a + (1-\nu)b$, valid for $a,b \ge 0$, $0 \le \nu \le 1$, we get

$$(Tu_p)^{p-1} \le \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} + (Tu_p)^p \tag{3.21}$$

then

$$\int_\Omega (Tu_p)^p dx \le \frac{1}{\gamma}\left(C + \Gamma\|\nabla g\|_{\infty,\Omega}\frac{\mathcal{L}(\Omega)}{p-1}\left(1-\frac{1}{p}\right)^p + 2\Gamma\|\nabla g\|_{\infty,\Omega}\int_\Omega (Tu_p)^p dx\right)$$

and the result follows. □

Until now we are essentially under the same hypothesis of Section 2. At this point it is necessary to restrict the class of problems we deal with. In particular, we need to reinforce the convergence results already proved. The following definition plays a role in this regard.

DEFINITION 3.5 (Evans and Gariepy (1987)). *Let $L : \mathbb{R}^{N\times m} \to \mathbb{R}$ be a given function, L is called uniformly strictly quasiconvex if*

$$\int_D (L(\xi) + \tau|\nabla\phi|^s) \leq \int_D L(\xi + \nabla\phi) \tag{3.22}$$

for some positive constant $\tau > 0$ and all open $D \subset \mathbb{R}^N$, $\xi \in \mathbb{R}^{N\times m}$, $\phi \in W^{1,s}(D, \mathbb{R}^m)$.
See Evans (1986) for examples of non convex functions satisfying (3.22).

COROLLARY 3.6. *Under assumptions* (3.13), (3.14), *if $u_p \rightharpoonup u_\infty$ weakly in $W^{1,s}(\Omega)$ and*

$$f(x, u_\infty(x), \cdot) \text{ is uniformly strictly quasiconvex a.e } x \in \Omega, \tag{3.23}$$

then $u_p \to u_\infty$ strongly in $W^{1,s}(\Omega)$.

Proof. see Evans and Gariepy (1987); Sychev (1998). □

The main result of this part is the existence of a Lagrange multiplier in the space of Radon measures. Despite the lack of regularity of the multiplier, we are able to extract useful information about it from the primal solutions that we can actually compute, in the sense given by the following definition.

DEFINITION 3.7. *The sequence $\{f_p\}$ converges in the bitting sense if there exists a sequence of non-increasing measurable sets E_k with $\mathcal{L}(E_k) \to 0$, a subsequence $\{p_j\}_{j\in\mathbb{N}}$ and $\bar{f} \in L^1(\Omega)$ such that for every fixed k, f_{p_j} converges weakly to $\bar{f}$ in $L^1(\Omega \setminus E_k)$.*

Now we prove, for the constraint $|\nabla u| \leq 1$ *a.e*, that regular cluster points of $\{u_p\}_{p\geq p_1}$ satisfy a Karush Kuhn Tucker type constrained optimality system.

THEOREM 3.8. *Let $T(x,\xi) = |\xi|$, and $g \in C^2(\bar{\Omega})$ be such that $\|\nabla g\|_{\infty,\Omega} < 1/2$. Let $u_\infty \in C^1(\Omega)$ be a cluster point of $\{u_p\}_{p\geq p_1}$ for the topology of $C(\overline{\Omega})$. Suppose that f satisfies* (3.13), (3.14) *and* (3.23). *There exists a nonnegative Radon measure multiplier μ such that:*

(i) *For a nonnegative Radon measure σ, and measurable non-negative functions λ and η,*

$$\mu = \lambda\nabla u_\infty\mathcal{L} + \eta\nabla u_\infty\sigma \tag{3.24}$$

Moreover, $\lambda \in L^1(\Omega)$.

(ii) *The primal-dual pair (u_∞, μ) satisfies the system*

$$-\text{div}(f_\xi(x, u_\infty, \nabla u_\infty) + \nabla u_\infty\mu) + f_s(x, u_\infty, \nabla u_\infty) = 0 \quad \text{in } \mathcal{D}'. \tag{3.25}$$
$$\lambda(x) \geq 0 \ \mathcal{L} - a.e \text{ in } \Omega, \qquad \eta(x) \geq 0 \ \sigma - a.e \text{ in } \Omega. \tag{3.26}$$
$$\lambda(x)(|\nabla u_\infty(x)| - 1) = 0 \ \mathcal{L} - a.e \text{ in } \Omega \tag{3.27}$$
$$\eta(x)(|\nabla u_\infty(x)| - 1) = 0 \ \sigma - a.e \text{ in } \Omega \tag{3.28}$$

(iii) *The sequence $\{|\nabla u_p|^{p-1}\}_{p\geq p_1}$ converges to λ in the bitting sense.*

Proof. Let $p_j \to \infty$ be a sequence such that

- $u_{p_j} \to u_\infty$ uniformly on $C(\bar{\Omega})$.
- $\nabla u_{p_j}(x) \to \nabla u_\infty(x)$ for a.e $x \in \Omega$ (by Corollary 3.6).

In the sequel we drop the index j for simplicity.

By the dominated convergence Theorem,

$$\int_\Omega |\langle f_s(x,u_p,\nabla u_p), u_p - u_\infty\rangle| dx + \int_\Omega |\langle f_\xi(x,u_p,\nabla u_p), \nabla u_p - \nabla u_\infty)\rangle| dx$$

converges to 0. Therefore, testing with $u_p - u_\infty$ in (3.19) we conclude that

$$\lim \int_\Omega |\nabla u_p|^{p-2}\langle \nabla u_p, \nabla u_p - \nabla u_\infty\rangle dx = 0. \tag{3.29}$$

Let $v_p = |\nabla u_p|^{p-2}\nabla u_p$. By Proposition 3.4, $\{v_p\}_{p\geq p_1}$ is bounded in $L^1(\Omega)$, therefore by Alibert and Bouchitté (1997, Theorem 2.5) there exists a subsequence, not relabeled, a nonnegative Radon measure σ and measurable families of finite measures $(\nu_x)_{x\in\Omega}$ and $(\nu_x^\infty)_{x\in\Omega}$ such that

$$\int_\Omega \left(\int |\xi| d\nu_x(\xi)\right) < \infty, \tag{3.30}$$

and, for every continuous function F verifying

$$F(x,\alpha\xi) = \alpha F(x,\xi) \quad \forall \alpha > 0 \tag{3.31a}$$

$$|F(x,\xi)| \leq \Gamma(1+|\xi|), \tag{3.31b}$$

it holds that

$$\int_\Omega F(x,v_p(x))dx \to \int_\Omega \left(\int F(x,\xi)\nu_x(d\xi)\right) dx + \int_\Omega \left(\int F(x,\xi)\nu_x^\infty(d\xi)\right)\sigma(dx).$$

In particular,

$$\int_\Omega |\nabla u_p|^{p-1} dx \to \int_\Omega \left(\int |\xi|\nu_x(d\xi)\right) dx + \int_\Omega \left(\int |\xi|\nu_x^\infty(d\xi)\right)\sigma(dx), \tag{3.32}$$

$$\begin{aligned} \int_\Omega \langle |\nabla u_p|^{p-2}\nabla u_p, \nabla u_\infty\rangle dx \ \to & \int_\Omega \left(\int \langle \nabla u_\infty(x), \xi\rangle \nu_x(d\xi)\right) dx \\ & + \int_\Omega \left(\int \langle \nabla u_\infty(x), \xi\rangle \nu_x^\infty(d\xi)\right)\sigma(dx). \end{aligned} \tag{3.33}$$

Let $h(x,\xi) = |\xi| - \langle \nabla u_\infty(x), \xi\rangle$. Since $|\nabla u_\infty(x)| \leq 1$ for every $x \in \Omega$, this function is non-negative for every (x,ξ) and vanishes only on the set

$$Z = \{(x, \alpha\nabla u_\infty(x)) : |\nabla u_\infty(x)| = 1, \alpha > 0\}.$$

Using (3.21) (or Hölder inequality) in (3.32), and replacing together with (3.33) in (3.29), we obtain

$$\int_\Omega \left(\int h(x,\xi)\nu_x(d\xi)\right) dx + \int_\Omega \left(\int h(x,\xi)\nu_x^\infty(d\xi)\right)\sigma(dx) = 0 \tag{3.34}$$

which leads to

$$\int h(x,\xi)\nu_x(d\xi) = 0 \quad \mathcal{L} - a.e\ x \in \Omega \tag{3.35}$$

$$\int h(x,\xi)\nu_x^\infty(d\xi) = 0 \quad \sigma - a.e\ x \in \Omega \tag{3.36}$$

since both terms in the zero sum are non-negative. This entails that, for $\mathcal{L} - a.e$ (resp. $\sigma - a.e$) $x \in \Omega$, ν_x (resp. ν_x^∞) is concentrated in $\{\alpha\nabla u_\infty, \alpha > 0\}$. Therefore, by the disintegration theorem (Dellacherie and Meyer, 1978; Chang and Pollard, 1997) there exist measures $(\tilde{\nu}_x)_{x\in\Omega}$ on $\mathbb{R}_+$ such that for any continuous function F and $\mathcal{L} - a.e$ x in Ω,

$$\int F(x,\xi)\nu_x(d\xi) = \int F(x, \alpha\nabla u_\infty)\tilde{\nu}_x(d\alpha).$$

Let $\lambda(x) = \int \alpha\tilde{\nu}_x(d\alpha)$. Then if F satisfies (3.31a) we have

$$\int F(x,\xi)\nu_x(d\xi) = \lambda(x)F(x, \nabla u_\infty), \tag{3.37}$$

and, by an analogous reasoning

$$\int F(x,\xi)\nu_x^\infty(d\xi) = \eta(x)F(x, \nabla u_\infty) \tag{3.38}$$

for a measurable non negative function $\eta(x)$.

Now by (3.35), (3.36), (3.37) and (3.38) we obtain, usign that h satisfies (3.31a)

$$\lambda(x)|\nabla u_\infty(x)| = \lambda(x)|\nabla u_\infty(x)|^2 \quad \text{and} \quad \eta(x)|\nabla u_\infty(x)| = \eta(x)|\nabla u_\infty(x)|^2,$$

$\mathcal{L} - a.e$ and $\sigma - a.e$ respectively, proving the complementarity property (3.27) and (3.28). Also, notice that (3.30) along with

$$\lambda(x) = \lambda(x)|\nabla u_\infty(x)| \quad \text{for } \mathcal{L} - a.e\ x \in \Omega \tag{3.39}$$

implies that $\lambda \in L^1(\Omega)$. Define μ according to (3.24). The Euler-Lagrange equation (3.25) is obtained taking limits in (3.19). Both terms involving f converge by dominated convergence. For the remaining term, we replace u_∞ by a generic $v \in C_0^\infty(\Omega)$ in (3.33), then use (3.37), (3.27) and (3.28). Assertion (iii) is a direct consequence of Alibert and Bouchitté (1997, Theorem 2.9) using $F(x,\xi) = |\xi|$, (3.37) and (3.39). □

Remark 6.

a) *The decomposition* (3.24) *is not the canonical singular decomposition w.r.t* $\mathcal{L}$*; the measure* σ *may have a non-trivial density w.r.t.* $\mathcal{L}$*. The Radon measure* σ *captures the 'concentrating' part of the limit, and* λ *captures the 'uniformly integrable' part, which explains (iii).*

b) *Part (iii) of the Theorem has strong implications from the computational viewpoint. It essentially says that away from an arbitrarily small set we can approximate the uniformly integrable part of the multiplier in the same way as we do for* L^∞ *multipliers.*

4. Numerical Issues. In this section we develop the numerical aspects of our method. We focus mainly on the regular problems treated in Section 3.1, and then explore the extensions of Section 3.2 by relaxing some of the hypothesis on the domain.

A practical application of our method leads to solving the quasilinear elliptic equations (3.3) for large values of p. Such problems have received a good deal of attention from the numerical analysis community since they capture the essential complexity of nonlinear, possibly degenerate, problems. For such nonlinear problems, the finite element method cannot be directly applied; the use of an iterative procedure is necessary. However, for large p the convergence and stability of such an iterative procedure is a delicate issue.

Let us consider the variational formulation for the p-Laplacian problem,

$$\min_{v \in H_0^1(\Omega)} \int_\Omega (|\nabla v|^p - hv) dx \tag{4.1}$$

The finite element approximation for this problem has been studied mainly by Ciarlet (1978); Barrett and Liu (1993); Bermejo and Infante (2000) and Huang et al. (2007), the later focused on the behaviour of the algorithms for large p. In this context, the steepest descent direction w^S for the discretization of Problem (4.1) at a point u is computed by solving the system

$$\int_\Omega \nabla w^S \nabla v = -\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla v \rangle + \int_\Omega hv \qquad \forall v \in \mathcal{V}^h, \tag{4.2}$$

where $\mathcal{V}^h$ stands for the finite element space under consideration. System (4.2) becomes very ill-conditioned for large values of p, for that reason Huang et al. (2007) proposed to use descent directions w^Q computed from the preconditioned system

$$\int_\Omega (\tau + |\nabla u|^{p-2}) \nabla w^Q \nabla v = -\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla v \rangle + \int_\Omega hv \qquad \forall v \in \mathcal{V}^h, \tag{4.3}$$

for some $\tau > 0$ intended to handle the degeneracy when $\nabla u = 0$. Notice that the preconditioner term in (4.3) coincides with the approximating multiplier. The convergence of a descent algorithm for the pure p-Laplacian problem using directions w^Q and exact line searches is proved in Huang et al. (2007). As a matter of fact, note that the previous algorithm fits into the category of *Quasi-Newton* methods, therefore a speedup with respect to steepest descent from linear to superlinear convergence rate is expected under appropriate conditions (Nocedal and Wright, 2006). The generalization of the previous idea to problems (3.2) is straightforward and leads to Algorithm 1, whose convergence is proved along the lines of Huang et al. (2007, Theorem 1).

Most of the computing time of Algorithm 1 is spent on step 2. The burden of computing the descent direction w^Q is comparable to that of computing the Newton direction, which guarantees quadratic convergence. The degeneracy of the p-Laplacian system at critical points have precluded the application of Newton's method to this problem. Moreover, for p in the medium-to-large range the term $|\nabla u|^{p-2}$ is close to zero not only at critical points, but also at any point such that $|\nabla u| < 1$. Nonetheless, for the problems under consideration the ellipticity condition (1.15) prevents the

Algorithm 1 Preconditioned algorithm for solving Problem (3.2) on a given mesh and for a fixed p.

Given $p > 2$ and an initial point $u_{p,0} \in \mathcal{V}^h$, choose c_1, ε.
Set $n := 0$ and iterate:

1. Compute the multiplier $\lambda_{p,n} = |\nabla u_{p,n}|^{p-2}$.
2. Find the descent direction w_n^Q by solving

$$\int_\Omega (W'(|\nabla u_{p,n}|^2) + \lambda_{p,n}) \nabla w_n^Q \nabla v = - \int_\Omega (W'(|\nabla u_{p,n}|^2) + \lambda_{p,n}) \nabla u_{p,n} \nabla v + \int_\Omega f v \quad \forall v \in \mathcal{V}^h \tag{4.4}$$

3. Perform a line-search with sufficient decrease condition, *i.e*,
find $\alpha_n > 0$ satisfying $J_p(u_{p,n} + \alpha_n w_n^Q) \leq J_p(u_{p,n}) + c_1 \alpha_n J'_p(u_{p,n})[w_n^Q]$
4. Set $u_{p,n+1} = u_{p,n} + \alpha_n w_n^Q$.
5. If $\|J'_p(u_{p,n+1})\| \leq \varepsilon$, stop. Otherwise update $n = n + 1$ and go to step 1.

Newton system from becoming singular at non-critical points. Taking advantage of this fact we propose to use the full Newton direction w^N, computed as

$$\int_\Omega (G(|\nabla u_{p,n}|^2) + (p-1)|\nabla u_{p,n}|^{p-2}) \nabla w_n^N \nabla v = - \int_\Omega (W'(|\nabla u_{p,n}|^2) + |\nabla u_{p,n}|^{p-2}) \nabla u_{p,n} \nabla v + \int_\Omega f v \quad \forall v \in \mathcal{V}^h, \tag{4.5}$$

where the function G is defined in (1.15). If the function G is such that $G(0) = 0$, it can be replaced in practice by $\tau + G$ or $\max\{G, \tau\}$ for some $\tau > 0$ small. For $G \equiv 0$ we recover the p-Laplacian problem. Note that in this case (4.5) differs from (4.4) by the term $(p-1)$, showing that the directions w^Q are not well scaled. This explains the fact that the unit step-length is never accepted in Algorithm 1 for p-Laplacian (Huang et al., 2007, Figure 8), obstructing the achievement of superior convergence rates and increasing the time spent in line searches.

Algorithm 2 Newton algorithm for solving Problem (3.2) on a given mesh and for a fixed p.

Given $p > 2$ and an initial point $u_{p,0} \in \mathcal{V}^h$, choose c_1, ε.
Set $n := 0$ and iterate:

1. Compute the multiplier $\lambda_{p,n} = |\nabla u_{p,n}|^{p-2}$.
2. Find the descent direction w_n^N by solving (4.5).
3. Perform a line-search with sufficient decrease condition, *i.e*,
find $\alpha_n > 0$ satisfying $J_p(u_{p,n} + \alpha_n w_n^N) \leq J_p(u_{p,n}) + c_1 \alpha_n J'_p(u_{p,n})[w_n^N]$
4. Set $u_{p,n+1} = u_{p,n} + \alpha_n w_n^N$.
5. If $\|J'_p(u_{p,n+1})\| \leq \varepsilon$, stop. Otherwise update $n = n + 1$ and go to step 1.

Algorithms 1 and 2 can be greatly improved by using adaptive mesh refinements

(Gago et al., 1983). Also, they can be sensitive with respect to the initial point, particularly for large p and/or when a non-homogeneous boundary condition is given. A possible way to get around this difficulty is to adopt a path-following strategy, which consists in running initially the algorithm on a coarse mesh and for a low value of p, and then increasing p and adaptively refining the mesh until reaching a target p. Some extra mesh refinements can be eventually performed once the target p has been achieved.

4.1. Results. We report on a numerical study conducted in order to validate our method and evaluate the algorithms introduced in this section. We solve numerically the elastoplastic torsion problem in a variety of domains. The problem

$$\min \left\{ J(u) := \frac{1}{2}\int |\nabla u|^2 - \int hu \;\middle|\; \begin{array}{c} |\nabla u| \leq 1 \; a.e \; in \; \Omega \\ u = g \text{ on } \partial\Omega \end{array} \right\} \tag{4.6}$$

is approximated by the sequence of unconstrained problems

$$\min \left\{ J_p(u) := \frac{1}{2}\int |\nabla u|^2 + \frac{1}{p}\int |\nabla u|^p - \int hu \;\middle|\; u = g \text{ on } \partial\Omega \right\} \tag{4.7}$$

which possess an unique regular solution. Let $\mathcal{V}_h$ stand for the space of continuous functions whose restriction to any element of a regular mesh of Ω is polynomial of degree 1 or 2. We implemented Algorithm 1 and Algorithm 2 with directions w^Q and w^N respectively in `C++` using the `deal.II` finite elements library (Bangerth et al., 2007), `v.8.0`. The line searches were performed using a quadratic interpolation algorithm implemented by ourselves. The adaptive refinements proceed by refining a percentage of the cells with the highest *a posteriori* gradient approximation error according to the estimator by Kelly et al. (1983) (see also Ainsworth and Oden, 1997) provided by the `deal.II` library. The descent directions are computed solving the systems by the conjugate gradient algorithm with a SSOR preconditioner.

Denote by D the unit disk of $\mathbb{R}^2$, i.e $D = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$. When $\Omega = D$ and h is constant, (4.6) has an explicit solution. If $h \equiv 4$ and $g \equiv 0$ the solution is given by (Glowinski et al., 1981):

$$u(x) = \begin{cases} 1 - r & if \; 1/2 \leq r \leq 1 \\ -r^2 + 3/4 & if \; 0 \leq r \leq 1/2 \end{cases} \tag{4.8}$$

where $r = \sqrt{x^2 + y^2}$. Since Ω is convex, in this case the multiplier λ is continuous (Brézis, 1972). In fact we obtained its explicit expression, which is given by

$$\lambda(x) = \begin{cases} 2r - 1 & if \; 1/2 \leq r \leq 1 \\ 0 & if \; 0 \leq r \leq 1/2. \end{cases} \tag{4.9}$$

The norm of the gradient of the computed solution and the multiplier are plot in Figure 4.1. In Table 4.1 we show the error with respect to the explicit solutions of the approximations computed by Algorithm 2 using the Newton directions w^N and finite elements of order 2.

The error on the primal solution is reported in the norms of the spaces $L^2(\Omega)$, $H^1(\Omega)$ and in the norm of $W_0^{1,\infty}(\Omega)$, understood as the $L^\infty(\Omega)$ norm error of the gradients of the solution. The error on the dual variable is reported in the norms of $L^1(\Omega)$ and $L^\infty(\Omega)$. For each p in $\{100, 200, 300, 400, 500\}$ we include the results on

Table 4.1

Error of u_p and λ_p with respect to the respective primal and dual analytical solutions of the limit problem given in (4.8) *and* (4.9) *in various norms, computed by Algorithm 2 with directions w^N and finite elements of order 2.*

	Mesh info		Primal error			Dual error	
p	#cells	#dofs	L^2	H^1	$W_0^{1,\infty}$	L^1	L^∞
	79712	353761	4.988e-03	1.705e-02	3.366e-02	2.543e-02	3.528e-02
100	202688	898897	4.988e-03	1.705e-02	3.366e-02	2.542e-02	3.499e-02
	324344	1433817	4.988e-03	1.705e-02	3.366e-02	2.542e-02	3.493e-02
	80264	354145	2.507e-03	8.808e-03	1.957e-02	1.272e-02	2.050e-02
200	205244	921313	2.507e-03	8.808e-03	1.957e-02	1.270e-02	2.012e-02
	328466	1480297	2.507e-03	8.808e-03	1.957e-02	1.270e-02	2.003e-02
	80024	344289	1.675e-03	5.956e-03	1.416e-02	8.623e-03	3.496e-02
300	203924	890625	1.674e-03	5.955e-03	1.414e-02	8.492e-03	1.463e-02
	326264	1403777	1.674e-03	5.956e-03	1.414e-02	8.489e-03	1.445e-02
	79940	352017	1.257e-03	4.504e-03	1.122e-02	6.413e-03	3.916e-02
400	205244	928009	1.257e-03	4.504e-03	1.120e-02	6.367e-03	2.703e-02
	328484	1504625	1.257e-03	4.504e-03	1.120e-02	6.361e-03	2.188e-02
	80036	354385	1.006e-03	3.624e-03	9.364e-03	5.170e-03	4.738e-02
500	205412	931689	1.006e-03	3.623e-03	9.330e-03	5.108e-03	2.808e-02
	328700	1509097	1.006e-03	3.623e-03	9.319e-03	5.100e-03	2.381e-02

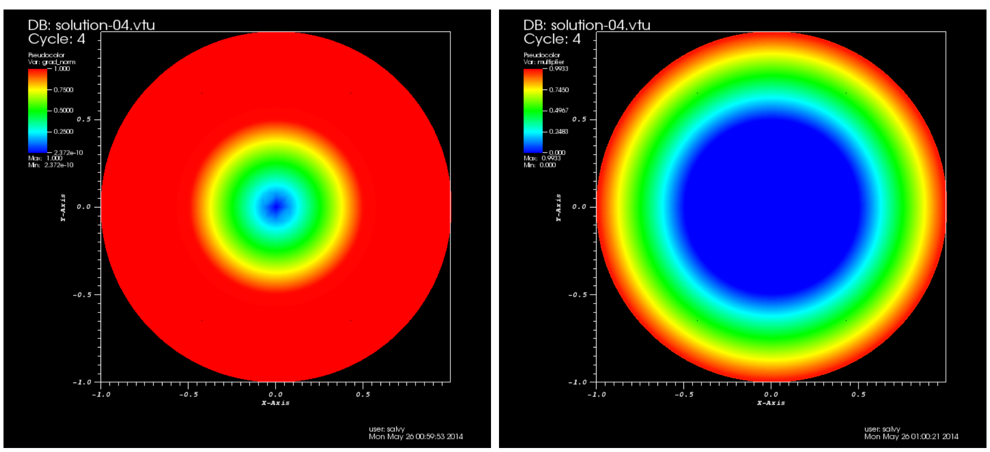
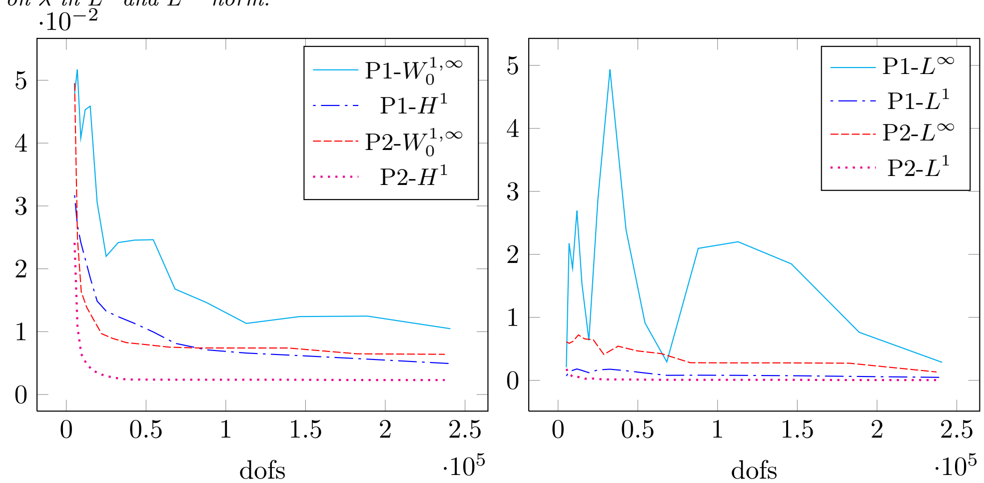


Fig. 4.1. *Plot of the norm of the gradient $|\nabla u_p|$ and the multiplier $\lambda_p = |\nabla u_p|^{p-2}$ on a circle.*

three adaptively refined meshes. For p in the range of a few hundreds the approximation of the primal solution is already satisfactory. Moreover, the error decreases each time p is increased or the mesh is refined, the former effect being more pronounced. The L^1 approximation error for the multiplier exhibits a similar behaviour, improving mostly with the increase of the penalty parameter. On the contrary, the L^∞ error is more sensitive with respect to the mesh, and can even worsen if p increases too much.

For comparison we plot in Figure 4.2 the approximation error for both primal and dual solution using Algorithm 2 and finite elements of order 1 and 2. For the comparison to make sense the error is plot in terms of the number of degrees of

Fig. 4.2. *Comparison of error evolution in terms of degrees of freedom for finite elements or order 1 (P1) and order 2 (P2). At the left, error on u in H^1 and $W_0^{1,\infty}$ norms; at the right error on λ in L^1 and L^∞ norm.*



freedom, since using higher order finite elements increases the size of the linear systems to solve. There is an advantage in computing primal solutions using order 2 elements, which achieve a lower approximation error for any given number of degrees of freedom. The evidence supporting the use of order 2 elements is much stronger looking at the error at computing the multiplier, which is quite sensitive with respect to the quality of the approximation of the gradients. The better approximation achieved using order 2 elements results in an increased stability of the method.

We solve the problem in different domains to confirm our intuition about the extensibility of Theorem 3.3 to more general situations. In Figures 4.3 and 4.4 we show the solutions of Problem 4.6 in a rectangle and a domain with an interior corner, respectively. The primal solution is well approximated in all the considered domains, which is consistent with the results of Section 2.

We also plot the approximate multipliers. It is seen that in the rectangle, a convex domain with piecewise smooth border, we are still able to compute satisfactorily both the solution and the multiplier. The gradients are uniformly bounded, and the multiplier belongs to $L^\infty(\Omega)$.

In the piecewise smooth nonconvex domain, even if the are able to compute the solution with a good accuracy, it is not enough to have the multiplier uniformly bounded. The difficulty relies on the concentration effect occuring near the interior corners. However, the plot with a truncated scale shows that away from the concentrations we compute the right multiplier, as anticipated by the bitting convergence result of Theorem 3.8.

5. Summary. We have presented a complete study of an approximation scheme for solving variational problems under uniform constraints on the gradient. We prove the existence of solutions and Lagrange multipliers under very general assumptions, and existence of Lagrange multipliers in L^∞ without requiring constraint qualification conditions. The numerical study confirms the applicability of our method. Also, our analysis shed some light on certain algorithms for computing solutions for the p-Laplacian. To the best of our knowledge, this is the first time that Lagrange mul-

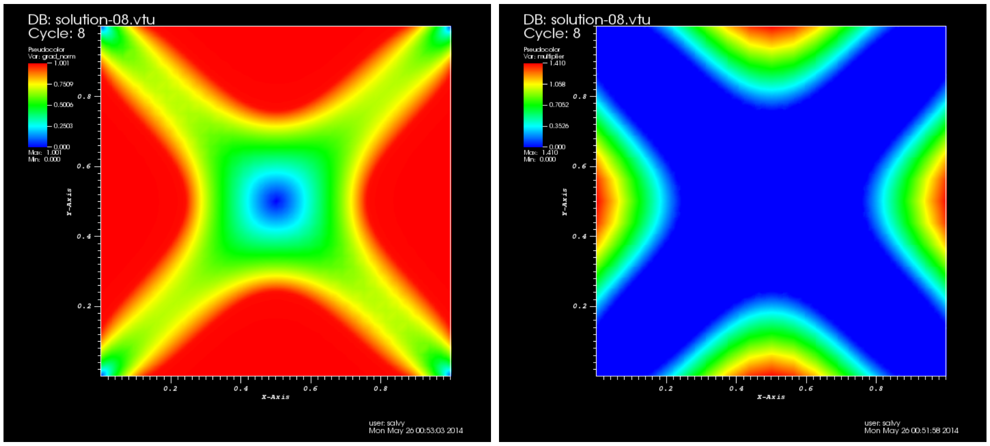


FIG. 4.3. *Plot of the norm of the gradient $|\nabla u_p|$ and the multiplier $\lambda_p = |\nabla u_p|^{p-2}$ on a rectangle.*

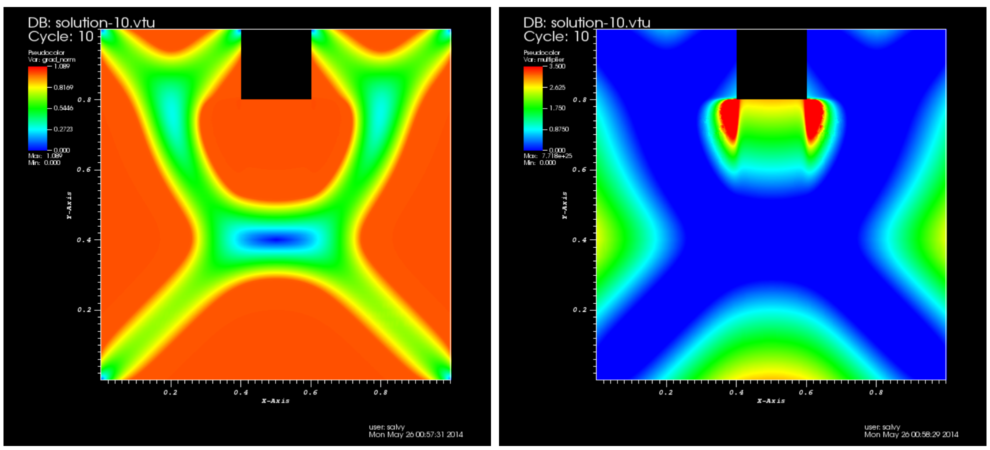


FIG. 4.4. *Plot of the norm of the gradient $|\nabla u_p|$ and the multiplier $\lambda_p = |\nabla u_p|^{p-2}$ on a domain with an interior corner. The scale in the plot of the multiplier is truncated.*

tipliers are computed for the considered class of problems.

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