université PARIS-SACLAY

Mémoire de Master 2

Mathématiques de l'Aléatoire

Hausdorff dimension of geodesic stars in random geometry

Pablo Zúñiga Rodríguez-Peña

Under the supervision of Armand Riera

Paris, France

Academic year 2024/2025

A veces no sé si estoy viviendo
la realidad, un sueño, ambas o ninguna

Acknowledgements

I feel like during this first year living in France, I went through all the possible emotions and feelings. But what is for sure is that I was never alone in this trip, and I want to dedicate a few words for those who were present in some manner during this time.

Quisiera comenzar agradeciendo enormemente a mi supervisor, Armand Riera, por su amabilidad, por la confianza que ha puesto en mí como estudiante de matemáticas, y por su apoyo absoluto en las gestiones de cara al doctorado. Armand me recibió muy bien desde el inicio, ofreciéndome temas de memoria muy interesantes y matemáticamente exigentes. A la par con este trabajo, me apoyó con toda su energía para conseguir una posición como doctorante, lo cual −después de una verdadera tormenta− resultó bien, y es hoy lo que para mí significa el comienzo de una importantísima etapa de mi vida. Gracias por todo aquello, y mis mejores deseos para el proyecto que hemos acordado. Je prolonge ce paragraphe pour remercier à M. Nicolas Broutin, qui sera également mon encadrant de thèse de doctorat.

Agradecer a Arie Wortsman y Pablo López, por ser protagonistas de mi llegada y adaptación a Francia, sin ustedes habría sido derechamente imposible. Gracias por su amistad, por cada momento, conversación y cerveza que compartimos, les debo montones y siento que me quedo corto incluso con estas palabras. Agradecer a Guido Tapia, quien fue mi primer amigo en el IMO y quien también ha estado presente durante todo este proceso, iluminaste el camino con tu simpatía y amabilidad. Agradecer a Ignacio Acevedo, por también ser súper apañador, motivado para las juntas y el tenis, y por la gran disposición que tuviste para ayudarme cuando lo necesité. Agradecer a Avelio Sepúlveda, quien me apoyó y ayudó muy activamente cuando las postulaciones no estaban resultando.

Quiero dedicarle unas palabras a Petra Zamorano, a quien conozco desde hace varios años y con quien he compartido muy lindos e importantes momentos de mi vida. Gracias por siempre estar ahí para apoyarme y escuchar mis inquietudes, pensamientos y sentimientos; son conversaciones de las que siempre me he llevado grandes palabras, consejos y calma. Estoy particularmente feliz por tenerte cerca en el preciso instante que termino mi primer año de estudios en Francia, es muy significativo considerando todo lo que caracteriza a nuestra amistad.

Agradecer a Pinhás Zamorano, quien también me ha ayudado mucho desde que llegué, y más particular y recientemente, con la obtención de mi nuevo departamento.

En general, quiero agradecer a todos mis amigos y conocidos, de Chile y de otras partes del mundo, que estando cerca o lejos de mí, me regalaron lindos momentos y apoyaron durante este primer año en Francia. Es verdaderamente un privilegio sentirse acompañado a pesar de estar tan lejos. Un gran saludo para todos ustedes.

I want to thank Simone Giancola, with whom I've shared lots of interesting conversations and jointly studied several times, instances where I significantly progressed in my understanding of the courses. He organized study sessions with a remarkable enthusiasm and responsibility. Thanks also for receiving me in Sicily and giving me the chance to share with your friends and family, they were amazing days there.

En général, je remercie à tous mes camarades du M2, pour me recevoir tellement bien et m'aider dans des plusieurs opportunités avec les cours. Je vous souhaite la meilleur continuation avec vos études, je suis sûr que vous serez des excellents mathématiciens et scientifiques. J'espère aussi que vous profitez d'une bonne lecture de ce document.

Je remercie à tout l'équipe de la Fondation Mathématique Jacques Hadamard pour le financement de ma première année d'études en France. Vous faites un travail exceptionnel dans le soutien aux étudiants de mathématiques. Je remercie à M. Nicolas Curien, pour être un professeur excellent, pour son travail exceptionnel comme responsable du M2 Mathématiques de l'Aléatoire, et pour son soutien à mes candidatures aux financements de thèse.

Quisiera agradecer a toda mi familia, esta memoria es de ustedes también. A pesar de la distancia, siento su cariño y apoyo todos los días, lo que significa para mí una gran fuente de energía para seguir adelante. Espero pronto estar con ustedes para contarles toda la experiencia que ha sido esto.

Finalmente, agradecer a la música por ambientar mi mundo, mis experiencias, mis momentos e inspirarme de esa manera; y agradecer a Colo-Colo, club al que amo y apoyo, que si bien ha tenido un año muy difícil, nada quitará su grandeza en términos deportivos, sociales e históricos. Vamos Colo-Colo.

Contents

Introduction

The Brownian sphere is a model for random surfaces with the topology of the two-dimensional sphere, chosen "uniformly at random". It appears as the scaling limit of a large class of random planar maps, all with the common feature that the faces of such maps disappear in the scaling limit. Despite its appearance as such limit, which gives us a way to construct it, we can also construct the Brownian sphere as a purely continuum object by using canonical stochastic processes (namely, Brownian excursions and Gaussian processes) that will rule its geometric properties. This gives a suitable framework to study many of its topological properties and observables.

Geodesics in the Brownian sphere are central in its geometric study. From its construction, one can see that the distances towards a typical point can always be traveled using a geodesic, so that in a sense, to understand geodesics means to understand distances in the Brownian sphere. They also provide information about how atypical points can be connected, and appear naturally in the discrete setting and in the scaling limit proofs. In this direction, a large amount of works have been released, giving us, at this point, a very complete understanding of geodesics in the Brownian sphere.

In this document, we present a survey on the Brownian sphere, including its main properties and some proofs (Chapter 1), and a review of the proof of the formula for the Hausdorff dimension of the so-called *geodesic m-stars*, that are points from which m disjoint geodesics emerge (Chapter 3). The lower bound of such Hausdorff dimension is due to Le Gall [7], where first and second moment estimates for an approximated version of the geodesic stars, along with standard potential theory techniques, give the desired result. The upper bound is due to Miller and Qian [15], where fine estimates for the exponent of the probability of geodesics towards a point near a typical point give the desired bound. A "warm-up" chapter on the Hausdorff dimension is included, where we review the main definitions and the proof of the formula for the Hausdorff dimension of the Brownian cone points. In Chapter 4, we introduce the *stable gasket and carpet*, a variant of the Brownian sphere that appears as the scaling limit of random planar maps with "large" faces. Finally, we sketch some ideas to compute the Hausdorff dimension of geodesic stars in this new object.

Notation index

Sets

- $\mathbb{N} = \{0, 1, 2, \dots\}$.
- $\mathbb{R}_+ = [0, \infty)$.
- For a metric space (E, d) and $A \subseteq E$, $\mathrm{diam}(A) = \sup\{d(x, y) : x, y \in A\}$ = diameter of A.

Measure theory and probability

- $\#A$ = counting measure of A.
- a.s. = almost surely.
- a.e. = almost everywhere.
- $\phi_* \mu = \mu \circ \phi^{-1}$ = pushforward measure (of μ by ϕ).

Real trees

- $\sigma(g) = \sup\{s > 0 : g(s) \neq 0\}$ = lifetime of g.
- d_g = pseudo-distance associated to g (1.1).
- $\mathcal{T}(g) = [0, \sigma(g)]/\{d_g = 0\}$ real tree associated to g.
- p_g = canonical projection from $[0, \sigma(g)]$ to $\mathcal{T}(g)$.
- $\rho_g = p_g(0)$ root of $\mathcal{T}(g)$.
- Vol_g = volume measure on $\mathcal{T}(g)$.
- d_{GHP} = Gromov-Hausdorff-Prokhorov distance (1.3).
- $\mathbb{K}^\bullet$ = equivalence classes of isometric compact pointed measure metric spaces.

- $\mathbb{K}^{\bullet\bullet}$ = equivalence classes of isometric compact two-pointed measure metric spaces.

- $\mathbb{K}^{\bullet b}$ = equivalence classes of isometric compact measure metric spaces with distinguished point and boundary.

- $\mathbb{K}^{bb}$ = equivalence classes of isometric compact measure metric spaces with two distinguished boundaries.

Brownian geometry

- ζ = lifetime function of the Brownian snake.

- $\widehat{W}$ = tip function of the Brownian snake.

- W_* = global minima of the Brownian snake (1.5).

- $\mathcal{S}_x$ = snake trajectories with initial value x (Definition 1.7).

- $\mathbb{N}_x$ = Brownian snake measure on $\mathcal{S}_x$.

- $\mathbb{N}_x^{(s)} = \mathbb{N}_x(\,\cdot\,|\sigma = s)$.

- $\mathbb{N}_x^{[r]} = \mathbb{N}_x(\,\cdot\,|W_* < -r)$.

- $\mathbb{N}_x^{\{a\}} = \mathbb{N}_x(\,\cdot\,|W_* = -a)$.

- $\mathbf{m}_\infty$ = Brownian sphere (Definition 1.14).

- Vol = volume measure on $\mathbf{m}_\infty$.

- $B^{\bullet(y)}(x,r)$ = complement of the connected component of $\mathbf{m}_\infty \setminus B(x,r)$ not containing y (Definition 1.16).

- $\mathcal{P}$ = Brownian plane.

- $\mathfrak{H}_{r,z}$ = hull of radius r and perimeter z.

- $\mathbb{D}_z$ = Brownian disk of perimeter z.

Other

- $\lesssim$ denotes inequality up to a positive constant that does not depend on relevant variables (context is always important for this).

- $x \asymp y$ means that $x \lesssim y$ and $y \lesssim x$ simultaneously.

Chapter 1

Brownian sphere

The Brownian sphere is a model for geodesic metric spaces with the topology of the two-dimensional sphere, chosen "uniformly at random". It is a universal object in the sense that it appears as the scaling limit of a large class of models of random planar maps with small faces, such as uniform triangulations and quadrangulations.

In this chapter, we introduce the Brownian sphere and state its main properties, with special focus on those that will allow us to study the set of geodesic m-stars, which are points where m disjoint geodesics emerge. We will not comment on how the Brownian sphere appears as the scaling limit of random planar maps, but only focus on its construction and features as a continuum object.

1.1 Construction

Let us start with a deterministic construction of real trees. After choosing an appropriate randomness on them, and gluing its points according to a certain Gaussian process, they give raise to the Brownian sphere.

1.1.1 Real trees

The notion of real trees allows us to extrapolate the abstract graph structure of a tree to a continuum object, where the edges are replaced by continuum line segments (with some length). Furthermore, from the construction it is easy to endow real trees with both a distance and volume measure. Formally, a real tree is defined as follows.

Definition 1.1. *A* ***real tree*** $(\mathcal{T}, d)$ *is a compact metric space such that for all* $a, b \in \mathcal{T}$*,*

1. *there is a unique isometry* $\phi : [0, d(a, b)] \to \mathcal{T}$ *such that* $\phi(0) = a$ *and* $\phi(d(a, b)) = b$*, and*

2. *if* $q : [0, 1] \to \mathcal{T}$ *is injective with* $q(0) = a$ *and* $q(1) = b$*, then* $q([0, 1]) = \phi([0, d(a, b)])$*.*

Let us now construct deterministic real trees. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous positive function, such that $g(0) = 0$ and $\sigma(g) := \sup\{s > 0 : g(s) \neq 0\} \in (0, \infty)$. Such a g is called an *excursion* and $\sigma(g)$ is its *lifetime*. For all $s, t \in [0, \sigma(g)]$, define

$$d_g(s,t) := g(s) + g(t) - 2 \min_{[s\wedge t, s\vee t]} g. \tag{1.1}$$

Proposition 1.2. *The function d_g defines a pseudo-distance in $[0, \sigma(g)]$.*

Proof. Write $\sigma = \sigma(g)$. First, for all $s, t \in [0, \sigma]$,

$$d_g(s,t) = g(s) - \min_{[s\wedge t, s\vee t]} g + g(t) - \min_{[s\wedge t, s\vee t]} g \geq g(s) - g(s) + g(t) - g(t) = 0.$$

For the triangle inequality, if $s, t, r \in [0, \sigma]$ are such that $s < r < t$, we have that

$$d_g(s,t) + d_g(r,t) - d_g(s,r) = 2\left(g(t) - \min_{[s,t]} g - \min_{[r,t]} g + \min_{[s,r]} g\right) \geq 2\left(g(t) - \min_{[r,t]} g\right) \geq 0,$$

where we used that $[s,r] \subseteq [s,t]$ implies $\min_{[s,r]} g \geq \min_{[s,t]} g$, so that $d_g(s,r) \leq d_g(s,t) + d_g(r,t)$. The inequality $d_g(r,t) \leq d_g(s,r) + d_g(s,t)$ is obtained analogously. Similarly, we have

$$d_g(s,r) + d_g(r,t) - d_g(s,t) = 2\left(g(r) - \min_{[s,r]} g - \min_{[r,t]} g + \min_{[s,t]} g\right).$$

Note that $\min_{[s,t]} g = \min_{[s,r]} g$ or $\min_{[s,t]} g = \min_{[r,t]} g$. In any case, we see that the quantity above is positive, so that $d_g(s,t) \leq d_g(s,r) + d_g(r,t)$. □

Evidently, $d_g(s,s) = 0$, but $d_g(s,t) = 0$ does not imply $s = t$. Instead, note that $d_g(s,t) = 0$ is equivalent to $g(s) = g(t) = \min_{[s\wedge t, s\vee t]} g$. If we define the relation $\sim$ on $[0, \sigma(g)]$ by

$$s \sim t \quad \text{if, and only if} \quad g(s) = g(t) = \min_{[s\wedge t, s\vee t]} g,$$

then the quotient space $\mathcal{T}(g) := [0, \sigma(g)]/\sim$ with the distance defined by

$$\widetilde{d}_g(p_g(s), p_g(t)) := d_g(s,t), \qquad \text{for all } s, t \in [0, \sigma(g)], \tag{1.2}$$

is a compact metric space, where $p_g : [0, \sigma(g)] \to \mathcal{T}(g)$ is the canonical projection.[1] Moreover, $(\mathcal{T}(g), d_g)$ is a real tree in the sense of Definition 1.1. Geometrically, this quotient space means that we glue points of the graph of g whenever there is a positive excursion of g above them, see Figure 1.1. Define the *root* of $\mathcal{T}(g)$ by $\rho_g := p_g(0)$ and the *volume measure on* $\mathcal{T}(g)$, denote by Vol_g, as the pushforward of the Lebesgue measure on $[0, \sigma(g)]$ under p_g.

In the literature, when a tree is given (in the real or pure graph sense) with a root, the function recording the distances to the root during a clockwise exploration of it is called *contour function.* In our construction, g is the contour function of $(\mathcal{T}(g), d_g)$.

[1] For all $s \in [0, \sigma(g)]$, $p_g(s)$ is the equivalence class of s for $\sim$.

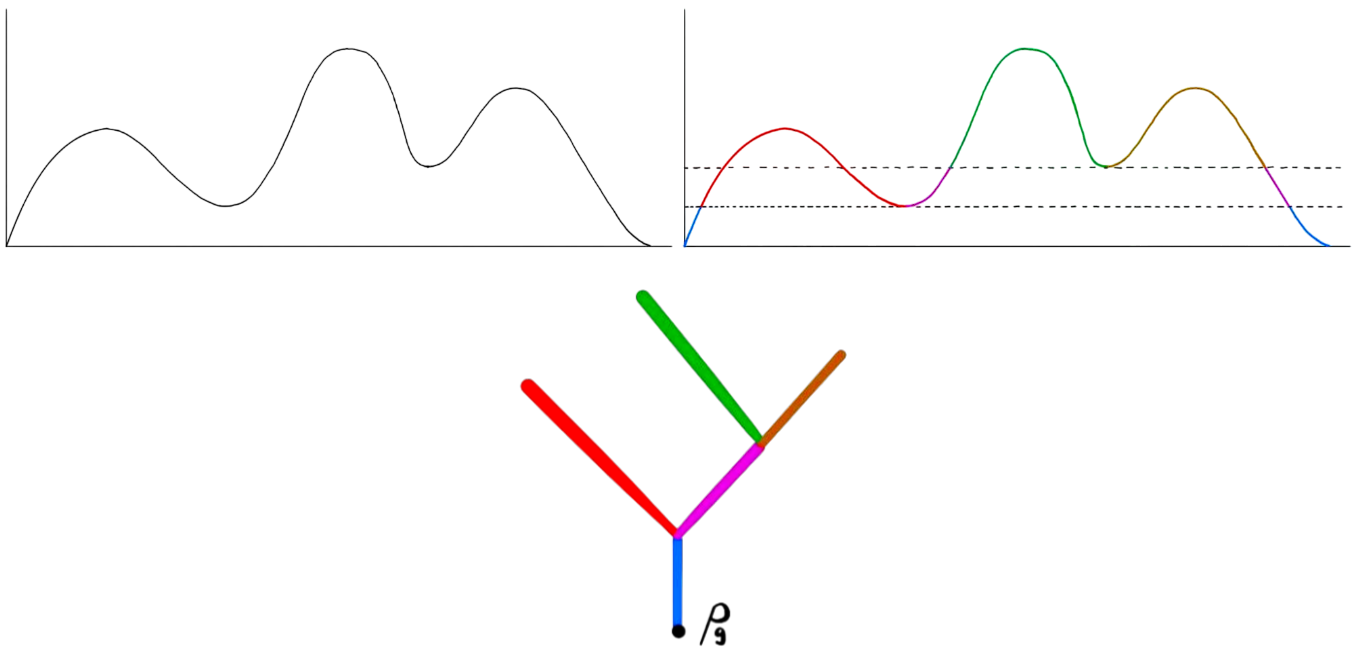

Figure 1.1: In the first picture, an example of a continuous positive function g is drawn in black. In the second picture, the dotted lines are tangent to the graph of g in the local minima. This makes easier to see how we glue the points on the graph of g, according to the identification through the relation $\sim$. In the third picture we see the representation of $\mathcal{T}(g)$ as a real tree.

For notational simplicity, we will never make explicit reference to the equivalence classes of $\sim$, and we will simply denote $\widetilde{d}_g$ by d_g. The context will always make clear when we are working with elements of $[0, \sigma(g)]$ or $\mathcal{T}(g)$. Also, when quotienting spaces by a pseudo-distance like before, we will use the notation $\mathcal{T}(g) = [0, \sigma(g)]/\{d_g = 0\}$.

The previous construction defines a map $g \mapsto (\mathcal{T}(g), d_g, \mathrm{Vol}_g, \rho_g)$ from the set of excursions to the space of 4-tuples (X, d, μ, x_*), where (X, d) is a compact metric space, μ is a Borel measure on X and x_* is a point of X, referred to as the *distinguished point* of X. Each such (X, d, μ, x_*) is called *compact pointed measure metric space*. We can give a meaning to the continuity of this mapping by defining a distance on the latter space. This is the so-called *Gromov-Hausdorff-Prokhorov* distance: if (X, d, μ, x_*) and (Y, ρ, ν, y_*) are compact pointed measure metric spaces, then

$$d_{\mathrm{GHP}}((X, d, \mu, x_*), (Y, \rho, \nu, y_*)) := \inf_{\phi, \phi', E} d_{\mathrm{H}}(\phi(X), \phi'(Y)) \vee d_{\mathrm{P}}(\phi_*\mu, \phi'_*\nu) \vee d_E(\phi(x_*), \phi'(y_*)), \quad (1.3)$$

where the infimum is taken over all the isometric embeddings $\phi : X \to E$ and $\phi' : Y \to E$ for some common Polish metric space (E, d_E), and $\phi_*\mu = \mu \circ \phi^{-1}$ denotes the pushforward measure. Here, d_{H} is the *Hausdorff distance*,

$$d_{\mathrm{H}}(A, B) := \inf\{\varepsilon > 0 : A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\}, \quad \text{for } A, B \subseteq E,$$

and d_{P} is the *Prokhorov distance*, that is, for all μ_1 and μ_2 finite Borel measures on (E, d_E),

$$d_{\mathrm{P}}(\mu_1, \mu_2) := \inf\{\varepsilon > 0 : \mu_1(A_\varepsilon) \leq \mu_2(A) + \varepsilon \text{ and } \mu_2(A_\varepsilon) \leq \mu_1(A) + \varepsilon \text{ for all } A \text{ measurable}\},$$

where $A_\varepsilon := \bigcup_{x \in A} B(x, \varepsilon)$ is the ε-neighborhood of A.

Note, however, that $d_{\mathrm{GHP}}((X, d, \mu, x_*), (Y, \rho, \nu, y_*)) = 0$ does not imply that (X, d, μ, x_*) and (Y, ρ, ν, y_*) are the same, but only isometric in the sense that there is an isometry $\Phi : X \to Y$ such that $\Phi(x_*) = y_*$ and $\Phi_*\mu = \nu$. Therefore, d_{GHP} is only a pseudo-distance on the compact pointed measure metric spaces, and we define $\mathbb{K}^\bullet$ as the quotient of the latter space for the relation $\{d_{\mathrm{GHP}} = 0\}$. We endow $\mathbb{K}^\bullet$ with the distance induced by d_{GHP} (as in (1.2)), so that $(\mathbb{K}^\bullet, d_{\mathrm{GHP}})$ is a Polish metric space (see Theorem 2.3 in [2]).

In this setting, the following continuity property for real trees holds, which is key in order to add randomness in the function g of the previous construction.

Proposition 1.3 (Proposition 3.3 in [2]). *For all $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ excursions, we have that*

$$d_{\mathrm{GHP}}((\mathcal{T}(g), d_g, \mathrm{Vol}_g, \rho_g), (\mathcal{T}(h), d_h, \mathrm{Vol}_h, \rho_h)) \leq 6||g - h||_\infty + |\sigma(g) - \sigma(h)|.$$

Let us end this deterministic construction with the definition of intervals in $\mathcal{T}(g)$. For $s, t \in [0, \sigma(g)]$ such that $s < t$, we use the convention $[t, s] := [0, s] \cup [t, \sigma(g)]$ (and $[s, t]$ is the usual interval). Recall that $p_g : [0, \sigma(g)] \to \mathcal{T}(g)$ denotes the canonical projection. For any $a, b \in \mathcal{T}(g)$, we can find $s, t \in [0, \sigma(g)]$ such that $p_g(s) = a$, $p_g(t) = b$ and $[s, t]$ has minimal length. In this case, we define

$$[a, b] := p_g([s, t]).$$

Note that $[a, b]$ and $[b, a]$ are different. The notation for the interval in a real trees will not change, since the context will always make clear when we are working with elements in $[0, \sigma(g)]$ or $\mathcal{T}(g)$.

1.1.2 Labeling a real tree

Let us now describe how can we assign (random) labels to each point in $\mathcal{T}(g)$ for a fixed excursion g, in such a way that the stochastic process that is seen when exploring a branch is a Brownian motion. It turns out that the function

$$m_g(s, t) := \min_{[s \wedge t, s \vee t]} g, \tag{1.4}$$

defined for all $s, t \in [0, \sigma(g)]$, which appears in the definition of d_g (1.1), has all the properties of a covariance function.

Proposition 1.4. *The function $m_g(\cdot,\cdot)$ is symmetric and positive-definite, that is, for all $n \in \mathbb{N}$, $s_1, \dots, s_n \in [0, \sigma(g)]$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, we have*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j m_g(s_i, s_j) \geq 0.$$

Consequently, there exists a centered Gaussian process $Z = (Z_t)_{t\in[0,\sigma(g)]}$ with covariance $m_g(\cdot,\cdot)$. In particular, for all $s, t \in [0, \sigma(g)]$ we have that

$$\mathbb{E}[(Z_s - Z_t)^2] = d_g(s,t).$$

Proof. Let $t \geq 0$. Define the relation $\sim$ on $\{i \in \{1, \dots, n\} : g(s_i) \geq t\}$ as

$$i \sim j \iff m_g(s_i, s_j) \geq t.$$

It is easy to verify that $\sim$ defines an equivalence relation on $\{i \in \{1, \dots, n\} : g(s_i) \geq t\}$. Then,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j \mathbb{1}_{\{t \leq m_g(s_i,s_j)\}} = \sum_{\mathscr{C}} \left(\sum_{i \in \mathscr{C}} \lambda_i^2 \right) \geq 0.$$

where the sum is over the equivalence classes $\mathscr{C}$ of $\sim$. Integrating on t yields the first statement. By Kolmogorov's extension theorem, there exists a centered Gaussian process $Z = (Z_t)_{t\in[0,\sigma(g)]}$ such that for all $s, t \in [0, \sigma(g)]$, we have $\mathbb{E}[Z_s Z_t] = m_g(s,t)$. In particular, a direct computation gives

$$\mathbb{E}[|Z_s - Z_t|^2] = \mathbb{E}[(Z_s)^2] + \mathbb{E}[(Z_t)^2] - 2\mathbb{E}[Z_s Z_t] = g(s) + g(t) - 2 \min_{[s\wedge t, s\vee t]} g = d_g(s,t).$$

□

Remark 1.5. *If g is β-Hölder continuous, then the associated process Z satisfies*

$$\mathbb{E}[|Z_s - Z_t|^2] = g(s) + g(t) - 2 \min_{[s\wedge t, s\vee t]} g \lesssim |s-t|^{\beta}.$$

More generally, using that $(Z_s - Z_t)d_g(s,t)^{-1/2}$ is normally distributed with mean 0 and variance 1, we have that for all integers $p \geq 1$,

$$\mathbb{E}[|Z_s - Z_t|^{2p}] = C(p) d_g(s,t)^p \lesssim |s-t|^{\beta p},$$

where $C(p)$ is the p-th moment of a Normal random variable with mean 0 and variance 1. By Kolmogorov's continuity theorem, Z has a continuous modification with $(\beta p - 1)/(2p)$-Hölder continuous sample paths with full probability, for all $p \geq 1$. Passing to the limit $p \to \infty$, we conclude that for all $\varepsilon \in (0, \beta/2)$, Z has $(\beta/2 - \varepsilon)$-Hölder continuous sample paths with full probability.

Note that we can see Z as a process indexed by $\mathcal{T}(g)$. In fact, the condition $d_g(s,t) = 0$ is equivalent to $Z_s = Z_t$ thanks to the last statement in the previous proposition. Using this, we introduce the labels on $\mathcal{T}(g)$ as follows.

Definition 1.6. *For each $a \in \mathcal{T}(g)$, denote $\ell_a := Z_s$, where $s \in [0, \sigma(g)]$ is any number such that $p_g(s) = a$. The number ℓ_a is called **label** of a.*

1.1.3 Snake trajectories

In this section, we introduce the formalism of the snake trajectories, which gives a suitable encoding of the Brownian sphere. Roughly speaking, the pairs of the form (g, Z) constructed in the previous section can be coded using a path-valued stochastic process, whose observables determine the geometric properties of the Brownian sphere. However, we stay at the deterministic level for the moment, the randomness will be added in the next section.

Let $\mathcal{W}$ be the space of all continuous functions of the form $\mathrm{w} : [0, \zeta] \to \mathbb{R}$ with $0 \leq \zeta < \infty$. For each such w, $\zeta = \zeta(\mathrm{w})$ is called *lifetime of* w and $\widehat{\mathrm{w}} := \mathrm{w}(\zeta)$ is called the *tip of* w. For each $x \in \mathbb{R}$, let $\mathcal{W}_x := \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$. Define $d_{\mathcal{W}}$ for all $\mathrm{w}, \mathrm{w}' \in \mathcal{W}$ by

$$d_{\mathcal{W}}(\mathrm{w}, \mathrm{w}') := \sup_{s \geq 0} |\mathrm{w}(s \wedge \zeta(\mathrm{w})) - \mathrm{w}'(s \wedge \zeta(\mathrm{w}'))| + |\zeta(\mathrm{w}) - \zeta(\mathrm{w}')|.$$

Then, $d_{\mathcal{W}}$ defines a distance on $\mathcal{W}$, and furthermore $(\mathcal{W}, d_{\mathcal{W}})$ is a Polish metric space. We are ready to formally define the snake trajectories.

Definition 1.7. *Let $x \in \mathbb{R}$. A* ***snake trajectory with initial point*** *x is a mapping $\omega = (\omega_s)_{s \geq 0}$ from $[0, \infty)$ to $\mathcal{W}_x$ such that the two following properties hold:*

- *$\omega_0 = x$ is the constant function, and $\sigma(\omega) := \sup\{s \geq 0 : \omega_s \neq x\}$ is finite.*
- *Snake property: For all $s, t \geq 0$ such that $s \leq t$, $\omega_s(r) = \omega_t(r)$ for all $r \in [0, \min_{[s,t]} \zeta(\omega_\cdot)]$.*

For each $x \in \mathbb{R}$, we denote $\mathcal{S}_x$ the set of snake trajectories with initial point x and $\mathcal{S} := \bigcup_{x \in \mathbb{R}} \mathcal{S}_x$.

There is an obvious abuse of notation when defining $\sigma(\omega)$, since we used the same notation for $\sigma(g)$. This will be justified a few lines later. Define for all $\omega, \omega' \in \mathcal{S}$,

$$d_{\mathcal{S}}(\omega, \omega') := \sup_{s \geq 0} d_{\mathcal{W}}(\omega_s, \omega'_s) + |\sigma(\omega) - \sigma(\omega')|.$$

It can be verified that $d_{\mathcal{S}}$ is a distance on $\mathcal{S}$, and that $(\mathcal{S}, d_{\mathcal{S}})$ is a Polish metric space.

The terminology snake is justified by the fact that these function-valued mappings can be visualized as functions continuously growing and being erased from the tip, starting and ending in a fixed point. See Figure 1.2. Let us now establish the bijection between snake trajectories and pairs as constructed in the previous section. To do so, we introduce the *tree-like paths.*

Definition 1.8. *A* ***tree-like path*** *is a pair (g, f) where $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $f : \mathbb{R}_+ \to \mathbb{R}$ are continuous functions satisfying:*

- *$g(0) = 0$ and $\sigma(g) \in (0, \infty)$ (that is, g is an excursion).*
- *For all $s, t \in [0, \sigma(g)]$, $d_g(s, t) = 0$ implies $f(s) = f(t)$.*

For each $x \in \mathbb{R}$, we denote $\mathbb{T}_x := \{(g, f) : f(0) = x\}$ and $\mathbb{T} := \bigcup_{x\in\mathbb{R}} \mathbb{T}_x$.

In the definition of tree-like paths, the function f has the role of the process Z of the previous section. Define $d_{\mathbb{T}}$ for all $(g, f), (g', f') \in \mathbb{T}$ by

$$d_{\mathbb{T}}((g, f), (g', f')) := \sup_{s\geq 0}(|g(s) - g'(s)| + |f(s) - f'(s)|) + |\sigma(g) - \sigma(g')|.$$

Then (again), $d_{\mathbb{T}}$ is a distance on $\mathbb{T}$, and $(\mathbb{T}, d_{\mathbb{T}})$ is a Polish metric space. The following result formalizes the coding of tree-like paths by snake trajectories (and viceversa) that we anticipated. In fact, the metric spaces $(\mathcal{S}, d_{\mathcal{S}})$ and $(\mathbb{T}, d_{\mathbb{T}})$ are homeomorphic.

Proposition 1.9 (Proposition 8 in [1])**.** *For each $\omega \in \mathcal{S}$, define $\Delta(\omega) = (g, f)$, where $g(s) = \zeta(\omega_s)$ and $f(s) = \widehat{\omega}_s$ for all $s \geq 0$. Then, Δ defines an homeomorphism from $(\mathcal{S}, d_{\mathcal{S}})$ to $(\mathbb{T}, d_{\mathbb{T}})$.*

Let us prove the bijection part of the previous proposition. For the full proof, see Theorem 2.1 in [14]. Given a snake trajectory $\omega = (\omega_s)_{s\geq 0}$, the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $g(s) = \zeta(\omega_s)$ is positive continuous with $g(0) = 0$ and $\sigma(g) = \sigma(\omega) \in (0, \infty)$. On the other hand, suppose that $0 \leq s \leq t$ and $d_g(s, t) = 0$. Then, $\min_{[s,t]} \zeta(\omega_\cdot) = \zeta(\omega_s) = \zeta(\omega_t)$, and by the snake property with $r = \min_{[s,t]} \zeta(\omega_\cdot)$ we have $f(s) = \widehat{\omega}_s = \omega_s(r) = \omega_t(r) = \widehat{\omega}_t = f(t)$. This justifies that Δ is well-defined as a mapping from $\mathcal{S}$ to $\mathbb{T}$. Conversely, let $(g, f) \in \mathbb{T}$. For each $s \in [0, \sigma(g)]$, define the continuous function $\omega_s : [0, g(s)] \to \mathbb{R}$ by $\omega_s(t) = f(\sup\{r \leq s : g(r) = t\})$ for all $t \in [0, g(s)]$. The mapping $\omega = (\omega_s)_{s\geq 0}$ constructed this way is a snake trajectory with initial point $f(0)$. Moreover, it holds that $\sigma(\omega) = \sigma(g)$, $\zeta(\omega_s) = g(s)$ and $\widehat{\omega}_s = f(s)$ for all $s \in [0, \sigma]$. In this case, we say that g is the *lifetime function* and f is the *tip function* of ω.

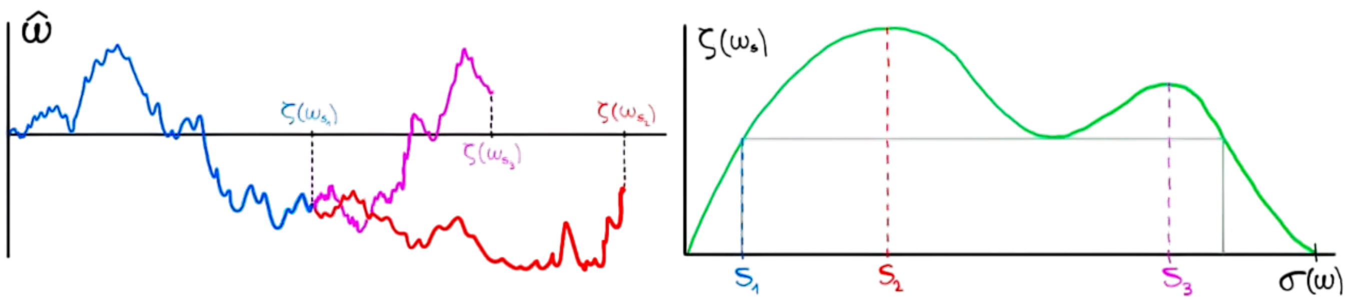

Figure 1.2: Representation of a snake trajectory and its lifetime function. It is possible to infer the shape of the tip function on $[0, \sigma(\omega)]$, which we left as an exercise for the reader.

If $\Delta(\omega) = (\zeta, \widehat{W})$, we will denote $\zeta(\omega) = (\zeta_s(\omega))_{s\geq 0}$ where $\zeta_s(\omega) := \zeta(\omega_s)$, and $W_s(\omega) := \omega_s$ for all $s \in [0, \sigma(\omega)]$. In fact, the Brownian sphere will be obtained by putting an appropriate randomness on $\mathcal{S}_x$ for fixed $x \in \mathbb{R}$, so that ζ and $\widehat{W}$ become random continuous functions.

Additionally, for future reference we introduce

$$W_*(\omega) := \min\{\widehat{W}_s(\omega) : s \in [0, \sigma(\omega)]\} = \min\{\ell_a(\omega) : a \in \mathcal{T}(\zeta)\}. \tag{1.5}$$

We will also drop the notation Z and keep $\widehat{W}$ when referring to the tip function or the labels $(\ell_a)_{a\in\mathcal{T}(\zeta)}$ of Definition 1.6. In particular, we will write $\ell_a(\omega) = \widehat{W}_s(\omega)$, where $s \in [0, \sigma(\omega)]$ is such that $p_\zeta(s) = a$.

1.1.4 Definition of the Brownian sphere

The Brownian sphere is defined as a quotient space of $\mathcal{T}(g)$ when g is a Brownian excursion, with a metric induced by the labels found in Proposition 1.4 and Definition 1.6. Formally, let $\mathbf{n}_+$ be the Itô measure of (positive) Brownian excursions, normalized in such a way that for all $\varepsilon > 0$,

$$\mathbf{n}_+\left(\sup_{t\geq 0} e(t) > \varepsilon\right) = \frac{1}{2\varepsilon}.$$

Recall that there exists a family of measures $(\mathbf{n}_+^{(s)} : s \geq 0)$, such that for all measurable subset A in the space of excursions[2] we have

$$\mathbf{n}_+(A) = \int_0^\infty \frac{\mathbf{n}_+^{(s)}(A)}{\sqrt{2\pi s^3}}\mathrm{d}s. \tag{1.6}$$

The measure $\mathbf{n}_+^{(s)}$ is naturally interpreted as $\mathbf{n}_+(\,\cdot\,|\sigma = s)$.

Definition 1.10. *Let $x \in \mathbb{R}$. We define the **Brownian snake measure** $\mathbb{N}_x$ on the space of snake trajectories $\mathcal{S}_x$ to be such that*

(i) the lifetime function $\zeta = (\zeta_s)_{s\geq 0}$ is a Brownian excursion with distribution $\mathbf{n}_+$, and

(ii) conditionally on ζ, the tip function $\widehat{W} = (\widehat{W}_s)_{s\geq 0}$ is a Gaussian process with mean x and covariance $m_\zeta(\cdot,\cdot)$ (recall the definition given in (1.4)).

*The random snake ω obtained from the random tree-like path $(\zeta, \widehat{W})$ is called **Brownian snake**.*

Similarly, for each $s \geq 0$ we denote $\mathbb{N}_x^{(s)}$ for the Brownian snake measure when ζ is sampled from $\mathbf{n}_+^{(s)}$. An immediate consequence of the previous definition is the following continuity property.

Proposition 1.11. *Let ω be a Brownian snake and $(\zeta, \widehat{W})$ be the associated tree-like path. Then, for all $\varepsilon \in (0, 1/4)$, $\widehat{W}$ has a modification with $(1/4 - \varepsilon)$-Hölder continuous sample paths with full probability.*

Proof. Follows from the the analog computation done in Remark 1.5 conditioning on ζ and the fact that for all $\varepsilon \in (0, 1/2)$, ζ has $(1/2 - \varepsilon)$-Hölder continuous sample paths with full probability. □

[2] The space of excursions is endowed with the σ-algebra generated by the projections $e \mapsto e(t)$, for each $t \geq 0$.

There is a special point on $\mathcal{T}(\zeta)$ that will play a fundamental role in the study of the Brownian sphere, where the global minima of the labels recorded by the Brownian snake is attained. The surprising and highly non-trivial fact is that such point is unique. We refer to the original reference for the proof of the following result.

Proposition 1.12 (Proposition 2.5 in [13])**.** *$\mathbb{N}_x$-a.e. there exists a unique $s_* \in (0, \sigma(\zeta))$ such that $\widehat{W}_{s_*} = W_*$ (recall* (1.5)*). We denote $a_* := p_\zeta(s_*)$.*

Let ω be a Brownian snake and $(\zeta, \widehat{W})$ be the associated tree-like path. We aim to obtain a metric structure in $\mathcal{T}(\zeta)$, for which we will use the labeling of this tree induced by $\widehat{W}$ as in Definition 1.6. For all $a, b \in \mathcal{T}(\zeta)$ define

$$D^\circ(a,b) := \ell_a + \ell_b - 2\max\left\{\min_{[a,b]} \ell, \min_{[b,a]} \ell\right\}. \tag{1.7}$$

It is clear that D° is a positive symmetric function. However, D° does not satisfy the triangle inequality. To fix this problem, let $D : \mathcal{T}(\zeta) \times \mathcal{T}(\zeta) \to \mathbb{R}_+$ be the largest symmetric function satisfying the triangle inequality that is bounded above by D°, that is,

$$D(a,b) = \inf\left\{\sum_{k=0}^{k-1} D^\circ(a_i, a_{i+1}) : k \geq 1,\ a_0, \ldots, a_k \in \mathcal{T}(\zeta), \text{ and } a_0 = a,\ a_k = b\right\}.$$

In the next proposition we record some important properties of D° and D. In particular, the function D satisfies the properties that we require.

Proposition 1.13.

- *For all $a, b \in \mathcal{T}(\zeta)$, we have $D^\circ(a,b) \geq |\ell_a - \ell_b|$.*
- *For all $a, b \in \mathcal{T}(\zeta)$, we have $D(a,b) = 0$ if, and only if $D^\circ(a,b) = 0$.*
- *The function D defines a pseudo-distance in $\mathcal{T}(\zeta)$.*

The first assertion comes from $D^\circ(a,b) \geq \ell_a + \ell_b - 2\max\{\ell_a, \ell_b\} = |\ell_a - \ell_b|$. The third assertion is straightforward from the definition of D. However, the left to right implication in the second point is very difficult and we refer to Theorem 3.4 in [10] for a proof.

Thanks to the third point of Proposition 1.13, we are almost ready to define the Brownian sphere. Define $\mathbf{m}_\infty := \mathbf{m}_\infty(\omega) := \mathcal{T}(\zeta)/\{D = 0\}$, and endow it with the distance induced by D, still denoted by D. Let $\Pi : \mathcal{T}(\zeta) \to \mathbf{m}_\infty$ be the canonical projection. The *volume measure on* $\mathbf{m}_\infty$, denoted Vol, is defined as the pushforward of the volume measure on $\mathcal{T}(\zeta)$ under Π. Define the *root* of $\mathbf{m}_\infty$ as $x_0 := \Pi(\rho_\zeta)$, and $x_* = \Pi(a_*)$.

Additionally, let $\mathbb{K}^{\bullet\bullet}$ be the space of compact *two-pointed* measure metric spaces, that is, the equivalence classes for $\{d_{\mathrm{GHP}} = 0\}$ of 5-tuples of the form (X, d, μ, x, y), where (X, d) is a compact metric space, μ is a finite Borel measure on X and x and y are two points of X, called the distinguished points of (X, d, μ, x, y) (or simply X when there is no possible confusion). Each such (X, d, μ, x, y) is called a compact *two-pointed* measure metric space. Here, d_{GHP} stands for the natural extension of (1.3) to the two-pointed context.

Definition 1.14. *The* ***(free) Brownian sphere*** *is defined as the random compact two-pointed measure metric space* $(\mathbf{m}_\infty, D, \mathrm{Vol}, x_0, x_*)$ *under the measure* $\mathbb{N}_0$. *More generally, the Brownian sphere can be sampled from* $\mathbb{N}_0^{(s)}$ *for any* $s \geq 0$*, and the context will make clear when this is the case.*

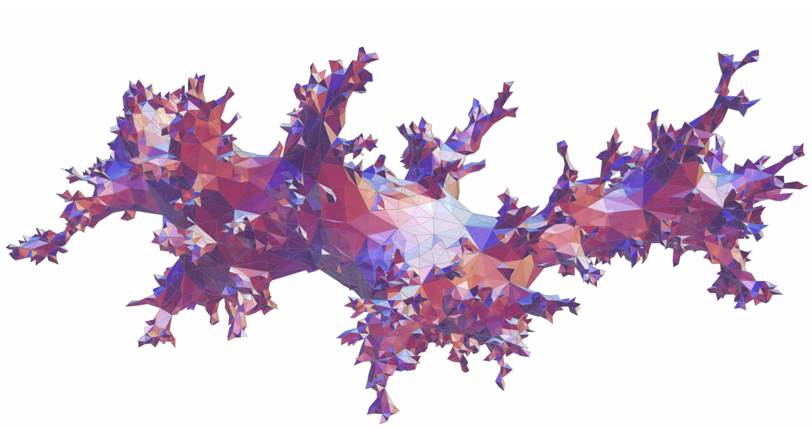

Figure 1.3: Approximation of the Brownian sphere via triangulations of the sphere. Simulation by Nicolas Curien taken from his website: https://www.imo.universite-paris-saclay.fr/∼nicolas.curien/simulation.html.

Note from (1.7) that $D^\circ(a, a_*) = \ell_a - \ell_{a_*}$ for all $a \in \mathcal{T}(\zeta)$, so we deduce that $D(x_0, x_*) = -W_*$. This relation is very useful since it will allow us to condition on the distance between x_0 and x_*, thanks to the following proposition. We refer to Section VI in [9] for a proof.

Proposition 1.15. *For* $x, y \in \mathbb{R}$ *such that* $y < x$*, we have*

$$\mathbb{N}_x(W_* \leq y) = \frac{3}{2(y-x)^2}.$$

Using this, we define the probability measure $\mathbb{N}_0^{[r]} := \mathbb{N}_0(\cdot | W_* < -r)$ for $r > 0$. As a consequence of the previous observations, the Brownian sphere sampled from $\mathbb{N}_0^{[r]}$ will satisfy $D(x_0, x_*) > r$.

A very useful way to visualize $\mathbf{m}_\infty$ is through its *cactus representation*. It amounts to take x_* as reference point and draw $\mathbf{m}_\infty$ vertically, according to the distance of each point to x_*, in such a way that each horizontal cut of the picture represents points that are equally distant from x_*. Better than words, Figure 1.4 is self-explanatory. The choice of putting x_* at the bottom makes the distances to be exactly the difference of the labels assigned by $\widehat{W}$. One should keep in mind this picture in the next sections and chapters, specially when drawing geodesics, that will be discussed in great detail.

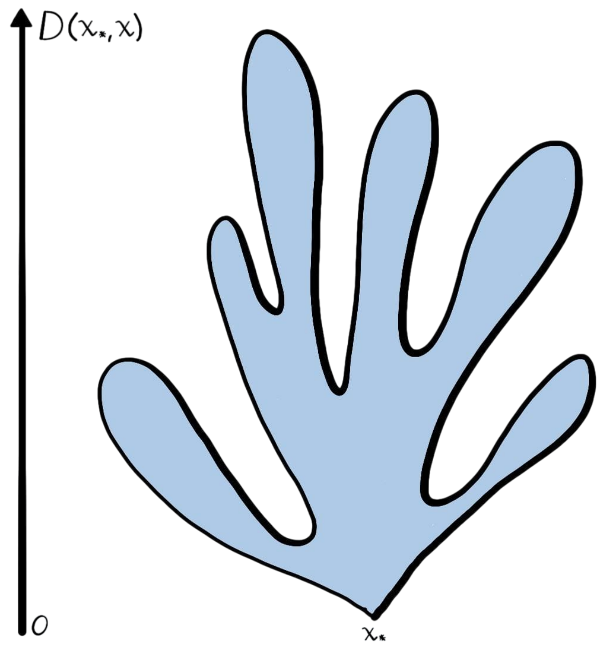

Figure 1.4: Cactus representation of $\mathbf{m}_\infty$.

1.2 Basic properties

In this section, we state and prove some geometric properties of the Brownian sphere. This includes scaling and symmetry properties, estimates for the volume of balls, compactness properties and the exit local time of the Brownian snake.

For all $x \in \mathbf{m}_\infty$ and $r > 0$, we write $B(x,r)$ for the usual open ball of center x and radius r,

$$B(x,r) = \{y \in \mathbf{m}_\infty : D(x,y) < r\}.$$

A very important subset of $\mathbf{m}_\infty$ is the following, which regarded under the appropriate conditioning, has the distribution of a *hull*, to be defined in Section 1.3.3.

Definition 1.16. *Let $x, y \in \mathbf{m}_\infty$ and $r \in (0, D(x,y))$. We define $B^{\bullet(y)}(x,r)$ as the closure of the complement of the connected component of the complement of $B(x,r)$ that contains y.*

Informally, $B^{\bullet(y)}(x,r)$ is $B(x,r)$ plus all the connected components of its complement except for the one that contains y. Note that $B(x,r) \subseteq B^{\bullet(y)}(x,r)$ and $\partial B^{\bullet(y)}(x,r) \subseteq \partial B(x,r)$, see Figure 1.5.

1.2.1 Symmetries

Let us first introduce some invariance properties of the Brownian snake that will find their expression in the Brownian sphere later. Namely, we discuss invariance properties of the Brownian snake under *re-rooting* and *scaling* operations on snakes. Formally, they are defined as follows:

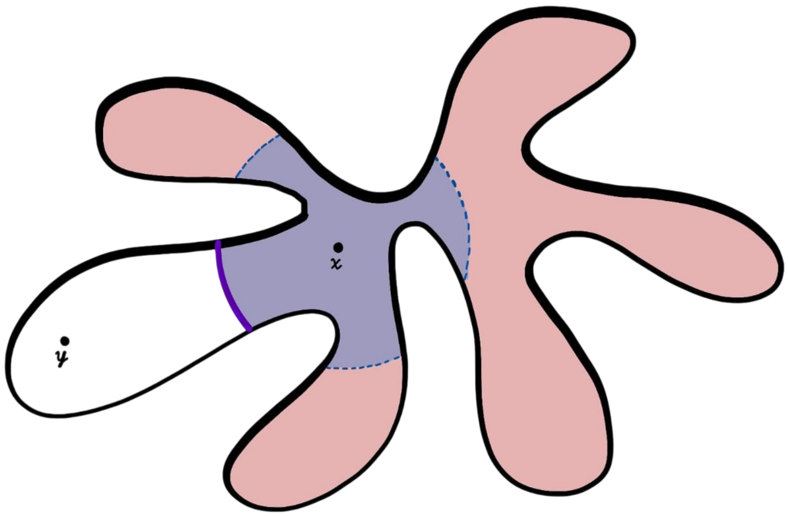

Figure 1.5: The blue shaded area is $B(x,r)$. The union of the blue and red shaded areas is $B^{\bullet(y)}(x,r)$. The purple line is $\partial B^{\bullet(y)}(x,r)$. Be aware that the balls are not simply connected as drawn here, they contain many holes. This is no taken into account in order to keep the picture simple.

- **Re-rooting**: Let $\omega \in \mathcal{S}_0$ and $r \in [0,\sigma(\omega)]$. Define $\omega^{[r]} \in \mathcal{S}_0$ to be such that

$$\zeta_s(\omega^{[r]}) = d_{\zeta(\omega)}(r, r\oplus s) \quad \text{and} \quad \widehat{W}_s(\omega^{[r]}) = \widehat{W}_{r\oplus s}(\omega) - \widehat{W}_r(\omega),$$

  where $r\oplus s = r+s$ if $r+s \le \sigma(\omega)$ and $r\oplus s = r+s-\sigma(\omega)$ if $r+s>\sigma(\omega)$. By the bijection part of Proposition 1.9, these definitions completely determine $\omega^{[r]}$. See Figure 1.6.

- **Scaling**: For $\lambda>0$ and $\omega\in\mathcal{S}_x$, define the snake trajectory $\theta_\lambda(\omega)\in\mathcal{S}_{x\sqrt{\lambda}}$ by $\theta_\lambda(\omega)=\omega'$, where

$$\omega'_s(t) := \sqrt{\lambda}\omega_{s/\lambda^2}(t/\lambda), \quad \text{for } s\ge 0 \text{ and } 0\le t\le \zeta'_s := \lambda\zeta_{s/\lambda^2}.$$

Concerning the previous two operations, the following invariances hold in the Brownian snake.

Proposition 1.17. *Let $t,\lambda>0$.*

1. *If $r\in[0,t]$, the measure $\mathbb{N}_0^{(t)}$ is invariant under the re-rooting operation $\omega\mapsto\omega^{[r]}$.*
2. *The pushforward of $\mathbb{N}_x$ under θ_λ is $\lambda\mathbb{N}_{x\sqrt{\lambda}}$.*

Let us now turn to the symmetries of the Brownian sphere. The following proposition states that the points x_0 and x_* are independent and uniformly distributed over $\mathbf{m}_\infty$. Informally, this means that the Brownian sphere seen from two uniformly chosen points always looks statistically the same.

Proposition 1.18 (Proposition 3 in [7]). *Let $F:\mathbb{K}^{\bullet\bullet}\to\mathbb{R}_+$ be a positive measurable function. Then,*

$$\mathbb{N}_0(F(\mathbf{m}_\infty, D, \mathrm{Vol}, x_0, x_*)) = \mathbb{N}_0\left(\iint \frac{\mathrm{Vol}(\mathrm{d}x)}{\sigma}\frac{\mathrm{Vol}(\mathrm{d}y)}{\sigma} F(\mathbf{m}_\infty, D, \mathrm{Vol}, x, y)\right).$$

The same identity holds if we replace $\mathbb{N}_0$ by $\mathbb{N}_0^{(s)}$, for any $s\ge 0$.

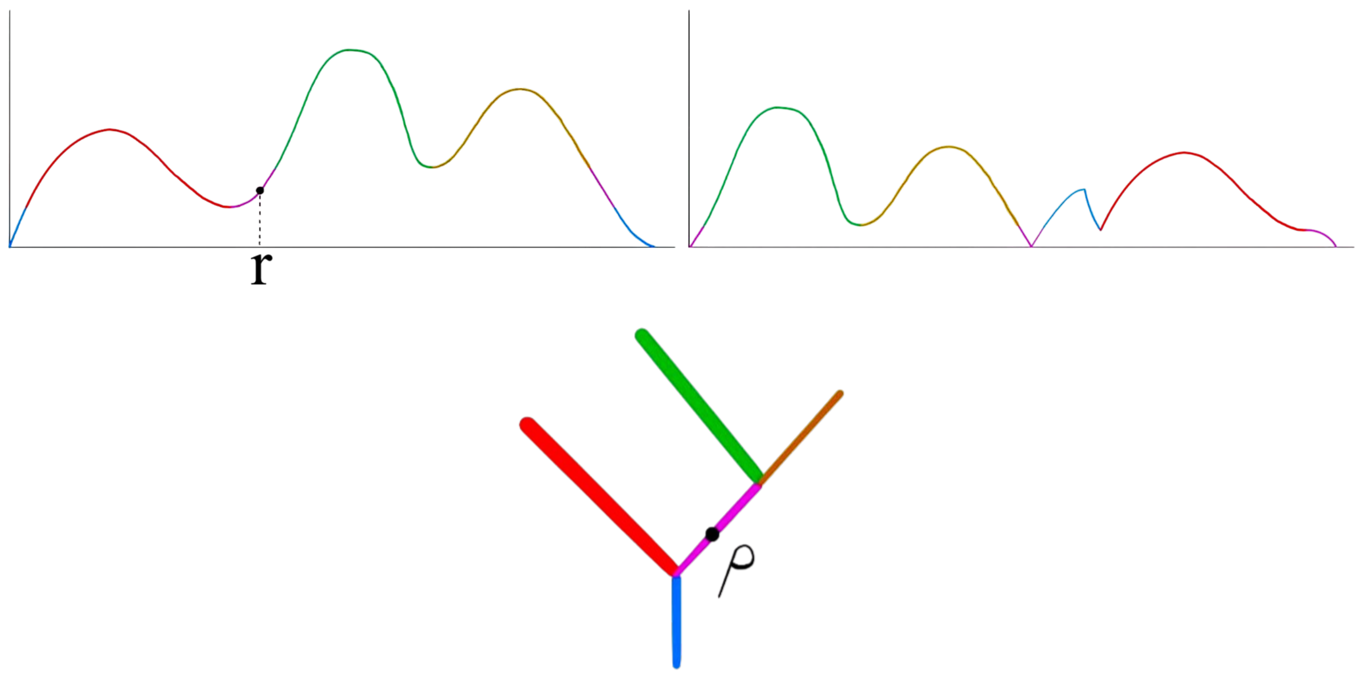

Figure 1.6: In the first picture, the very same function of Figure 1.1 and a marked time $\mathrm{r} \in [0, \sigma(\omega)]$. In the second picture, the re-rooted lifetime function $\zeta(\omega^{[\mathrm{r}]})$. In the third picture we see $\mathcal{T}(\zeta(\omega^{[\mathrm{r}]}))$ with its root, that can be drawn the same way as $\mathcal{T}(\zeta(\omega))$ before the re-rooting, but only relocating the root. Colors corresponding to the branches of $\mathcal{T}(\zeta(\omega))$ are left to see how $\zeta(\omega^{[\mathrm{r}]})$ corresponds to the counterclockwise exploration of $\mathcal{T}(\zeta(\omega))$ starting from the new root (ρ in the picture).

Proof. By (1.6), it is enough to prove the result for $\mathbb{N}_0^{(s)}$, with fixed $s \geq 0$. By Proposition 1.17, $\mathbb{N}_0^{(s)}$ is invariant under $\omega \mapsto \omega^{[r]}$ for any $r \in [0, s]$. Secondly, note that $\mathbf{m}_\infty$ is left unchanged if we change ω by $\omega^{[r]}$, and that the minimal label is attained at the same point of $\mathcal{T}(\zeta(\omega))$ and $\mathcal{T}(\zeta(\omega^{[r]}))$. The preceding considerations give that

$$\mathbb{N}_0^{(s)}(F(\mathbf{m}_\infty, D, \mathrm{Vol}, x_0, x_*)) = \mathbb{N}_0\left(\int \frac{\mathrm{Vol}(\mathrm{d}x)}{s} F(\mathbf{m}_\infty, D, \mathrm{Vol}, x, x_*)\right).$$

Then, an application of (an extension of) Theorem 8.1 in [8] concludes the proof. □

As a corollary, the following three symmetries hold for the Brownian sphere:

- $\mathbb{N}_0(F(\mathbf{m}_\infty, D, \mathrm{Vol}, x_0, x_*)) = \mathbb{N}_0(F(\mathbf{m}_\infty, D, \mathrm{Vol}, x_*, x_0))$.
- $\mathbb{N}_0^{[r]}(F(\mathbf{m}_\infty, D, \mathrm{Vol}, x_0, x_*)) = \mathbb{N}_0^{[r]}(F(\mathbf{m}_\infty, D, \mathrm{Vol}, x_*, x_0))$, for any $r \geq 0$.
- If F is defined on (the equivalence classes of) compact three-pointed measure metric spaces, then

$$\mathbb{N}_0\left(\int \frac{\mathrm{Vol}(\mathrm{d}x)}{\sigma} F(\mathbf{m}_\infty, D, \mathrm{Vol}, x_0, x_*, x)\right) = \mathbb{N}_0\left(\int \frac{\mathrm{Vol}(\mathrm{d}x)}{\sigma}\frac{\mathrm{Vol}(\mathrm{d}y)}{\sigma}\frac{\mathrm{Vol}(\mathrm{d}z)}{\sigma} F(\mathbf{m}_\infty, D, \mathrm{Vol}, x, y, z)\right).$$

1.2.2 Volume of balls

The encoding by stochastic processes of the Brownian sphere allows us also to estimate the volume of balls in this metric space. In fact, the following proposition gives a simple formula for the moments of the ball $B(x_*, r)$, for which the scaling and the properties of $\widehat{W}$ are key.

Proposition 1.19 (Lemma 4 in [7]). *Let $p \geq 1$ be an integer. There exists a constant C_p such that for all $r > 0$,*

$$\mathbb{N}_0 \left(\mathrm{Vol}(B(x_*, r))^p \right) = C_p r^{4p-2}.$$

Consequently, for every integer $p \geq 1$ and $\eta \in (0,1)$, there exists a constant $C_{p,\eta} > 0$ such that for all $r \in (0,1)$,

$$\mathbb{N}_0^{[1]} \left(\mathrm{Vol}(B(x_*, r))^p \right) \leq C_{p,\eta} r^{4p-\eta}.$$

Proof. By scaling, we have

$$\mathbb{N}_0 \left(\mathrm{Vol}(B(x_*, r))^p \right) = \mathbb{N}_0 \left(\left(\int_0^\sigma \mathbb{1}_{\{\widehat{W}_s - W_* \leq r\}} \mathrm{d}s \right)^p \right) = r^{4p-2} \mathbb{N}_0 \left(\left(\int_0^\sigma \mathbb{1}_{\{\widehat{W}_s - W_* \leq 1\}} \mathrm{d}s \right)^p \right).$$

We need to verify that the constant multiplying r^{4p-2} is finite. To this end, write

$$\mathbb{N}_0 \left(\left(\int_0^\sigma \mathbb{1}_{\{\widehat{W}_s - W_* \leq 1\}} \mathrm{d}s \right)^p \right) = \int_0^\infty \frac{1}{2\sqrt{2\pi t^3}} \mathbb{N}^{(t)} \left(\left(\int_0^t \mathbb{1}_{\{\widehat{W}_s - W_* \leq 1\}} \mathrm{d}s \right)^p \right) \mathrm{d}t,$$

and split the integral in the regions $[0,1)$ and $[1,\infty)$. The first part is bounded by $\int_0^1 \frac{t^p}{2\sqrt{2\pi t^3}} \mathrm{d}t$, which is finite. For the second part, scaling and Lemma 6.1 in [8] gives that for all $\delta \in (0,1)$ and $t \geq 1$,

$$\mathbb{N}^{(t)} \left(\left(\int_0^t \mathbb{1}_{\{\widehat{W}_s - W_* \leq 1\}} \mathrm{d}s \right)^p \right) = t^p \mathbb{N}^{(t)} \left(\left(\int_0^1 \mathbb{1}_{\{\widehat{W}_s - W_* \leq t^{-1/4}\}} \mathrm{d}s \right)^p \right) \lesssim t^{\delta/4},$$

which gives a finite upper bound again. For the second assertion, by Cauchy-Schwartz we have

$$\mathbb{N}_0^{[1]} \left(\mathrm{Vol}(B(x_*, r))^p \right) \leq \mathbb{N}_0^{[1]} \left(\mathrm{Vol}(B(x_*, r))^{qp} \right)^{1/q} \leq \left(\frac{2}{3} \mathbb{N}_0 \left(\mathrm{Vol}(B(x_*, r))^{qp} \right) \right)^{1/q} \lesssim r^{4p-2/q},$$

where the implicit constant depends on p and q. We conclude by taking q large so that $2/q < \eta$. $\square$

Even more explicitly, we have an almost sure estimate for the volume of a ball with sufficiently small radius. Note how the label process plays a role with its Hölder continuity property.

Proposition 1.20 (Lemma 2.2 in [15]). *$\mathbb{N}_0$-a.e. the following property holds. For all $\eta > 0$, there exists $\varepsilon_0 > 0$ such that for all $x \in \mathbf{m}_\infty$ and $\varepsilon \in (0, \varepsilon_0)$,*

$$\varepsilon^{4+\eta} \leq \mathrm{Vol}(B(x, \varepsilon)) \leq \varepsilon^{4-\eta}.$$

Proof. By scaling, we can argue under $\mathbb{N}_0^{(1)}$. The upper bound follows from the control of the moments of the volume of balls in $\mathbf{m}_\infty$ and Borel-Cantelli's lemma (see Lemma 2.1 in [15]). For the lower bound, fix $\eta > 0$ and let $s \in [0,1]$. By Proposition 1.5, we have for all $b \in (0, 1/4)$,

$$D(\Pi(p_\zeta(s)), \Pi(p_\zeta(s + \varepsilon^{4+\eta}))) \leq C\varepsilon^{(4+\eta)(1/4-b)} = C\varepsilon^{1+\eta/4-b(4+\eta)}.$$

for some random constant $C > 0$. Therefore, if b is sufficiently small, we see that there exists $\varepsilon_0 > 0$ (random) such that for all $\varepsilon \in (0, \varepsilon_0)$, $\Pi(p_\zeta([s, s+\varepsilon^{4+\eta}])) \subseteq B(\Pi(p_\zeta(s)), \varepsilon)$. Taking the volume measure gives $\varepsilon^{4+\eta} \leq \mathrm{Vol}(B(\Pi(p_\zeta(s)), \varepsilon))$ and the result follows since s is arbitrary. □

1.2.3 Compactness

In this section, we briefly state propositions concerning covering properties of the Brownian sphere.

Proposition 1.21 (Lemma 2.3 in [15])**.** $\mathbb{N}_0$-*a.e. the following property holds. Let* $(x_i)_{i\in\mathbb{N}}$ *be an iid sequence sampled from* $\mathrm{Vol}(\cdot)$*. For all* $a > 0$*, there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$*, we have*

$$\mathbf{m}_\infty \subseteq \bigcup_{i=1}^{\varepsilon^{-4-a}} B(x_i, \varepsilon).$$

Proof. By scaling, we argue under $\mathbb{N}_0^{(1)}$. Fix $a > 0$ and let $\varepsilon_0 > 0$ be given by Proposition 1.21. For each $k \in \mathbb{N}$, let $N_k := 2^{(k+1)(4+2a)}$ and define the event

$$A_k := \{\text{For all } x \in \mathbf{m}_\infty, \text{ there exists } i \in \{1, \ldots, N_k\} \text{ such that } x_i \in B(x, 2^{-k})\}.$$

If we show that $\sum_{k\in\mathbb{N}} \mathbb{P}(A_k^c | \varepsilon_0 \geq 2^{-k})$ is finite, then the result follows by Borel-Cantelli's lemma. To see this, note that

$$\mathbb{P}(D(x_{N_k+1}, x_i) \geq 2^{-(k+1)} \text{ for all } i \in \{1, \ldots, N_k\} \,|\, A_k^c \cap \{\varepsilon_0 \geq 2^{-k}\}) \geq 2^{-(k+1)(4+a)},$$

which follows from the fact if x_{N_k+1} lies at distance at most $2^{-(k+1)}$ from some $x \in \mathbf{m}_\infty$ given by A_k^c, this implies that $D(x_{N_k+1}, x_i) \geq 2^{-(k+1)}$ for all $i \in \{1, \ldots, N_k\}$, since $(x_i)_{i\in\mathbb{N}}$ is iid sampled from $\mathrm{Vol}(\cdot)$ and $\mathrm{Vol}(B(z, 2^{-(k+1)})) \geq 2^{-(k+1)(4+a)}$ in this conditioning. Noting that

$$\mathbb{P}(D(x_{N_k+1}, x_i) \geq 2^{-(k+1)} \text{ for all } i \in \{1, \ldots, N_k\}) \leq (1 - 2^{-(k+1)(4+a)})^{N_k} \leq \exp(-2^{(k+1)a}),$$

we have

$$\mathbb{P}(A_k^c \cap \{\varepsilon_0 \geq 2^{-k}\}) \leq 2^{(k+1)(4+a)} \exp(-2^{(k+1)a}),$$

and then $\mathbb{P}(A_k^c | \varepsilon_0 \geq 2^{-k}) \leq 2^{(k+1)(4+a)} \exp(-2^{(k+1)a})(1 + o(1))$, which is summable in k, proving the claim and the result. □

The following two propositions will be very useful. We refer to the original reference for the proofs. Here, $\mathrm{diam}(\mathbf{m}_\infty) := \sup\{D(x,y) : x, y \in \mathbf{m}_\infty\}$ stands for the *diameter* of $\mathbf{m}_\infty$.

Proposition 1.22 (Lemma 4.4 in [15])**.** $\mathbb{N}_0$*-a.e. for all* $R \in (0, \mathrm{diam}(\mathbf{m}_\infty))$*, there exists* $r_0 > 0$ *such that for all* $r \in (0, r_0)$ *and* $z \in \mathbf{m}_\infty$*, there is at most one connected component of* $\mathbf{m}_\infty \setminus B(z, r)$ *with diameter at least* R*.*

Proposition 1.23 (Lemma 7.1 in [15])**.** $\mathbb{N}_0$*-a.e. there exists* $r_0 > 0$ *such that for all* $r \in (0, r_0)$ *and* $z \in \mathbf{m}_\infty$*, if* $U_{z,r}$ *denotes the connected component of* $\mathbf{m}_\infty \setminus B(z, r)$ *with the largest diameter, then* $\mathrm{Vol}(U_{z,r}) \geq \mathrm{Vol}(\mathbf{m}_\infty)/2$*.*

1.2.4 Exit measures and boundary length of hulls

Let $x, y \in \mathbb{R}$ with $y < x$, ω be a Brownian snake and $(\zeta, \widehat{W})$ be the associated tree-like path. We can make sense of the "time spent" by ω at y through the *exit local time process* $(L^y_t)_{t\geq 0}$, defined for all $t \geq 0$ as

$$L^y_t := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_0^t \mathrm{d}s \mathbb{1}_{\{\tau_y(W_s)=\infty, \widehat{W}_s < y+\varepsilon\}}.$$

where $\tau_y(f) := \inf\{s \geq 0 : f(s) = y\}$, with the convention $\inf(\emptyset) = \infty$. The process $(L^y_t)_{t\geq 0}$ is constant for $t \geq \sigma(\zeta)$. The *exit measure at* y is defined as $Z_y := L^y_\infty = L^y_{\sigma(\zeta)}$. Using a description of the measure $\mathbb{N}_0(\cdot | W_* = -u)$ (which we will not present here) and a decomposition of the Brownian snake at its minimum, we can make sense of the exit measure at $W_* + r$ for $r > 0$. Explicitly,

$$Z_{W_*+r} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_0^\sigma \mathrm{d}s \mathbb{1}_{\{\tau_{W_*+r}(W_s)=\infty, \widehat{W}_s < W_*+r+\varepsilon\}} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathrm{Vol}(B^{\bullet(x_0)}(x_*, r+\varepsilon)^c \cap B(x_*, r)).$$

where the last equality holds by definition of the volume measure on $\mathbf{m}_\infty$ as successive pushforwards of the Lebesgue measure on $[0, \sigma(\omega)]$ and then on $\mathcal{T}(\zeta)$. Thanks to the last display, it is not difficult to interpret Z_{W_*+r} as the *boundary length* of the hull $B^{\bullet(x_*)}(x_0, D(x_0, x_*) - r)$ (recall that it only makes sense under $\mathbb{N}_0(\cdot | W_* < -r)$).

Now, we aim to describe the law of the boundary length process associated to $B^{\bullet(x_*)}(x_0, D(x_0, x_*) - r)$, where r varies from 0 to $D(x_0, x_*)$. To do so, we need to introduce the so-called *continuous state branching processes.*

Interlude: Stable continuous state branching processes

We aim to briefly discuss some facts concerning the so-called continuous state branching processes (CSBP from now on) towards their application to the study of the Brownian sphere. The presentation of this interlude is based on [5]. Let $Y = (Y_t)_{t\geq 0}$ be a strong Markov process and let $(\mathbb{P}_x)_{x\geq 0}$ be the family of probability measures satisfying $\mathbb{P}_x(Y_0 = x) = 1$ for each $x \geq 0$. Write $\mathbb{E}_x$ for the expectation under $\mathbb{P}_x$. Such Y is called CSBP if it has càdlàg[3] sample paths and satisfies the *branching property*, that is, for all $x, y \geq 0$ and $\theta \geq 0$, we have

[3] *Càdlàg* stands for *continu à droit, limite à gauche*, which means right-continuous with left-limits.

$$\mathbb{E}_{x+y}\left[e^{-\theta Y_t}\right] = \mathbb{E}_x\left[e^{-\theta Y_t}\right]\mathbb{E}_y\left[e^{-\theta Y_t}\right]. \tag{1.8}$$

Define the *Laplace exponent of* Y as $u_t(\theta) := -\log(\mathbb{E}_1[e^{-\theta Y_t}])$, for all $t, \theta \geq 0$. From (1.8) we can derive that

$$\mathbb{E}_x\left[e^{-\theta Y_t}\right] = e^{-x u_t(\theta)}. \tag{1.9}$$

From (1.9) and the Markov property, we can derive $u_{t+s}(\theta) = u_t(u_s(\theta))$. This means that $(u_t(\theta))_{t\geq 0}$ satisfies the semigroup property. The following theorem, for which we refer to [cite] for a proof, states that each CSBP is related to some Lévy process.

Theorem 1.24. *Suppose that* $u_t(\theta)$*,* $t, \theta \geq 0$*, is the Laplace exponent of a CSBP. Then,* $t \mapsto u_t(\theta)$ *is differentiable, and satisfies the ordinary differential equation*

$$\frac{\partial u_t}{\partial t}(\theta) + \psi(u_t(\theta)) = 0, \tag{1.10}$$

with initial condition $u_0(\theta) = \theta$*, and for all* $\lambda \geq 0$*,* $\psi(\lambda) = \log(\mathbb{E}[e^{-\lambda X_1}])$ *where* $X = (X_t)_{t\geq 0}$ *is either a spectrally positive Lévy process or a killed subordinator (see Section 2.6.2 of [5] to see the meaning of these objects).*

Such a function ψ given in the previous theorem is called *branching mechanism* of the corresponding CSBP. The following theorem goes further and states a bijection between CSBP and Lévy processes, through the so-called *Lamperti transform.*

Theorem 1.25. *Let* ψ *be any branching mechanism.*

1. *Suppose that* $X = (X_t)_{t\geq 0}$ *is a Lévy process with no negative jumps, killed at an exponentially distributed time with parameter* $q \geq 0$*, such that* $\psi(\lambda) = \log(\mathbb{E}[e^{-\lambda X_1}])$*. Define for all* $t \geq 0$*,*

   $$Y_t = X_{\theta_t \wedge \tau_0^-},$$

   where $\tau_0^- = \inf\{t > 0 : X_t < 0\}$ *and*

   $$\theta_t = \inf\left\{s > 0 : \int_0^s \frac{\mathrm{d}u}{X_u} > t\right\}.$$

   Then, for all $x \geq 0$*, under* $\mathbb{P}_x$*,* $Y = (Y_t)_{t\geq 0}$ *is a CSBP with branching mechanism* ψ*.*

2. *Conversely, suppose that* $Y = (Y_t)_{t\geq 0}$ *is a CSBP with branching mechanism* ψ *such that* $Y_0 = x \geq 0$*. Define* $X_t = Y_{\varphi_t}$ *for all* $t \geq 0$*, where*

   $$\varphi_t = \inf\left\{s > 0 : \int_0^s Y_u \mathrm{d}u > t\right\}.$$

   Then, $X = (X_t)_{t\geq 0}$ *is a Lévy process with no negative jumps and* $X_0 = x$*, stopped in the first entry into* $(-\infty, 0)$ *and killed at an exponentially distributed time with some parameter* $q \geq 0$*, such that* $\psi(\lambda) = \log(\mathbb{E}[e^{-\lambda X_1}])$ *for all* $\lambda \geq 0$*.*

We are ready to define the stable CSBP. Let $Y = (Y_t)_{t\geq 0}$ be a CSBP and $\alpha \in (1, 2)$. Such Y is additionally called *α-stable* if the branching mechanism of Y is given by $\psi(\lambda) = c\lambda^\alpha$, for all $\lambda \geq 0$, where $c > 0$ is some constant. Note that in this case we can solve directly (1.10) to get that for all $t, \theta \geq 0$,

$$u_t(\theta) = (\theta^{1-\alpha} + c(\alpha - 1)t)^{\frac{1}{1-\alpha}}. \tag{1.11}$$

Using the previous facts, two properties of α-stable CSBP can be derived. Namely, we have:

- **Scaling**: For all $C > 0$, the process $(CY_t)_{t\geq 0}$ has the same distribution as $(Y_{C^{\alpha-1}t})_{t\geq 0}$.
- **Extinction time**: Let $T = \inf\{t \geq 0 : Y_t = 0\}$ be the extinction time of Y. Plugin (1.11) into (1.9) gives

$$\mathbb{P}_x(T > t) = 1 - \exp(-x(c(\alpha - 1)t)^{\frac{1}{1-\alpha}}). \tag{1.12}$$

End of the interlude - Reverse exploration of the Brownian sphere

We can identify the law of the boundary length corresponding to a *reverse exploration from x_0 to x_**. More precisely, for $r > 0$ and conditioning on $D(x_0, x_*) > r$, define Y_t as the boundary length of $B^{\bullet(x_0)}(x_*, D(x_0, x_*) - t)$. Then, it is proven in [16] that $(Y_t)_{t\geq 0}$ has the law of a 3/2-stable CSBP. The same holds true if we replace x_0 and x_* by typical points[4].

1.2.5 Geodesics in the Brownian sphere

Geodesics are central in the study of surfaces. In particular, geodesics in the Brownian sphere are fundamental to understand the behavior of atypical points in $\mathbf{m}_\infty$, and they naturally appear in the scaling limits results. Let us formally define the framework to study geodesics.

Definition 1.26. *Let (E, d) be a metric space.*

- *Let $a, b \in E$. A path $\gamma : [0, d(a, b)] \to E$ such that $\gamma(0) = a$ and $\gamma(d(a, b)) = b$ is called* ***geodesic (from** a **to** b**)*** *if $d(\gamma(s), \gamma(t)) = |s - t|$, for all $s, t \in [0, d(a, b)]$.*
- *(E, d) is called a* ***geodesic space*** *if between any pair of points there exists a geodesic.*

The fact that the Brownian sphere is a geodesic space is discussed in great detail in [4]. Let us now heuristically construct a geodesic from x_0 to x_* in $\mathbf{m}_\infty$. In fact, we see from $D(\Pi(a), x_*) = \ell_a - \ell_{a_*}$ that a geodesic from x_0 to x_* is obtained by *following the running infimum* of $(\ell_a)_{a\in\mathcal{T}(\zeta)}$ along the branches of $\mathcal{T}(\zeta)$. Formally, this path is written as the function $\gamma : [0, -W_*] \to \mathbf{m}_\infty$ defined by

$$\gamma(t) := \Pi(p_\zeta(\inf\{s \in [0, \sigma] : \widehat{W}_s = -t\})). \tag{1.13}$$

[4] A *typical point* is point that can be obtained as a sample of the volume measure. On the contrary, an *atypical point* cannot arise as the sample of the volume measure.

It is clear that $\gamma(0) = x_0$ and $\gamma(-W_*) = x_*$, and the isometry property follows easily by the definition of γ. Since the Brownian sphere has no "orientation", an equivalent way of writing γ is

$$\gamma(t) = \Pi(p_\zeta(\sup\{s \in [0,\sigma] : \widehat{W}_s = -t\})).$$

These two writings will be relevant when defining another quotient space of $\mathcal{T}(\zeta)$ called *slice*, see Section 1.3.4 (in fact, they will represent two different geodesics between typical points in the slice). It also makes sense to ask if there are more geodesics connecting x_0 and x_*. The answer is given in the following result, whose proof can be found in [8].

Proposition 1.27. *The path γ defined in* (1.13) *is the unique geodesic between x_0 and x_*.*

More generally, between any pair of typical points in $\mathbf{m}_\infty$ there is exactly one geodesic. In fact, there are "not many" geodesics in the Brownian sphere. An illustration of this phenomenon is the following confluence result, that states that two geodesics starting at different points in $\mathbf{m}_\infty$ towards x_0, must coalesce near x_0.

Proposition 1.28 (Corollary 7.7 in [8])**.** *Almost surely, for all $\delta > 0$ there exists $\alpha \in [0, \delta]$ such that the following holds. If $x, y \in \mathbf{m}_\infty$ are such that $D(x, x_0) \geq \delta$ and $D(y, x_0) \geq \delta$, and if γ_1 is a geodesic from x to x_0 and γ_2 is a geodesic from y to x_0, then $\gamma_1(t) = \gamma_2(t)$ for all $t \in [0, \alpha]$.*

The following result can be seen as an extension of Proposition 1.28. It is called *strong confluence of geodesics*, and describes the interaction of two different geodesics with enough duration and sufficiently close as subsets of the Brownian sphere. Informally, if two geodesics are close enough, then they coincide everywhere except in some neighborhoods of their endpoints.

Theorem 1.29 (Theorem 1.1 in [15])**.** $\mathbb{N}_0$*-a.e. the following statement holds. For all $u > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, if $\eta_1 : [0, T_1] \to \mathbf{m}_\infty$ and $\eta_2 : [0, T_2] \to \mathbf{m}_\infty$ are geodesics with* $\min\{T_1, T_2\} \geq 2\varepsilon^{1-u}$ *and $d_{\mathrm{H}}(\eta_1([0, T_1]), \eta_2([0, T_2])) \leq \varepsilon$, then*

$$\eta_1([\varepsilon^{1-u}, T_1 - \varepsilon^{1-u}]) \subseteq \eta_2([0, T_2]) \quad \textit{and} \quad \eta_2([\varepsilon^{1-u}, T_2 - \varepsilon^{1-u}]) \subseteq \eta_1([0, T_1]).$$

A further evidence of the little amount of geodesics in the Brownian sphere is the following result, that tells us that geodesics cannot "bounce" into another. The lack of bouncing behavior reduces the number of possible configurations of geodesics between two points. Such configurations are known in the literature as *geodesic networks*.

Theorem 1.30 (Theorem 1.3 in [15])**.** $\mathbb{N}_0$*-a.e. if $\eta_1 : [0, T_1] \to \mathbf{m}_\infty$ and $\eta_2 : [0, T_2] \to \mathbf{m}_\infty$ are geodesics, then $\{t \in (0, T_1) : \eta_1(t) \in \eta_2([0, T_2])\}$ and $\{t \in (0, T_2) : \eta_2(t) \in \eta_1([0, T_1])\}$ are connected sets.*

However, pairs of atypical points in $\mathbf{m}_\infty$ connected by more than one geodesic do exist. Moreover, the following result quantifies the "amount" of such points using the Hausdorff dimension (see Section 2.1). Here, consider the fact that $\dim_{\mathrm{H}}(\mathbf{m}_\infty) = 4$ (see [8]), so that $\dim_{\mathrm{H}}(\mathbf{m}_\infty \times \mathbf{m}_\infty) = 8$.

Theorem 1.31 (Theorem 1.6 in [15]). $\mathbb{N}_0$-*a.e. the following holds. Let* Φ_i *be the set of pairs* $(u, v) \in \mathbf{m}_\infty \times \mathbf{m}_\infty$ *with* $u \neq v$*, such that* u *and* v *are connected by exactly* i *geodesics. If* $i \geq 10$, $\Phi_i = \emptyset$*. On the contrary,* Φ_7, Φ_8 *and* Φ_9 *are countably infinite, and the Hausdorff dimensions for* $i \in \{1, \ldots, 9\}$ *are:*

i	1	2	3	4	5	6	7	8	9
$\dim_{\mathrm{H}}(\Phi_i)$	8	6	4	4	2	2	0	0	0

In fact, Chapter 3 will be devoted to the computation of the Hausdorff dimension of another set of atypical points in $\mathbf{m}_\infty$, called *geodesic* m*-stars*. These points are such that m disjoint geodesics emerge from them, and they are not typical since we already discussed that geodesics towards typical points must coalesce before reaching them (one can replace x_0 in Proposition 1.28 by a typical point).

1.3 Related constructions

As claimed at the beginning of this chapter, the Brownian sphere is an universal model for surfaces with the topology of the two-dimensional sphere chosen uniformly at random. However, we can change the topological constraint of the sphere to other planar topologies, such as the plane, half-plane, disk, among others. The random metric spaces constructed using similar procedures to the one used to construct the Brownian sphere give raise to a whole family of canonical models of planar surfaces. This framework is now known as *Brownian geometry*. In the next sections, we briefly describe how can we obtain some variants of the Brownian sphere.

1.3.1 Brownian plane

The Brownian plane is the plane topology version of the Brownian sphere, and it is obtained by essentially replacing the Brownian excursion by a Bessel process in $\mathbb{R}$. Formally, let $(R_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ be two independent Bessel processes started at 0. Define $(X_t)_{t\in\mathbb{R}}$ by setting $X_t = R_t$ if $t \geq 0$ and $X_t = L_t$ if $t < 0$. For all $s, t \in \mathbb{R}$, define

$$d_X(s,t) := \begin{cases} X_s + X_t - 2 \inf\limits_{[s\wedge t, s\vee t]} X, & \text{if } t \text{ and } s \text{ have the same sign,} \\ X_s + X_t - 2 \inf\limits_{(-\infty, s\wedge t]\cup[s\vee t,\infty)} X, & \text{otherwise.} \end{cases}$$

Analogously to Proposition 1.2, $d_X : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defines a random pseudo-distance in $\mathbb{R}$. Consider the non-compact real tree $\mathcal{T}(\infty) := \mathbb{R}/\{d_X = 0\}$, endowed with the distance induced by d_X, still denoted by d_X. Now, we assign labels to $\mathcal{T}(\infty)$ and obtain a metric structure from them. As in Section 1.1.2 define for all $s, t \in \mathbb{R}$

$$m_X(s,t) := \begin{cases} \inf_{[s\wedge t, s\vee t]} X, & \text{if } t \text{ and } s \text{ have the same sign,} \\ \inf_{(-\infty, s\wedge t]\cup[s\vee t, \infty)} X, & \text{otherwise.} \end{cases}$$

Conditionally on X, let $Z = (Z_t)_{t\in\mathbb{R}}$ be a centered Gaussian process with covariance $m_X(\cdot,\cdot)$ (still defines a symmetric positive-definite function). For all $s, t \in \mathbb{R}$ define

$$D^\circ_\infty(s,t) := Z_s + Z_t - 2 \min_{[s\wedge t, s\vee t]} Z,$$

and extend this function to $\mathcal{T}(\infty)$ by setting for all $a, b \in \mathcal{T}(\infty)$,

$$D^\circ_\infty(a,b) := \inf\{D^\circ_\infty(s,t) : s,t \in \mathbb{R} \text{ such that } p_X(s) = a,\ p_X(t) = b\}.$$

The function $D^\circ_\infty : \mathcal{T}(\infty)\times\mathcal{T}(\infty) \to \mathbb{R}_+$ is positive and symmetric, but it does not satisfy the triangle inequality. Let D_∞ be the largest pseudo-distance bounded by D°_∞ and set $\mathcal{P} := \mathcal{T}(\infty)/\{D_\infty = 0\}$. Endow $\mathcal{P}$ with the distance induced by D_∞, still denoted by D_∞. The *Brownian plane* is defined as the random metric space $(\mathcal{P}, D_\infty)$. See Figure 1.7.

1.3.2 Brownian disk

The Brownian disk is the unit disk topology version of the Brownian plane, and it is obtained by considering killed Brownian motions at a fixed negative level. Fix $z > 0$, let $B = (B_t)_{t\geq 0}$ be a Brownian motion and define $I_t := \inf_{[0,t]} B$. Let $Y^\circ = (Y^\circ_t)_{t\geq 0}$ be such that, conditionally on B, Y° is a centered Gaussian process with covariance given for all $s, t \geq 0$ by

$$\mathbb{E}[Y^\circ_s Y^\circ_t | B] = \inf_{r\in[s\wedge t, s\vee t]} (B_r - I_r).$$

Let $\mathrm{b} = (\mathrm{b}_t)_{0\leq t\leq z}$ be a Brownian bridge with duration z and $\mathrm{b}_0 = \mathrm{b}_r = 0$, independent of (X, Y°). Let $T_z := \inf\{t \geq 0 : B_t = -z\}$ and define for all $t \in [0, T_z]$,

$$Y_t = Y^\circ_t + \sqrt{3}\mathrm{b}_{-X_t}.$$

In this construction, the process $(Y_t)_{t\in[0,T_z]}$ represents the labels assigned to the real tree induced by the process $(B_t - I_t)_{t\in[0,z]}$. Using this, define for all $s, t \in [0, T_z]$,

$$D^\circ_\partial(s,t) := Y_s + Y_t - 2\max\left\{\min_{[s\wedge t, s\vee t]} Y, \min_{[0,s\wedge t]\cup[s\vee t, T_z]} Y\right\}.$$

Let D_∂ be the largest pseudo-distance bounded by D°_∂ and set $\mathbb{D}_z := [0, T_z]/\{D_\partial = 0\}$. Endow $\mathbb{D}_z$ with the distance induced by D_∂, still denoted by D_∂. The *Brownian disk of perimeter z* is defined as the random metric space $\mathbb{D}_z := [0, T_z]/\{D_\partial = 0\}$. See Figure 1.7.

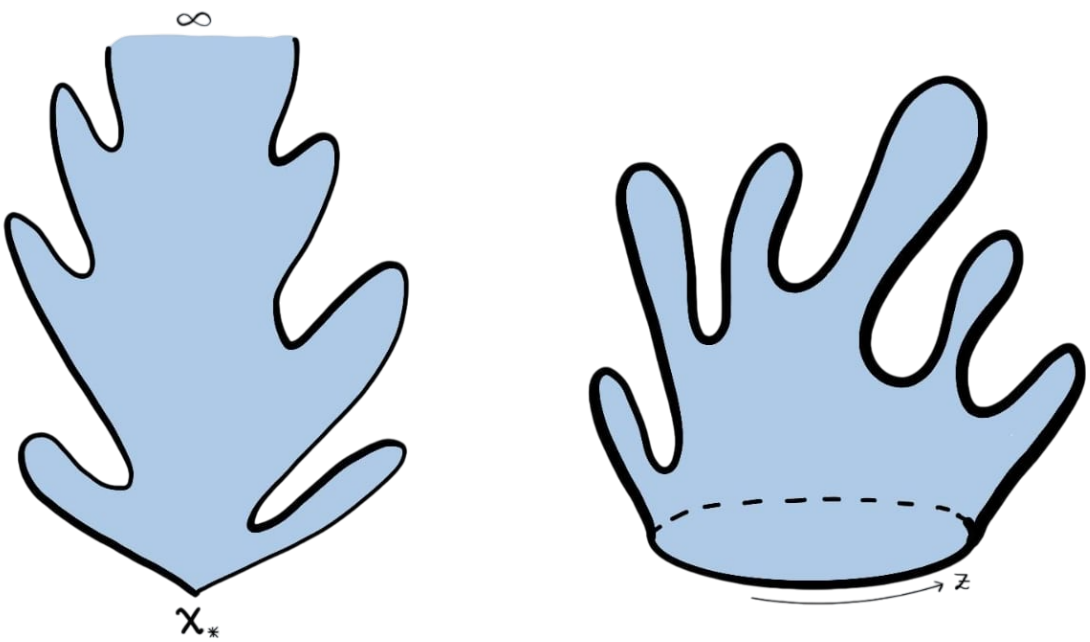

Figure 1.7: Cactus representation of the Brownian plane and disk.

1.3.3 Hulls

Let $r, z > 0$ be fixed. Let $\mathcal{N} = \sum_{i \in I} \delta_{(t_i, \omega_i)}$ be a Poisson point measure with intensity

$$\mathbb{1}_{[0,z]}(t)\mathrm{d}t \otimes \mathbb{N}_0(\mathrm{d}\omega \cap \{W_* > -r\}),$$

where $t_i \in [0, z]$ and we write $\zeta_i = \zeta(\omega_i)$. Let ω_* be sampled from $\mathbb{N}_0(\cdot \cap \{W_* = -r\})$, write $\zeta_* = \zeta(\omega_*)$ and U_* be uniformly distributed over $[0, z]$. Assume that $\mathcal{N}$, ω_* and U_* are independent. Let $\mathbf{H}$ be derived from the disjoint union

$$[0, z] \cup \left(\bigcup_{i \in I} \mathcal{T}(\zeta_i) \right) \cup \mathcal{T}(\zeta_*),$$

by identifying 0 with z, the root of $\mathcal{T}(\zeta_i)$ with t_i, for each $i \in I$, and the root of $\mathcal{T}(\zeta_*)$ with U_*. Define

$$\Sigma = \sum_{i \in I} \sigma(\zeta_i) + \sigma(\zeta_*).$$

Now we construct a pseudo-distance in $\mathbf{H}$. We proceed in steps as follows:

- *Labels in* $\mathbf{H}$: Let us define the labels $(\Lambda_a)_{a \in \mathbf{H}}$ on $\mathbf{H}$. For $a \in [0, z]$ set $\Lambda_a = 0$. For $a \in \mathcal{T}(\zeta_i)$ with $i \in I$, define $\Lambda_a = \ell_a(\omega_i)$. For $a \in \mathcal{T}(\zeta_*)$, define $\Lambda_a = \ell_a(\omega_*)$.

- *Cyclic exploration of* $\mathbf{H}$: Let $(\mathcal{E}_s)_{s \in [0, \Sigma]}$ be a cyclic exploration of $\mathbf{H}$ such that $\mathcal{E}_0 = \mathcal{E}_\Sigma = 0$ and the exploration discovers in counterclockwise order each tree $\mathcal{T}(\zeta_i)$ attached to $[0, z]$ in the order prescribed by $\{t_i\}_{i \in I} \cup \{U_*\}$ (see Figure 1.8).

- *Intervals in* $\mathbf{H}$: For $s,t \in [0,\Sigma]$, we let $[s,t] = [s,\Sigma] \cup [0,t]$ if $t < s$, otherwise it denotes the usual interval. For each $a,b \in \mathbf{H}$, we can find $s,t \in [0,\Sigma]$ such that $\mathcal{E}_s = a$, $\mathcal{E}_t = b$ and $[s,t]$ is small as possible. In this situation, we set $[a,b]_{\mathbf{H}} := \{\mathcal{E}_r : r \in [s,t]\}$.

- *Distance in* $\mathbf{H}$: For all $a,b \in \mathbf{H}$, set

$$D^{\circ}_{\mathbf{H}}(a,b) := \Lambda_a + \Lambda_b - 2\max\left\{\min_{c\in[a,b]_{\mathbf{H}}} \Lambda, \min_{c\in[b,a]_{\mathbf{H}}} \Lambda_c\right\}. \tag{1.14}$$

  The function $D^{\circ}_{\mathbf{H}} : \mathbf{H} \times \mathbf{H} \to \mathbb{R}_+$ is positive and symmetric, but it does not satisfy the triangle inequality. Let $D_{\mathbf{H}}$ be the largest pseudo-distance bounded by $D^{\circ}_{\mathbf{H}}$ and set $\mathfrak{H}_{r,z} := \mathbf{H}/\{D_{\mathbf{H}} = 0\}$. Endow $\mathfrak{H}_{r,z}$ with the distance induced by $D_{\mathbf{H}}$, still denoted $D_{\mathbf{H}}$.

The *hull of radius* r *and perimeter* z is defined as the random metric space $(\mathfrak{H}_{r,z}, D_{\mathbf{H}})$.

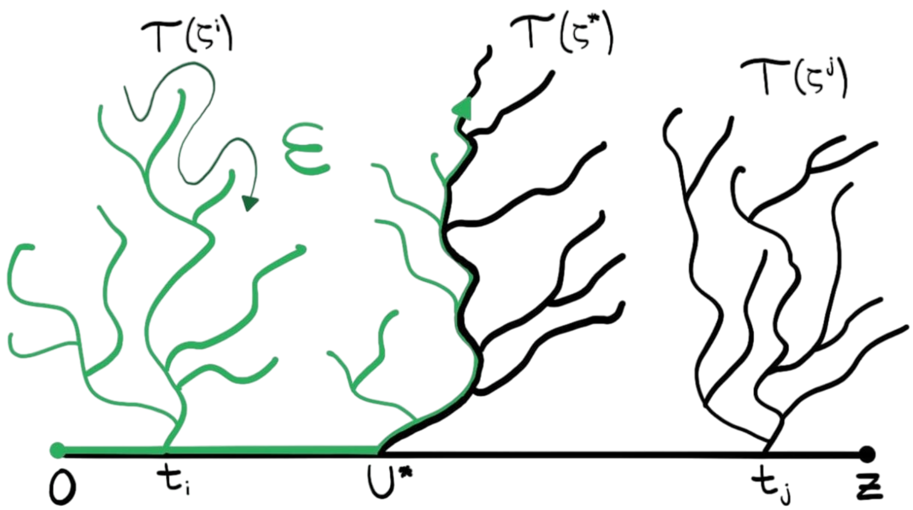

Figure 1.8: Representation of $\mathbf{H}$ and the exploration $(\mathcal{E}_s)_{s\in[0,\Sigma]}$.

Let $a_* \in \mathcal{T}(\zeta_*)$ be the unique point in $\mathbf{H}$ with label $-r$ and $b_* = \Pi_{\mathbf{H}}(a_*)$, where $\Pi_{\mathbf{H}} : \mathbf{H} \to \mathfrak{H}_{r,z}$ is the canonical projection. We will define geodesics from an arbitrary point of $\mathfrak{H}_{r,z}$ to b_*, for which the heuristic is the same of Section 1.2.5. Let $x \in \mathfrak{H}_{r,z}$, and let $a \in \mathbf{H}$ be such that $\Pi_{\mathbf{H}}(a) = x$, and $s \in [0,\Sigma]$ be such that $\mathcal{E}_s = a$.

- If $s \in [0, s_*]$, define for all $t \in [0, \Lambda_a + r]$,

$$\gamma_s(t) := \Pi_{\mathbf{H}}\left(\mathcal{E}_{\inf\{u\geq s:\Lambda_{\mathcal{E}_u}=\Lambda_a - t\}}\right). \tag{1.15}$$

- If $s \in [s_*, \Sigma]$, define for all $t \in [0, \Lambda_a + r]$,

$$\gamma_s(t) := \Pi_{\mathbf{H}}\left(\mathcal{E}_{\sup\{u\leq s:\Lambda_{\mathcal{E}_u}=\Lambda_a - t\}}\right). \tag{1.16}$$

In both cases, it can be verified that γ_s is a geodesic from x to b_*.

1.3.4 Slices

Slices are defined in a similar way to the Brownian sphere, with the slight difference that this metric space will have two boundaries corresponding to a "cut" of the geodesic connecting x_0 with x_*. Formally, let ω be a Brownian snake and $(\zeta, \widehat{W})$ be the associated tree-like path. For all $s, t \in [0, \sigma(\zeta)]$, define

$$\widetilde{D}^\circ(s,t) = \widehat{W}_s + \widehat{W}_t - 2 \min_{[s\wedge t, s\vee t]} \widehat{W}, \tag{1.17}$$

and extend this function to $\mathcal{T}(\zeta)$ by setting for all $a, b \in \mathcal{T}(\zeta)$,

$$\widetilde{D}^\circ(a,b) := \inf\{\widetilde{D}^\circ(s,t) : s,t \in [0,\sigma(\zeta)] \text{ such that } p_\zeta(s) = a,\ p_\zeta(t) = b\}.$$

Let $\widetilde{D}$ be the maximal pseudo-distance bounded above by $\widetilde{D}^\circ$ and set $\mathbf{S} := \mathbf{S}(\omega) := \mathcal{T}(\zeta)/\{\widetilde{D} = 0\}$, that we endow with the distance induced by $\widetilde{D}$, still denoted by $\widetilde{D}$. The *slice* is defined as the random metric space $(\mathbf{S}, \widetilde{D})$.

Let us justify what we claimed at the beginning. Let $\widetilde{\Pi} : \mathcal{T}(\zeta) \to \mathbf{S}$ be the canonical projection and define $\widetilde{x}_0 := \widetilde{\Pi}(\rho_\zeta)$ and $\widetilde{x}_* := \widetilde{\Pi}(a_*)$. Recall that, under $\mathbb{N}_0(\cdot | W_* = -h)$, the unique geodesic connecting x_0 and x_* in $\mathbf{m}_\infty$ is the path $\gamma = (\gamma(r))_{r\in[0,h]}$ defined by

$$\gamma(r) = \Pi(p_\zeta(\inf\{s \in [0,\sigma(\zeta)] : \widehat{W}_s = -r\})) = \Pi(p_\zeta(\sup\{s \in [0,\sigma(\zeta)] : \widehat{W}_s = -r\})).$$

However, in $\mathbf{S}$ we have the following result.

Lemma 1.32. *For all $a, b \in \mathcal{T}(\zeta)$, $\widetilde{D}(a,b) = 0$ implies $D(a,b) = 0$. However, $D(a,b) = 0$ implies $\widetilde{D}(a,b) = 0$ whenever a and b do not belong to the range of γ.*

Proof. Note that $\widetilde{D}^\circ(a,b) \geq D^\circ(a,b)$, implying $\widetilde{D}(a,b) \geq D(a,b)$ so that $\widetilde{D}(a,b) = 0$ implies $D(a,b) = 0$. On the other hand, assume that $D(a,b) = 0$. By Proposition 1.13, we have $D^\circ(a,b) = 0$, so that without loss of generality we can assume

$$\ell_a = \ell_b = \min_{[a,b]} \ell. \tag{1.18}$$

Let $s, t \in [0, \sigma]$ be such that $p_\zeta(s) = a$, $p_\zeta(t) = b$ and $[s,t]$ small as possible, with the convention $[s,t] = [0,t] \cup [s,\sigma]$ if $t < s$. If $s \leq t$, (1.18) is equivalent to $\widetilde{D}(a,b) = 0$. If $t < s$, (1.18) is equivalent to

$$\widehat{W}_s = \widehat{W}_t, \quad \widehat{W}_s = \min_{[s,\sigma]} \widehat{W} \quad \text{and} \quad \widehat{W}_t = \min_{[0,t]} \widehat{W},$$

so that $a, b \in \gamma([0,h])$. However, it turns out that $\widetilde{D}(a,b) > 0$ by definition of $\widetilde{D}$. □

The conclusion of the previous lemma is that every point in $\mathbf{m}_\infty \setminus \gamma([0,h])$ corresponds to a unique point in $\mathbf{S}$, and every point in $\gamma([0,r])$ corresponds to two points in $\mathbf{S}$. As a consequence, if we define $\gamma' = (\gamma'(r))_{r\in[0,h]}$ and $\gamma'' = (\gamma''(r))_{r\in[0,h]}$ by

$$\gamma'(r) := p_\zeta(\inf\{s \in [0,\sigma(\zeta)] : \widehat{W}_s = -r\}), \tag{1.19}$$

$$\gamma''(r) := p_\zeta(\sup\{s \in [0,\sigma(\zeta)] : \widehat{W}_s = -r\}), \tag{1.20}$$

then γ' and γ'' are geodesics from $\widetilde{x}_0$ to $\widetilde{x}_*$ such that $\gamma'((0,h))$ and $\gamma''((0,h))$ are disjoint, called *left boundary* and *right boundary* of $\mathbf{S}$, respectively.

1.4 Markov property

Suppose that we have launched an exploration from x_* to x_0 by continuously following the boundaries of the hulls $B^{\bullet(x_0)}(x_*,r)$, with $r > 0$ increasing up to $D(x_0,x_*)$. The Markov property of the Brownian sphere states that conditionally on the boundary length, $B^{\bullet(x_0)}(x_*,r)$ and the closure of its complement are independent, with $B^{\bullet(x_0)}(x_*,r)$ distributed as a hull and (the closure of) its complement distributed as a Brownian disk. Let us first discuss some concepts that appear in the statement of the Markov property.

- First, we need to introduce the *intrinsic distance*. If O is an open connected subset of $\mathbf{m}_\infty$, the intrinsic distance D^O_{int} is defined as follows. For $x,y \in O$, $D^O_{\text{int}}(x,y)$ is equal to the infimum of the lengths corresponding to the set of continuous paths staying in O connecting x and y.

- We need an extension of the spaces $\mathbb{K}^\bullet$ and $\mathbb{K}^{\bullet\bullet}$ used before. We let $\mathbb{K}^{\bullet b}$ be the space of equivalence classes for $\{d_{\text{GHP}} = 0\}$ of 5-tuples of the form (X,d,μ,x_*,F), where (X,d) is a compact metric space, μ is a finite Borel measure on X, x_* is the distinguished point and F is a fixed subset of X called *distinguished boundary*. Here, d_{GHP} stands for the natural extension of (1.3) to this context.

With these objects, the Markov property of the Brownian sphere can be stated as follows. We refer to the original reference for a proof.

Theorem 1.33 (Theorem 8 in [7]). *With $\mathbb{N}_0^{[r]}$-probability one, $D_{\text{int}}^{B^{\bullet(x_0)}(x_*,r)}$ has a continuous extension to $B^{\bullet(x_0)}(x_*,r)$, which is a metric on $B^{\bullet(x_0)}(x_*,r)$. Similarly, $D_{\text{int}}^{\mathbf{m}_\infty\setminus B^{\bullet(x_0)}(x_*,r)}$ has a continuous extension to $\overline{\mathbf{m}_\infty \setminus B^{\bullet(x_0)}(x_*,r)}$, which is a metric on $\overline{\mathbf{m}_\infty \setminus B^{\bullet(x_0)}(x_*,r)}$. Consider both $B^{\bullet(x_0)}(x_*,r)$ and $\overline{\mathbf{m}_\infty \setminus B^{\bullet(x_0)}(x_*,r)}$ as metric spaces for these extended intrinsic metrics. Then, $B^{\bullet(x_0)}(x_*,r)$ equipped with the restricted volume measure on $\mathbf{m}_\infty$, distinguished point x_* and distinguished boundary $\partial B^{\bullet(x_0)}(x_*,r)$ is a random variable with values in $\mathbb{K}^{\bullet b}$, and the same holds for $\overline{\mathbf{m}_\infty \setminus B^{\bullet(x_0)}(x_*,r)}$. Furthermore, for all positive measurable functions F and G defined on $\mathbb{K}^{\bullet b}$, we have for all $r,z>0$ that*

$$\mathbb{N}_0^{[r]}\left(F\left(B^{\bullet(x_0)}(x_*,r)\right) G\left(\overline{\mathbf{m}_\infty \setminus B^{\bullet(x_0)}(x_*,r)}\right) \middle| Z_{W_*+r} = z\right) = \mathbb{E}[F(\mathfrak{H}_{r,z})]\mathbb{E}[G(\mathbb{D}_z)].$$

The following two-pointed version will be also very useful. Let $Z_r^{x_*(x_0)}$ be the boundary length of $B^{\bullet(x_0)}(x_*, r)$ (equal to Z_{W_*+r}) and $Z_r^{x_0(x_*)}$ be the boundary length of $B^{\bullet(x_*)}(x_0, r)$. Denote $\Theta_{r,z}$ the distribution of the hull $\mathfrak{H}_{r,z}$ and

$$\mathcal{C}_r^{x_*,x_0} := \mathbf{m}_\infty \setminus (B^{\bullet(x_0)}(x_*, r) \cup B^{\bullet(x_*)}(x_0, r)). \tag{1.21}$$

Analogously to the extension from $\mathbb{K}^{\bullet b}$ to $\mathbb{K}^{\bullet b}$, we let $\mathbb{K}^{bb}$ be the space of (equivalence classes for $\{d_{\mathrm{GHP}} = 0\}$ of) compact measure metric spaces with two distinguished boundaries. We refer to the original reference for a proof of the following statement.

Theorem 1.34 (Corollary 9 in [7])**.** *With full $\mathbb{N}_0(\cdot \cap \{D(x_0, x_*) > 2r\})$-measure, $D_{\mathrm{int}}^{\mathcal{C}_r^{x_*,x_0}}$ has a continuous extension to $\overline{\mathcal{C}_r^{x_*,x_0}}$, which is a metric on this space. We equip this metric space with the restriction of the volume measure on $\mathbf{m}_\infty$, and the distinguished boundaries $\partial B^{\bullet(x_0)}(x_*, r)$ and $\partial B^{\bullet(x_*)}(x_0, r)$. Then, this space is a random variable with values in $\mathbb{K}^{bb}$. Furthermore, for all positive measurable functions F_1, F_2 and G defined on $\mathbb{K}^{bb}$, we have for all $r, z > 0$ that*

$$\mathbb{N}_0^{[2r]}\left(F_1\left(B^{\bullet(x_0)}(x_*, r)\right) F_2\left(B^{\bullet(x_*)}(x_0, r)\right) G\left(\overline{\mathcal{C}_r^{x_*,x_0}}\right)\right) = \mathbb{N}_0^{[2r]}\left(\Theta_{r,Z_r^{x_*(x_0)}}(F_1)\Theta_{r,Z_r^{x_0(x_*)}}(F_2) G\left(\overline{\mathcal{C}_r^{x_*,x_0}}\right)\right).$$

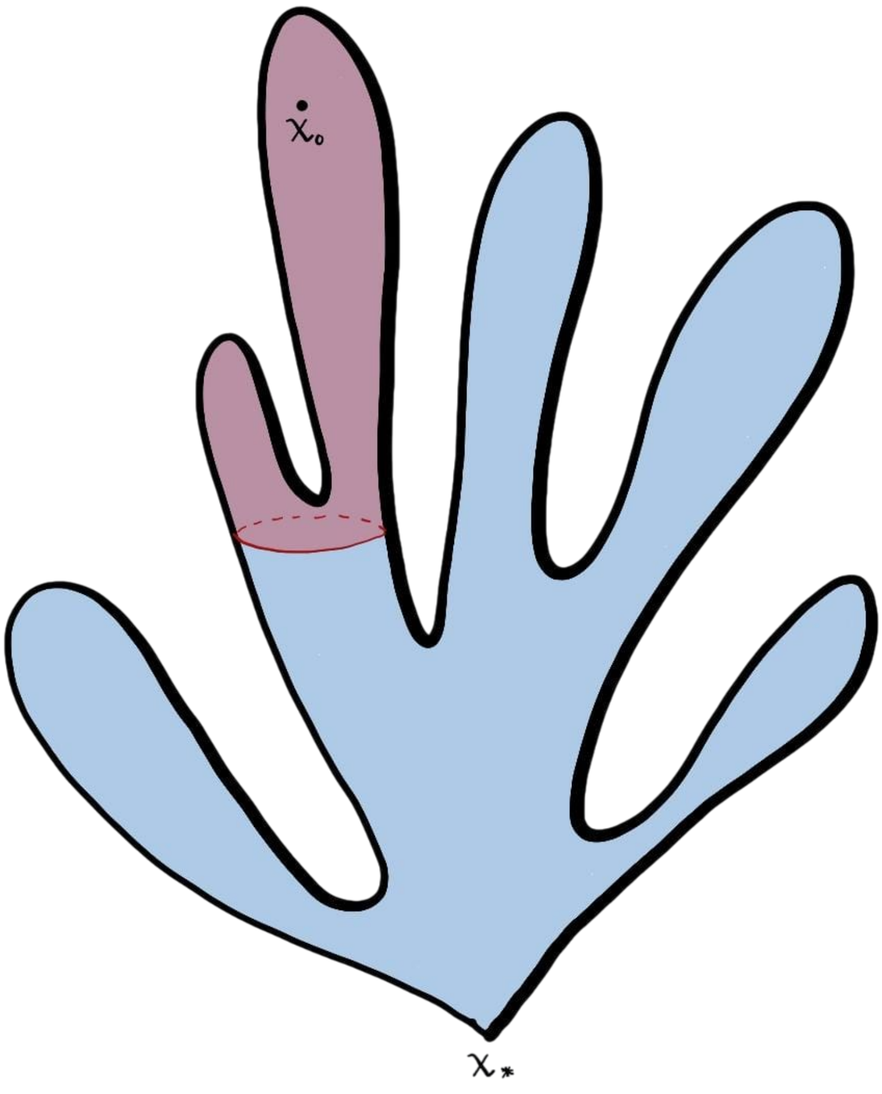

Figure 1.9: The Markov property of the Brownian sphere states that, conditionally on the length of the red "loop" in the picture, the blue and purple shaded parts of $\mathbf{m}_\infty$ are independent and distributed as a hull and a Brownian disk, respectively.

Chapter 2

Preliminaries on the Hausdorff dimension

In this chapter, we define the Hausdorff dimension of a bounded metric space and its relation with finite measures, namely, through Frostman lemma and the energy method. Then we state and present the proof of Evans theorem on the Hausdorff dimension of the Brownian cone points, as a warm-up towards the next chapter concerning the Hausdorff dimension of a subset of the Brownian sphere. The content of this chapter is completely based on [17].

2.1 Definition and main properties

We recall the main concepts used to define the Hausdorff dimension of a set. From now on, fix (E, d) a bounded metric space, that is, $\sup\{d(x,y) : x, y \in E\} < \infty$. The *$\alpha$-Hausdorff content* of E is defined as

$$\mathcal{H}^{\alpha}_{\infty}(E) := \inf \left\{ \sum_{k \in \mathbb{N}} \operatorname{diam}(A_k)^{\alpha} : (A_k)_{k \in \mathbb{N}} \text{ covering of } E \right\}.$$

Here, the quantity $\sum_{k\in\mathbb{N}} \operatorname{diam}(A_k)^{\alpha}$ is called α-**value** of $(A_k)_{k\in\mathbb{N}}$. Observe that if $0 \le \alpha \le \beta$, then $\mathcal{H}^{\alpha}_{\infty}(E) = 0$ implies $\mathcal{H}^{\beta}_{\infty}(E) = 0$. Thus we define the *Hausdorff dimension* as

$$\dim_{\mathrm{H}}(E) := \inf\{\alpha \ge 0 : \mathcal{H}^{\alpha}_{\infty}(E) = 0\} = \sup\{\alpha \ge 0 : \mathcal{H}^{\alpha}_{\infty}(E) > 0\} \tag{2.1}$$

Additionally, for all $\delta \ge 0$ we introduce

$$\mathcal{H}^{\alpha}_{\delta}(E) := \inf \left\{ \sum_{k \in \mathbb{N}} \operatorname{diam}(A_k)^{\alpha} : (A_k)_{k \in \mathbb{N}} \text{ covering of } E \text{ with } \operatorname{diam}(A_k) \le \delta \text{ for all } k \in \mathbb{N} \right\},$$

and the *Hausdorff measure* defined by $\mathcal{H}^{\alpha}(E) := \lim_{\delta \searrow 0} \mathcal{H}^{\alpha}_{\delta}(E)$. We can express the Hausdorff dimension in terms of the Hausdorff measure, thanks to the following result. We refer to [17] for a proof.

Proposition 2.1 (Proposition 4.9 in [17]). *For each $\alpha \geq 0$, $\mathcal{H}^\alpha_\infty(E) = 0$ if, and only if $\mathcal{H}^\alpha(E) = 0$. Consequently,*

$$\begin{aligned}\dim_{\mathrm{H}}(E) &= \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(E) = 0\} = \sup\{\alpha \geq 0 : \mathcal{H}^\alpha(E) < \infty\} \\ &= \sup\{\alpha \geq 0 : \mathcal{H}^\alpha(E) > 0\} = \sup\{\alpha \geq 0 : \mathcal{H}^\alpha(E) = \infty\}.\end{aligned}$$

We record two simple but useful properties of the Hausdorff dimension, namely, monotonicity and countable stability. The latter allows us to simplify the computation of the Hausdorff dimension of a set built as a countable union, by computing the individual Hausdorff dimensions of the sets in the union.

Proposition 2.2.

- *(Monotonicity) If $A, B \subseteq E$ are such that $A \subseteq B$, then* $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{H}}(B)$.
- *(Countable stability) Let $E_1, E_2, \dots \subseteq E$. Then, it holds that*

$$\dim_{\mathrm{H}}\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sup_{k\in\mathbb{N}} \dim_{\mathrm{H}}(E_k). \tag{2.2}$$

Proof. For the monotonicity, if $A \subseteq B$ and $(B_k)_{k\in\mathbb{N}}$ is a covering of B, then $(B_k)_{k\in\mathbb{N}}$ is also a covering of A. For any $\alpha \geq 0$, this gives that $\mathcal{H}^\alpha_\infty(A) \leq \sum_{k\in\mathbb{N}} \operatorname{diam}(B_k)^\alpha$ and therefore $\mathcal{H}^\alpha_\infty(A) \leq \mathcal{H}^\alpha_\infty(B)$. From this inequality, $\{\alpha \geq 0 : \mathcal{H}^\alpha_\infty(B) = 0\} \subseteq \{\alpha \geq 0 : \mathcal{H}^\alpha_\infty(A) = 0\}$ and $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{H}}(B)$ follows from (2.1).

For the countable stability, suppose first that both sides of (2.2) are finite. The inequality

$$\dim_{\mathrm{H}}\left(\bigcup_{k\in\mathbb{N}} E_k\right) \geq \sup_{k\in\mathbb{N}} \dim_{\mathrm{H}}(E_k)$$

follows by monotonicity. For the converse inequality, if $(A_i^k)_{i\in\mathbb{N}}$ is a covering of E_k for each $k \in \mathbb{N}$, then $(A_i^k)_{i,k\in\mathbb{N}}$ is a covering of $\bigcup_{k\in\mathbb{N}} E_k$. If $\alpha > 0$ is given, it holds that

$$\mathcal{H}^\alpha_\infty\left(\bigcup_{k\in\mathbb{N}} E_k\right) \leq \sum_{k\in\mathbb{N}} \sum_{i\in\mathbb{N}} \operatorname{diam}(A_i^k)^\alpha.$$

Taking infimum for each $k \in \mathbb{N}$, over all the coverings $(A_i^k)_{i\in\mathbb{N}}$ of E_k, gives

$$\mathcal{H}^\alpha_\infty\left(\bigcup_{k\in\mathbb{N}} E_k\right) \leq \sum_{k\in\mathbb{N}} \mathcal{H}^\alpha_\infty(E_k).$$

If $\alpha > \sup_{k\in\mathbb{N}} \dim_{\mathrm{H}}(E_k)$, we see from (2.1) that all the terms in the last sum vanish, so that $\mathcal{H}^\alpha_\infty\left(\bigcup_{k\in\mathbb{N}} E_k\right) = 0$ and $\dim_{\mathrm{H}}\left(\bigcup_{k\in\mathbb{N}} E_k\right) \leq \alpha$. Letting $\alpha \searrow \sup_{k\in\mathbb{N}} \dim_{\mathrm{H}}(E_k)$ gives (2.2). From the proof, we can easily see how to adapt it to the cases where one side of (2.2) is infinite. □

It follows from (2.1) that to give an upper bound for the Hausdorff dimension of a set, it suffices to find a suitable covering and compute its α-value. Now, we will present methods to give lower bounds for the Hausdorff dimension. The first method will be the *mass distribution principle*, that allows us to give a lower bound for the Hausdorff dimension by finding a measure that is dominated by the diameter to some power.

Theorem 2.3 (Mass distribution principle, Theorem 4.19 in [17])**.** *Let $\alpha \geq 0$. Suppose there exists a nonzero finite measure μ on E and constants $C, \delta > 0$ such that for all $V \subseteq E$ closed such that* $\operatorname{diam}(V) \leq \delta$, $\mu(V) \leq C \operatorname{diam}(V)^\alpha$. *Then,*

$$\mathcal{H}^\alpha_\delta(E) \geq \frac{\mu(E)}{C} > 0.$$

In particular, $\dim_{\mathrm{H}}(E) \geq \alpha$.

Proof. Let $(E_k)_{k\in\mathbb{N}}$ be a covering of E such that $\operatorname{diam}(E_k) \leq \delta$ for all $k \in \mathbb{N}$. Then,

$$0 < \mu(E) \leq \sum_{k\in\mathbb{N}} \mu(\overline{E_k}) \leq C \sum_{k\in\mathbb{N}} \operatorname{diam}(\overline{E_k})^\alpha = \sum_{k\in\mathbb{N}} \operatorname{diam}(E_k)^\alpha.$$

Taking infimum over $(E_k)_{k\in\mathbb{N}}$ gives $\mathcal{H}^\alpha_\delta(E) \geq \mu(E)/C > 0$ and $\dim_{\mathrm{H}}(E) \geq \alpha$ follows by Proposition 2.1. □

We can state a converse of the mass distribution principle known as Frostman lemma. The metric space taken here is $\mathbb{R}^d$ with any distance, say, the Euclidian distance. We refer to [17] for a proof.

Theorem 2.4 (Frostman lemma, Theorem 4.30 in [17])**.** *If $A \subseteq \mathbb{R}^d$ is a closed set with $\mathcal{H}^\alpha(A) > 0$, then there exists a probability measure μ on A and a constant $C > 0$ such that $\mu(D) \leq C \operatorname{diam}(D)^\alpha$ for all $D \subseteq A$ Borel measurable.*

The last method to lower bound the Hausdorff dimension that we will present needs the concept of energy of a measure. Given a nonzero finite measure μ on E, the α**-energy of** μ is defined as

$$I_\alpha(\mu) := \iint \frac{\mu(\mathrm{d}x)\mu(\mathrm{d}y)}{d(x,y)^\alpha}.$$

Using this object, the energy method allows us to lower bound the Hausdorff dimension by finding a measure with finite energy.

Theorem 2.5 (Energy method, Theorem 4.27 in [17])**.** *Let $\alpha \geq 0$ and μ be a nonzero finite measure on E. Then, for all $\delta > 0$,*

$$\left(\int_{d(x,y)<\delta} \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{d(x,y)^\alpha} \right)^{-1} \mu(E)^2 \leq \mathcal{H}^\alpha_\delta(E). \tag{2.3}$$

In particular, $I_\alpha(\mu) < \infty$ implies $\dim_{\mathrm{H}}(E) \geq \alpha$.

Proof. Let $\delta > 0$ and $(A_k)_{k\in\mathbb{N}}$ be a pairwise disjoint covering of E. Then,

$$\int_{d(x,y)<\delta} \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{d(x,y)^\alpha} \geq \sum_{k\in\mathbb{N}} \frac{\mu(A_k)^2}{\mathrm{diam}(A_k)^\alpha}.$$

On the other hand, by Cauchy-Schwartz inequality we have

$$\begin{aligned}\mu(E)^2 \leq \left(\sum_{k\in\mathbb{N}} \mu(A_k)\right)^2 &\leq \left(\sum_{k\in\mathbb{N}} \mathrm{diam}(A_k)^\alpha\right)\left(\sum_{k\in\mathbb{N}} \frac{\mu(A_k)^2}{\mathrm{diam}(A_k)^\alpha}\right) \\ &\leq \left(\sum_{k\in\mathbb{N}} \mathrm{diam}(A_k)^\alpha\right) \int_{d(x,y)<\delta} \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{d(x,y)^\alpha}\end{aligned}$$

If $\varepsilon > 0$ is given and $(A_k)_{k\in\mathbb{N}}$ is a pairwise disjoint covering such that $\mathrm{diam}(A_k) \leq \delta$ for each $k \in \mathbb{N}$ and satisfying $\sum_{k\in\mathbb{N}} \mathrm{diam}(A_k)^\alpha \leq \mathcal{H}^\alpha_\delta(E) + \varepsilon$, we deduce from the above computations that

$$\mu(E)^2 \leq (\mathcal{H}^\alpha_\delta(E) + \varepsilon) \int_{d(x,y)<\delta} \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{d(x,y)^\alpha}.$$

Letting $\varepsilon \searrow 0$ gives (2.3). If $I_\alpha(\mu) < \infty$, then the left-hand side of (2.3) diverge as $\delta \searrow 0$, so that $\mathcal{H}^\alpha(E) = \infty$ and therefore $\dim_{\mathrm{H}}(E) \geq \alpha$ by Proposition 2.1. □

2.2 Hausdorff dimension of Brownian cone points

In this section we state and sketch the proof of the result on the so-called Brownian cone points. Let us fix some notation. For $\alpha, \xi \in [0, 2\pi)$, we define the *cone of direction* ξ *and opening angle* α as

$$C[\alpha, \xi] := \{re^{i(\theta+\xi)} : r > 0, |\theta| \leq \alpha/2\}.$$

We will consider cones of the form $x + C[\alpha, \xi]$, in which case x is called the tip of the cone. The dual cone of $x + C[\alpha, \xi]$ is defined as $x + C[2\pi - \alpha, \xi]$.

Throughout this chapter, $W = (W(t))_{t\geq 0}$ will always denote a Brownian motion, $\mathbb{P}_x$ is the law of W starting from x, and $\mathbb{E}_x$ is the expectation with respect to $\mathbb{P}_x$.

Definition 2.6. *Let $W = (W(t))_{t\geq 0}$ be a Brownian motion started at the origin. A point $z = W(t)$, for some $t \in [0, 1]$, is called α-**cone point** if there exists $\xi \in [0, 2\pi)$ and $\varepsilon > 0$ such that*

$$W([0, 1]) \cap B(z, \varepsilon) \subseteq z + C[\alpha, \xi].$$

Informally, an α-cone point is a point z of the Brownian path for which there exists a sufficiently small neighborhood of z, such that the restriction of the path to it is contained in some cone of opening angle α.

The α-cone points are atypical points, since with full probability the Brownian motion does an infinite number of windings around typical points of its trajectory. However, their existence is a non-trivial question was completely answered by Evans in the following theorem.

Theorem 2.7. *Almost surely, the following properties hold:*

1. *If $\alpha \in [0, \pi)$, α-cone points do not exist.*

2. *If $\alpha \in [\pi, 2\pi)$, α-cone points exist, and moreover,*

$$\dim_{\mathrm{H}}(\{z \in \mathbb{R}^2 : z \text{ is an } \alpha\text{-cone point}\}) = 2 - \frac{2\pi}{\alpha}.$$

Note that cone points exist for the critical dimension $\alpha = \pi$, but they have Hausdorff dimension equal to 0. In the next sections, we will present the proof of Theorem 2.7 using the following scheme:

1. **Approximation of the cone points:** Since the α-cone points are atypical, it is better to introduce an approximation of them to prove the result. It will be clear from the pictures why it is the good approximation. In fact, the cone points can be easily recovered by intersecting approximate cone points.

2. **Estimates for the approximation:** It is possible to compute explicit bounds for the probability of being an approximate cone point using the strong Markov property of W.

3. **Upper bound:** Using the estimates for the approximation, we estimate the probability that any dyadic cube contains an approximate cone point. This yields estimates for both the probability of the approximate cone points to be a non-empty set when $\alpha \in (0, \pi)$, and the γ-value of this covering when $\alpha \in [\pi, 2\pi)$. In the latter case, this gives the upper bound for the Hausdorff dimension as previously discussed.

4. **Lower bound:** We rely on an auxilliary theorem concerning the Hausdorff dimension of random sets. Applied to the approximate cone points, this gives that the lower bound of the Hausdorff dimension holds true with positive probability for a subset of the α-cone points, which after Blumenthal's law it turns out that it holds almost surely.

2.2.1 Approximation of the cone points

For all $z \in \mathbb{R}^2$, $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$, define

$$\begin{aligned} T_\delta(z) &:= \inf\{t \geq 0 : W(t) \in B(z, \delta)\}, \\ S_{\delta,\varepsilon}(z) &:= \inf\{t \geq T_{\delta/2}(z) : W(t) \notin B(z, \varepsilon)\}, \\ S_\varepsilon^{(r)}(z) &:= \inf\{t \geq r : W(t) \notin B(z, \varepsilon)\}. \end{aligned}$$

Definition 2.8. *Let $\alpha, \xi \in [0, 2\pi)$ and $\varepsilon > 0$. For each $\delta \in (0, \varepsilon)$, we call $z \in \mathbb{R}^2$ a (δ, ε)-**approximate cone point with direction** ξ if*

$$W([0, T_\delta(z)]) \subseteq z + C[\alpha, \xi] \quad \text{and} \quad W([T_{\delta/2}(z), S_{\delta,\varepsilon}(z)]) \subseteq z + C[\alpha, \xi].$$

Note that a (δ, ε)-approximate α-cone point is not necessarily a point of the Brownian path. From this definition we see that the approximation is well chosen, since an (actual) α-cone point is roughly a (δ, ε)-approximate α-cone point for all $\delta > 0$ and for some $\varepsilon > 0$ and direction $\xi \in [0, 2\pi)$ which is hidden in these definitions. Therefore, the estimates on the probability of being an approximate cone point will give estimates of the actual cone points through a limit argument.

2.2.2 Estimates for the approximation

Now we estimate the probability of z to be an approximate cone point. To do so, we study separately the trajectories in the definition of approximate cone points, thanks to the Markov property.

Lemma 2.9. *For some constants depending only on α, we have that for all $\delta > 0$ and $z \in \mathbb{R}^2$ with $0 \in z + C[\alpha/2, \xi]$,*

$$\mathbb{P}_0\left(W([0, T_\delta(z)]) \subseteq z + C[\alpha, \xi]\right) \asymp \left(\frac{\delta}{|z|}\right)^{\frac{\pi}{\alpha}}.$$

Proof. First, we derive an explicit formula for $\mathbb{P}_0(W([0, T_\delta(z)]) \subseteq z + C[\alpha, \xi])$. Write $z = |z|e^{i\theta}$. Apply skew-product representation (Theorem 7.26 in [17]) to the Brownian motion $(z - W(t))_{t \geq 0}$ to write $W(t) = z - R(t)e^{i\theta(t)}$ for all $t \geq 0$, where

- $R(t) = e^{W_1(H(t))}$,
- $\theta(t) = W_2(H(t))$,
- W_1 and W_2 are Brownian motions with $W_1(0) = \log(|z|)$ and $W_2(0) = \theta$,
- $H(t) = \inf\left\{\int_0^u \exp(2W_1(s))\mathrm{d}s > t\right\}$.

Using this, we have $H(T_\delta(z)) = \inf\{u \geq 0 : W_1(u) \leq \log(\delta)\} = \tau_{\log(\delta)}$ and therefore,

$$\{W([0, T_{\delta/2}(z)]) \subseteq z + C[\alpha, \xi]\} = \{|W_2(u) + \pi - \xi| \leq \alpha/2 \text{ for all } u \in [0, \tau_{\log(\delta)}]\}.$$

Using the independence of W_1 and W_2, the Laplace transform of $\tau_{\log(\delta)}$ and Theorem 7.45 in [17] (more precisely, equation (7.15)), we see that the probability of the right-hand side is equal to

$$\left(\frac{\delta}{|z|}\right)^{\frac{\pi}{\alpha}} \times \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi(\alpha/2 + \xi - \pi - \theta)}{\alpha}\right)\left(\frac{\delta}{|z|}\right)^{2n\frac{\pi}{\alpha}}. \tag{2.4}$$

For the upper bound, if $|z| \leq 2\delta$ we can bound directly the probability using the Beurling estimate, see [6]. If $|z| \geq 2\delta$, we can bound the series in (2.4) by a summable series. For the lower bound, let $\kappa \in (0, 1)$ be such that

$$\sum_{n=1}^{\infty} \frac{\kappa^{2n\frac{\pi}{\alpha}}}{(2n+1)} < \sin(\pi/4).$$

If $\kappa|z| < \delta$, then the desired bound follows from Brownian scaling. If $\kappa|z| \geq \delta$, since $0 \in z + C[\alpha/2, \xi]$, we have $|\theta + \pi - \xi| \leq \alpha/4$, so that the series in (2.4) is bounded below by

$$\frac{4}{\pi}\sin(\pi/4) - \sum_{n=1}^{\infty} \frac{4\kappa^{2n\frac{\pi}{\alpha}}}{(2n+1)\pi} > 0.$$

□

The very same proof allows us to prove the following lemma concerning the second part of the trajectory for an approximate cone point.

Lemma 2.10. *For some constants depending only on α, we have that for all $\varepsilon, \delta > 0$ with $\delta < \varepsilon$ and $x, z \in \mathbb{R}^2$ with $|x - z| = \delta/2$ and $x - z \in C[\alpha/2, \xi]$, we have*

$$\mathbb{P}_x \left(W([0, S_\varepsilon^{(0)}(z)]) \subseteq z + C[\alpha, \xi] \right) \asymp \left(\frac{\delta}{\varepsilon} \right)^{\frac{\pi}{\alpha}}.$$

Now, we use lemmas 2.9 and 2.10 to upper bound the probability of a point to be an approximate cone point.

Lemma 2.11. *There exists $C_0 > 0$ such that for all $z \in \mathbb{R}^2$,*

$$\mathbb{P}_0(z \textit{ is a } (\delta, \varepsilon)\textit{-approximate cone point}) \leq C_0 |z|^{-\frac{\pi}{\alpha}} \varepsilon^{-\frac{\pi}{\alpha}} \delta^{\frac{2\pi}{\alpha}}.$$

Proof. We use the strong Markov property on $T_{\delta/2}(z)$,

$$\begin{aligned}
&\mathbb{P}(z \text{ is a } (\delta, \varepsilon)\text{-approximate cone point}) \\
&\leq \mathbb{E}\left[\mathbb{1}_{\{W([0,T_\delta(z)]) \subseteq z + C[\alpha,\xi]\}} \mathbb{P}_{W(T_{\delta/2}(z))}(W([0, S_\varepsilon^{(0)}(z)]) \subseteq z + C[\alpha, \xi]\}) \right] \\
&\leq C \left(\frac{\delta}{|z|} \right)^{\frac{\pi}{\alpha}} \times C \left(\frac{\delta}{\varepsilon} \right)^{\frac{\pi}{\alpha}} = C^2 |z|^{-\frac{\pi}{\alpha}} \varepsilon^{-\frac{\pi}{\alpha}} \delta^{\frac{2\pi}{\alpha}}.
\end{aligned}$$

□

2.2.3 Upper bound

Define $M(\alpha, \xi, \varepsilon)$ to be the set of (δ, ε)-approximate α-cone points with direction ξ for all $\delta \in (0, \varepsilon)$. By continuity of the Brownian motion, we can write

$$M(\alpha, \xi, \varepsilon) = \{z \in \mathbb{R}^2 : z = W(t) \text{ for some } t > 0 \text{ and } W([0, S_\varepsilon^{(t)}]) \subseteq z + C[\alpha, \xi]\}.$$

Lemma 2.12. *Almost surely, we have that*

- *if $\alpha \in (0, \pi)$, then $M(\alpha, \xi, \varepsilon) = \emptyset$.*
- *if $\alpha \in [\pi, 2\pi)$, then $\dim_{\mathrm{H}}(M(\alpha, \xi, \varepsilon)) \leq 2 - 2\pi/\alpha$.*

Proof. Let Cube $= x_0 + [0,1]^2$ for some $x_0 \in \mathbb{R}^2$. We will prove that $M(\alpha,\xi,\varepsilon) \cap \text{Cube} = \emptyset$ a.s. in the case $\alpha \in (0,\pi)$ and $\dim_{\text{H}}(M(\alpha,\xi,\varepsilon) \cap \text{Cube}) \le 2 - 2\pi/\alpha$ in the case $\alpha \in [\pi, 2\pi)$. Let us start with a geometric construction.

For each dyadic subcube $D \subseteq$ Cube of side-length 2^{-k}, let D^* be a concentric circumference of radius $(1+\sqrt{2})2^{-k}$. Define the focal point $x(D)$ as follows:

- If $\alpha \in (0,\pi)$, let $x(D)$ be such that the two branches of $x(D) + C[\alpha,\xi]$ are tangent to D^*.
- If $\alpha \in [\pi, 2\pi)$, let $x(D)$ be such that the two branches of the dual of $x(D)+C[\alpha,\xi]$ are tangent to D^*.

In this setting, we can verify that

- for all k sufficiently large and $D \in \mathcal{D}_k$, $B(x,\varepsilon/2) \subseteq B(y,\varepsilon)$, and
- there exists constants $c_1, c_2 > 0$ depending only in α such that $c_1 < c_2$ and

$$y \in B(x, c_1c_2 2^{-k}) \subseteq B(y, c_2 2^{-k}/2) \subseteq B(y, c_2 2^{-k}) \subseteq B(x, c_2^2 2^{-k}).$$

See Figure 2.1 for an illustration of the affirmations above. From these observations, we have that if D contains a $(c_2 2^{-k}, \varepsilon)$-approximate cone point, then

$$W([0, T_{c_2^2 2^{-k}}(x)]) \subseteq x(D) + C[\alpha,\xi] \text{ and } W([T_{c_1c_2 2^{-k}}(x), S_{c_1c_2 2^{-k},\varepsilon/2}(x)]) \subseteq x(D) + C[\alpha,\xi].$$

As in Lemma 2.11, we have that for some constant $C_1 > 0$,

$$\mathbb{P}(D \text{ contains a } (c_2 2^{-k},\varepsilon)\text{-approximate cone point}) \le C_1 |x(D)|^{-\frac{\pi}{\alpha}} \varepsilon^{-\frac{\pi}{\alpha}} 2^{-k\frac{2\pi}{\alpha}}.$$

Moreover, for k sufficiently large and uniformly on $D \in \mathcal{D}_k$, $x(D)$ is far from the origin, so that

$$\mathbb{P}(D \text{ contains a } (c_2 2^{-k},\varepsilon)\text{-approximate cone point}) \lesssim 2^{-k\frac{2\pi}{\alpha}}.$$

where the implicit constant depends on α and ε. If $\alpha \in (0,\pi)$, then for all k sufficiently large,

$$\mathbb{P}(M(\alpha,\xi,\varepsilon) \cap \text{Cube} \neq \emptyset) \le \sum_{\substack{D \in \mathcal{D}_k \\ D \subseteq \text{Cube}}} \mathbb{P}(D \text{ contains a } (c_2 2^{-k},\varepsilon)\text{-approximate cone point}) \lesssim 2^{k(2-\frac{2\pi}{\alpha})},$$

and this quantity converge to 0 as $k \to \infty$.
If $\alpha \in [\pi,2\pi)$, we cover $M(\alpha,\xi,\varepsilon)$ with the collection of $D \in \mathcal{D}_k$ such that D contains a $(c_2 2^{-k},\varepsilon)$-approximate cone point, for each $k \ge 0$. For $\gamma > 2 - 2\pi/\alpha$, we have

$$\mathbb{E}\left[\sum_{D \in \mathcal{D}_k} \mathbb{1}_{\{D \text{ contains a } (c_2 2^{-k},\varepsilon)\text{-approximate cone point}\}} \operatorname{diam}(D)^\gamma\right] \lesssim 2^{k\left(2-\frac{2\pi}{\alpha}-\gamma\right)},$$

and this quantity converge to 0 as $k \to \infty$. Therefore, $\dim_{\text{H}}(M(\alpha,\xi,\varepsilon) \cap \text{Cube}) \le \gamma$ almost surely, and we conclude taking the limit $\gamma \searrow 2 - 2\pi/\alpha$. $\square$

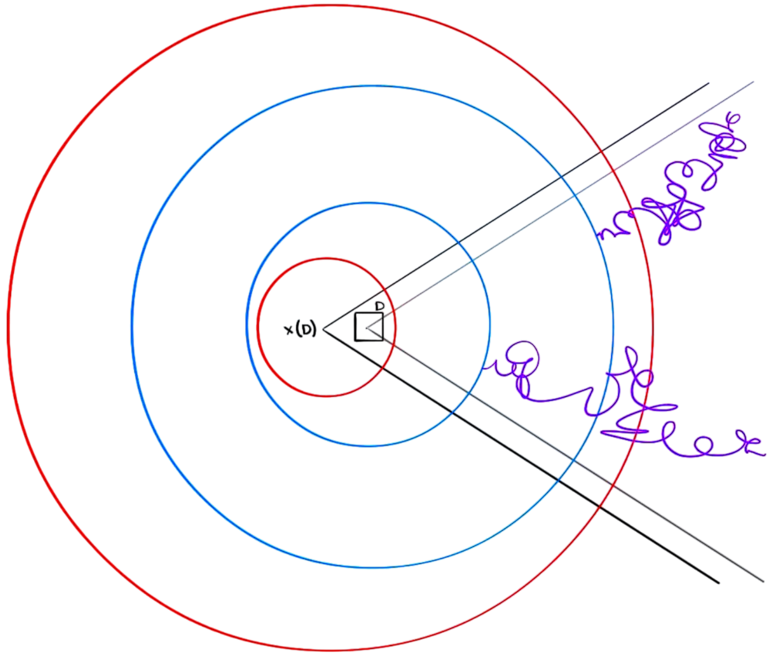

Figure 2.1: Nested balls around $x(D)$ and a point in D. A Brownian trajectory is drawn in the event where D contains a $(c_2 2^{-k}, \varepsilon)$-approximate cone point.

We are now ready to prove the upper bound of Theorem 2.7.

Proof of the upper bound in Theorem 2.7. Let $z \in \mathbb{R}^2$ be an α-cone point and $u > 0$, and let $t \in (0,1)$ be such that $z = B(t)$. Then, there exists rational numbers $q, \xi, \varepsilon > 0$ such that

$$B([q,t]) \subseteq B(t) + C[\alpha + u, \xi] \text{ and } B([t, S_\varepsilon^{(t)}(x)]) \subseteq B(t) + C[\alpha + u, \xi].$$

If $\alpha \in [0, \pi)$, we can pick $u > 0$ such that $\alpha + u < \pi$ and we already shown in Lemma 2.12 that these points do not exist, so there are no α-cone points. If $\alpha \in [\pi, 2\pi)$, by the previous argument we have covered the set of α-cone points with a countable union of (δ, ε)-approximate cone points. By the countable stability of the Hausdorff dimension and Lemma 2.12, it follows that the Hausdorff dimension of the α-cone points is at most $2 - 2\pi/(\alpha + u)$. We conclude by taking the limit $u \searrow 0$.

2.2.4 Lower bound

The proof of the lower bound requires an auxilliary theorem regarding the Hausdorff dimension of a certain type of random set, often called *limsup fractal*, that is constructed as follows. Let $\{Z(I) : I \in \mathcal{D}\}$ be a collection of random variables with values in $\{0, 1\}$ and let

$$A := \bigcap_{k \geq 1} \bigcup_{\substack{D \in \mathcal{D}_k \\ Z(I)=1}} I. \tag{2.5}$$

We have the following theorem regarding the Hausdorff dimension of A.

Theorem 2.13. *Suppose that there exists $\gamma > 0$ such that the following conditions hold:*

1. *For all $I, J \in \mathcal{D}$, $I \subseteq J$ and $Z(I) = 1$ imply $Z(J) = 1$.*
2. *For all $I \in \mathcal{D}$, $\mathbb{E}[Z(I)] \asymp \operatorname{diam}(I)^\gamma$.*
3. *For all $k \in \mathbb{N}$ and $I, J \in \mathcal{D}_k$ with $d(I,J) > 0$, $\mathbb{E}[Z(I)Z(J)] \lesssim \operatorname{diam}(I)^{2\gamma} d(I,J)^{-\gamma}$.*

Then, for all $\lambda > \gamma$ and $\Lambda \subseteq \mathrm{Cube}$ with $\mathcal{H}^\lambda(\Lambda) > 0$, there exists $p > 0$ such that

$$\mathbb{P}(\dim_{\mathrm{H}}(A \cap \Lambda) \geq \lambda - \gamma) \geq p.$$

Proof. We fix $\lambda > \gamma$ and $\Lambda \subseteq \mathrm{Cube}$ with $\mathcal{H}^\lambda(\Lambda) > 0$. We will show that for some $p > 0$, the probability that there exists a probability measure μ with finite β-energy for all $\beta \in (0, \lambda - \gamma)$ is at least p.

By Theorem 2.4, there exists a probability measure ν supported on Λ such that $\nu(D) \lesssim \operatorname{diam}(D)^\lambda$. For each $n \in \mathbb{N}$, let $A_n := \bigcup_{\substack{D \in \mathcal{D}_n \\ Z(I)=1}} I$ and define the measure

$$\mu_n(B) = 2^{\gamma n} \nu(B \cap A_n).$$

We will do a first and second moment estimate, as well as the estimate for the β-energy of μ_n.

<u>First moment estimate.</u> Observe that $\mathbb{P}(Z(I) = 1) = \mathbb{E}[Z(I)]$ and $\operatorname{diam}(I) = d^{1/2} 2^{-n}$ for each $I \in \mathcal{D}_n$ and $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$ we have

$$\mathbb{E}[\mu_n(A_n)] = 2^{\gamma n} \sum_{I \in \mathcal{D}_n} \nu(I) \mathbb{P}(Z(I) = 1) \gtrsim 2^{\gamma n} \sum_{I \in \mathcal{D}_n} \nu(I) \operatorname{diam}(I)^\gamma \gtrsim 2^{\gamma n} \sum_{I \in \mathcal{D}_n} \nu(I) 2^{-n\gamma} \gtrsim 1.$$

<u>Second moment estimate.</u> For all $n \in \mathbb{N}$, we have

$$\mathbb{E}[\mu_n(A_n)^2] = 2^{2n\gamma} \sum_{I,J \in \mathcal{D}_n} \mathbb{E}[Z(I)Z(J)] \nu(I) \nu(J).$$

Now we separate the sum in the pairs $I, J \in \mathcal{D}_n$ such that $d(I,J) = 0$ and $d(I,J) > 0$. Using $Z(J) \leq 1$ and the property of ν, the first sum can be bounded as

$$2^{2n\gamma} \sum_{\substack{I,J \in \mathcal{D}_n \\ d(I,J)=0}} \mathbb{E}[Z(I)Z(J)] \nu(I) \nu(J) \lesssim 2^{2n\gamma} 2^{-n\lambda} \sum_{\substack{I,J \in \mathcal{D}_n \\ d(I,J)=0}} \mathbb{E}[Z(I)] \nu(I) \lesssim 2^{n(\gamma - \lambda)} \lesssim 1,$$

since $\gamma - \lambda < 0$. For the second sum, we use the hypothesis to bound this term as

$$2^{2n\gamma} \sum_{\substack{I,J \in \mathcal{D}_n \\ d(I,J)>0}} \mathbb{E}[Z(I)Z(J)] \nu(I) \nu(J) \lesssim \sum_{\substack{I,J \in \mathcal{D}_n \\ d(I,J)>0}} d(I,J)^{-\gamma} \nu(I) \nu(J) \lesssim \iint |x - y|^{-\gamma} \nu(\mathrm{d}x) \nu(\mathrm{d}y),$$

where we used that for $I, J \in \mathcal{D}_n$ such that $d(I,J) > 0$, $|x-y| \lesssim d(I,J)$ for all $x \in I$ and $y \in J$. To bound the last integral, note that for all $r > 0$,

$$\begin{aligned}\int_{B(x,r)} |x-y|^{-\gamma}\nu(\mathrm{d}y) &= \int_0^\infty \nu(\{y \in B(x,r) : |x-y|^{-\gamma} > s\})\mathrm{d}s \\ &\lesssim \int_{r^{-\gamma}}^\infty \nu(B(x, s^{-1/\gamma}))\mathrm{d}s + \int_0^{r^{-\gamma}} \nu(B(x,r))\mathrm{d}s \\ &\lesssim \int_{r^{-\gamma}}^\infty s^{-\lambda/\gamma}\mathrm{d}s + r^{\lambda-\gamma} \lesssim r^{\lambda-\gamma}.\end{aligned}$$

Therefore,

$$\iint |x-y|^{-\gamma}\nu(\mathrm{d}x)\nu(\mathrm{d}y) \le \iint_{B(x,1)} |x-y|^{-\gamma}\nu(\mathrm{d}y)\nu(\mathrm{d}x) \lesssim 1.$$

Energy estimate. For all $n \in \mathbb{N}$, we have

$$\mathbb{E}[I_\beta(\mu_n)] = 2^{2n\gamma} \sum_{I,J \in \mathcal{D}_n} \mathbb{E}[Z(I)Z(J)] \iint_{I\times J} |x-y|^{-\beta}\nu(\mathrm{d}x)\nu(\mathrm{d}y).$$

Again, we separate the sum in the pairs I, J such that $d(I,J) = 0$ and $d(I,J) > 0$. For the first sum, we have

$$2^{2n\gamma} \sum_{\substack{I,J\in\mathcal{D}_n\\ d(I,J)=0}} \mathbb{E}[Z(I)Z(J)] \iint_{I\times J} |x-y|^{-\beta}\nu(\mathrm{d}x)\nu(\mathrm{d}y) \lesssim 2^{n\gamma} \sum_{\substack{I,J\in\mathcal{D}_n\\ d(I,J)=0}} \int_I\int_J |x-y|^{-\beta}\nu(\mathrm{d}y)\nu(\mathrm{d}x).$$

Just as shown before, we have that

$$\int_I\int_J |x-y|^{-\beta}\nu(\mathrm{d}y)\nu(\mathrm{d}x) \lesssim 2^{-n(\lambda-\beta)}\nu(I),$$

so that the first sum is bounded up to positive constants by $2^{n(\gamma-\lambda+\beta))}$ which is less than 1 since $\gamma - \lambda + \beta < 0$. For the second sum,

$$2^{2n\gamma} \sum_{\substack{I,J\in\mathcal{D}_n\\ d(I,J)>0}} \mathbb{E}[Z(I)Z(J)] \iint_{I\times J} |x-y|^{-\beta}\nu(\mathrm{d}x)\nu(\mathrm{d}y) \lesssim \sum_{\substack{I,J\in\mathcal{D}_n\\ d(I,J)=0}} d(I,J)^{-\gamma} \int_I\int_J |x-y|^{-\beta}\nu(\mathrm{d}x)\nu(\mathrm{d}y).$$

Using that for $I, J \in \mathcal{D}_n$ such that $d(I,J) > 0$ we have $d(I,J)^{-\gamma} \lesssim |x-y|^{-\gamma}$ for all $x \in I$ and $y \in J$, we can bound the previous term by

$$\iint |x-y|^{-(\gamma+\beta)}\nu(\mathrm{d}x)\nu(\mathrm{d}y).$$

Since $\gamma + \beta < \lambda$, this integral is finite.

As a result of the previous estimates, we see that there exists constants $c_1, c_2, c_3 > 0$ such that for all $n \in \mathbb{N}$, $\mathbb{E}[\mu_n(A_n)] \ge c_1$, $\mathbb{E}[\mu_n(A_n)^2] \le c_2$ and $\mathbb{E}[I_\beta(\mu_n)] \le c_3$. By Paley-Zigmund inequality, we have

$$\mathbb{P}(\mu_n(A_n) \ge c_1/2) \ge \mathbb{P}(\mu_n(A_n) \ge \mathbb{E}[\mu_n(A_n)]/2) \ge \frac{1}{4}\frac{\mathbb{E}[\mu_n(A_n)]^2}{\mathbb{E}[\mu_n(A_n)^2]} \ge \frac{c_1^2}{4c_2}.$$

Define $p := c_1^2/(8c_2)$. Combining with the β-energy estimate and using Markov's inequality, we can find $\ell = \ell(\beta, p) > 0$ sufficiently large so that

$$\mathbb{P}(I_\beta(\mu_n) \geq \ell) \leq \frac{\mathbb{E}[I_\beta(\mu_n)]}{\ell} \leq \frac{c_3}{\ell} \leq p.$$

This gives that for all $n \in \mathbb{N}$, $\mathbb{P}(I_\beta(\mu_n) \leq \ell$ and $\mu_n(A_n) \geq c_1) \geq p$, and therefore

$$\mathbb{P}(\limsup\{I_\beta(\mu_n) \leq \ell \text{ and } \mu_n(A_n) \geq c_1/2\}) \geq p.$$

In the event $\limsup\{I_\beta(\mu_n) \leq \ell$ and $\mu_n(A_n) \geq c_1\}$ we can pick a subsequence of μ_n converging to some μ. Such μ is supported by A, satisfies $\mu(\text{Cube}) \geq c_1/2$ and has finite β-energy, concluding the proof. □

We will use the previous theorem with a family $\{Z(I) : I \in \mathcal{D}\}$ related to the Brownian cone points defined as follows. Let $\text{Cube} = x_0 + [0,1]^2 \subseteq C[\alpha/2, 0]$ and $R > 2$ sufficiently large such that $\text{Cube} \subseteq B(0, R/2)$. For all $k \in \mathbb{N}$, define

$$r_k = R - \sum_{j=1}^{k} 2^{-j}.$$

Note that $r_k \searrow R - 1 > R/2$. For each $k \in \mathbb{N}$ and $I \in \mathcal{D}_k$, denote by z the center of I and define

$$Z(I) := \begin{cases} 1, & \text{if } z \text{ is a } (2^{-k}, r_k)\text{-approximate cone point with } \xi = \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.14. *The family $\{Z(I) : I \in \mathcal{D}\}$ satisfies the hypothesis of Theorem 2.13 with $\gamma = 2\pi/\alpha$.*

We refer to lemmas 10.46 and 10.47 in [17] for the proof. We are now ready to prove the lower bound in Theorem 2.7.

Proof of the upper bound in Theorem 2.7. We will find a subset of the α-cone points with Hausdorff dimension larger that $2-2\pi/\alpha$. The limsup fractal A obtained by (2.5) with the family $\{Z(I) : I \in \mathcal{D}\}$ defined above satisfies by continuity of the Brownian motion,

$$A \subseteq \widetilde{A} := \{W(t) : W([0,t]) \subseteq W(t) + C[\alpha, \xi] \text{ and } W([t, S_{R/2}^{(t)}(W(t)]) \subseteq W(t) + C[\alpha, \xi]\}.$$

By Theorem 2.13, $\dim_{\mathrm{H}}(\widetilde{A}) \geq 2 - 2\pi/\alpha$ with positive probability. Let $\delta, r > 0$. Define the sequence of stopping times $(\tau_k^{(\delta)})_{k\in\mathbb{N}}$ by $\tau_0^{(\delta)} = 0$, and

$$\tau_{k+1}^{(\delta)} := S_{\delta r}^{(\tau_k^{(\delta)})}(B(\tau_k^{(\delta)})), \quad \text{for all } k \in \mathbb{N}.$$

Define $\eta = R/(2r)$ For all $\delta > 0$ and $k \in \mathbb{N}$, define

$$A_k^{(\delta)} := \{B(t) : \tau_k^{(\delta)} \leq t < \tau_{k+1}^{(\delta)} \text{ and } W([\tau_k^{(\delta)}, S_{\eta r}^{(t)}(W(t))]) \subseteq W(t) + C[\alpha, \xi]\}.$$

Then, we have

$$\widetilde{A} \subseteq \bigcup_{k\in\mathbb{N}} A_k^{(\delta)}$$

Let $\beta < 2-2\pi/\alpha$. By the Markov property, the events $\{\dim_{\mathrm{H}}(A_k^{(\delta)}) \geq \beta\}$ have all the same probability $p^{(\delta)}$, which is strictly positive since otherwise it contradicts the lower bound of the dimension of $\widetilde{A}$. In particular, with probability at least $p^{(\delta)}$ it holds that

$$\dim_{\mathrm{H}}(\{W(t) : 0 \leq t < S_{\delta r}^{(0)}(0) \text{ and } W([0, S_{\eta r}^{(t)}(W(t))]) \subseteq W(t) + C[\alpha, \xi]\}) \geq \beta.$$

By scaling, this event does not depend on $r > 0$ and neither $p^{(\delta)}$. Therefore, by Kolmogorov's 0-1 law we have $p^{(\delta)} = 1$ and by taking $\beta \nearrow 2 - 2\pi/\alpha$ we have almost surely that

$$\dim_{\mathrm{H}}(\{W(t) : 0 \leq t < S_{\delta}^{(0)}(0) \text{ and } W([0, S_{\eta}^{(t)}(B(t))]) \subseteq W(t) + C[\alpha, \xi]\}) \geq 2 - \frac{2\pi}{\alpha}.$$

Finally, for each $\varepsilon \in (0, 1)$, we can find $\delta, \eta > 0$ such that $S_{\delta}^{(0)}(0) < 1$ and $S_{\eta}^{(t)}(W(t))$ for all $t \in [0, 1]$ with probability $1 - \varepsilon$. It follows that with probability at least $1 - \varepsilon$,

$$\dim_{\mathrm{H}}(\{W(t) : 0 \leq t < 1 \text{ and } W([0, 1]) \subseteq W(t) + C[\alpha, \xi]\}) \geq 2 - \frac{2\pi}{\alpha},$$

so that if we take $\varepsilon \searrow 0$, the previous event holds almost surely. Such set is contained in the set of α-cone points, concluding the proof.

Chapter 3

Hausdorff dimension of geodesic stars

In this chapter, we present the techniques applied to the Brownian sphere in order to study the so-called geodesic m-stars, specifically its Hausdorff dimension. The lower bound technique will be the one developed by Le Gall in [7], based on first and second moment estimates for an approximation of the geodesic m-stars. The upper bound technique will be the one developed by Miller and Qian in [15], based on the direct computation of the Hausdorff measure of the geodesic m-stars, for which fine estimates on the probability of the existence of m-stars are required.

Let us formally define what is a (geodesic) m-star in a metric space.

Definition 3.1. *Let (E, d) be a metric space. A point $x \in E$ is called* ***(geodesic) m-star*** *if m disjoint (except at x) geodesics emerge from x.*

In the Brownian sphere, the existence of m-stars for $m \in \{1, 2\}$ is clear from the fact that it is a geodesic space and that any point lying in a geodesic is a 2-star. The existence of 3-stars can be seen from the fact that two different geodesics from typical points towards x_0 coalesce before reaching x_0 (recall Proposition 1.28), and the point where they coalesce is a 3-star. The existence of 4-stars has not been established explicitly before, but we will see that this set has Hausdorff dimension 1, implying its existence. The existence of 5-stars remains unknown, but according to the theorem that we will present, their Hausdorff dimension is 0 in any case.

The theorem that we are referring to, which is the main result of this chapter, is the following.

Theorem 3.2 (Le Gall, Miller & Qian)**.** *For all integers $m \geq 1$, let $\mathfrak{G}_m$ be the set of geodesic m-stars of the Brownian sphere.*

- *If $m \in \{1, 2, 3, 4, 5\}$, $\dim_{\mathrm{H}}(\mathfrak{G}_m) = 5 - m$ a.s.*
- *If $m \geq 6$, the set $\mathfrak{G}_m$ is a.s. empty.*

3.1 Lower bound

The lower bound for $\dim_{\mathrm{H}}(\mathfrak{G}_m)$ in Theorem 3.2 is the consequence of first and second moment estimates for measures supported in an approximation of the geodesic stars. To prove these estimates, the main tool is a spatial Markov property of the Brownian sphere, identifying the law of a hull and its complement, conditionally on the boundary. Then, standard techniques as the ones described for the lower bound of the Brownian cone points can be implemented to conclude the result. From now on, we present the proof of the following result.

Theorem 3.3. *If* $m \in \{1, 2, 3, 4\}$*,* $\dim_{\mathrm{H}}(\mathfrak{G}_m) \geq 5 - m$ *a.s.*

Note that $\mathfrak{G}_1 = \mathbf{m}_\infty$ since the Brownian sphere is a geodesic space. Considering that $\dim_{\mathrm{H}}(\mathbf{m}_\infty) = 4$ (see [8]), Theorem 3.3 holds trivially in the case $m = 1$. Therefore, we will exclude $m = 1$ from the proof of the lower bound for convenience.

3.1.1 First moment estimate

In this section, we state and present the main ingredients of the *first moment estimate* proved by Le Gall in [7]. The proofs strongly depend on the symmetries, the spatial Markov property, the slices defined in Section 1.3.4, and the explicit formulas for the moments of the volume of balls and exit measures in the Brownian sphere. However, due to its technicalities, we refer to the original reference for the full proofs.

Slices in hulls

We start with some lemmas relating hulls and slices defined in sections 1.3.3 and 1.3.4, respectively. These results are the toolbox for the first moment estimate, see Proposition 3.9. Specifically, these lemmas collect information about hulls containing an approximated geodesic star, and this information is enough to obtain a first moment estimate, after an application of the Markov property.

We start with the following result, that can be deduced from the fact that γ' and γ'' are, except at their endpoints, disjoint curves, as claimed in Section (1.3.4). In particular, the sequence $(\delta_k)_{k\in\mathbb{N}}$ appearing here will be used to argue that a sequence of measures supported on the approximated geodesic stars converge weakly to a measure on actual geodesic stars, see Proposition 3.12.

Lemma 3.4 (Lemma 10 in [7])**.** *Let* $\varepsilon \in (0, 1/4)$*. There exists a deterministic sequence* $(\delta_k)_{k\in\mathbb{N}}$ *such that for all* $h \in [1-\varepsilon, 1]$*, the probability under* $\mathbb{N}_0(\cdot|W_* = -h)$ *of the event where*

$$\inf_{s,t\in[1-2^{-k},1-2^{-k-3}]} \widetilde{D}(\gamma'(s), \gamma''(t)) \geq \delta_k, \quad \text{for all } k \geq 1 \text{ such that } 2^{-k-4} > \varepsilon,$$

is at least $9/10$*.*

Now, we implement the machinery of the slices to obtain fine estimates on the distances between disjoint geodesics towards a geodesic star. For $z, \varepsilon > 0$ and $m \in \{1, 2, 3\}$, let us consider the event where at least m snakes in the hull $\mathfrak{H}_{1,z}$ (recall its definition from Section 1.3.3) have a global minima strictly below of $-1 + \varepsilon$, namely,

$$E_\varepsilon^m := \{\#\{i \in I : W_*(\omega_i) < -1 + \varepsilon\} \geq m\}. \tag{3.1}$$

By construction of the hull, $\mathbb{P}(E_\varepsilon^m)$ is the probability of a Poisson random variable to be larger than m, with parameter given by

$$\int_{[0,z]\times\{\omega\in\mathcal{S}_0 : W_*(\omega)<-1+\varepsilon\}} \mathrm{d}t \otimes \mathbb{N}_0(\mathrm{d}\omega \cap \{W_* > -1\}) = z\mathbb{N}_0(-1 < W_* < -1 + \varepsilon) = \frac{3z}{2}((1-\varepsilon)^{-2} - 1),$$

where we used Proposition 1.15 for the distribution of W_*. Let us now argue in the event E_ε^m. Let $i_1, \ldots, i_m \in I$ be the indices such that $W_*(\omega_i) < -1 + \varepsilon$ and recall the notation $\zeta_i = \zeta(\omega_i)$. Set

$$\mathbf{R}_j := \Pi_{\mathbf{H}}(\mathcal{T}(\zeta_{i_j})), \text{ for } j \in \{1, \ldots, m\}, \quad \text{and} \quad \mathbf{R}_0 := \Pi_{\mathbf{H}}(\mathcal{T}(\zeta_*)).$$

If there are more than m such indices, we keep those giving the smallest values of t_i. For each $j \in \{1, \ldots, m\}$, let $\mathbf{S}_j$ be the slice obtained from $\mathcal{T}(\zeta_{i_j})$, and let $\mathbf{S}_0$ be the slice obtained from $\mathcal{T}(\zeta_*)$, as in the Section 1.3.4. Let us see that $\mathbf{R}_j$ is identified with $\mathbf{S}_j$ for $j \in \{1, \ldots, m\}$. Note that this is false for $j = 0$, as we shall comment.

Lemma 3.5. *For all $a, b \in \mathcal{T}(\zeta_{i_1})$, $D_{\mathbf{H}}(a, b) = 0$ if, and only if $\widetilde{D}(a, b) = 0$. Here, $\widetilde{D}$ stands for the distance in the slice $\mathbf{S}_1$.*

Proof. Let $a, b \in \mathcal{T}(\zeta_{i_1})$ and suppose that $\widetilde{D}(a, b) = 0$. Then, there exists $s, t \in [0, \sigma(\omega_{i_1})]$ such that $s \leq t$, $p_{\zeta_{i_1}}(s) = a$, $p_{\zeta_{i_1}}(t) = b$ and

$$\widehat{W}_s(\omega_{i_1}) = \widehat{W}_t(\omega_{i_1}) = \min_{r\in[s,t]} \widehat{W}_r(\omega_{i_1}). \tag{3.2}$$

If $[u_1, u_1 + \sigma(\omega_{i_1})] \subseteq [0, \Sigma]$ is the interval where the cyclic exploration $(\mathcal{E}_r)_{r\in[0,\Sigma]}$ of $\mathbf{H}$ explores $\mathcal{T}(\zeta_{i_1})$, then $[a, b]_{\mathbf{H}} \subseteq \{\mathcal{E}_{u_1+r} : s \leq r \leq t\}$ since $[a, b]_{\mathbf{H}}$ is defined s and t as before with the requirement that $[s, t]$ is as small as possible. Therefore, $\min_{r\in[s,t]} \widehat{W}_r(\omega_{i_1}) \leq \min_{c\in[a,b]_{\mathbf{H}}} \Lambda_c$ so that (3.2) is equivalent to

$$\Lambda_a = \Lambda_b = \min_{c\in[a,b]_{\mathbf{H}}} \Lambda_c, \tag{3.3}$$

which means $D_{\mathbf{H}}(a, b) = 0$. On the other hand, suppose $D_{\mathbf{H}}(a, b) = 0$ and assume without loss of generality that (3.3) holds. Note that this is possible only if $[a, b]_{\mathbf{H}} \subseteq \mathcal{T}(\zeta_{i_1})$, since otherwise $b_* \in [a, b]_{\mathbf{H}}$ and (3.3) falls down. Again, since $[a, b]_{\mathbf{H}}$ is defined with the requirement that the corresponding interval in $[0, \Sigma]$ is the smallest possible, we have that

$$\min_{c\in[a,b]_{\mathbf{H}}} \Lambda_c = \max\left\{\min_{r\in[s,t]} \widehat{W}_r(\omega_{i_1}) : s, t \in [0, \sigma(\omega_{i_1})] \text{ such that } p_{\zeta_{i_1}}(s) = a, p_{\zeta_{i_1}}(t) = b\right\},$$

so that for the optimal choice of s and t, we see that (3.3) holds. $\square$

The conclusion of Lemma 3.5 is that $\mathbf{R}_1$ is identified as a set with $\mathbf{S}_1$, since the equivalence relation used to define them identify the same points. This result remains true for any $j \in \{2, \ldots, m\}$. Moreover, note that for all $a, b \in \mathcal{T}(\zeta_{i_1})$,

$$D_{\mathbf{H}}(a, b) \leq \widetilde{D}(a, b), \tag{3.4}$$

which can be seen by definition, recall (1.14) and (1.17). Since $(\mathbf{R}_1, D_{\mathbf{H}})$ and $(\mathbf{S}_1, \widetilde{D})$ are compact metric spaces, (3.4) implies that they are homeomorphic.

Consider the geodesics γ' and γ'' defined in (1.19) and (1.20), respectively, corresponding to $\mathbf{S}_1$. Defining $\Gamma' := \gamma'([0, 1])$ and $\Gamma'' := \gamma''([0, 1])$, we see that $\Gamma' \cup \Gamma''$ is identified with the topological boundary of $\mathbf{R}_1$. In fact, a point in $a \in \mathcal{T}(\zeta_{i_1})$ is identified with a point in $\mathbf{H} \setminus \mathcal{T}(\zeta_{i_1})$ if, and only if $a \in \Gamma' \cup \Gamma''$. Using this fact, we will further denote $\Gamma' = \partial_\ell \mathbf{R}_1$ and $\Gamma'' = \partial_r \mathbf{R}_1$, and call them *left* and *right boundaries* of $\mathbf{R}_1$, respectively. The sets $\partial_\ell \mathbf{R}_j$ and $\partial_r \mathbf{R}_j$ are defined analogously for $j \in \{2, \ldots, m\}$. Finally, define $\mathrm{Int}(\mathbf{R}_1) = \mathbf{R}_1 \setminus (\Gamma' \cup \Gamma'')$.

The next lemma states that the length of any continuous path that crosses $\mathbf{R}_1$ from Γ' to Γ'' and coincides with γ' and γ'' the rest of the times, is the same for $D_{\mathbf{H}}$ and $\widetilde{D}$. We refer to the original reference for a proof.

Lemma 3.6 (Lemma 11 in [7]). *Let $\phi = (\phi(t))_{0 \leq t \leq 1}$ be a continuous path in $\mathbf{R}_1$ satisfying:*

(H) *There exists $u, v \in [0, 1]$ such that $u \leq v$, $\phi([0, u]) \subseteq \partial_\ell \mathbf{R}_1$, $\phi((u, v)) \subseteq \mathrm{Int}(\mathbf{R}_1)$ and $\phi([v, 1]) \subseteq \partial_r \mathbf{R}_1$.*

Then, the length of ϕ with respect to $D_{\mathbf{H}}$ coincides with its length with respect to $\widetilde{D}$.

Now, we combine lemmas 3.4, 3.5 and 3.6 to bound the probability of the event where any path satisfying (H) for a specific choice of u and v depending on $k \in \mathbb{N}$, has length at least δ_k.

Lemma 3.7 (Lemmas 12 and 13 in [7]). *Let $\varepsilon \in (0, 1/4)$ and $j \in \{0, \ldots, m\}$. The following event, denoted $D_\varepsilon^{m,j}$, holds with probability at least $9/10$ under $\mathbb{P}(\cdot | E_\varepsilon^m)$. For every integer $k \geq 1$ such that $2^{-k-4} > \varepsilon$, for every continuous path $(\phi(t))_{0 \leq t \leq 1}$ in $\mathbf{R}_j$ such that $\phi(0) \in \gamma'([1 - 2^{-k}, 1 - 2^{-k-3}])$ and $\phi(1) \in \gamma''([1 - 2^{-k}, 1 - 2^{-k-3}])$, the length of ϕ (with respect to $D_{\mathbf{H}}$) is at least δ_k.*

Proof. The case $j = 0$ is more delicate and we will justify it later. For $j \in \{1, \ldots, m\}$, this is a direct application of Lemma 3.6. Indeed, if ϕ is a path like in the statement of the lemma, then extend ϕ to a path $\widetilde{\phi}$ by gluing to $\phi(0)$ the range set $\gamma'([0, t_1])$ where $\gamma'(t_1) = \phi(0)$, and analogously glue to $\phi(1)$ the range set $\gamma''([t_2, 1])$ where $\gamma''(t_2) = \phi(1)$. Then $\widetilde{\phi}$ satisfies (H) and the length of this path is bounded below by $\widetilde{D}(\phi(0), \phi(1)) \geq \delta_k$. Since the length of $\widetilde{\phi}$ for $D_{\mathbf{H}}$ and $\widetilde{D}$ coincide, the result follows by Lemma 3.6.

For $j = 0$, we cannot apply Lemma 3.6 directly. In fact, by confluence of geodesics towards a typical point in the Brownian sphere, in $\mathbf{S}_0$ the range sets $\gamma'([u,v])$ and $\gamma''([u,v])$ are disjoint if, and only if $0 \le u \le v < \mu_0$, where $\mu_0 := \sup\{-W_*(\omega_i) : i \in I\}$ (note that $\mu_0 \in [1-\varepsilon, 1)$). This is the reason why $\mathbf{R}_0$ and $\mathbf{S}_0$ are not identified. However, let us introduce slight modifications to the previous lemmas in order to make the statement valid for $j = 0$. Namely:

- In Lemma 3.4, suppose that $\delta_k < 2^{-k-5}$ for all $k \in \mathbb{N}$.
- In Lemma 3.6, add to (H) the restriction $D_{\mathbf{H}}(\phi(t), b_*) > \varepsilon$ for all $t \in [0,1]$.

Both assumptions do not alter the validity of the cases $j \in \{1, \dots, m\}$. However, by adding these assumptions, we are considering paths that do not visit $\{x \in \mathbf{R}_0 : D_{\mathbf{H}}(x, x_*) \le \varepsilon\}$ and the proof in this case is analogous to the cases $j \in \{1, \dots, m\}$. □

The last ingredient of the first moment estimate is the following lemma.

Lemma 3.8. *For all $\varepsilon \in (0, 1/4)$ and $z > 0$, define $\mathcal{A}_\varepsilon^m$ as the event where there exists $m+1$ geodesics $\eta_0, \dots, \eta_m$ from the boundary of $\mathfrak{H}_{1,z}$ to b_*, such that the sets $\eta_j([0, 1-\varepsilon])$ for $j \in \{0, \dots, m\}$ are disjoint, and moreover*

$$D_{\mathbf{H}}(\eta_i(t), \eta_j(t)) \ge \delta_k,$$

for all $i, j \in \{0, \dots, m\}$ with $i < j$, for all $t \in [1-2^{-k-1}, 1-2^{-k-2}]$ and for all $k \ge 1$ such that $2^{-k-4} > \varepsilon$. Then, there exists a constant $c > 0$ such that $\mathbb{P}(\mathcal{A}_\varepsilon^m) \ge c\varepsilon^m$.

Proof. Recall the event E_ε^m defined in (3.1). We have $\mathbb{P}(E_\varepsilon^m) \gtrsim \varepsilon^m$ by construction, so that if we verify $\mathbb{P}(\mathcal{A}_\varepsilon^m | E_\varepsilon^m) \ge c'$ for some constant $c' > 0$, the claim follows. For each $j \in \{0, \dots, m\}$ set $B_\varepsilon^{m,j} := E_\varepsilon^m \cap D_\varepsilon^{m,j}$ (recall that $D_\varepsilon^{m,j}$ was defined in Lemma 3.7) and $B_\varepsilon^m := \bigcap_{j=1}^m B_\varepsilon^{m,j}$. We have that $\mathbb{P}((B_\varepsilon^{m,j})^c | E_\varepsilon^m) \le 1/10$, so that $\mathbb{P}(B_\varepsilon^m | E_\varepsilon^m) \ge 1/2$. It remains to show that $B_\varepsilon^m \subseteq \mathcal{A}_\varepsilon^m$. To do so, set

$$\begin{aligned} s'_j &:= \inf\{s \in [0, \Sigma] : \mathcal{E}_s = t_{i_j}\}, \quad s''_j := \sup\{s \in [0, \Sigma] : \mathcal{E}_s = t_{i_j}\}, \quad j \in \{1, \dots, m\}, \\ s'_0 &:= \inf\{s \in [0, \Sigma] : \mathcal{E}_s = U^*\}, \quad s''_0 := \sup\{s \in [0, \Sigma] : \mathcal{E}_s = U^*\}, \end{aligned}$$

and consider the geodesics $\gamma_{s'_j}$ and $\gamma_{s''_j}$ defined in (1.15) and (1.16) relative to $\mathbf{R}_j$. For $j \in \{1, \dots, m\}$, note that $\gamma_{s'_j}([u,v])$ and $\gamma_{s''_j}([u,v])$ are disjoint whenever $0 \le u \le v < \mu_j := -W_*(\omega_{i_j})$. Then, we can see that $\gamma_{s'_j}([0, \mu_j]) = \partial_\ell \mathbf{R}_j$ and $\gamma_{s''_j}([0, \mu_j]) = \partial_r \mathbf{R}_j$ for all $j \in \{0, \dots, m\}$ (recall that $\mu_0 = \sup\{-W_*(\omega_i) : i \in I\}$). Assuming that B_ε^m holds, let us justify that we can take $\eta_j = \gamma_{s'_j}$ for $\mathcal{A}_\varepsilon^m$ to hold. In fact, by construction, the sets $\gamma_{s'_j}([0, 1-\varepsilon])$ for $j \in \{0, \dots, m\}$ are disjoint. Since we are assuming that $D_\varepsilon^{m,j}$ holds for all $j \in \{0, \dots, m\}$, for all integers $k \ge 1$ such that $2^{-k-4} > \varepsilon$, the length of any continuous path starting from $\gamma_{s'_j}([1-2^{-k}, 1-2^{-k-3}])$, ending in $\gamma_{s''_j}([1-2^{-k}, 1-2^{-k-3}])$ and staying in $\mathbf{R}_j$, is bounded below by δ_k. This implies that

$$D_{\mathbf{H}}(\gamma_{s'_i}(t), \gamma_{s'_j}(t)) \ge \delta_k,$$

for any different $i,j \in \{0,\dots,m\}$, for all $t \in [1-2^{-k-1}, 1-2^{-k-2}]$ and for all $k \geq 1$ such that $2^{-k-4} > \varepsilon$. This comes from the fact that for any different $i,j \in \{0,\dots,m\}$, a path starting from $\partial_\ell \mathbf{R}_i$ and reaching $\partial_\ell \mathbf{R}_j$ that stays away from b_*, always hits the $\partial_r \mathbf{R}_i$ or $\partial_r \mathbf{R}_j$ by planarity. Then, Lemma 3.7 implies that the distance between these left boundaries is bounded below by δ_k, hence the desired bound holds. This concludes the proof. □

Proof of the first moment estimate

We are ready to state and present the proof of the first moment estimate. To do so, fix $m \in \{1,2,3\}$ and consider the following objects:

- For $x,y \in \mathbf{m}_\infty$, $r > 0$ and $\varepsilon \in (0,r)$, define $G^{(m)}_{\varepsilon,r}(x,y)$ as the event where there exists $m+1$ geodesics $\eta_0,\eta_1,\dots,\eta_m$ such that for all $j \in \{0,\dots,m\}$, $\eta_j(0) \in \partial B^{\bullet(y)}(x,r)$, $\eta_j(r) = x$ and the range sets $\eta_j([0,r-\varepsilon])$ for $j \in \{0,\dots,m\}$ are disjoint.
- Consider the sequence $(\delta_k)_{k\in\mathbb{N}}$ of Lemma 3.4. Define $\widetilde{G}^{(m)}_{\varepsilon,r}(x,y)$ as $G^{(m)}_{\varepsilon,r}(x,y)$ with the additional condition that the geodesics $\eta_0,\dots,\eta_m$ must satisfy

$$D(\eta_i(t),\eta_j(t)) \geq \delta_k,$$

  for all $t \in [1-2^{-k-1}, 1-2^{-k-2}]$, $k \geq 1$ with $2^{-k-4} > \varepsilon$, and all distinct $i,j \in \{0,\dots,m\}$.
- Define $F^{(m)}_{\varepsilon,r}(x,y) = \mathbb{1}_{G^{(m)}_{\varepsilon,r}(x,y)}$ and $\widetilde{F}^{(m)}_{\varepsilon,r}(x,y) = \mathbb{1}_{\widetilde{G}^{(m)}_{\varepsilon,r}(x,y)}$.

The first bulllet point defines what we referred to as *approximated geodesics stars.*

Proposition 3.9 (Proposition 14 in [7]). *There exists a constant $c > 0$ such that for all $\varepsilon \in (0,1/4)$, we have*

$$\mathbb{N}_0\left(\int \frac{\mathrm{Vol}(\mathrm{d}x)}{\sigma}\mathbb{1}_{\{D(x,x_*)<2\}}\widetilde{F}^{(m)}_{\varepsilon,1}(x,x_*)\right) \geq c\varepsilon^m. \tag{3.5}$$

Proof. By Proposition 1.18, the left-hand side of (3.5) can be written as

$$\mathbb{N}_0\left(\mathbb{1}_{\{D(x_0,x_*)<2\}}\widetilde{F}^{(m)}_{\varepsilon,1}(x_*,x_0)\right) = \frac{3}{2}\mathbb{N}^{[1]}_0\left(\mathbb{1}_{\{D(x_*,x_0)<2\}}\widetilde{F}^{(m)}_{\varepsilon,1}(x_*,x_0)\right).$$

The indicator $\mathbb{1}_{\{D(x_*,x_0)<2\}}$ is a function of $\overline{\mathbf{m}_\infty \setminus B^{\bullet(x_0)}(x_*,1)}$ and $\widetilde{F}^{(m)}_{\varepsilon,1}(x_*,x_0)$ is a function of $B^{\bullet(x_0)}(x_*,1)$. By Theorem 1.33, we have that

$$\mathbb{N}^{[1]}_0\left(\mathbb{1}_{\{D(x_*,x_0)<2\}}\widetilde{F}^{(m)}_{\varepsilon,1}(x_*,x_0)\right) = \mathbb{N}^{[1]}_0\left(\theta(Z_{W_*+1})\mathbb{N}^{[1]}_0\left(\widetilde{F}^{(m)}_{\varepsilon,1}(x_*,x_0)\middle| Z_{W_*+1}\right)\right),$$

where $\theta(z)$ is the probability for a free pointed Brownian disk that the distance from the distinguished point to the boundary is less than 1, which is bounded below by a positive constant. Since, $B^{\bullet(x_0)}(x_*,1)$ is distributed as a hull of radius 1 and diameter z under $\mathbb{N}_0(\cdot|Z_{W_*+1}=z)$, we find that $\mathbb{N}^{[1]}_0\left(\widetilde{F}^{(m)}_{\varepsilon,1}(x_*,x_0)\middle| Z_{W_*+1}\right) \gtrsim \varepsilon^m$ by Lemma 3.8. This completes the proof. □

3.1.2 Second moment estimate

In this section, we state and present the main ingredients of the *second moment estimate* proved by Le Gall in [7]. Propositions 3.10 and 3.11 will give both the *second moment estimate* and the *energy estimate* (in the language of Theorem 2.13) during the construction of a measure supported on $\mathfrak{G}_m$. We keep the notations introduced in the previous section, namely, the functions $F^{(m)}_{\varepsilon,r}(x,y)$.

Proposition 3.10 (Lemma 15 in [7]). *Let $\delta \in (0,1)$. There exists a constant $C_{(\delta)} > 0$ such that for all $\varepsilon \in (0,1/8)$ and every integer $k \geq 1$ such that $2^{-k} > 2\varepsilon$,*

$$\mathbb{N}_0\left(\frac{1}{\sigma}\iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\mathbb{1}_{\{D(x,y)\in[2^{-k+2},2^{-k+3}]\}}F^{(m)}_{\varepsilon,1}(x,x_*)F^{(m)}_{\varepsilon,1}(y,x_*)\right) \leq C_{(\delta)}2^{-k(4-m-\delta)}\varepsilon^{2m}.$$

Sketch of proof. This proof is quite technical and we will briefly present only the arguments concerning the application of symmetries and the spatial Markov property of the Brownian sphere. Let us proceed in steps.

Step 1: Rewriting the left-hand side. Write $\Gamma_{\varepsilon,k}(x_*,x,y) = \mathbb{1}_{\{D(x,y)\in[2^{-k+2},2^{-k+3}]\}}F_{\varepsilon,1}(x,x_*)F_{\varepsilon,1}(y,x_*)$. Using Proposition 1.18, we can write the left-hand side as

$$\mathbb{N}_0\left(\sigma\iint \frac{\mathrm{Vol}(\mathrm{d}x)}{\sigma}\frac{\mathrm{Vol}(\mathrm{d}y)}{\sigma}\Gamma_{\varepsilon,k}(x_*,x,y)\right) = \mathbb{N}_0\left(\int \mathrm{Vol}(\mathrm{d}z)\Gamma_{\varepsilon,k}(z,x_*,x_0)\right) = A^1_{\varepsilon,k} + A^2_{\varepsilon,k},$$

where $A^1_{\varepsilon,k}$ is obtained by restricting the integral to $\mathcal{C}^{x_*,x_0}_{2^{-k}}$, which was introduced in (1.21).

Step 2.1: First estimates of $A^1_{\varepsilon,k}$. Observe that

$$\Gamma_{\varepsilon,k}(z,x_*,x_0) = \mathbb{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}F_{\varepsilon,1}(x_*,z)F_{\varepsilon,1}(x_0,z).$$

By definition, we have

$$F_{\varepsilon,1}(x_*,z) \leq F_{\varepsilon,2^{-k}}(x_*,z)F_{2^{-k+4},1}(x_*,z),$$
$$F_{\varepsilon,1}(x_0,z) \leq F_{\varepsilon,2^{-k}}(x_0,z).$$

Next, observe that $z \notin B^{\bullet(x_0)}(x_*,2^{-k})$ implies that z and x_0 are in the same connected component of $B(x_*,2^{-k})^c$. Similarly, $z \notin B^{\bullet(x_*)}(x_0,2^{-k})$ implies that z and x_* are in the same connected component of $B(x_0,2^{-k})^c$. Therefore, if $z \notin B^{\bullet(x_0)}(x_*,2^{-k}) \cap B^{\bullet(x_*)}(x_0,2^{-k})$, it holds that

$$B^{\bullet(z)}(x_*,2^{-k}) = B^{\bullet(x_0)}(x_*,2^{-k}) \quad \text{and} \quad B^{\bullet(z)}(x_0,2^{-k}) = B^{\bullet(x_*)}(x_0,2^{-k}).$$

Combining with the previous bounds, we obtain

$$\mathbb{1}_{\{D(x_*,x_0)>2^{-k}\}}F_{\varepsilon,2^{-k}}(x_*,z) \leq F_{\varepsilon,2^{-k}}(x_*,x_0),$$
$$\mathbb{1}_{\{D(x_*,x_0)>2^{-k}\}}F_{\varepsilon,2^{-k}}(x_0,z) \leq F_{\varepsilon,2^{-k}}(x_0,x_*).$$

It follows that $A^1_{\varepsilon,k}$ is bounded by

$$\mathbb{N}_0\left(\mathbb{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}F_{\varepsilon,2^{-k}}(x_*,x_0)F_{\varepsilon,2^{-k}}(x_0,x_*)\int_{\mathcal{C}^{x_*,x_0}_{2^{-k}}}\mathrm{Vol}(\mathrm{d}x)F_{2^{-k+4},1}(x_*,z)\right)$$

Step 2.2: First use of Markov property. Observe that $F_{\varepsilon,2^{-k}}(x_*,x_0)$ is a function of $B^{\bullet(x_0)}(x_*,2^{-k})$ and $F_{\varepsilon,2^{-k}}(x_0,x_*)$ is the same function applied to $B^{\bullet(x_*)}(x_0,2^{-k})$. On the other hand, the quantity

$$\mathbb{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\int_{\mathcal{C}^{x_*,x_0}_{2^{-k}}}\mathrm{Vol}(\mathrm{d}z)F_{2^{-k+4},1}(x_*,z),$$

is a function of $\overline{\mathcal{C}^{x_*,x_0}_{2^{-k}}}$ (for the precise argument, see [7]). An application of Theorem 1.34 gives that the previous display is equal to

$$\mathbb{N}_0\left(\mathbb{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\varphi_{\varepsilon,k}(Z^{x_*(x_0)}_{2^{-k}})\varphi_{\varepsilon,k}(Z^{x_0(x_*)}_{2^{-k}})\int_{\mathcal{C}^{x_*,x_0}_{2^{-k}}}\mathrm{Vol}(\mathrm{d}z)F_{2^{-k+4},1}(x_*,z)\right),$$

where $\varphi_{\varepsilon,k}(z)$ is the probability under the distribution of the hull $\mathfrak{H}_{2^{-k},z}$ that there exists $m+1$ geodesics starting at the boundary of $\mathfrak{H}_{2^{-k},z}$ ending at z, disjoint up to the time when they are at distance ε from z. Using the symmetries of $\mathbf{m}_\infty$ again, this can be written as

$$\mathbb{N}_0\left(F_{2^{-k+4},1}(x_*,x_0)\int\mathrm{Vol}(\mathrm{d}z)\mathbb{1}_{\{x_0\in\mathcal{C}^{x_*,z}_{2^{-k}}\}}\mathbb{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\varphi_{\varepsilon,k}(Z^{x_*(z)}_{2^{-k}})\varphi_{\varepsilon,k}(Z^{z(x_*)}_{2^{-k}})\right).$$

Similar to a previous argument, $x_0\in\mathcal{C}^{x_*,z}_{2^{-k}}$ implies $B^{\bullet(z)}(x_*,2^{-k})=B^{\bullet(x_0)}(x_*,2^{-k})$ and $B^{\bullet(x_*)}(z,2^{-k})=B^{\bullet(x_0)}(z,2^{-k})$. In particular, $Z^{x_*(z)}_{2^{-k}}=Z^{x_*(x_0)}_{2^{-k}}$, so that the last display is equal to

$$\mathbb{N}_0\left(F_{2^{-k+4},1}(x_*,x_0)\varphi_{\varepsilon,k}(Z^{x_*(x_0)}_{2^{-k}})\int\mathrm{Vol}(\mathrm{d}z)\mathbb{1}_{\{x_0\in\mathcal{C}^{x_*,z}_{2^{-k}}\}}\mathbb{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\varphi_{\varepsilon,k}(Z^{z(x_*)}_{2^{-k}})\right).$$

Step 2.3: Second use of the Markov property. Observe that

$$\varphi_{\varepsilon,k}(Z^{x_*(x_0)}_{2^{-k}})\int\mathrm{Vol}(\mathrm{d}z)\mathbb{1}_{\{x_0\notin B^{\bullet(z)}(x_*,2^{-k})\cup B^{\bullet(x_*)}(z,2^{-k})\}}\mathbb{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\varphi_{\varepsilon,k}(Z^{z(x_*)}_{2^{-k}})$$

is a function of $B^{\bullet(x_0)}(x_*,2^{-k+4})$, and $F_{2^{-k+4},1}(x_*,x_0)$ is a function of $\overline{\mathbf{m}_\infty\setminus B^{\bullet(x_0)}(x_*,2^{-k+4})}$. An application of Theorem 1.33 gives that the last display is equal to

$$\begin{aligned}\mathbb{N}^{[1]}_0\Big(&\mathbb{N}^{[1]}_0\left(F_{2^{-k+4},1}(x_*,x_0)\middle|Z^{x_*(x_0)}_{2^{-k+4}}\right)\varphi_{\varepsilon,k}(Z^{x_*(x_0)}_{2^{-k}})\\&\times\int\mathrm{Vol}(\mathrm{d}z)\mathbb{1}_{\{x_0\in\mathcal{C}^{x_*,z}_{2^{-k}}\}}\mathbb{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\varphi_{\varepsilon,k}(Z^{z(x_*)}_{2^{-k}})\Big).\end{aligned}$$

Now, using Lemma 16 in [7] gives that $A^1_{\varepsilon,k}$ is bounded, up to a constant depending only in m, by

$$\mathbb{N}^{[1]}_0\left((Z^{x_*(x_0)}_{2^{-k+4}})^{m/2}\varphi_{\varepsilon,k}(Z^{x_*(x_0)}_{2^{-k}})\int\mathrm{Vol}(\mathrm{d}z)\mathbb{1}_{\{x_0\in\mathcal{C}^{x_*,z}_{2^{-k}}\}}\mathbb{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\varphi_{\varepsilon,k}(Z^{z(x_*)}_{2^{-k}})\right).$$

Step 2.4: Ingredients for the final bound. The last display can be bounded using Cauchy-Schwartz inequality and the explicit bounds for the volume of balls in $\mathbf{m}_\infty$ and the moments of the exit local times. We refer to [7] for the computations. The term $A^2_{\varepsilon,k}$ can be bounded using similar arguments.
□

Proposition 3.11 (Lemma 17 in [7]). *Let $\alpha \in (0, 4-m)$. There exists a constant $C_\alpha > 0$ such that for all $\varepsilon \in (0, 1/2)$,*

$$\mathbb{N}_0 \left(\frac{1}{\sigma} \iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y) \mathbb{1}_{\{D(x,y)<\varepsilon\}} D(x,y)^{-\alpha} F^{(m)}_{\varepsilon,1}(x, x_*) \right) \leq C_\alpha \varepsilon^{2m}.$$

Proof. Write $\Gamma_\varepsilon(x_*, x, y) = \mathbb{1}_{\{D(x,y)<\varepsilon\}} D(x,y)^{-\alpha} F^{(m)}_{\varepsilon,1}(x, x_*)$. Using Proposition 1.18 and conditioning on $\{W_* < -1\}$, we can write the left-hand side as

$$\frac{3}{2} \mathbb{N}_0^{[1]} \left(F^{(m)}_{\varepsilon,1}(x_*, x_0) \int \mathrm{Vol}(\mathrm{d}z) \mathbb{1}_{\{D(x_*,z)<\varepsilon\}} D(x_*, z)^{-\alpha} \right).$$

Note that $F^{(m)}_{\varepsilon,1}(x_*, x_0)$ is a function of $\overline{\mathbf{m}_\infty \setminus B^{\bullet(x_0)}(x_*, \varepsilon)}$ and $\int \mathrm{Vol}(\mathrm{d}z) \mathbb{1}_{\{D(x_*,z)<\varepsilon\}} D(x_*, z)^{-\alpha}$ is a function of $B^{\bullet(x_0)}(x_*, \varepsilon)$. Using Theorem 1.33 we have that the last display is equal to

$$\frac{3}{2} \mathbb{N}_0^{[1]} \left(\mathbb{N}_0 \left(F^{(m)}_{\varepsilon,1}(x_*, x_0) \middle| Z_{W_*+\varepsilon} \right) \int \mathrm{Vol}(\mathrm{d}z) \mathbb{1}_{\{D(x_*,z)<\varepsilon\}} D(x_*, z)^{-\alpha} \right).$$

Again, Lemma 16 in [7] allows us to bound the previous term as

$$\frac{3}{2} C_m \mathbb{N}_0^{[1]} \left((Z_{W_*+\varepsilon})^{m/2} \int \mathrm{Vol}(\mathrm{d}z) \mathbb{1}_{\{D(x_*,z)<\varepsilon\}} D(x_*, z)^{-\alpha} \right).$$

Let $k(\varepsilon)$ be the first integer such that $2^{-k(\varepsilon)} < \varepsilon$ and $\kappa > 0$ such that $\alpha - (4-m) + \kappa < 0$. For each $k \geq k(\varepsilon)$, we have

$$\begin{aligned}
&\mathbb{N}_0^{[1]} \left((Z_{W_*+\varepsilon})^{m/2} \int \mathrm{Vol}(\mathrm{d}z) \mathbb{1}_{\{2^{-k} \leq D(x_*,z) < 2^{-k+1}\}} D(x_*, z)^{-\alpha} \right) \\
&\leq 2^{k\alpha} \mathbb{N}_0^{[1]} \left((Z_{W_*+\varepsilon})^{m/2} \mathrm{Vol}(B(x_*, 2^{-k+1})) \right) \\
&\leq 2^{k\alpha} \mathbb{N}_0^{[1]} \left((Z_{W_*+\varepsilon})^{m} \right)^{1/2} \mathbb{N}_0^{[1]} \left(\mathrm{Vol}(B(x_*, 2^{-k+1})) \right)^{1/2} \\
&\leq C_{m,\kappa} \varepsilon^m 2^{(\alpha-(4-\kappa))k}.
\end{aligned}$$

Summing over $k \geq k(\varepsilon)$ we have

$$\begin{aligned}
\mathbb{N}_0^{[1]} \left((Z_{W_*+\varepsilon})^{m/2} \int \mathrm{Vol}(\mathrm{d}z) \mathbb{1}_{\{D(x_*,z)<\varepsilon\}} D(x_*, z)^{-\alpha} \right) &\leq C_{m,\kappa} \sum_{k=k(\varepsilon)}^{\infty} \varepsilon^m 2^{(\alpha-(4-\kappa))k} \\
&\leq C_{m,\kappa} \varepsilon^m 2^{-mk(\varepsilon)} \sum_{k=k(\varepsilon)}^{\infty} 2^{(\alpha-(4-\kappa)+m)k} \\
&\leq C'_{m,\kappa} \varepsilon^{2m}.
\end{aligned}$$

□

3.1.3 Proof of the lower bound

Using the results of sections 3.1.1 and 3.1.2, we now present the conclusion of the proof of the lower bound in Theorem 3.2. This is a standard scheme to prove that the Hausdorff dimension is lower bounded, which was already illustrated in Theorem 2.13 of the previous chapter. Once the first and second moment estimates are proved, they give us a way to construct a (random) measure supported on the set of geodesic stars on an event with positive probability. By identification with the Brownian plane, a scaling argument and a Blumenthal zero-one law, we can find a subset of the geodesic stars that has Hausdorff dimension with the desired lower bound.

From now on, let us fix the following elements:

- For all $\varepsilon \in (0, 1/32)$, define the measure ν_ε on $\mathbf{m}_\infty$ by

$$\nu_\varepsilon(\mathrm{d}x) = \varepsilon^{-m} \widetilde{F}^{(m)}_{\varepsilon,1}(x, x_*) \mathbb{1}_{\{D(x,x*)<2\}} \mathrm{Vol}(\mathrm{d}x).$$

- $R^{\max} := \max\{D(x, x_*) : x \in \mathbf{m}_\infty\}$.
- Let $\widetilde{\mathbb{N}}_0$ be the measure with density $\frac{1}{\sigma}$ with respect to $\mathbb{N}_0$.
- $\widetilde{\mathbb{N}}^\star_0 := \widetilde{\mathbb{N}}_0(\cdot \cap \{R^{\max} \geq 1\})$.

Proposition 3.12. *Let $\delta \in (0, 4 - m)$. There exists $a_\delta, A_\delta > 0$ and $p_0 \in (0, 1)$, such that for all $\varepsilon \in (0, 1/32)$, we have*

$$\widetilde{\mathbb{N}}^\star_0 \left(\{a_\delta \leq \langle \nu_\varepsilon, 1\rangle \leq A_\delta\} \cap \left\{ \iint \nu_\varepsilon(\mathrm{d}x)\nu_\varepsilon(\mathrm{d}y) D(x, y)^{-(4-m-\delta)} \leq A_\delta \right\} \right) \geq p_0. \tag{3.6}$$

Consequently, with $\widetilde{\mathbb{N}}^\star_0$-measure at least p_0, there exists a measure ν_0 supported in $\mathfrak{G}_{m+1}$ with finite $(4 - m - \delta)$-energy. In particular, $\dim_{\mathrm{H}}(\mathfrak{G}_{m+1}) \geq 4 - m$ *with $\widetilde{\mathbb{N}}^\star_0$-measure at least p_0.*

Proof. We will proceed as in the proof of Theorem 2.13. Fix $\delta \in (0, 4 - m)$.

<u>*First moment estimate.*</u> By Proposition 3.9, for some constant $c > 0$ we have directly that

$$\widetilde{\mathbb{N}}^\star_0(\langle \nu_\varepsilon, 1\rangle) \geq c. \tag{3.7}$$

<u>*Energy estimate.*</u> Note that

$$\begin{aligned} &\iint \nu_\varepsilon(\mathrm{d}x)\nu_\varepsilon(\mathrm{d}y) D(x, y)^{-(4-m-\delta)} \\ &\leq \varepsilon^{-2m} \iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y) \mathbb{1}_{\{D(x,y)<4\}} F^{(m)}_{\varepsilon,1}(x, x_*) F^{(m)}_{\varepsilon,1}(y, x_*) D(x, y)^{-(4-m-\delta)}, \end{aligned}$$

where we used that $D(x, x_*) < 2$ and $D(y, x_*) < 2$ imply $D(x, y) < 4$, and $\widetilde{F}^{(m)}_{\varepsilon,1} \leq F^{(m)}_{\varepsilon,1}$. Let $k(\varepsilon)$ be the greatest integer such that $2^{-k} > 2\varepsilon$. We split the integral above according to the cases

$\{2^{-k(\varepsilon)+2} \leq D(x,y) < 4\}$ and $\{D(x,y) < 2^{-k(\varepsilon)+2}\}$. For the first part, we have by Proposition 3.10 that

$$\begin{aligned}
&\widetilde{\mathbb{N}}_0^{\star}\left(\varepsilon^{-2m} \iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\mathbb{1}_{\{2^{-k(\varepsilon)+2}\leq D(x,y)<4\}} F_{\varepsilon,1}^{(m)}(x,x_*)F_{\varepsilon,1}^{(m)}(y,x_*)D(x,y)^{-(4-m-\delta)}\right) \\
&= \sum_{k=1}^{k(\varepsilon)} \widetilde{\mathbb{N}}_0^{\star}\left(\varepsilon^{-2m} \iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\mathbb{1}_{\{2^{-k+2}\leq D(x,y)<2^{-k+3}\}} F_{\varepsilon,1}^{(m)}(x,x_*)F_{\varepsilon,1}^{(m)}(y,x_*)D(x,y)^{-(4-m-\delta)}\right) \\
&\leq \sum_{k=1}^{k(\varepsilon)} 2^{-(-k+2)(4-m-\delta)}\widetilde{\mathbb{N}}_0^{\star}\left(\varepsilon^{-2m} \iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\mathbb{1}_{\{2^{-k+2}\leq D(x,y)<2^{-k+3}\}} F_{\varepsilon,1}^{(m)}(x,x_*)F_{\varepsilon,1}^{(m)}(y,x_*)\right) \\
&\leq \sum_{k=1}^{k(\varepsilon)} 2^{(k-2)(4-m-\delta)} C_{(\delta/2)} 2^{-k(4-m-\delta/2)} \\
&\leq C_{(\delta/2)} 2^{-2(4-m-\delta)} \sum_{k=1}^{\infty} 2^{-k\delta/2} \\
&=: C'_{(\delta)}.
\end{aligned}$$

For the second part of the integral, since $2^{-k(\varepsilon)+2} = 2^{-(k(\varepsilon)+1)+3} \leq 16\varepsilon$ by definition of $k(\varepsilon)$, the fact that $F_{\varepsilon,1}^{(m)} \leq F_{16\varepsilon,1}^{(m)}$ and the trivial bound $F_{\varepsilon,1}^{(m)} \leq 1$, we have by Proposition 3.11 that

$$\begin{aligned}
&\widetilde{\mathbb{N}}_0^{\star}\left(\varepsilon^{-2m} \iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\mathbb{1}_{\{D(x,y)<2^{-k(\varepsilon)+2}\}} F_{\varepsilon,1}^{(m)}(x,x_*)F_{\varepsilon,1}^{(m)}(y,x_*)D(x,y)^{-(4-m-\delta)}\right) \\
&\leq \widetilde{\mathbb{N}}_0^{\star}\left(\varepsilon^{-2m} \iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\mathbb{1}_{\{D(x,y)<16\varepsilon\}} F_{16\varepsilon,1}^{(m)}(x,x_*)D(x,y)^{-(4-m-\delta)}\right) \\
&\leq C''_{(\delta)}.
\end{aligned}$$

Joining both parts of the first integral, we have shown that

$$\widetilde{\mathbb{N}}_0^{\star}\left(\iint \nu_\varepsilon(\mathrm{d}x)\nu_\varepsilon(\mathrm{d}y)D(x,y)^{-(4-m-\delta)}\right) \leq C'_{(\delta)} + C''_{(\delta)} =: K_{(\delta)}. \tag{3.8}$$

Second moment estimate. Analogously, we deduce from (3.8) that

$$\widetilde{\mathbb{N}}_0^{\star}(\langle \nu_\varepsilon, 1\rangle^2) = \widetilde{\mathbb{N}}_0^{\star}\left(\iint \nu_\varepsilon(\mathrm{d}x)\nu_\varepsilon(\mathrm{d}y)\right) \leq 64\widetilde{\mathbb{N}}_0^{\star}\left(\iint \nu_\varepsilon(\mathrm{d}x)\nu_\varepsilon(\mathrm{d}y)D(x,y)^{-(4-m-\delta)}\right) \leq 64K_{(\delta)}, \tag{3.9}$$

where we used that $D(x,y) < 4$ implies $D(x,y)^{4-m-\delta} \leq 64$.

The end of the argument is analogous to the conclusion of the proof of Theorem 2.13. Using Paley-Zigmund inequality, (3.7) and (3.9) we have that

$$\widetilde{\mathbb{N}}_0^{\star}\left(\langle \nu_\varepsilon, 1\rangle \geq \frac{c}{2\widetilde{\mathbb{N}}_0(R^{\max} \geq 1)}\right) \geq \widetilde{\mathbb{N}}_0^{\star}\left(\langle \nu_\varepsilon, 1\rangle \geq \frac{\widetilde{\mathbb{N}}_0^{\star}(\langle \nu_\varepsilon, 1\rangle)}{2\widetilde{\mathbb{N}}_0(R^{\max} \geq 1)}\right) \geq \frac{1}{2}\frac{\widetilde{\mathbb{N}}_0^{\star}(\langle \nu_\varepsilon, 1\rangle)^2}{\widetilde{\mathbb{N}}_0^{\star}(\langle \nu_\varepsilon, 1\rangle^2)} \geq \frac{c^2}{128K_{(\delta)}}.$$

We take $a_\delta := c/(2\widetilde{\mathbb{N}}_0(R^{\max} \geq 1))$ and $p_0 = c^2/(256K_{(\delta)})$. On the other hand, using Markov's inequality and (3.8) we can find $A_\delta > 0$ sufficiently large so that

$$\widetilde{\mathbb{N}}_0^\star\left(\iint \nu_\varepsilon(\mathrm{d}x)\nu_\varepsilon(\mathrm{d}y)D(x,y)^{-(4-m-\delta)} > A_\delta\right) \leq \frac{K_{(\delta)}}{A_\delta} \leq p_0,$$

and $\widetilde{\mathbb{N}}_0^\star(\langle \nu_\varepsilon, 1\rangle \leq A_\delta) \geq p_0$. These estimates give (3.6).

For the conclusion, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $(0, 1/32)$ converging to zero. By (3.6) applied to each n, we have that

$$\widetilde{\mathbb{N}}_0^\star\left(\{a_\delta \leq \langle \nu_{\varepsilon_n}, 1\rangle \leq A_\delta\} \cap \left\{\iint \nu_{\varepsilon_n}(\mathrm{d}x)\nu_{\varepsilon_n}(\mathrm{d}y)D(x,y)^{-(4-m-\delta)} \leq A_\delta\right\} \text{ infinitely often}\right) \geq p_0.$$

In the latter event, there exists a (random) subsequence of $(\nu_{\varepsilon_n})_{n\in\mathbb{N}}$ that converges weakly to a measure ν_0 with finite $(4-m-\delta)$-energy. We will abuse of notation by still denoting $(\nu_{\varepsilon_n})_{n\in\mathbb{N}}$ such subsequence.

Let us prove that ν_0 is supported on $\mathfrak{G}_{m+1}$. Let $x \in \mathbf{m}_\infty$ belonging to the topological support of ν_0 and V an open neighborhood of x. Then, $\nu_{\varepsilon_n}(V) > 0$ for all n sufficiently large, which implies by definition of ν_ε that for all n sufficiently large there exists $y \in V$ such that $\widetilde{F}^{(m)}_{\varepsilon_n,1}(y, x_*) = 1$. This means that there are sequences $(x_n)_{n\in\mathbb{N}}$ and $(\varepsilon'_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, there exists geodesics $\eta_0^{(n)}, \dots, \eta_m^{(n)}$ ending at x_n and such that for all $i, j \in \{0, \dots, m\}$,

$$D(\eta_i^{(n)}(t), \eta_j^{(n)}(t)) \geq \delta_k, \tag{3.10}$$

for all $t \in [1-2^{-k-1}, 1-2^{-k-2}]$, for all $k \geq 1$ such that $2^{-k-4} > \varepsilon'_n$. Up to extracting subsequences, we can assume that for all $j \in \{1, \dots, m\}$, $\eta_j^{(n)}(t) \to \eta_j^{(0)}(t)$ uniformly in $t \in [0,1]$, where $\eta_0^{(0)}, \dots, \eta_m^{(0)}$ are geodesics ending at x such that the sets $\{\eta_j^{(0)}(t) : t \in [3/4, 1]\}$ are disjoint because of (3.10). This means that $x \in \mathfrak{G}_{m+1}$, concluding that ν_0 is supported on $\mathfrak{G}_{m+1}$. Theorem 2.5 gives that $\dim_{\mathrm{H}}(\mathfrak{G}_{m+1}) \geq 4-m-\delta$ with $\widetilde{\mathbb{N}}_0^\star$-measure at least p_0. □

For all $a > 0$, let $\mathbb{N}_0^{\{a\}} = \mathbb{N}_0(\cdot | W_* = -a)$. The final step is to justify that the event $\dim_{\mathrm{H}}(\mathfrak{G}_{m+1}) \geq 4-m-\delta$ has full $\mathbb{N}_0^{\{a\}}$-probability. To do so, we use an isometric identification with the Brownian plane and a zero-one law that holds for this structure.

Lemma 3.13 (Lemma 18 in [7])**.** *On the same probability space, we can construct both a Brownian plane $\mathcal{P}$ and a Brownian sphere sampled from $\mathbb{N}_0^{\{1\}}$, in such a way that, for all $\varepsilon \in (0,1)$, there exists an event E_ε where the following holds: If $x_\mathcal{P}$ is the distinguished point of $\mathcal{P}$ and $B^\bullet(x_\mathcal{P}, 1-\varepsilon)$ is the complement of the unbounded connected component of the complement of $B(x_\mathcal{P}, 1-\varepsilon)$ in $\mathcal{P}$, then* $\mathrm{Int}(B^{\bullet(x_0)}(x_*, 1-\varepsilon))$ *equipped with its intrinsic distance is isometric to* $\mathrm{Int}(B(x_\mathcal{P}, 1-\varepsilon))$ *equipped with its intrinsic distance.*

The set of geodesic $(m+1)$-stars that lie in $\mathrm{Int}(B^{\bullet(x_0)}(x_*, 1-\varepsilon))$ has dimension at least $4-m-\delta$ with positive probability. By Lemma 3.13, the set of geodesic $(m+1)$-stars, denoted $\mathfrak{G}_{m+1}(\mathcal{P})$, that lie in $\mathrm{Int}(B^\bullet(x_\mathcal{P}, 1-\varepsilon))$ has dimension at least $4-m-\delta$ with (the same) positive probability. By scaling, for all $a > 0$ we have that with positive probability $\dim_\mathrm{H}(\mathfrak{G}_{m+1}(\mathcal{P}) \cap \mathrm{Int}(B^\bullet(x_\mathcal{P}, a))) \geq 4-m-\delta$, and therefore, with positive probability it holds that

$$\bigcap_{a>0} \{\dim_\mathrm{H}(\mathfrak{G}_{m+1}(\mathcal{P}) \cap \mathrm{Int}(B^\bullet(x_\mathcal{P}, a))) \geq 4-m-\delta\}.$$

Such event belongs to an asymptotic σ-field for the Brownian plane. By a Kolmogorov zero-one law, we conclude that $\dim_\mathrm{H}(\mathfrak{G}_{m+1}(\mathcal{P}) \cap \mathrm{Int}(B^\bullet(x_\mathcal{P}, a))) \geq 4-m-\delta$ almost surely. Taking $\delta \searrow 0$, we have $\dim_\mathrm{H}(\mathfrak{G}_{m+1}(\mathcal{P}) \cap \mathrm{Int}(B^\bullet(x_\mathcal{P}, a))) \geq 4-m$ almost surely. Using a coupling of the Brownian sphere and the Brownian plane, gives that the same holds for the Brownian sphere, concluding the lower bound.

3.2 Upper bound

In this section, we present the proof of the upper bound for the Hausdorff dimension of the geodesic stars as done by Miller and Qian in [15]. To do so, we cover the set of geodesic stars with a set of balls using Proposition 1.21. In order to estimate the Hausdorff measure of this covering, we need the probability for a given ball to actually intersect the set of geodesic stars. This information comes from a very non-trivial construction concerning geodesics towards a point in the neighborhood of a given typical point. After applying this result, the upper bound is proved easily.

3.2.1 Exponent for disjoint geodesics towards a point

The upper bound in Theorem 3.2 is the consequence of an estimate for the probability of the event where there are m geodesics starting from the boundary of a hull towards a point near a typical point. From now on, x and y will denote typical points of $\mathbf{m}_\infty$. We will present the proof of the following result:

Proposition 3.14. *Let $r, \rho > 0$ and $b_0 \in (0,1)$. Conditionally on $D(x,y) > r$, let Y_s be the boundary length of $B^{\bullet(y)}(x, D(x,y)-s)$ for all $s \geq r$, and define*

$$\tau_0 := \inf\{s \geq r : Y_s = \rho\}.$$

Define $E(\varepsilon, m, r, \rho)$ to be the event where $\tau_0 < \infty$ and there exists $z \in B(x,\varepsilon)$ for which there are m geodesics starting at $\partial B^{\bullet(y)}(x, D(x,y)-\tau_0)$ and disjoint except at their endpoint equal to z. Then, for all $\rho \in (\varepsilon^{2b_0}, \varepsilon^{-2b_0})$,

$$\mathbb{N}_0(E(\varepsilon, m, r, \rho) | D(x,y) > r) = O((\varepsilon \rho^{-1/2})^{m-1+o(1)}).$$

The proof of the previous proposition is quite technical and we aim to describe the main constructions and estimates that yield the result. The basic idea is to identify the geodesics of the event $E(\varepsilon, m, r, \rho)$ with geodesics towards a typical point using strong confluence (recall Proposition 1.29). By succeeding at that, the boundary lengths between geodesics evolve as independent 3/2-stable CSBP, for which explicit estimates yield the result. However, this identification cannot be done always, so we need to introduce bad behaved layers where this identification is not possible. The key fact about these bad layers is that, outside an event with negligible probability, there is a constant number of them.

From now on, let $m \in \mathbb{N}$, $r > 0$, $\varepsilon, b_0 \in (0,1)$ and $\rho \in (\varepsilon^{2b_0}, \varepsilon^{-2b_0})$. Fix a realization of the Brownian sphere lying in the event $E(\varepsilon, m, r, \rho)$, that we will call E for simplicity. Consider the following objects:

- Let $b_1 \in (0,1)$, which will tend to 1 at the end of the proof.
- Let $a \in (0, (1-b_1)/6)$ and $a_k := a(k+2)/(k+1)$ for each $k \in \mathbb{N}$.
- For each $k \geq 1$, define $\tau_k := \inf\{s \geq \tau_{k-1} : Y_s = Y_{\tau_{k-1}}/2\}$, where τ_0 is defined in the statement of Proposition 3.14. Note that $Y_{\tau_k} = 2^{-k}\rho$ for all $k \geq 1$ such that $\tau_k < \infty$.
- $k_0 := \inf\{k \in \mathbb{N} : 2^{-k}\rho \leq \varepsilon^{2b_1}\}$.
- For each $t \geq 0$, $\mathcal{L}_t := \partial B^{\bullet(y)}(x, D(x,y) - t)$.

Assume that we can select $z \in B(x, \varepsilon)$ and m geodesics $\eta_1, \ldots, \eta_m$ starting respectively at $z_1, \ldots, z_m \in \partial B^{\bullet(y)}(x, D(x,y) - \tau_0)$ in a measurable way (see [15]). Using this, we introduce further objects:

- For each $j \in \{1, \ldots, m\}$ and $t \geq 0$, $u_{t,j}$ is the closest point to z in $\eta_j \cap \mathcal{L}_t$. Analogously, $v_{t,j}$ is the furthest point to z in $\eta_j \cap \mathcal{L}_t$.

Note that since $\eta_1, \ldots, \eta_m$ are geodesics, $v_{t,j}$ (resp. $u_{t,j}$) is the first (resp. last) point of contact between η_j and $\mathcal{L}_t$, for each $j \in \{1, \ldots, m\}$ and $t \geq 0$.

Construction of bad layers

In this section we define the bad and good layers, which will play a fundamental role in the proof.

Definition 3.15. *For each $\beta > 0$, $k \in \{1, \ldots, k_0\}$ is called a β-**fat layer** if*

$$\tau_k - \tau_{k-1} \geq \varepsilon^{-\beta} 2^{-(k-1)/2} \rho^{1/2}.$$

Informally, a fat layer is such that the time it takes for the boundary length to drop by a factor of 2 is too long. Note that if $0 < \beta_1 < \beta_2$, then any β_2-fat layer is a β_1-fat layer (provided that $\varepsilon \in (0,1)$, which is the case here).

Definition 3.16. *A layer $k \in \{1, \ldots, k_0\}$ is called* ***crossing layer*** *if there exists $j \in \{1, \ldots, m\}$ such that the clockwise and counterclockwise boundary length between $u_{\tau_k,j}$ and $v_{\tau_k,j}$ is larger than $\varepsilon^{2a_k} 2^{-k} \rho$.*

Informally, a crossing layer is such that there is some geodesic among $\eta_1, \ldots, \eta_m$ such that the first and last points of contact with the layer are too far apart.

We need further objects to define the last type of bad layer, for which we will proceed inductively.

- For each $j \in \{1, \ldots, m\}$, let w_j^0 be a point sampled uniformly from the boundary interval centered at z_j of length $2\varepsilon^{2a_0}\rho$ on $\partial B^{\bullet(y)}(x, D(x,y) - \tau_0)$.
- For each $j \in \{1, \ldots, m\}$, let γ_j^0 be the geodesic joining w_j^0 with x.
- For each $j \in \{1, \ldots, m\}$ and $t \geq 0$, let $e_{t,j}^0$ be the intersection point of γ_j^0 with $\mathcal{L}_t$ (the intersection is a unique point in this case).

Now, we will define points $w_1^\ell, \ldots, w_m^\ell$ for ℓ in a certain subset of $\{1, \ldots, k_0\}$ as follows:

1. If $k = 1$ is not a fat and crossing layer, then we have two possibilities:
   - For all $j \in \{1, \ldots, m\}$, $v_{\tau_1,j} = e_{\tau_1,j}^0$, in which case we set $w_j^1 = e_{\tau_1,j}^0$.
   - Otherwise, $k = 1$ is called a *non-merging layer.*
2. If $k = 1$ is a fat, crossing or non-merging layer, let $\ell \geq k+1 = 2$ be the first layer that is not fat and not crossing. For each $j \in \{1, \ldots, m\}$, do the following construction: Let $w_1, \ldots, w_N \in \mathcal{L}_{\tau_\ell}$ be a minimal collection such that the intervals with centers $e_{\tau_\ell,j}^0, w_1, \ldots, w_N$ and boundary length $\varepsilon^{2a_\ell} 2^{-\ell} \rho$ cover the set
   $$\{w \in \mathcal{L}_{\tau_\ell} : D(w, w_j^0) \leq \tau_\ell - \tau_0 + \varepsilon^{a_0} \rho^{1/2}\}.$$
   Set w_j^ℓ as one of the points $e_{\tau_\ell,j}^0, w_1, \ldots, w_N$ that has boundary length distance at most $\varepsilon^{2a_\ell} 2^{-\ell} \rho$ to $v_{\tau_\ell,j}$.
3. Proceed inductively applying 1 or 2 to the next layer of the last layer ℓ for which the points $w_1^\ell, \ldots, w_m^\ell$ were successfully defined, by taking into account new geodesics $\gamma_1^\ell, \ldots, \gamma_m^\ell$ joining $w_1^\ell, \ldots, w_m^\ell$ to x, respectively. For each $j \in \{1, \ldots, m\}$, $e_{t,j}^\ell$ is the intersection point of γ_j^ℓ with $\mathcal{L}_t$. If $\ell + 1$ lies in the case 2 and $\ell' \geq \ell + 2$ is the first layer that is not fat and crossing, the set to be covered for each $j \in \{1, \ldots, m\}$ is
   $$\{w \in \mathcal{L}_{\tau_{\ell'}} : D(w, w_j^\ell) \leq \tau_{\ell'} - \tau_\ell + \varepsilon^{a_\ell} 2^{-\ell/2} \rho^{1/2}\}.$$

We formalize the definition of non-merging layers using the construction above as follows.

Definition 3.17. *If $\ell \in \{1, \ldots, k_0\}$ is a layer for which the points $w_1^\ell, \ldots, w_m^\ell$ were successfully defined, then $\ell + 1$ is a* ***non-merging layer*** *if it is not fat, not crossing and there is some $j \in \{1, \ldots, m\}$ such that $e^\ell_{\tau_{\ell+1}, j} \neq v_{\tau_\ell, j}$.*

The fat, crossing and non-merging layers are called bad layers, while the rest are called good layers. Informally, a non-merging layer is a layer where the geodesics towards x coming from the marked points of the last good layer do not merge with the geodesics towards z.

Estimates for bad layers

Now we estimate the probability of observing each kind of bad layer. These estimates will ensure that outside and event with negligible probability, the number of each kind of bad layer is of constant order, something that will be crucial for the conclusion of the proof. Let us start with the fat layers.

Lemma 3.18. *For each $\beta > 0$ and $k \in \{1, \ldots, k_0\}$ we have*

$$\mathbb{P}(k \text{ is a } \beta\text{-fat layer}) = O(\varepsilon^{2\beta}).$$

Proof. Let $k \in \{1, \ldots, k_0\}$. Conditionally on $\tau_{k-1} < \infty$, the process $(2^{k-1}\rho^{-1}Y_t)_{t \geq \tau_{k-1}}$ starts at 1 and has the same distribution as $(Y_{2^{-(k-1)/2}\rho^{1/2}t})_{t \geq \tau_{k-1}}$ by scaling for a 3/2-stable CSBP. If T is extinction time of a 3/2-stable CSBP started at 1, the probability of survival until time $t > 0$ is $O(t^{-2})$ by (1.12). By the Markov property, we have

$$\mathbb{P}(k \text{ is a } \beta\text{-fat layer}) = \mathbb{P}(2^{-(k-1)/2}\rho^{1/2}(\tau_k - \tau_{k-1}) \geq \varepsilon^{-\beta}) = \mathbb{P}(T > \varepsilon^{-\beta}) = O(\varepsilon^{2\beta}).$$

□

Using the previous lemma, we can easily estimate the probability of observing a set of fat layers. To do so, let $K_0 \in \mathbb{N}$ and $H = \{h_1, \ldots, h_{K_0}\}$ be such that $1 \leq h_1 < \cdots < h_{K_0} \leq k_0$. For $\beta > 0$, define the events

$$F(\beta, H) := \{h_1, \ldots, h_{K_0} \text{ are } \beta\text{-fat layers and the rest are not}\}, \tag{3.11}$$
$$F(\beta, \emptyset) := \{\text{There are no } \beta\text{-fat layers}\}.$$

Now we use Lemma 3.18 to estimate the probability of $F(\beta, H)$.

Proposition 3.19. *For $\beta > 0$ and H as above, we have*

$$\mathbb{P}(F(\beta, H)) = O(\varepsilon^{2\beta|H|}).$$

Proof. Direct from the Markov property of the 3/2-stable CSBP and Lemma 3.18. □

Now we estimate the probability of observing a crossing layer, as well as a set of crossing layers.

Lemma 3.20. *For each $k \in \{1, \dots, k_0\}$, we have*

$$\mathbb{P}(k \textit{ is a crossing layer}) = O(\varepsilon^{\frac{4}{3}(1-b_1-6a)+o(1)}).$$

Proof. The event of being a crossing layer implies that there is some $j \in \{1, \dots, m\}$ such that the boundary length distance between $u_{\tau_k,j}$ and $v_{\tau_k,j}$ is at least $\varepsilon^{2a_k}2^{-k}\rho$. However, note that

$$\varepsilon \geq D(x,z) \geq |D(u_{\tau_k,j}, z) - D(u_{\tau_k,j}, x)| = |D(u_{\tau_k,j}, z) - (D(x,y) - \tau_k)|,$$

and analogously, $|D(v_{\tau_k,j}, z) - (D(x,y) - \tau_k)| \leq \varepsilon$. Therefore, as $u_{\tau_k,j}$ and $v_{\tau_k,j}$ belong to the same geodesic, we have

$$D(u_{\tau_k,j}, v_{\tau_k,j}) = |D(u_{\tau_k,j}, z) - D(v_{\tau_k,j}, z)| \leq 2\varepsilon.$$

Rescaling the distance in $B^{\bullet(y)}(x, D(x,y) - \tau_{k-1}) \setminus B^{\bullet(y)}(x, D(x,y) - \tau_{k+1})$ by $2^{(k-1)/2}\rho^{-1/2}$, Lemma 6.2 of [15] implies that

$$\mathbb{P}(k \text{ is a crossing layer}) = O((\varepsilon 2^{(k-1)/2}\rho^{-1/2})^{4/3}\varepsilon^{-4a_k+o(1)}).$$

We conclude by noting that $2^{-(k-1)}\rho \geq \varepsilon^{2b_1}$ and $a_k \leq 2a$. □

Let $K_1 \in \mathbb{N}$ and $I = \{i_1, \dots, i_{K_1}\}$ be such that $1 \leq i_1 < \cdots < i_{K_1} \leq k_0$. Define the event

$$C(I) := \{i_1, \dots, i_{K_2} \text{ are crossing layers and the rest are not}\}.$$

Proposition 3.21. *For I as above we have*

$$\mathbb{P}(E \cap C(I)) = O(\varepsilon^{\frac{2}{3}(1-b_1-6a)|I|+o(1)}).$$

Proof. Given $I \subseteq \{1, \dots, k_0\}$, there are at least $|I|/2$ layers which are all odd or even. In any of these cases, the conditional independence between the metric bands and Lemma 3.20 imply that the probability of having such quantity of crossing layers is $O(\varepsilon^{\frac{4}{3}(1-b_1-6a)\cdot\frac{|I|}{2}+o(1)})$, which is the desired estimate. □

Now we will prove that we can restrict the required statement of the proof to cases where there are at most a constant number of fat and crossing layers. We start with the analysis for the fat layers. From now on, we fix $\beta_0 := (1+b_0)(m-1)/2$.

Proposition 3.22. *For all $\rho \in (\varepsilon^{2b_0}, \varepsilon^{-2b_0})$, we have*

$$\mathbb{P}(F(\beta_0, \emptyset)^c) = O((\varepsilon\rho^{-1/2})^{m-1+o(1)})$$

Proof. Note that $k_0 = O(\varepsilon^{o(1)})$. Then,

$$\mathbb{P}(F(\beta_0, \emptyset)^c) \leq \sum_{k=1}^{k_0} \mathbb{P}(k \text{ is a } \beta_0\text{-fat layer}) = O(\varepsilon^{2\beta_0+o(1)}) = O(\varepsilon^{(1+b_0)(m-1)+o(1)}).$$

We conclude by noting that $\varepsilon^{b_0} < \rho^{-1/2}$. □

For $N \in \mathbb{N}$ and $\beta > 0$, define $\mathcal{F}(\beta, N) := \bigcup_{|H|>N} F(\beta, H)$.

Proposition 3.23. *Let $\beta \in (0, \beta_0]$ and $N_0 := \lceil m(1+b_0)/(2\beta) \rceil$. Then, for all $\rho \in (\varepsilon^{2b_0}, \varepsilon^{-2b_0})$ we have*

$$\mathbb{P}(\mathcal{F}(\beta, N_0)) = O((\varepsilon\rho^{-1/2})^{m-1+o(1)}).$$

Proof. By union bound,

$$\mathbb{P}(\mathcal{F}(\beta, N_0)) \leq \sum_{|H|>N_0} \mathbb{P}(F(\beta, H)) \leq \sum_{n=N_0+1}^{k_0} k_0^n O(\varepsilon^{2\beta n}).$$

Using that $k_0 = O(\varepsilon^{o(1)})$ and the definition of N_0, we have that

$$\sum_{n=N_0+1}^{k_0} \varepsilon^{(2\beta+o(1))n} = \varepsilon^{(2\beta+o(1))(N_0+1)} \cdot \frac{1-\varepsilon^{(2\beta+o(1))(k_0+N_0)}}{1-\varepsilon^{2\beta+o(1)}} = O(\varepsilon^{(1+b_0)(m-1)+o(1)}),$$

and we conclude as in the previous proposition. □

Observe that the two previous propositions allows us to consider an event smaller than E, simplifying the proof of Proposition 3.14. In fact, for $\beta \in (0, \beta_0]$ and N_0 as defined previously, we have that

$$\begin{aligned}\mathbb{P}(E) &= \mathbb{P}(E \cap \mathcal{F}(\beta, N_0)^c \cap F(\beta_0, \emptyset)) + \mathbb{P}(E \cap (\mathcal{F}(\beta, N_0) \cup F(\beta_0, \emptyset)^c)) \\ &= \mathbb{P}(E \cap \mathcal{F}(\beta, N_0)^c \cap F(\beta_0, \emptyset)) + O((\varepsilon\rho^{-1/2})^{m-1+o(1)}).\end{aligned}$$

Note that by definition (3.11), for all H with $|H| > N_0$ we have

$$\mathcal{F}(\beta, N_0)^c \subseteq F(\beta, H)^c = \bigcup F(\beta, (H \setminus H_1) \cup H_2),$$

where the union is taken over all $H_1 \subseteq H$, $H_2 \subseteq \{1, \ldots, k_0\} \setminus H$ such that $H_1 \neq \emptyset$, $H_2 \neq \emptyset$ and $|(H \setminus H_1) \cup H_2| \leq N_0$. Note that the number of choices for a set $H' \subseteq \{1, \ldots, k_0\}$ such that $|H'| \leq N_0$ is at most $\sum_{k=0}^{N_0} k_0^n = \varepsilon^{o(1)}$. Plugging this in the probability computed before, we see that to prove Proposition 3.14 it suffices to prove that for all H with $|H| \leq N_0$ and $\beta \in (0, \beta_0]$, we have

$$\mathbb{P}(E \cap F(\beta, H) \cap F(\beta_0, \emptyset)) = O((\varepsilon\rho^{-1/2})^{m-1+o(1)}). \tag{3.12}$$

We can do an analogous argument to bound the amount of crossing layers, considering the following proposition. Define $\mathcal{C}(I) := \bigcup_{|I|>N_1} C(I)$.

Proposition 3.24. *Let $N_1 := \lceil \frac{3}{2}m(1+b_0)/(1-b_1-6a) \rceil$. Then, for all $\rho \in (\varepsilon^{2b_0}, \varepsilon^{-2b_0})$ we have*

$$\mathbb{P}(E \cap \mathcal{C}(I)) = O((\varepsilon\rho^{-1/2})^{m-1+o(1)}).$$

Proof. By union bound,

$$\mathbb{P}(E \cap \mathcal{C}(I)) \leq \sum_{|I|>N_1}^{k_0} \mathbb{P}(E \cap C(I)) \leq \sum_{n=N_1+1}^{k_0} O(\varepsilon^{(\frac{2}{3}(1-b_1-6a)+o(1))n})$$
$$= O(\varepsilon^{\frac{2}{3}(1-b_1-6a)N_1+o(1)}) = O(\varepsilon^{(1+b_0)(m-1)+o(1)}).$$

□

Arguing the same way as for the fat layers, we have that to prove (3.12) it suffices to prove that for all H and I as above with $|H| \leq N_0$ and $|I| \leq N_1$ and $\beta \in (0, \beta_0]$,

$$\mathbb{P}(E \cap F(\beta, H) \cap F(\beta_0, \emptyset) \cap C(I)) = O((\varepsilon\rho^{-1/2})^{m-1+o(1)}). \tag{3.13}$$

The estimates for non-merging layers require further developments, so we postpone its analysis. By the moment, let us introduce the following event. Let $K_3 \in \mathbb{N}$ and $J = \{j_1, \ldots, j_{K_2}\}$ be such that $1 \leq j_1 < \cdots < j_{K_2} \leq k_0$. Define

$$N(J) := \{j_1, \ldots, j_{K_2} \text{ are non-merging layers and the rest are not}\}.$$

Definition of a Markovian exploration

In this section we define an exploration from $\mathcal{L}_{\tau_0}$ towards the root x as done with the points (w_j^ℓ). However, recall that the points (w_j^ℓ) do not define a Markovian exploration since they are defined using geodesics towards z. To do this, the exploration process in a given layer will depend only in the randomness of the previous layer and independent and uniform choices of the children.

Let $K \in \mathbb{N}$ and $S = \{s_1, \ldots, s_K, s_{K+1}\}$ be such that $1 = s_1 < \cdots < s_K < s_{K+1} = k_0$. We define points z_J, with J a vector of integers, for all the layers in S as follows:

1. Let $z_1^0, \ldots, z_m^0 \in \mathcal{L}_{\tau_0}$ be iid sampled uniformly according to the boundary measure, and consider them as counterclockwise ordered.

2. For each $j \in \{1, \ldots, m\}$, let η_{z_j} be the unique geodesic from z_j to x and let z_{j0} be the intersection point of η_{z_j} with $\mathcal{L}_{s_1}$. Conditionally on $\mathcal{F}_{s_1}$, let $z_{j1}, \ldots, z_{jN} \in \mathcal{L}_{\tau_1}$ be a minimal collection chosen independently of $B^{\bullet(y)}(x, D(x,y) - \tau_{s_1})$ such that the intervals with centers $z_{j0}, z_{j1}, \ldots, z_{jN}$ and boundary length $\varepsilon^{2a_{s_1}} 2^{-s_1}\rho$ cover the set

$$\{w \in \mathcal{L}_{\tau_{s_1}} : D(w, z_j) \leq \tau_{s_1} - \tau_0 + \varepsilon^{a_0}\rho^{1/2}\}.$$

3. Proceed inductively applying 2 until the points z_J, with J a vector of integers, are successfully defined for all the layers in S. We remark that the set to be covered at step n is of the form

$$\{w \in \mathcal{L}_{\tau_{s_n}} : D(w, z_J) \leq \tau_{s_n} - \tau_{s_{n-1}} + \varepsilon^{a_{s_n}} 2^{-s_n/2}\rho^{1/2}\},$$

for all J vector of integers obtained in the step $n-1$.

At each step, each point defined in a layer s_n give origin to a set of points in the next layer s_{n+1} that we call children.

Definition 3.25. *We define points* $(x_j^{s_n})$ *for* $n \in \{1, \ldots, K\}$ *and* $j \in \{1, \ldots, m\}$ *as follows:*

1. *For all* $j \in \{1, \ldots, m\}$*, set* $x_j^{s_0} = z_j$*.*
2. *Inductively, for all* $n \in \{1, \ldots, K\}$ *and* $j \in \{1, \ldots, m\}$*, let* $x_j^{s_{n+1}}$ *be sampled uniformly and independently of everything else from the set of children of* $x_j^{s_n}$*.*

The points $(x_j^{s_n})$ constitute the Markovian exploration. Now, we aim to restrict ourselves to an event where the amount of children of (z_J) (and therefore, the possibilities for $(x_j^{s_n})$) is bounded, as done with the number of fat and crossing layers. The following estimate is a direct consequence of Lemma 4.8 in [15].

Proposition 3.26. *Let* $H \subseteq \{1, \ldots, k_0\}$ *be such that* $|H| \leq N_0$*. Let* J *be a vector of integers such that* z_J *is defined in the layer* $\mathcal{L}_{\tau_n}$*, for some* $n \in \{1, \ldots, K\}$*. For each integer* $C \geq 1$*, we have*

$$\mathbb{P}(F(\beta_0, \emptyset) \cap F(\beta, H) \cap \{z_J \textit{ has at most } C \textit{ children}\}) = O(\varepsilon^{(C-1)(a_{s_n} - a_{s_{n+1}}) - a_{s_{n+1}} - \beta_0 + o(1)}).$$

For each $n \in \{1, \ldots, K\}$, let $C_n = \lceil ((1 + b_0)m + 2a + \beta_0)/(a_{s_{n+1}} - a_{s_n}) \rceil + 1$ and define the event

$$\mathscr{C} := \{\text{for all } n \in \{1, \ldots, K\} \text{ and } J \in \mathbb{N}^n, z_J \text{ has at most } C_n \text{ children}\}.$$

Repeating the exact same argument done the fat and crossing layers after propositions 3.23 and 3.24, we can use the previous estimate to show that, it is enough to prove that for all H, I and J as considered for each type of bad layer with $|H| \leq N_0$ and $|I| \leq N_1$, we have

$$\mathbb{P}(E \cap F(\beta, H) \cap F(\beta_0, \emptyset) \cap C(I) \cap \mathscr{C}) = O((\varepsilon \rho^{-1/2})^{m-1+o(1)}). \tag{3.14}$$

Coupling of the Markovian exploration

Let $H = \{h_1, \ldots, h_{K_0}\}$, $I = \{i_1, \ldots, i_{K_1}\}$ and $J = \{j_1, \ldots, j_{K_2}\}$ be subsets of $\{1, \ldots, k_0\}$ labeled in increasing order such that the following conditions hold:

(i) $(H \cup I) \cap J = \emptyset$.

(ii) For all $n \in \{1, \ldots, K_0\}$, $h_n + 1 \notin J$.

(iii) For all $n \in \{1, \ldots, K_1\}$, $i_n + 1 \notin J$.

(iv) For all $n \in \{1, \ldots, K_2\}$, $j_n + 1 < j_{n+1}$.

The first condition is the fact that we define the non-merging layers to be neither fat or crossing. The second, third and fourth condition is the fact that if k is a fat, crossing or non-merging layer, then the next non-merging layer comes after the first good layer after m, which has index at least $m+2$. For such H, I and J, define the event

$$B(H, I, J) = E \cap F(\beta_0, \emptyset) \cap F(\beta, H) \cap C(I) \cap N(J).$$

Note that for given H and I as above,

$$E \cap F(\beta_0, \emptyset) \cap F(\beta, H) \cap C(I) \subseteq \bigcup B(H, I, J),$$

where the union is taken over all J satisfying the previous conditions. Define

$$\begin{aligned} S &:= \{k \in \{1, \dots, k_0\} \setminus (H \cup I \cup J) : k+1 \in H \cup I \cup J \text{ or } k-1 \in H \cup I \cup J\}, \\ S_+ &:= \{k \in \{1, \dots, k_0\} \setminus (H \cup I \cup J) : k+1 \in H \cup I \cup J\}, \\ S_- &:= S \setminus S_+. \end{aligned}$$

Write $S = \{s_1, \dots, s_K, s_{K+1}\}$ with $0 = s_0 < s_1 < \dots < s_K < s_{K+1} = k_0$. We will show that we will show that the Markovian exploration process can be coupled with the points defined $(w_{k,j})$ for the good layers. Proceed as follows assuming that the event $B(H, I, J) \cap \mathscr{C}$ holds:

1. Recall that the points x_j^0 with $j \in \{1, \dots, m\}$ are iid sampled uniformly according to the boundary length measure. In particular, the event where for all $j \in \{1, \dots, m\}$, $x_{0,j}$ has boundary distance at most $\varepsilon^{2a_0}\rho$ has probability $(2\varepsilon^{2a_0})^m$. Conditionally on this event, we can choose $x_j^0 = w_j^0$ for all $j \in \{1, \dots, m\}$.

2. Inductively, assume that we have chosen $x_j^{s_n} = w_j^{s_n}$ for all $j \in \{1, \dots, m\}$ for some $n \in \{1, \dots, K\}$. Then,

   (a) If $s_n \in S_+$, then s_{n+1} is the first good layer after s_n. For all $j \in \{1, \dots, m\}$, choose the set of children of $w_j^{s_n}$ in $\mathcal{L}_{\tau_{s_n}}$ to be the same as the set of children of $x_j^{s_n}$, and set $x_j^{s_{n+1}} = w_j^{s_{n+1}}$ for all $j \in \{1, \dots, m\}$. The latter event has probability at least L_n^{-m}.

   (b) If $s_n \in S_-$, then $\{s_n, s_n + 1, \dots, s_{n+1}\}$ are all good layers and $w_j^{s_{n+1}} = v_{\tau_{s_{n+1}}, j}$ for all $j \in \{1, \dots, m\}$. The probability of choosing $x_j^{s_{n+1}} = w_j^{s_{n+1}}$ for all $j \in \{1, \dots, m\}$ is again at least L_n^{-m}.

Using this coupling, define the event

$$G(H, I, J) := \bigcap \{x_j^{s_n} = w_j^{s_n}\} \cap B(H, I, J) \cap \mathscr{C},$$

where the intersection is taken over all $n \in \{1, \dots, K+1\}$ and $j \in \{1, \dots, m\}$. The coupling constructed in 1 and 2 allows us to prove the following useful estimate.

Proposition 3.27. *There is a constant depending only on a, b_0, β_0 and m such that*

$$\mathbb{P}(B(H,I,J)\cap\mathscr{C}) \leq O\left(\varepsilon^{-2a_0 m+o(1)}\right)\mathbb{P}(G(H,I,J)).$$

Proof. By the construction of the coupling, we have

$$\mathbb{P}(G(H,I,J)|B(H,I,J)\cap\mathscr{C}) \geq (C_1\dots C_K)^{-m}(2\varepsilon^{2a_0})^m.$$

Note that for all $n\in\{1,\dots,K\}$, $a_{s_{n+1}}-a_{s_n}=a(m+1)^{-1}(m+2)^{-1}$ so that

$$C_n \leq \widetilde{C}\log_2(\varepsilon^{-1})^2,$$

where $\widetilde{C}$ depends only on a, b_0, β_0 and m. Then, for sufficiently small ε,

$$\mathbb{P}(G(H,I,J)|B(H,I,J)\cap\mathscr{C}) \geq \widetilde{C}\varepsilon^{2a_0 m+o(1)},$$

and we conclude by developing the conditional probability. □

Bounding the non-merging layers

The following proposition gives the (order of the) probability of a given layer to be non-merging. The proof can be found in the original reference. Such exponent allows us to bound the amount of non-merging layers, as we did with the fat and crossing layers in propositions 3.23 and 3.24.

Proposition 3.28. *Let $k\in\{1,\dots,k_0\}$ be such that $k+1\in J$. Conditionally on $\mathcal{F}_k$ and $x_j^m=w_{m,j}$, the probability that $m+1$ is a non-merging layer is $O(\varepsilon^{a_k-\beta+o(1)})$.*

Take $\beta=\min\{a/2,\beta_0\}$. As a direct consequence of the previous proposition, we have

$$\mathbb{P}(G(H,I,J)) = O(\varepsilon^{\frac{a}{2}|J|}),$$

which implies by Proposition 3.27 that

$$\mathbb{P}(B(H,I,J)\cap\mathscr{C}) = O(\varepsilon^{-2a_0 m+\frac{a}{2}|J|+o(1)}).$$

Repeating the exact same argument done the fat and crossing layers after propositions 3.23 and 3.24, we can use the previous estimate to show that, if $N_2:=\lceil 2m(a_0+1+b_0)/a\rceil$, then to prove (3.14) it is enough to prove that for all H, I and J as considered for each type of bad layer with $|H|\leq N_0$, $|I|\leq N_1$ and $|J|\leq N_2$, we have

$$\mathbb{P}(B(H,I,J)\cap\mathscr{C}) = O((\varepsilon\rho^{-1/2})^{m-1+o(1)}). \tag{3.15}$$

Conclusion of the proof

Let H, I and J be as in the previous section and such that $|H| \leq N_0$, $|I| \leq N_1$ and $|J| \leq N_2$. In this setting, we will prove the following proposition.

Proposition 3.29. $\mathbb{P}(B(H, I, J)) = O((\varepsilon\rho^{-1/2})^{m-1+o(1)})$.

Note that this implies (3.15), so this ends the proof of Proposition 3.14.

Proof. Consider the set S_- as defined in the previous section and fix $n \in \{1, \ldots, K\}$. By construction, for each $s_n \in S_-$, all the layers in $\{s_n, \ldots, s_{n+1}\}$ are good layers and the geodesic $\gamma_j^{s_n}$ from $w_j^{s_n}$ to x merge with the geodesic η_j from z_j to z in the layer $s_n + 1$. Therefore, the geodesics $\eta_{s_n,j}$ for $j \in \{1, \ldots, m\}$ are disjoint in the metric band $B^{\bullet(y)}(x, D(x,y) - \tau_{s_n}) \setminus B^{\bullet(y)}(x, D(x,y) - \tau_{s_{n+1}})$.

For each $j \in \{1, \ldots, m\}$ and $t \geq 0$, let

$$Z_t^{s_n,j} = \text{counterclockwise boundary length from } e^{s_n}_{\tau_{s_n}+t,j} \text{ to } e^{s_n}_{\tau_{s_n}+t,j+1},$$

where we take the convention $e^{s_n}_{\tau_{s_n}+t,m+1} = e^{s_n}_{\tau_{s_n}+t,1}$. Then, the processes $Z^{s_n,1}, \ldots, Z^{s_n,m}$ evolve as independent 3/2-stable CSBP. Using this, define the event

$$A_n := \{\text{None of the processes } Z^{s_n,1}, \ldots, Z^{s_n,m} \text{ hit } 0 \text{ before } \tau_{s_{n+1}} - \tau_{s_n}\}.$$

By construction, we have

$$B(H, I, J) \subseteq \bigcap_{\substack{n \in \{1,\ldots,K\} \\ s_n \in S_-}} A_n.$$

Finally, noting that $\mathbb{P}(A_n) \lesssim 2^{-\frac{1}{2}(s_{n+1}-s_n)(m-1)}$ using the independence of the m independent stable CSBPs and 1.12, we have by conditional independence of the bands that

$$\mathbb{P}(B(H, I, J)) \lesssim \prod_{\substack{n \in \{1,\ldots,K\} \\ s_n \in S_-}} 2^{-\frac{1}{2}(s_{n+1}-s_n)(m-1)}.$$

Note that

$$\sum_{\substack{n \in \{1,\ldots,K\} \\ s_n \in S_-}} (s_{n+1} - s_n) \geq k_0 - 2(N_0 + N_1 + N_2) \geq \log_2(\varepsilon^{-2b_1}\rho) - 2(N_0 + N_1 + N_2),$$

so we have

$$\mathbb{P}(B(H, I, J)) = O((\varepsilon^{b_1}\rho^{-1/2})^{m-1}).$$

Since b_1 can be made arbitrarily close to 1, this ends the proof. □

3.2.2 Proof of the upper bound

We are ready to prove the upper bound in Theorem 3.2. In fact, it follows rapidly by bounding the Hausdorff measure of a well chosen covering of the geodesic m-stars, where Proposition 3.14 plays a fundamental role in the involved exponents.

Theorem 3.30.

- *If* $m \in \{1,2,3,4,5\}$, $\dim_{\mathrm{H}}(\mathfrak{G}_m) \leq 5-m$ *a.s.*
- *If* $m \geq 6$, *the set* $\mathfrak{G}_m$ *is a.s. empty.*

Proof. Fix $a, M > 0$. It suffices to prove for $\mathrm{diam}(\mathbf{m}_\infty) \leq M$. By propositions 1.22 and 1.23, there exists $r_0 > 0$ such that for all $r \in (0, r_0)$ and $z \in \mathbf{m}_\infty$, if $U_{z,r}$ denotes the connected component with the largest diameter of $\mathbf{m}_\infty \setminus B(z,r)$, then $\mathrm{diam}(U_{z,r}) \geq \mathrm{diam}(\mathbf{m}_\infty)/2$ and $\mathrm{Vol}(U_{z,r}) \geq \mathrm{Vol}(\mathbf{m}_\infty)/2$.

For each $r_1 \in (0, r_0)$, let $\mathfrak{G}_{m,r}$ be the set of points z such that there are m disjoint geodesics emerging from z towards $\partial B^{\bullet(w)}(z, r_1)$, where w is any point in U_{z,r_1}. It follows that

$$\mathfrak{G}_m \subseteq \bigcup_{r \in (0,r_0)} \mathfrak{G}_{m,r}.$$

Noting that $\mathfrak{G}_{m,r}$ is decreasing as set in the parameter r, so that we can see the previous union as a countable union. By countable stability of the Hausdorff dimension (Proposition 2.2), it suffices to show that for all $r \in (0, r_0)$, $\dim_{\mathrm{H}}(\mathfrak{G}_{m,r}) \leq 5-m$.

Let $b > 0$ and set $N_\varepsilon = \varepsilon^{-4-b}$ and $M_\varepsilon = \varepsilon^{-b}$. Let $(x_i)_{i \in \mathbb{N}}$ and $(y_j)_{j \in \mathbb{N}}$ be iid points sampled according to $\mathrm{Vol}(\cdot)$. Then, by Proposition 1.21, there is some (random) $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$,

$$\mathbf{m}_\infty \subseteq \bigcup_{i=1}^{N_\varepsilon} B(x_i, \varepsilon), \text{ and } \mathbf{m}_\infty \subseteq \bigcup_{i=1}^{M_\varepsilon} B(y_j, \mathrm{diam}(\mathbf{m}_\infty)/10).$$

The goal now is to estimate the probability that the i-th ball of the first covering has a non-empty intersection with $\mathfrak{G}_{m,r}$. Choose $\varepsilon' \in (0, r_1/2)$ and let $\varepsilon \in (0, \varepsilon_0 \wedge \varepsilon')$. Define

$$I_{\varepsilon,m,r_1} = \{i \in \{1, \ldots, N_\varepsilon\} : \mathfrak{G}_{m,r_1} \cap B(x_i, \varepsilon) \neq \emptyset\}.$$

Observe that if $i \in I_{\varepsilon,m,r_1}$ and $z \in \mathfrak{G}_{m,r_1} \cap B(x_i, \varepsilon)$, then there exists $j \in \{1, \ldots, M_\varepsilon\}$ such that $y_j \in U_{z,r_1}$ and $D(z, y_j) \geq \mathrm{diam}(\mathbf{m}_\infty)/4$. Moreover, there are m geodesics starting at z towards $\partial B^{\bullet(y_j)}(x_i, r_1/2)$. If we set $t_n = nr_1/4$ for $n \in \{1, \ldots, \lfloor 4M/r_1 \rfloor\}$, Lemma 7.2 in [15] tells us that the following event holds outside an event with negligible probability:

$$E(x_i, y_j) := \bigcup_{n=1}^{\lfloor 4M/r_1 \rfloor} \left(E(\varepsilon, m, t_n, \varepsilon^{2a}) \cap \{\mathrm{diam}(\mathbf{m}_\infty) \leq M\} \cap \{D(x_i, y_j) > t_n\} \right).$$

where $E(\varepsilon, m, t_n, \varepsilon^{2a})$ is the event of Proposition 3.14. We also have that

$$\{i \in I_{\varepsilon,m,r_1}\} \cap \{\mathrm{diam}(\mathbf{m}_\infty) \leq M\} \subseteq \bigcup_{j=1}^{M_\varepsilon} E(x_i, y_j),$$

By union bound, we see that

$$\begin{aligned}
&\mathbb{N}_0\left(\{i \in I_{\varepsilon,m,r_1}\} \cap \{\mathrm{diam}(\mathbf{m}_\infty) \leq M\}\right) \\
&\leq \sum_{j=1}^{M_\varepsilon} \sum_{n=1}^{\lfloor 4M/r_1 \rfloor} \mathbb{N}_0\left(E(\varepsilon, m, t_n, \varepsilon^{2a}) \cap \{\mathrm{diam}(\mathbf{m}_\infty) \leq M\} \cap \{D(x_i, y_j) > t_n\}\right) \\
&\leq \sum_{j=1}^{M_\varepsilon} \sum_{n=1}^{\lfloor 4M/r_1 \rfloor} \mathbb{N}_0\left(E(\varepsilon, m, t_n, \varepsilon^{2a}) | D(x_i, y_j) > t_n\right) \mathbb{N}_0(D(x_i, y_j) > t_n) \\
&\lesssim \varepsilon^{-b+(m-1)(1-a)+o(1)}.
\end{aligned}$$

Since $a > 0$ was arbitrary, this shows that $\mathbb{N}_0\left(\{i \in I_{\varepsilon,m,r_1}\} \cap \{\mathrm{diam}(\mathbf{m}_\infty) \leq M\}\right) = O(\varepsilon^{-b+m-1+o(1)})$. We use this to compute the α-value of the covering I_{ε,m,r_1} of the set $\mathfrak{G}_{m,r}$. We have that

$$\mathbb{N}_0\left[\sum_{i \in I_{\varepsilon,m,r_1}} \mathrm{diam}(B(x_i, \varepsilon))^\alpha \mathbb{1}_{\{\varepsilon < \varepsilon_0\}} \mathbb{1}_{\{\mathrm{diam}(\mathbf{m}_\infty) \leq M\}}\right] \lesssim \varepsilon^{\alpha - 2b + m - 5 + o(1)}.$$

Assume that $\alpha > 5 - m$ and that b is sufficiently small so that the resulting exponent is positive. This gives that the above quantity goes to zero as $\varepsilon \to 0$, proving that $\dim_{\mathrm{H}}(\mathfrak{G}_{m,r}) \leq 5 - m$ for $m \leq 5$. On the other hand, if $m \geq 6$ and $\alpha \leq 5 - m$,

$$\begin{aligned}
&\mathbb{N}_0(\{\mathfrak{G}_{m,r} \neq \emptyset\} \cap \{\varepsilon < \varepsilon_0\} \cap \{\mathrm{diam}(\mathbf{m}_\infty) \leq M\}) \\
&\leq \sum_{i=1}^{N_\varepsilon} \mathbb{N}_0\left(\{i \in I_{\varepsilon,m,r_1}\} \cap \{\varepsilon < \varepsilon_0\} \cap \{\mathrm{diam}(\mathbf{m}_\infty) \leq M\}\right) \\
&\lesssim \varepsilon^{-2b+m-5+o(1)}.
\end{aligned}$$

If we take $b \in (0, 1/2)$, the exponent is positive and this quantity goes to zero as $\varepsilon \to 0$. This completes the proof of Theorem 3.30. □

Chapter 4

Stable geometry

At the beginning of Chapter 1, we claimed that the Brownian map appears as the scaling limit of uniform random planar maps with small faces. The latter restriction can be weakened to consider models of planar maps with large faces. In this case, a different limit space appears, which is called *α-stable gasket or carpet*, depending on the value of $\alpha \in (1, 2)$. The canonical way of doing this is to consider a model of bipartite planar maps where the weight sequence is in the domain of attraction of a stable law [3]. This gives raise to the so-called stable geometry, which is the stable version of the Brownian geometry presented in Chapter 1.

4.1 Construction

Again we focus on the construction of the α-stable geometry as a continuum object, rather than its emergence as scaling limit of planar maps. For a complete presentation of this, we refer to [3].

The stable geometry is a construction where the building blocks are Lévy excursions and Gaussian processes, unlike Brownian geometry where we use only Brownian excursions and Gaussian processes. In this sense, the Brownian geometry is a particular case of the stable geometry. However, the mathematical treatment of the latter is sometimes very different from the Brownian setting, as we shall remark.

The natural space for Lévy processes is $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, the space of càdlàg functions. On $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, we consider the canonical process $X = (X_t)_{t \geq 0}$ and endow $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ with different measures that will define the distribution of X. Namely:

- For each $v \geq 0$, we let $\mathbf{N}^{(v)}$ be the measure on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ under which X is a normalized excursion with lifetime v of an α-stable Lévy process with no-negative jumps and Laplace exponent $\lambda \mapsto \lambda^\alpha$ (see the Interlude in Section 1.2.4).

- We let $\mathbf{N}$ to be the *excursion measure*, which can be defined by the disintegration formula

$$\mathbf{N}(A) := \int_0^\infty \mathbf{N}^{(v)}(A)\frac{\mathrm{d}v}{\alpha\Gamma(1-\frac{1}{\alpha})v^{\frac{1}{\alpha}+1}},$$

  for each measurable subset of A. Informally, a sample of the excursion measure is composed of lifetime σ with distribution $v \mapsto 1/(\alpha\Gamma(1-\frac{1}{\alpha})v^{\frac{1}{\alpha}+1})$ and a Lévy excursion sampled from $\mathbf{N}^{(\sigma)}$.

Now we construct the stable geometry. Fix $\alpha \in (1,2)$ and endow $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ with $\mathbf{N}$. Denote the lifetime of X by $\sigma = \sup\{t \geq 0 : X_t > 0\}$. Let us introduce further notation. For $s,t \in [0,\sigma]$,

- Set $\Delta_t := X_t - X_{t-}$ and $I_{s,t} := \inf_{[s,t]} X$.
- Write $s \preccurlyeq t$ if $s \leq t$ and $I_{s,t} \geq X_{s-}$. In such case, we say that s is an *ancestor* of t. If both inequalities are strict, we say that s is a *strict ancestor* of t.
- For s ancestor of t, define $x_{s,t} = I_{s,t} - X_{s-}$.
- $s \curlywedge t := \sup\{r \leq s \wedge t : r \preccurlyeq s \text{ and } r \preccurlyeq t\}$ is the most recent ancestor of s and t (well-defined since $\preccurlyeq$ is a partial order on $[0,\sigma]$).
- Set $\mathrm{Branch}(s,t) := \{r \in [0,\sigma] : s \curlywedge t \prec r \prec s\} \cup \{r \in [0,1] : s \curlywedge t \prec r \prec t\} \cup \{s,t\}$.

Similarly to the construction of real trees, we now identify times in $[0,\sigma]$ using the excursion X. However, the jumps of X make this a bit more complicated, but still possible in the following way. In fact, it is easier to introduce the equivalence relation first, rather than the metric, on which we will not comment. Define the equivalence relation $\sim$ on $[0,\sigma]$ to be such that for all $s,t \in [0,\sigma]$,

$$s \sim t \quad \text{if, and only if} \quad I_{s\wedge t, s\vee t} = X_{(s\wedge t)-} = X_{s\vee t}.$$

The equivalence relation $\sim$ is in fact induced by a pseudo-distance d on $[0,\sigma]$, that we are not introducing here (see [3]). We then consider the quotient space $\mathcal{L}_\alpha := [0,\sigma]/\sim$ with the metric induced by d. Topologically, this space is not a real tree in the sense of Definition 1.1, but a *looptree*. Informally, a looptree a concatenation of tangent closed curves (normally drawn as circumferences) that form the structure of a tree. The equivalence relation $\sim$ is constructed in a way such that each jump of X corresponds to a loop in the looptree. Better than words, Figure 4.1 is self-explanatory.

Now we assign labels to the looptree in a similar way to the Brownian sphere. To do so, assume further that there is a probability space $(\Omega,\mathcal{F},P)$ supporting an iid sequence $(\mathrm{b}_i)_{i\in\mathbb{N}}$ of Brownian bridges with lifetime 1, starting and ending at 0, independent of X. Let $(t_i)_{i\in\mathbb{N}}$ be a measurable enumeration of the set of jump times $\{t \in [0,\sigma] : \Delta_t > 0\}$. For each $t \in [0,\sigma]$, define

$$Z_t := \sum_{i\in\mathbb{N}: t_i \preccurlyeq t} \Delta_{t_i}^{1/2}\mathrm{b}_i\left(\frac{x_{t_i,t}}{\Delta_{t_i}}\right).$$

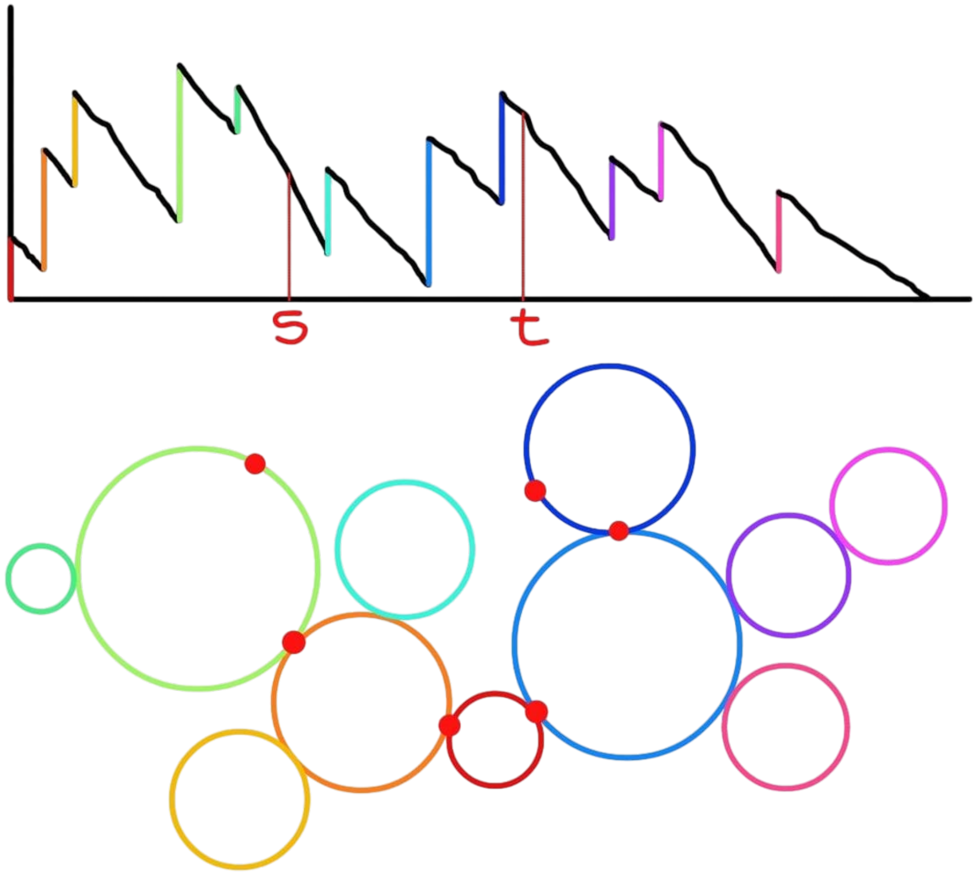

Figure 4.1: In the first picture, an approximated graph of X and two times $s, t \in [0, 1]$. In the second picture, the associated looptree drawn at scale. The red dots are the set $\mathrm{Branch}(s, t)$.

where the series is proven to converge in L^2, and the process $t \mapsto Z_t$ has a continuous modification with a.s. $(\frac{1}{2\alpha} - \varepsilon)$-Hölder continuous sample paths, for all $\varepsilon \in (0, \frac{1}{2\alpha})$, see propositions 5 and 6 in [12]. We consider such a modification from now on. Similarly to the label process of the Brownian sphere, the process $Z = (Z_t)_{0 \le t \le \sigma}$ can be seen as indexed by $\mathcal{L}_\alpha$. Equivalently, this process can be seen as the Gaussian Free Field in $\mathcal{L}_\alpha$, see Section 3.1 in [3]. The construction of the α-stable gasket and carpet now continues as we already have seen for the Brownian sphere. Using the convention $[s, t] = [0, t] \cup [s, 1]$ if $t < s$, define for all $s, t \in [0, \sigma]$,

$$D^\circ_\alpha(s, t) := Z_s + Z_t - 2 \max \left\{ \min_{[s,t]} Z, \min_{[t,s]} Z \right\}.$$

Consider the D_α to be the largest pseudo-distance bounded by D°_α, and define $\mathcal{S}_\alpha := [0, \sigma] \setminus \{D_\alpha = 0\}$ with the distance induced by D_α, still denoted D_α. Let $\Pi_\alpha : [0, \sigma] \to \mathcal{S}_\alpha$ be the canonical projection and define the *root* as $\rho_\alpha := \Pi_\alpha(0)$. Finally, define the *volume measure* Vol_α on $\mathcal{S}_\alpha$ as the pushforward of the Lebesgue measure on $[0, \sigma]$ by Π_α.

Definition 4.1. *The compact pointed measure metric space* $(\mathcal{S}_\alpha, D_\alpha, \mathrm{Vol}_\alpha, \rho_\alpha)$ *is called* ***α-stable gasket (resp. carpet)*** *if* $\alpha \in (1, 3/2)$ *(resp.* $\alpha \in [3/2, 2)$*).*

The terminology changes with the value of α since there is a phase transition for the topology of $(\mathcal{S}_\alpha, D_\alpha)$, as shown in Section 8 of [3]. For $\alpha \in [3/2, 2)$, the loops of $\mathcal{L}_\alpha$ are converted in the holes of S_α, whose boundaries are simple and non-intersecting loops. Furthermore, the topology of $(\mathcal{S}_\alpha, D_\alpha)$ in this phase is that of the Sierpinski carpet. For $\alpha \in (1, 3/2)$, the boundaries of the holes in S_α do intersect, and it is conjectured that the topology is random in this phase.

The previous facts represent the very first remarkable difference between the stable gasket/carpet and the Brownian sphere, since the latter is homeomorphic to the 2-dimensional sphere. Another inmediate consequence of the topology of $\mathcal{S}_\alpha$ is that there is no notion of exit measures. However, the holes in $\mathcal{S}_\alpha$ also facilitate other computations, such as the identification of the equivalence classes of $\{D = 0\}$. There is actually a trade-off in terms of geometric and probabilistic properties associated to the change from the Brownian geometry to the stable geometry, that the reader is invited to reflect on.

The construction of $(\mathcal{S}_\alpha, D_\alpha, \mathrm{Vol}_\alpha, \rho_\alpha)$ is the same when $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ is endowed with $\mathbf{N}^{(v)}$. Note that the distribution of the stable gasket/carpet is $\mathbf{N} \otimes P$. However, we will abuse of notation by always dropping from the notation the measure P, so that the distribution of the stable gasket/carpet is denoted just by $\mathbf{N}$. We still denote $B(x, r) = \{y \in \mathcal{S}_\alpha : D_\alpha(x, y) < r\}$.

Let us briefly comment on the point with minimal label of Z. It turns out that such a point is almost surely unique as in the Brownian sphere, and it is a typical point in $\mathcal{S}_\alpha$. We record this in the following proposition, for which we refer to the original reference for the proof.

Proposition 4.2 (Propositions 4.3 and 7.1 in [3])**.**

- *The (two-sided) local minima of Z are distinct. Denote by $t_* \in [0, 1]$ the (a.s. unique) time such that $Z_{t_*} = \min_{[0,1]} Z$ and let $\rho_* := \Pi_\alpha(t_*)$.*
- *Re-rooting property: Conditionally on σ,*

$$(\mathcal{S}_\alpha, D, \mathrm{Vol}_\alpha, \rho_\alpha) \overset{(d)}{=} (\mathcal{S}_\alpha, D, \mathrm{Vol}_\alpha, \rho_*) \overset{(d)}{=} (\mathcal{S}_\alpha, D, \mathrm{Vol}_\alpha, \Pi_\alpha(U)),$$

  where U is an uniform random variable on $[0, \sigma]$.

4.2 Heuristics for geodesics stars

In this section, we aim to give some heuristics to prove the extension of Theorem 3.2 to stable geometry. One should not expect to replicate the exact same computations done for the Brownian sphere. This is mainly because of the presence of faces and the weaker Markovian structure in stable geometry. However, we can still try to translate some objects in Brownian geometry to the stable language. Let $\mathfrak{G}_m^\alpha$ be the set of geodesic m-stars in $\mathcal{S}_\alpha$.

Conjecture 4.1. $\dim_{\mathrm{H}}(\mathfrak{G}_m^\alpha) = 2\alpha + 1 - m$.

Note that if $\alpha \to 2$, the conjecture matches Theorem 3.2. In fact, this is expected since the basic underlying process ruling the Brownian sphere is the Brownian motion, whose Laplace exponent is $\lambda \mapsto \lambda^2$. For the α-stable geometry with $\alpha \in (1,2)$, the latter is replaced by $\lambda \mapsto \lambda^\alpha$, so that α is changed only in a linear way. There are also other quantities that show this linear dependence of observables with respect to α, such as the Hausdorff dimension of the α-stable gasket/carpet itself: we have $\dim_{\mathrm{H}}(\mathcal{S}_\alpha) = 2\alpha$, while $\dim_{\mathrm{H}}(\mathbf{m}_\infty) = 4$.

Towards a first moment estimate

Let us sketch some ideas to prove the lower bound of the previous conjecture, and more specifically, a first moment estimate as done in 3.9 for the Brownian sphere. To do so, let us identify the key ingredients used in the latter that allowed to prove such an estimate.

Note from the proof of 3.9 that the first moment estimate comes from two essential properties of the Brownian sphere, namely, its Markov property and the Poissonian structure governing hulls. In fact, after applying the Markov property, the estimate is simplified to analize the probability of events related to independent parts of $\mathbf{m}_\infty$, where the hull is easily handled thanks to its Poissonian structure (recall Section 1.3.3). Then, it makes sense to pose the question:

Can we find a structure in the stable gasket/carpet with nice Markovian and Poissonian properties?

A first answer might come from the *spinal decomposition* of $\mathcal{S}_\alpha$, discussed in Section 6.2 of [3]. First, define the measure $\mathbf{N}^\bullet$ by the relation

$$\mathbf{N}^\bullet(F(X,Z,t^\bullet)) = \mathbf{N}\left(\int_0^\sigma \mathrm{d}t F(X,Z,t)\right),$$

for all $F : \mathbb{D}(\mathbb{R}_+,\mathbb{R}) \times C(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R}_+ \to \mathbb{R}_+$ measurable. The measure $\mathbf{N}^\bullet$ can be seen as $\mathbf{N}$ together with a distinguished uniformly distributed point $t^\bullet$. The spinal decomposition describes the interaction between the loops visited along the branch connecting 0 and $t^\bullet$ and the remaining parts of $\mathcal{S}_\alpha$. Roughly speaking, when performing such an exploration and conditionally on the labels across the branch, for each visited loop we have a Poissonian structure for the labeled looptrees attached to them, see Figure 4.2. To state this property formally, we need to introduce further objects.

- For each fixed $t \geq 0$, let
$$H_t := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathrm{d}s \mathbb{1}_{\{X_s < I_{s,t} + \varepsilon\}},$$
where the limit is proved to exist in probability, see [11]. The process $(H_t)_{t\geq 0}$ has a continuous modification, that we consider from now and still denote it by $(H_t)_{t\geq 0}$. We call $(H_t)_{t\geq 0}$ the *height process associated to X*.

- For each $t \geq 0$ and $r \in [0, H_t)$, define

$$\xi_t(r) := \inf\{s \leq t : H_u > r \text{ for all } u \in (s, t]\},$$

  and set $\xi_t(r) = t$ for all $r \geq H_t$.

- For $t \geq 0$ such that $\Delta_t > 0$, let $f_t(s) := \inf\{r \geq t : X_r = X_t - s\Delta_t\}$ for all $s \in [0, 1]$.

Let us now argue under $\mathbf{N}^\bullet$ and let $t^\bullet$ be the distinguished uniformly distributed point sampled from such measure. For each $r \in [0, H_{t^\bullet}]$, denote $Y_r^\bullet := Z_{\xi_{t^\bullet}(r)}$. We have that the process $Y^\bullet = (Y_r^\bullet)_{r \in [0, H_{t^\bullet}]}$ keeps track of the labels seen across $\text{Branch}(0, t^\bullet)$, and that if r is a jump time of $Y^\bullet$, then $\xi_{t^\bullet}(r-)$ is a jump time of X (see Lemma 2.5 in [3]). Using this, we aim to describe the structure around the loop of $\xi_{t^\bullet}(r-)$ associated to a fixed jump time r of $Y^\bullet$. To do so, we introduce the quantities

$$\hat{\mathfrak{G}}_r := X_{\xi_{t^\bullet}(r-)} - X_{\xi_{t^\bullet}(r)}, \quad \text{and} \quad \mathfrak{G}_r := X_{\xi_{t^\bullet}(r)} - X_{\xi_{t^\bullet}(r-)-}.$$

Note that $\Delta_{\xi_{t^\bullet}(r-)} = \hat{\mathfrak{G}}_r + \mathfrak{G}_r$. Define also the labels seen across the segments of lengths $\hat{\mathfrak{G}}_r$ and $\mathfrak{G}_r$ respectively by

$$\hat{B}_s := Z_{f_{\xi_{t^\bullet}(r-)}\left(\frac{s}{\hat{\mathfrak{G}}_r + \mathfrak{G}_r}\right)}, \text{ for } s \in [0, \hat{\mathfrak{G}}_r], \quad \text{and} \quad B_s := Z_{f_{\xi_{t^\bullet}(r-)}\left(\frac{\hat{\mathfrak{G}}_r + \mathfrak{G}_r - s}{\hat{\mathfrak{G}}_r + \mathfrak{G}_r}\right)}, \text{ for } s \in [0, \mathfrak{G}_r].$$

Now we code the labeled looptrees attached to the loop corresponding to r. Let $((u_i, v_i))_{i \in \mathbb{N}}$ be the connected components of the set $\{t \in [f_{\xi_{t^\bullet}(r-)}(0), f_{\xi_{t^\bullet}(r-)}(1)] : X_t > I_{\xi_{t^\bullet}(r-), t}\}$. Using this, define the excursions processes

$$X_s^i := X_{(u_i + s) \wedge v_i} - X_{u_i} \quad \text{and} \quad Z_s^i := Z_{(u_i + s) \wedge v_i} - Z_{u_i}, \quad \text{for all } s \geq 0.$$

Finally, introduce the point measures

$$\hat{\mathcal{P}}_r := \sum_{i \geq 1 : u_i < \xi_{t^\bullet}(r)} \delta_{X_{\xi_{t^\bullet}(r-)} - X_{u_i}, X^i, Z^i} \quad \text{and} \quad \mathcal{P}_r := \sum_{i \geq 1 : u_i < \xi_{t^\bullet}(r)} \delta_{X_{u_i} - X_{\xi_{t^\bullet}(r-)-}, X^i, Z^i}. \tag{4.1}$$

We are ready to state the spinal decomposition of the stable gasket/carpet. Here, $\mathbb{P}_{a \to b}^{(T)}$ denotes the law of a Brownian bridge from a to b with lifetime T.

Proposition 4.3 (Proposition 6.2 in [3]). *Under $\mathbf{N}^\bullet$ and conditionally on $r \mapsto Y_r^\bullet$, the collection $(\hat{\mathfrak{G}}_r, \mathfrak{G}_r, \hat{B}^{(r)}, B^{(r)}, \hat{\mathcal{P}}_r, \mathcal{P}_r)$ for r such that $\Delta Y_r^\bullet := Y_r^\bullet - Y_{r-}^\bullet \neq 0$, are independent and their conditional distribution can be determined as follows. First, the law of $(\hat{\mathfrak{G}}_r, \mathfrak{G}_r)$ is proportional to*

$$\mathbb{1}_{\ell_1, \ell_2 > 0} |\Delta Y_r^\bullet|^{2\alpha - 1} \frac{(\ell_1 + \ell_2)^{-\alpha - \frac{1}{2}}}{\sqrt{2\pi \ell_1 \ell_2}} \exp\left(-(\Delta Y_r^\bullet)^2 \frac{\ell_1 + \ell_2}{2\ell_1 \ell_2}\right) \mathrm{d}\ell_1 \mathrm{d}\ell_2.$$

Then, conditionally on $((Y_r^\bullet, \hat{\mathfrak{G}}_r, \mathfrak{G}_r) : r \geq 0)$, the variables $((\hat{B}^{(r)}, B^{(r)}, \hat{\mathcal{P}}_r, \mathcal{P}_r) : r \geq 0)$ are independent and for all r jumping time of $Y^\bullet$ we have:

- *The law of* $(\hat{B}^{(r)}, B^{(r)})$ *is* $\mathbb{P}^{(\hat{\mathfrak{S}}_r)}_{Y^\bullet_{r-}\to Y^\bullet_r} \otimes \mathbb{P}^{(\mathfrak{S}_r)}_{Y^\bullet_{r-}\to Y^\bullet_r}(\mathrm{d}\hat{B}\mathrm{d}B)$.

- $(\hat{\mathcal{P}}_r, \mathcal{P}_r)$ *are independent Poisson measures, independent of* $(\hat{B}^{(r)}, B^{(r)})$*, with respective intensities*

$$\mathbb{1}_{[0,\hat{\mathfrak{S}}_r]}(t)\mathrm{d}t \otimes \mathbf{N}(\mathrm{d}X\mathrm{d}Z) \quad \textit{and} \quad \mathbb{1}_{[0,\mathfrak{S}_r]}(t)\mathrm{d}t \otimes \mathbf{N}(\mathrm{d}X\mathrm{d}Z).$$

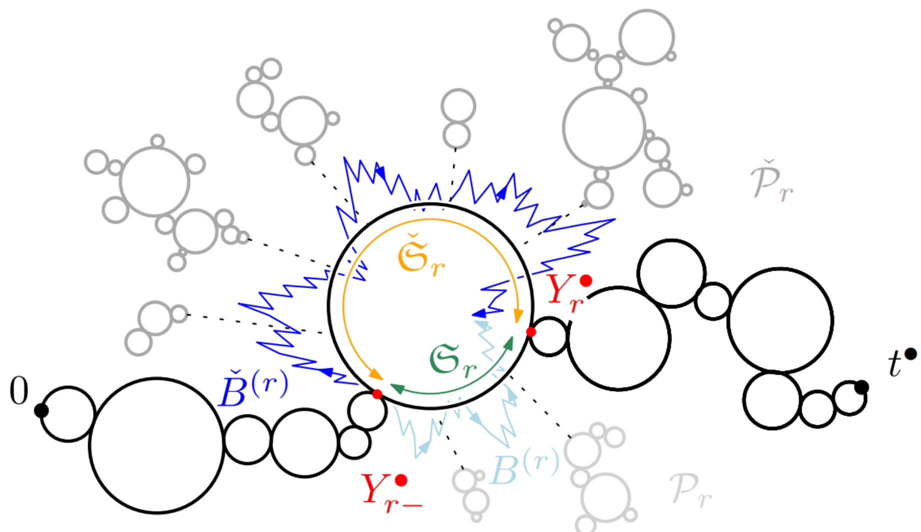

Figure 4.2: Figure 22 in [3]. Illustration of Proposition 4.3.

Using the previous fact, we can define a version of the approximated geodesic stars, that were coded by the function $F^{(m)}_{\varepsilon,r}(x,y)$ in the Brownian sphere, recall the definitions predecing Proposition 3.9. In this setting, we propose to define $F^{(\alpha,m)}_{\varepsilon,r}(x,y)$ in the following way.

- Given $r>0$, $x,y\in\mathcal{S}_\alpha$ and $s,t\in[0,\sigma]$ such that $\Pi_\alpha(s)=x$ and $\Pi_\alpha(t)=y$, let $\ell(x,y,r)$ be the loop associated to the furthest pinch point in Branch(s,t) at distance r from s in $(\mathcal{L}_\alpha, d)$.

- For $\varepsilon, r>0$, $x,y\in\mathcal{S}_\alpha$ and $s,t\in[0,\sigma]$ as before, let $G^{(\alpha,m)}_{\varepsilon,r}(x,y)$ be the event where there exists m geodesics $\eta_1,\dots,\eta_m$ starting from $\ell(x,y,r)$ towards y, disjoint outside $B(y,\varepsilon)$.

- Finally, define $F^{(\alpha,m)}_{\varepsilon,r}(x,y) := \mathbb{1}_{G^{(\alpha,m)}_{\varepsilon,r}(x,y)}$.

Now, recall that to estimate the proportion of geodesic stars in the Brownian sphere we used the point with minimal label as a reference point, since we can then consider the geodesics following the running infimum along the neighboring labeled trees (recall Lemma 3.8). In this setting, the goal is to implement the same idea, for which we would need a version of the spinal decomposition for the branch connecting 0 with t_*. In fact, this would allow us to study the event $G^{(\alpha,m)}_{\varepsilon,r}(\rho_\alpha,\rho_*)$, since it can be translated to an event concerning the properties of the labeled looptrees attached to $\ell(\rho_\alpha,\rho_*,r)$ which we expect to behave in a Poissonian way (recall the event 3.1), as in Proposition 4.3.

Heuristically, assume that a version of Proposition 4.3 holds with $t^\bullet$ replaced by t_*. By Proposition 4.2, we expect that an identity of the form

$$\mathbf{N}\left(\int_{\mathcal{S}_\alpha} \frac{\mathrm{Vol}_\alpha(\mathrm{d}x)}{\sigma} F^{(\alpha,m)}_{\varepsilon,r}(\rho_\alpha, x)\right) = \mathbf{N}\left(F^{(\alpha,m)}_{\varepsilon,r}(\rho_\alpha, \rho_*)\right),$$

holds, and there we should have a suitable Poissonian structure to make computations using the conjectured spinal decomposition. In fact, by the previous discussion we can relate this quantity to the probability of the event

$$\mathcal{A}^\alpha_{\varepsilon,m} := \{\#\{i \geq 1 : Z_{t_*} < \min Z^i + \hat{B}^{(r)}_{s_i} < Z_{t_*} + \varepsilon\} \geq m\}.$$

where $s_i \in [0, \hat{\mathfrak{G}}_r + \mathfrak{G}_r]$ is defined to be such that $f_{\xi_{t^\bullet}(r-)}\left(\frac{s_i}{\hat{\mathfrak{G}}_r + \mathfrak{G}_r}\right) = u_i$ (this corresponds to adding the label at the root to recover the actual labels in $\mathcal{S}_\alpha$). If we define

$$\mathcal{A}^{1,\alpha}_{\varepsilon,m} := \{\#\{i \geq 1 \text{ such that } u_i < \xi_{t_*}(r) : Z_{t_*} < \min Z^i + \hat{B}^{(r)}_{s_i} < Z_{t_*} + \varepsilon\} = m\},$$
$$\mathcal{A}^{2,\alpha}_{\varepsilon,m} := \{\#\{i \geq 1 \text{ such that } u_i > \xi_{t_*}(r) : Z_{t_*} < \min Z^i + B^{(r)}_{s_i} < Z_{t_*} + \varepsilon\} = m\},$$

which corresponds to separate the looptrees emerging from both sides of the loop associated to $\xi_{t^\bullet}(r-)$ (see Figure 4.2), then we can write

$$\mathcal{A}^\varepsilon_m = \bigcup_{k \geq m} \bigcup_{\substack{m_1, m_2 \in \mathbb{N} \\ m_1 + m_2 = k}} \left(\mathcal{A}^{1,\alpha}_{\varepsilon,m_1} \cap \mathcal{A}^{2,\alpha}_{\varepsilon,m_2}\right).$$

Conditionally on the labels seen across Branch$(0, t_*)$, $\hat{\mathfrak{G}}_r$ and $\mathfrak{G}_r$, the probability of each term in the previous writing of $\mathcal{A}^\varepsilon_m$ can be determined as the probability related to a Poisson random variable, thanks to the (expected) Poissonian mechanism ruling the labeled looptrees emerging from $\ell(\rho_\alpha, \rho_*, r)$. By estimating properly such probabilities, we could obtain a first moment estimate for the approximated geodesic stars in the α-stable gasket/carpet.

Final comments

In this document, we have reviewed the Brownian sphere along with a proof involving many of its properties, remarkably its Markov property that gives the Poissonian mechanism of the hull part. The latter fact simplifies the seemingly difficult estimation of the probability for the existence of geodesics towards a point, disjoint outside a small neighborhood of it. This is the notion of approximated geodesic stars, that in the limit to zero of the radius of such neighborhood, turns out to be actual geodesic stars in the sense of Chapter 3 (just like remarked for the Brownian cone points in Chapter 2). This gives, after some additional arguments, the lower bound of the Hausdorff dimension of the geodesic stars. On the other hand, the upper bound is obtained as −essentially− a consequence of the strong confluence of geodesics. In fact, the probability for the existence of geodesics towards a point in a neighborhood of a typical point, is estimated by identifying these geodesics with geodesics towards the typical point in all but a constant number of layers across the Brownian sphere. Here, the independence properties of the Brownian sphere again play a fundamental role in the computations.

For our purposes, we pose the question of how can we adapt these techniques to the stable geometry setting. As we saw, there is less independence in this structure, due to the jumps of the underlying Lévy excursion. However, such a process within its labels (on the associated looptree) still Markovian when performing appropriate explorations, and the Poissonian behavior still holds in a weaker form, since many additional conditionings have to be made. In this sense, if we think about adapting the arguments given for the Brownian sphere to the stable geometry setting, a good starting point might be to find a spinal decomposition for the branch connecting the root and the point with minimal label in the stable geometry setting. Here, we think that the analog structure of hulls in the Brownian sphere is any loop associated to a jump time of the label process in such a branch, to which labeled looptrees are attached according to independent Poisson measures. This gives a way to define and also mathematically treat the notion of approximated geodesic stars in the stable gasket/carpet, in case that we let such geodesics to start at the loop. These ideas are left as future work.

Bibliography

[1] Céline Abraham and Jean-François Le Gall. "Excursion theory for Brownian motion indexed by the Brownian tree". In: *Journal of the European Mathematical Society* 20.12 (2018), pp. 2951–3016.

[2] Romain Abraham, Jean-François Delmas, and Patrick Hoscheit. "A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces". In: *Electronic Journal of Probability* 18.none (Jan. 2013). ISSN: 1083-6489. DOI: `10.1214/ejp.v18-2116`. URL: `http://dx.doi.org/10.1214/EJP.v18-2116`.

[3] Nicolas Curien, Grégory Miermont, and Armand Riera. "The scaling limit of planar maps with large faces". In: *arXiv preprint arXiv:2501.18566* (2025).

[4] Jean-François Le Gall and Grégory Miermont. "Scaling limits of random trees and planar maps". In: *arXiv preprint arXiv:1101.4856* (2011).

[5] Andreas E Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications*. Springer Science & Business Media, 2006.

[6] Gregory F Lawler. *Conformally invariant processes in the plane*. 114. American Mathematical Soc., 2008.

[7] Jean-François Le Gall. "Geodesic stars in random geometry". In: *The Annals of Probability* 50.3 (2022), pp. 1013–1058.

[8] Jean-François Le Gall. "Geodesics in large planar maps and in the Brownian map". In: (2010).

[9] Jean-François Le Gall. *Spatial branching processes, random snakes and partial differential equations*. Springer Science & Business Media, 1999.

[10] Jean-François Le Gall. "The topological structure of scaling limits of large planar maps". In: *Inventiones mathematicae* 169.3 (2007), pp. 621–670.

[11] Jean-Francois Le Gall and Yves Le Jan. "Branching processes in Lévy processes: the exploration process". In: *The Annals of Probability* 26.1 (1998), pp. 213–252.

[12] Jean-François Le Gall and Grégory Miermont. "On the scaling limit of random planar maps with large faces". In: *XVIth International Congress On Mathematical Physics: (With DVD-ROM)*. World Scientific. 2010, pp. 470–474.

[13] Jean-François Le Gall and Mathilde Weill. "Conditioned brownian trees". In: *Annales de l'Institut Henri Poincare (B) Probability and Statistics*. Vol. 42. 4. Elsevier. 2006, pp. 455–489.

[14] Jean-François Marckert and Abdelkader Mokkadem. "States spaces of the snake and its tour-convergence of the discrete snake". In: *Journal of Theoretical Probability* 16.4 (2003), pp. 1015–1046.

[15] Jason Miller and Wei Qian. "Geodesics in the Brownian map: Strong confluence and geometric structure". In: *arXiv preprint arXiv:2008.02242* (2020).

[16] Jason Miller and Scott Sheffield. "An axiomatic characterization of the Brownian map". In: *Journal de l'École polytechnique—Mathématiques* 8 (2021), pp. 609–731.

[17] Peter Mörters and Yuval Peres. *Brownian motion*. Vol. 30. Cambridge University Press, 2010.