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Eigenvalues of Set-Valued Operators in Banach Spaces

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Abstract. This work deals with the spectral analysis of set-valued operators from a Banach space X into its dual space X^*. The main goal of the paper is to study semicontinuity properties of the spectrum operator.

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1. Introduction

In this paper we study different semicontinuity properties of the spectrum and eigenset of a set-valued operator defined on a real Banach space X with values into its topological dual X^*. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X and X^* and $\|\cdot\|_*$ the dual norm in X^* associated with the initial norm $\|\cdot\|$. Given a set-valued operator $F\colon X \rightrightarrows X^*$ and a single-valued operator $B\colon X \to X^*$ we say that a real number λ is a B-eigenvalue of F if there exists a nonzero element $x \in X$, such that

$$\lambda B(x) \in F(x). \tag{1.1}$$

The element x satisfying the above inclusion is called a B-eigenvector of F associated with λ. The set of all B-eigenvalues of F is called the B-spectrum of F. The set of all $x \in X$ satisfying (1.1) is the B-eigenset of F associated with λ.

Spectral theory of set-valued operators in Hilbert spaces has been studied by several authors in the last years. For a clear introduction and a brief historic account on the eigenvalue analysis of set-valued operators, the reader can consult Seeger [25, 26] and Seeger and Lavilledieu [17]. The important case when the operator is a convex process is studied by Leizarowitz [18] and by Aubin, Frankowska and Olech [3] where the eigenvalue problem is related with the controllability of a differential inclusion; the book of Aubin and Frankowska [4] also deals with this

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subject. In [10] Correa and Seeger give a localization formula for the eigenvalues and a variational characterization of the spectral radius.

Inclusion (1.1) in Banach spaces, when F is a single valued operator, has been considered by several authors in the study of nonlinear elliptic eigenvalue problems. We mention as examples the works of Browder [7, 8] and Amann [1] that follow the ideas of Lusternik and Schnirelman [19] to establish the existence of infinitely many distinct B-eigenvalues of F associated with λ.

A first motivation for the study of the B-spectrum and the B-eigenset of a set-valued operator F is the equivalence between (1.1) and the variational inequality

$$\begin{aligned} &x \in C, \\ &\langle T(x), y - x\rangle \geqslant \lambda\langle Bx, y - x\rangle \quad \text{for all } y \in C, \end{aligned} \tag{1.2}$$

where C is a closed convex set in X and T is a set-valued operator from X into X^*. In fact, the equivalence follows immediately if we define $F = T + N_C$ where $N_C(x)$ is the normal cone to C at x defined by

$$N_C(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leqslant 0 \text{ for all } y \in C\}.$$

The general variational inequality (1.2) in Banach spaces has been studied by Benci and Micheletti [6] among others. It is important to note that even in the classical variational problem (1.2) where T is single-valued, the operator $F = T + N_C$ in (1.1) will be a set-valued operator.

As we will see now, for the study of some important eigenvalue problems, it is useful to extend (1.1) to the case where the operator B is set-valued. In this case, inclusion (1.1) can be seen as

$$0 \in (F - \lambda B)(x) \tag{1.3}$$

and the definition of B-eigenvalue and B-eigenvector of F are refered to the inclusion (1.3). In fact, let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a closed proper convex function and f_λ be its the Moreau envelope defined by

$$f_\lambda(y) = \inf_{x\in X}\left\{ f(x) + \frac{\lambda}{2}\|x - y\|^2\right\}. \tag{1.4}$$

It is well known that for any fixed $y \in X$ the unique point x where the infimus in (1.4) is achieved verifies the inclusion (1.3), where F is the set-valued operator from X into X^* defined by

$$F(x) = -\partial f(x) = \{x^* : f(y) \geqslant f(x) + \langle -x^*, y - x\rangle \text{ for all } y \in X\}$$

(opposite of the so called subdifferential of f at x) and B is the set-valued operator defined by $B(x) = I(x - y)$, with

$$I = \partial(\tfrac{1}{2}\|\cdot\|^2), \tag{1.5}$$

that is, x is the unique element of the B-eigenset of $-\partial f$ associated with λ. Envelope (1.4) plays an important role in convex and variational analysis, see, for example, the excellent books of Attouch [2] and Rockafellar and Wets [24].

Let us give a last example where inclusion (1.1) appears in a natural way. It corresponds to the generalization to Banach spaces of the classical differential inclusion

$$\dot{u}(t) \in F(u(t)), \tag{1.6}$$

where F is defined in a Hilbert space X into itself and where u is a function from $T \subset \mathbb{R}$ into X, with $\dot{u}(t)$ the usual derivative of u at t. For this purpose let us recall that the duality map $I\colon\ X \rightrightarrows X^*$ defined in (1.5) is a natural generalization of identity in Hilbert spaces, as can be seen from equality (2.1). For further properties and applications of the duality map, see, for example, the books of Barbu [5] and Megginson [20]. As it is well known the important "single-valuedness property" of the duality map holds for a large class of Banach spaces. Then, if F is a set-valued operator from such a space into its topological dual, we can generalize the differential inclusion (1.6) in a natural way by

$$I(\dot{u}(t)) \in F(u(t)). \tag{1.7}$$

In the study of differential inclusions, the case where F is a process, that is $F(\alpha x) = \alpha F(x)$ for all $\alpha > 0$ and $0 \in F(0)$, plays an important role. With this assumption, a nontrivial solution of (1.7) is given by the function $u(t) = \mathrm{e}^{\lambda t}x$, where x is an I-eigenvector of F associated with λ. In fact, if $\lambda I(x) \in F(x)$, from the positive homogeneity of I and F we obtain $I(\dot{u}(t)) = \lambda \mathrm{e}^{\lambda t} I(x) \in \mathrm{e}^{\lambda t} F(x) = F(u(t))$.

In this paper we focus our attention to inclusion (1.3) with $B = I$, and often we work in Banach spaces where I is single valued. The I-spectrum of F will be denoted $\sigma(F)$ and the I-eigenset associated with λ will be denoted by $E_\lambda(F)$. In X we work with the norm topology and in X^* with both, the norm and the $*$-weak topologies. Section 2 recalls some important properties of the duality map $I\colon\ X \rightrightarrows X^*$ and gives some closeness properties of the operator $F - \lambda I$ and of the eigenset $E_\lambda(F)$. Section 3.1 is devoted to the study of semicontinuity properties of the eigenset operator $(F, \lambda) \rightrightarrows E_\lambda(F)$. An in-depth analysis of the upper semicontinuity of the point spectrum operator $F \rightrightarrows \sigma(F)$ and an extension of the results obtained by Lavilledieu and Seeger [16] is given in Section 3.2. Finally, Section 3.3 shows that the lower semicontinuity of the spectrum operator is difficult to obtain. In order to avoid this difficulty we extend the idea of *enlargement of the spectrum* to Banach spaces. This idea was originally introduced for linear operators by Landau [14, 15] and then extended by Gajardo and Seeger [11] in the frame of set-valued operators defined on Hilbert spaces.

2. Eigenvalues and Eigenvectors

In this section, we first give some properties of the duality map $I\colon\ X \rightrightarrows X^*$ defined by (1.5); these properties will be constantly used in the study of different continuity notions of the eigenvalue and eigenset operators. Throughout the paper, X will denote a real Banach space whose dual unit ball is sequentially $*$-weak compact. Such compactness property is known to hold (see [21]) for Asplund spaces and Banach spaces with Gâteaux differentiable renorms, so in particular for separable Banach spaces.

Let us give now an important characterization of the duality map (see [5]); this formula is often given as the definition of the duality map

$$I(x) = \{x^* \in X^* : \|x\|^2 = \|x^*\|_*^2 = \langle x, x^* \rangle\}, \tag{2.1}$$

where $\|\cdot\|_*$ is the dual norm in X^* associated with the initial norm $\|\cdot\|$.

From equality (2.1) we immediately obtain the following two properties:

(i) $\mathrm{Dom}(I) := \{x \in X : I(x) \neq \emptyset\} = X$. This is a clear consequence of Hahn–Banach theorem.
(ii) $\alpha I(x) = I(\alpha x)\ \forall x \in X, \forall \alpha \in \mathbb{R}$.

DEFINITION 2.1. Given $(E_1, \mathcal{T}_1)$ and $(E_2, \mathcal{T}_2)$ two linear topological spaces, and $F\colon E_1 \rightrightarrows E_2$, we say that F is $\mathcal{T}_2 - \mathcal{T}_1$ (sequentially) closed at $x \in E_1$ if for any sequences $x_k \overset{\mathcal{T}_1}{\to} x$, $y_k \overset{\mathcal{T}_2}{\to} y$ with $y_k \in F(x_k)$ we have that $y \in F(x)$. Moreover, F is said to be locally bounded at $x \in E_1$ if there exists a bounded neighborhood V of x such that $F(V)$ is bounded in E_2.

PROPOSITION 2.1. *Consider x_k a sequence in X such that $x_k \to x$, $x_k^* \in I(x_k)$. Then, the sequence x_k^* is bounded and for all x^* such that there exists $x_{k_j}^* \overset{*}{\rightharpoonup} x^*$, we have*

$$\lim_{j\to\infty} \|x_{k_j}^*\|_* = \|x^*\|_* \quad \text{and } x^* \in I(x).$$

This means that I is $\omega^* - \|\cdot\|$ closed and locally bounded.

Proof. From (2.1), the sequence x_k^* is bounded due to $\|x_k^*\| = \|x_k\| \to \|x\|$. Suppose $x \neq 0$ and $x_{k_j}^* \overset{*}{\rightharpoonup} x^*$. Passing to the limit in $\|x_{k_j}^*\|_* = \langle x_{k_j}, x_{k_j}^* \rangle / \|x_{k_j}\|$, we obtain

$$\lim_{j\to\infty} \|x_{k_j}^*\|_* = \frac{\langle x, x^* \rangle}{\|x\|},$$

therefore $\lim_{j\to\infty} \|x_{k_j}^*\|_* \leqslant \|x^*\|_*$, and the limit exists. On the other hand, $\|x^*\|_* = \sup_{\|x\|=1} \langle x, x^* \rangle$ is equivalent to say that for all $\varepsilon > 0$, there exists $\tilde{x}$ with $\|\tilde{x}\| = 1$ such that $\|x^*\|_* - \varepsilon \leqslant \langle \tilde{x}, x^* \rangle$, hence

$$\lim_{j\to\infty} \|x_{k_j}^*\|_* \geqslant \lim_{j\to\infty} \langle \tilde{x}, x_{k_j}^* \rangle = \langle \tilde{x}, x^* \rangle \geqslant \|x^*\|_* - \varepsilon \quad \text{for each } \varepsilon > 0,$$

that is to say,

$$\lim_{j\to\infty} \|x_{k_j}^*\|_* \geqslant \|x^*\|_*.$$

This yields to the equalities

$$\lim_{j\to\infty} \|x^*_{k_j}\|_* = \|x^*\|_*,$$
$$\|x\|^2 = \|x^*\|_*^2 = \langle x, x^*\rangle,$$

implying $x^* \in I(x)$, as required. The case $x = 0$ is trivial. In fact, if $\|x^*_k\|_* = \|x_k\| \to 0$ then, $x^*_k \to x^* = 0$. □

An important question for the remaining of this work is to characterize when the duality map is a function (a single-valued operator). The simplest case is when X is a Hilbert space, here the duality map is just the identity function. Corollary 2.1 below gives the final answer to this question.

DEFINITION 2.2. A *point of smoothness* in a Banach space X is an element $x \neq 0$ such that there exists a unique point x^* in the unit ball B_{X^*} satisfying $\langle x, x^*\rangle = \|x\|$. A Banach space X is called *smooth* if all elements in $X \setminus \{0\}$ are points of smoothness.

PROPOSITION 2.2. *X is smooth if and only if $I\colon X \rightrightarrows X^*$ is a $\omega^* - \|\cdot\|$ continuous function.*

Proof. It is clear that the smoothness of X is equivalent to say that I is single valued. The continuity follows from Proposition 2.1, indeed, the boundedness and $\omega^* - \|\cdot\|$ closeness of a single valued operator is equivalent to its $\omega^* - \|\cdot\|$ continuity. □

PROPOSITION 2.3 (Banach 1932). *The Banach space X is smooth if and only if its norm is Gâteaux differentiable* (*outside of* 0).

Proof. See [20], Chapter 5. □

COROLLARY 2.1. *$I\colon X \rightrightarrows X^*$ is a $\omega^* - \|\cdot\|$ continuous function if and only if the norm in X is Gâteaux differentiable* (*outside of* 0).

The next example provides an illustration of the result in Corollary 2.1.

EXAMPLE 2.1. Let $L^p(\Omega)$, $1 \leqslant p < \infty$, denote the usual Banach space of Lebesgue measurable functions from an open set $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}$. Then, for $1 < p < \infty$ the duality map $I\colon L^p(\Omega) \to L^{p'}(\Omega)$, $((1/p) + (1/p') = 1)$ is given by

$$I(x)(z) = \frac{x(z)|x(z)|^{p-2}}{\|x\|^{p-2}_{L^p(\Omega)}} \quad \text{for all } x \neq 0, \text{ a.e. } z \in \Omega$$

and for $p = 1$ the duality map $I\colon L^1(\Omega) \rightrightarrows L^\infty(\Omega)$ is given by

$$I(x)(z) = \|x\|_{L^1(\Omega)}\, \text{sign}[x(z)],$$

where

$$\operatorname{sign}[\alpha] = \begin{cases} -1 & \text{for } \alpha < 0, \\ [-1,1] & \text{for } \alpha = 0, \\ 1 & \text{for } \alpha > 0. \end{cases}$$

PROPOSITION 2.4. *$I\colon X \rightrightarrows X^*$ is a $\|\cdot\|_* - \|\cdot\|$ continuous function if and only if the norm in X is Fréchet differentiable (outside of 0).*

Proof. See [20], Chapter 5. □

OBSERVATION 2.1. *Recall that every reflexive Banach space admits an equivalent norm that is Fréchet differentiable.*

PROPOSITION 2.5. *If $F\colon X \rightrightarrows X^*$ is a $\omega^* - \|\cdot\|$ closed operator, then for all $\lambda \in \mathbb{R}$ the operator $(F - \lambda I)$ is $\omega^* - \|\cdot\|$ closed.*

Proof. Let $x_k \to x$ and $x_k^* \overset{*}{\rightharpoonup} x^*$ be two sequences satisfying $x_k^* \in (F - \lambda I)(x_k)$, $y_k^* \in I(x_k)$ and $x_k^* + \lambda y_k^* \in F(x_k)$. By Proposition 2.1 there exists a subsequence $y_{k_j}^*$ such that $x_{k_j}^* + \lambda y_{k_j}^* \overset{*}{\rightharpoonup} x^* + \lambda y^*$, where $y^* \in I(x)$. Then we have that $x^* + \lambda y^* \in F(x)$, i.e., $x^* \in (F - \lambda I)(x)$. □

In order to obtain the same result for the norm topology in X^*, we must work in a Banach space where the norm is Fréchet differentiable.

PROPOSITION 2.6. *If the norm in the Banach space X is Fréchet differentiable and if $F\colon X \rightrightarrows X^*$ is a $\|\cdot\|_* - \|\cdot\|$ closed operator, then for each $\lambda \in \mathbb{R}$, the operator $(F - \lambda I)$ is $\|\cdot\|_* - \|\cdot\|$ closed.*

Proof. Let be $x_k \to x$ and $x_k^* \to x^*$ two sequences such that $x_k^* + \lambda I(x_k) \in F(x_k)$. By Proposition 2.4 we know that $I(x_k) \to I(x)$ and hence $x^* + \lambda I(x) \in F(x)$, as required. □

We finish this section defining the notions of eigenvalues and eigenvectors for set-valued operators defined in a Banach space and we give a condition for closeness of the eigenset $E_\lambda(F)$.

DEFINITION 2.3. The real number λ is called an *eigenvalue* of the operator $F\colon X \rightrightarrows X^*$ if there exists an element $x \in X \setminus \{0\}$ such that $0 \in (F - \lambda I)(x)$. In such a case, we say that x is an *eigenvector* associated with the eigenvalue λ.

DEFINITION 2.4. The *spectrum* of the operator $F\colon X \rightrightarrows X^*$ is the set defined by

$$\sigma(F) := \{\lambda \in \mathbb{R} : 0 \in (F - \lambda I)(x) \text{ for some } x \neq 0\} \tag{2.2}$$

and the *eigenset* of F associated with $\lambda \in \mathbb{R}$ is

$$E_\lambda(F) := \{x \in X : 0 \in (F - \lambda I)(x)\} = (F - \lambda I)^{-1}(0). \tag{2.3}$$

As a consequence of Propositions 2.5 and 2.6 we obtain:

PROPOSITION 2.7. *If* $F\colon X \rightrightarrows X^*$ *is a* $\omega^* - \|\cdot\|$ *closed operator, then* $E_\lambda(F)$ *is a closed set for each real* λ.

PROPOSITION 2.8. *Let* $F\colon X \rightrightarrows X^*$ *be a* $\|\cdot\|_* - \|\cdot\|$ *closed operator and* X *a Banach space with Fréchet differentiable norm, then* $E_\lambda(F)$ *is a closed set for all real* λ.

3. Semicontinuity Properties of the Eigenset Operator $(F, \lambda) \rightrightarrows E_\lambda(F)$ and the Spectrum $F \rightrightarrows \sigma(F)$

The purpose of this section is to analyze semicontinuity properties of the operators $F \rightrightarrows \sigma(F)$ and $(F, \lambda) \rightrightarrows E_\lambda(F)$. More precisely, defining appropriate convergence notions for sequences of operators and sets, we study inclusions of the form

$$\text{(Upper semicontinuity inclusions)} \begin{cases} \limsup\limits_{k\to\infty} E_{\lambda_k}(F_k) \subset E_\lambda(\limsup\limits_{k\to\infty} F_k), \\ \limsup\limits_{k\to\infty} \sigma(F_k) \subset \sigma(\limsup\limits_{k\to\infty} F_k), \end{cases}$$

$$\text{(Lower semicontinuity inclusions)} \begin{cases} E_\lambda(\liminf\limits_{k\to\infty} F_k) \subset \liminf\limits_{k\to\infty} E_{\lambda_k}(F_k), \\ \sigma(\liminf\limits_{k\to\infty} F_k) \subset \liminf\limits_{k\to\infty} \sigma(F_k). \end{cases}$$

Before starting with the analysis we need to define the limit notions for a sequence of operators F_k and for a sequence of sets A_k.

DEFINITION 3.1. Given $(E_1, \mathcal{T}_1)$ and $(E_2, \mathcal{T}_2)$ two topological spaces, and a sequence of operators $F_k\colon E_1 \rightrightarrows E_2$, we define the sequential upper and lower limits:

$$(\mathcal{T}_2 - \mathcal{T}_1 \limsup_{k\to\infty} F_k)(x) := \{y \in E_2 : \exists x_{k_j} \xrightarrow{\mathcal{T}_1} x, \exists y_{k_j} \xrightarrow{\mathcal{T}_2} y \text{ such that}$$
$$y_{k_j} \in F_{k_j}(x_{k_j})\},$$
$$(\mathcal{T}_2 - \mathcal{T}_1 \liminf_{k\to\infty} F_k)(x) := \{y \in E_2 : \exists x_k \xrightarrow{\mathcal{T}_1} x, \exists y_k \xrightarrow{\mathcal{T}_2} y \text{ such that}$$
$$y_k \in F_k(x_k)\}.$$

The above is the so called *graphical convergence* in [24].

DEFINITION 3.2. Given a topological space $(E, \mathcal{T})$ and a sequence of sets A_k in E, we define

$$\limsup_{k\to\infty} A_k := \{x \in E : \exists x_{k_j} \xrightarrow{\mathcal{T}} x \text{ with } x_{k_j} \in A_{k_j}\},$$

$$\liminf_{k\to\infty} A_k := \{x \in E : \exists x_k \xrightarrow{\mathcal{T}} x \text{ with } x_k \in A_k\}.$$

This convergence is known as *Kuratowski convergence* for set sequences.

3.1. SEMICONTINUITY PROPERTIES OF $(F, \lambda) \rightrightarrows E_\lambda(F)$

The next result gives us the $\omega^* - \|\cdot\|$ upper-semicontinuity for the operator $(F, \lambda) \rightrightarrows E_\lambda(F)$.

PROPOSITION 3.1. *Let $F_k\colon X \rightrightarrows X^*$ be a sequence of operators and $\lambda_k \to \lambda$, then*

$$\limsup_{k\to\infty} E_{\lambda_k}(F_k) \subset E_\lambda\Big(\omega^* - \|\cdot\| \limsup_{k\to\infty} F_k\Big). \tag{3.1}$$

Proof. If $x \in \limsup_{k\to\infty} E_{\lambda_k}(F_k)$, then there exists $x_{k_j} \to x$ such that

$$0 \in (F_{k_j} - \lambda_{k_j} I)(x_{k_j}),$$

and this implies the existence of $x^*_{k_j} \in I(x_{k_j})$ with $\lambda_{k_j} x^*_{k_j} \in F_{k_j}(x_{k_j})$. From Proposition 2.1, we can assume without lost of generality, that $x^*_{k_j} \overset{*}{\rightharpoonup} x^* \in I(x)$ and then

$$x_{k_j} \longrightarrow x, \qquad \lambda_{k_j} x^*_{k_j} \overset{*}{\longrightarrow} \lambda x^*, \quad \lambda_{k_j} x^*_{k_j} \in F_{k_j}(x_{k_j}),$$

this means that $\lambda x^* \in (\omega^* - \|\cdot\| \limsup_{k\to\infty} F_k)(x)$. Besides $x^* \in I(x)$, we obtain

$$0 \in \Big[\omega^* - \|\cdot\| \limsup_{k\to\infty} F_k - \lambda I\Big](x)$$

that is

$$x \in E_\lambda\Big(\omega^* - \|\cdot\| \limsup_{k\to\infty} F_k\Big)$$

as required. □

Restricting ourselves to Banach spaces with Fréchet differentiable norm, will allow us to obtain the $\|\cdot\|_* - \|\cdot\|$ upper-semicontinuity,

PROPOSITION 3.2. *Let $F_k\colon X \rightrightarrows X^*$ be a sequence of operators and $\lambda_k \to \lambda$. If the norm in the Banach space X is Fréchet differentiable, then*

$$\limsup_{k\to\infty} E_{\lambda_k}(F_k) \subset E_\lambda\Big(\|\cdot\|_* - \|\cdot\| \limsup_{k\to\infty} F_k\Big). \tag{3.2}$$

Proof. Analogous to Proposition 3.1, using Proposition 2.4 instead of Proposition 2.1. □

The following two propositions show what happen if we change lim sup by lim inf in the above propositions.

PROPOSITION 3.3. *Let $F_k\colon X \rightrightarrows X^*$ be a sequence of operators and $\lambda_k \to \lambda$. If the norm in the Banach space X is Gâteaux differentiable, then*

$$\liminf_{k\to\infty} E_{\lambda_k}(F_k) \subset E_\lambda\Big(\omega^* - \|\cdot\| \liminf_{k\to\infty} F_k\Big). \tag{3.3}$$

Proof. If $x \in \liminf_{k\to\infty} E_{\lambda_k}(F_k)$, there exists $x_k \to x$ such that

$$0 \in (F_k - \lambda_k I)(x_k) \quad \forall k \in \mathbb{N}.$$

By Corollary 2.1 we can write $\lambda_k I(x_k) \in F_k(x_k)$ for each $k \in \mathbb{N}$ and $I(x_k) \overset{*}{\rightharpoonup} I(x)$. Then

$$\lambda I(x) \in \Big(\omega^* - \|\cdot\| \liminf_{k\to\infty} F_k\Big)(x)$$

and we conclude that

$$x \in E_\lambda\Big(\omega^* - \|\cdot\| \liminf_{k\to\infty} F_k\Big). \qquad \square$$

PROPOSITION 3.4. *Let $F_k\colon X \rightrightarrows X^*$ be a sequence of operators and $\lambda_k \to \lambda$. If the norm in the Banach space X is Fréchet differentiable, then*

$$\liminf_{k\to\infty} E_{\lambda_k}(F_k) \subset E_\lambda\Big(\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k\Big). \tag{3.4}$$

Proof. Analogous to Proposition 3.3, using Proposition 2.4 instead of Corollary 2.1. $\square$

Concerning the lower semicontinuity of $(F, \lambda) \rightrightarrows E_\lambda(F)$, the following example shows that even when X is a Hilbert space and F_k is a sequence of single valued linear operators, inclusion

$$E_\lambda\Big(\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k\Big) \subset \liminf_{k\to\infty} E_{\lambda_k}(F_k)$$

can fail.

EXAMPLE 3.1. Let $F_k\colon L^2[0,1] \to L^2[0,1]$ be the sequence of symmetric continuous linear operators defined by

$$F_k(x)(t) = \frac{t}{k} x(t) \quad t \in [0,1].$$

For every $k \in \mathbb{N}$ it is clear that $\sigma(F_k) = \emptyset$, therefore $E_\lambda(F_k) = \{0\}$ for all λ real. Also

$$\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k = 0,$$

which implies for $\lambda = 0$ that

$$E_\lambda\Big(\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k\Big) = L^2[0,1],$$

hence, for all $\lambda_k \to 0$,

$$E_0\Big(\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k\Big) \not\subseteq \liminf_{k\to\infty} E_{\lambda_k}(F_k).$$

With the purpose of repairing the negative result exhibited in the previous example, a natural way is trying to enlarge the set $E_\lambda(F)$. In the remaining of this section, we propose an idea in this direction.

DEFINITION 3.3. Given an operator $F\colon X \rightrightarrows X^*$, for any $\varepsilon > 0$ and $\lambda \in \mathbb{R}$ we define the enlargement $E_\lambda^\varepsilon(F)$ of $E_\lambda(F)$ by

$$E_\lambda^\varepsilon(F) = \{x \in X : \exists x^* \in I(x) \text{ such that } \lambda x^* \in F(x) + \mathbb{B}_\varepsilon^*\}, \tag{3.5}$$

where $\mathbb{B}_\varepsilon^* = \{x^* \in X^* : \|x^*\|_* \leqslant \varepsilon\}$.

PROPOSITION 3.5. *Let X be a Banach space such that its norm is Fréchet differentiable* (*outside of* 0). *If $F_k\colon X \rightrightarrows X^*$ is a sequence of operators and $\lambda_k \to \lambda \in \mathbb{R}$, then for all $\varepsilon > 0$ the following inclusions hold:*

(i) $E_\lambda(\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k) \subset \liminf_{k\to\infty} E_{\lambda_k}^\varepsilon(F_k)$;
(ii) $E_\lambda(\|\cdot\|_* - \|\cdot\| \limsup_{k\to\infty} F_k) \subset \limsup_{k\to\infty} E_{\lambda_k}^\varepsilon(F_k)$.

Proof. Let us prove the first of these inclusions, the other one follows similarly. Let be $x \in E_\lambda(\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k)$. Since I is a function, we have

$$\lambda I(x) \in \Big(\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k\Big)(x)$$

and there exist $y_k^* \to I(x)$ and $x_k \to x$ such that

$$\lambda_k y_k^* \in F_k(x_k) \quad \forall k \in \mathbb{N}.$$

By the $\|\cdot\|_* - \|\cdot\|$ continuity of I we get that for some $k_0 \in \mathbb{N}$

$$\|\lambda_k y_k^* - \lambda_k I(x_k)\|_* \leqslant \varepsilon \quad \forall k \geqslant k_0,$$

that is, $\lambda_k I(x_k) \in F_k(x_k) + \mathbb{B}_\varepsilon^*$ for all $k \geqslant k_0$, which means that

$$x_k \in E_{\lambda_k}^\varepsilon(F_k) \quad \forall k \geqslant k_0$$

as required. □

3.2. UPPER SEMICONTINUITY PROPERTIES OF THE OPERATOR $F \rightrightarrows \sigma(F)$

The upper semicontinuity of $\sigma(\cdot)$ is the main subject studied in [16] when X is a Hilbert space and X^* is endowed with the dual norm topology. Our aim in this section is to extend these results to set-valued operators defined in Banach spaces. Studying the case when the dual space X^* is endowed by the $*$-weak topology, is of our interest as well.

Given a sequence of operators $F_k\colon X \rightrightarrows X^*$ we analyze the following inclusion:

$$\limsup_{k\to\infty} \sigma(F_k) \subset \sigma\Big(\tau - \|\cdot\| \limsup_{k\to\infty} F_k\Big),$$

where τ is the $*$-weak or the norm topology in X^*. The next condition provides a basic hypothesis in order to obtain such inclusion.

We will consider sequences of operators $F_k\colon X \rightrightarrows X^*$ satisfying the following condition:

(H) For each $\lambda \in \limsup_{k\to\infty} \sigma(F_k)$ we have that

(i) there exist $\lambda_{k_j} \to \lambda$,
(ii) there exist $x_{k_j} \to x \neq 0$,
(iii) there exist $x^*_{k_j} \in I(x_{k_j})$

such that $\lambda_{k_j} x^*_{k_j} \in F_{k_j}(x_{k_j})$ for some subsequence F_{k_j}.

Note that if we erase "$\to x \neq 0$" in (H) (ii), we obtain the definition of $\lambda \in \limsup_{k\to\infty} \sigma(F_k)$.

Examples of operators satisfying condition (H) will be given later.

PROPOSITION 3.6. *If $F_k\colon X \rightrightarrows X^*$ satisfies the condition (H), then*

$$\limsup_{k\to\infty} \sigma(F_k) \subset \sigma\Big(\omega^* - \|\cdot\| \limsup_{k\to\infty} F_k\Big). \tag{3.6}$$

Proof. Fix any $\lambda \in \limsup_{k\to\infty} \sigma(F_k)$. By (H) we know that there exist

$$\lambda_{k_j} \to \lambda, \qquad x_{k_j} \to x \neq 0 \quad \text{and } x^*_{k_j} \in I(x_{k_j}) \text{ such that } \lambda_{k_j} x^*_{k_j} \in F_{k_j}(x_{k_j}).$$

From Proposition 2.1 we may suppose, without loss of generality that $x^*_{k_j} \overset{*}{\rightharpoonup} x^* \in I(x)$, therefore

$$\lambda x^* \in \Big(\omega^* - \|\cdot\| \limsup_{k\to\infty} F_k\Big)(x),$$

that is, $\lambda \in \sigma(\omega^* - \|\cdot\| \limsup_{k\to\infty} F_k)$. □

PROPOSITION 3.7. *If $F_k\colon X \rightrightarrows X^*$ satisfies the condition (H) and the norm in the the Banach space X is Fréchet differentiable, then*

$$\limsup_{k\to\infty} \sigma(F_k) \subset \sigma\Big(\|\cdot\|_* - \|\cdot\| \limsup_{k\to\infty} F_k\Big). \tag{3.7}$$

Proof. Similar to the above but using Proposition 2.4 instead of Proposition 2.1. □

OBSERVATION 3.1. *In* [16] *the hypothesis, on the sequence F_k, to obtain inclusion* (3.7)*, when X is a Hilbert space, is the following:*

$$(\tilde{H}) \qquad \begin{cases} \text{There exists } k_0 \in \mathbb{N} \text{ such that } \bigcup_{k \geqslant k_0} \mathrm{Dom}(F_k) \text{ is a relatively} \\ \text{compact set, and} \\ 0 \notin (\limsup_{k\to\infty} F_k)(0). \end{cases}$$

It is easy to prove that $(\tilde{H})$ implies (H). In fact, if $\lambda \in \limsup_{k\to\infty} \sigma(F_k)$, there exist

$$\lambda_{k_j} \to \lambda, \qquad x_{k_j} \in \bigcup_{k_j \geqslant k_0} \mathrm{Dom}(F_{k_j}) \quad \text{and } x^*_{k_j} \in I(x_{k_j}) \text{ such that}$$

$$\lambda_{k_j} x^*_{k_j} \in F_{k_j}(x_{k_j}),$$

and without loss of generality, from condition $(\tilde{H})$, we can assume $x_{k_j} \to x$. If $x = 0$ we obtain from (2.1) *that $x^*_{k_j} \to 0$ which implies that $0 \in (\limsup_{k\to\infty} F_k)(0)$, that is, a contradiction with $(\tilde{H})$.*

The following result is a direct consequence of Proposition 3.6.

PROPOSITION 3.8. *Suppose that the Banach space X is finite-dimensional and $F_k\colon X \rightrightarrows X^*$ is a sequence of positively homogeneous operators. Then, $\{F_k\}_{k\in\mathbb{N}}$ satisfies the condition (H) and therefore:*

$$\limsup_{k\to\infty} \sigma(F_k) \subset \sigma\Big(\limsup_{k\to\infty} F_k\Big).$$

Proof. Let be $\lambda \in \limsup_{k\to\infty} \sigma(F_k)$. Then, there must exist subsequences $x_{k_j} \neq 0$, $x^*_{k_j} \in I(x_{k_j})$ and $\lambda_{k_j} \to \lambda$ such that

$$\lambda_{k_j} x^*_{k_j} \in F_{k_j}(x_{k_j}).$$

As for each $k \in \mathbb{N}$, both F_k and I are positively homogeneous we may assume that $\|x_{k_j}\| = 1$ for all j and that $x_{k_j} \to x \neq 0$, that is, the sequence F_k satisfies the condition (H). □

EXAMPLE 3.2. Let us consider the sequence of positively homogeneous operators $F_k\colon X \rightrightarrows X^*$ defined by $F_k(x) = \|x\| d_k$. If d_k is a sequence in X^* such that $\|d_k\| = 1$ and $d_k \overset{*}{\rightharpoonup} 0$, it is easy to check that (H) does not hold for F_k and that $\sigma(F_k) = \{-1, 1\}$, then (3.6) and (3.7) fail. In fact

$$\{-1, 1\} = \limsup_{k\to\infty} \sigma(F_k) \not\subseteq \sigma(\omega^* - \|\cdot\| \limsup_{k\to\infty} F_k) = \{0\}.$$

Observe that the above upper limits are indeed limits.

OBSERVATION 3.2. *As pointed out in Observation* 3.1*, condition* $(\tilde{H})$ *is stronger than* (H). *Some interesting classes of sequences, obtained from sequences satisfying* $(\tilde{H})$*, such that* (3.7) *holds, are given in* [16]. *The first class is obtained by homogenization of the sequence* F_k *(satisfying* $(\tilde{H})$*). Given a sequence* C_k *of real sets, intersecting a fixed real number different from zero, we can define the sequence* $C_k \bullet F_k : X \rightrightarrows X^*$ *by its graph*

$$\mathrm{Gr}(C_k \bullet F_k) := \{t(x, y) : t \in C_k,\ y \in F_k(x)\}.$$

In our context whenever F_k *satisfies* (H)*, the new sequence* $C_k \bullet F_k$ *also satisfies this hypothesis and, a fortiori, inclusion* (3.6). *Inclusion* (3.7) *holds if the norm in the Banach space* X *is Fréchet differentiable, in particular when* X *is a Hilbert space. The latter can be easily checked from the equality* $\sigma(C \bullet F) = \sigma(F)$.

Another interesting class of sequences – given in [16] – satisfying (3.7) but not $(\tilde{H})$, is obtained by using pseudo-similarity transformations Q. Since we want to work with operators $F\colon X \rightrightarrows X^*$ where X is different to X^*, we give a slightly different notion of pseudo-similarity by considering a pair of operators $Q\colon X \to X$ and $\tilde{Q}\colon X^* \to X^*$ and the modified sequence $\tilde{Q}^{-1} \circ F_k \circ Q$.

PROPOSITION 3.9. *Consider a sequence of operators* $F_k\colon X \rightrightarrows X^*$ *satisfying* (H) *and let* $Q\colon X \to X$ *and* $\tilde{Q}\colon X^* \to X^*$ *be two functions for which:*

(i) *There exists* $\xi\colon \mathbb{R} \to \mathbb{R}$ *such that* $\tilde{Q}(tx^*) = \xi(t)\tilde{Q}(x^*)$ *for all* $x^* \in X^*$*, and* $t \in \mathbb{R}$*, with* ξ *continuous and locally bijective at each point of the set* $\limsup_{k\to\infty} \sigma(\tilde{Q}^{-1} \circ F_k \circ Q)$*,*
(ii) $\tilde{Q}(x^*) \in I(Q(x))$ *for each* $x,\ x^*$ *such that* $x^* \in I(x)$*,*
(iii) $Q(x) = 0 \Rightarrow x = 0$*,*
(iv) $Q(x_k) \to y \neq 0 \Rightarrow$ *there exists a sequence* $x_{k_j} \to x \neq 0$*,*
(v) Q *and* $\tilde{Q}$ *are surjective.*

Then the sequence $S_k = \tilde{Q}^{-1} \circ F_k \circ Q$ *also satisfies the condition* (H).

Proof. Let $\lambda \in \limsup_{k\to\infty} \sigma(S_k)$. There exist $\lambda_{k_j} \to \lambda$, $x_{k_j} \neq 0$ and $x^*_{k_j} \in I(x_{k_j})$ such that

$$\lambda_{k_j} x^*_{k_j} \in S_{k_j}(x_{k_j});$$

additionally, from (i) we get

$$\xi(\lambda_{k_j})\tilde{Q}(x^*_{k_j}) \in F_{k_j}(Q(x_{k_j})).$$

From (i), (ii) and (iii) we know that $\lim_{j\to\infty} \xi(\lambda_{k_j}) = \xi(\lambda)$, $\tilde{Q}(x^*_{k_j}) \in I(Q(x_{k_j}))$ and $Q(x_{k_j}) \neq 0$, respectively. Therefore, we have

$$\xi(\lambda) \in \limsup_{k\to\infty} \sigma(F_k).$$

By condition (H), there exist $\mu_{k_j} \to \xi(\lambda)$, $y_{k_j} \to y \neq 0$ and $y^*_{k_j} \to y^* \neq 0$ with $y^*_{k_j} \in I(y_{k_j})$ such that

$$\mu_{k_j} y^*_{k_j} \in F_{k_j}(y_{k_j}).$$

The hypothesis (v) ensures the existence of z_{k_j} and $z^*_{k_j}$ such that $Q(z_{k_j}) = y_{k_j}$ and $\tilde{Q}(z^*_{k_j}) = y^*_{k_j}$, that is

$$\mu_{k_j} \tilde{Q}(z^*_{k_j}) \in F_{k_j}(Q(z_{k_j})). \qquad (*)$$

The hypothesis (i) implies the existence of $\tilde{\lambda_{k_j}}$ such that

$$\xi(\tilde{\lambda_{k_j}}) = \mu_{k_j}$$

and from the inclusion $(*)$ we can write $\tilde{\lambda_{k_j}} z^*_{k_j} \in S_{k_j}(z_{k_j})$. Since $Q(z_{k_j}) \to y \neq 0$, from (iv) we can suppose that $z_{k_j} \to z \neq 0$ and therefore the sequence S_k satisfies (H). □

OBSERVATION 3.3. *The above proposition is not a generalization of Theorem* 4.7 *in* [16] *where* $\tilde{Q} = Q$ *does not satisfy conditions* (iii) *and* (v)*, and where* F_k *satisfy the stronger condition* $(\tilde{H})$*. If in Proposition* 3.9 *we do not assume* (iii) *and* (v) *and if we assume a stronger hypothesis for the sequence* F_k *(as, for example,* $(\tilde{H})$*) we can obtain the same conclusion.*

3.3. LOWER SEMICONTINUITY PROPERTIES OF THE OPERATOR $F \rightrightarrows \sigma(F)$

In this section we study the following inclusion

$$\sigma\left(\tau - \|\cdot\| \liminf_{k\to\infty} F_k\right) \subset \liminf_{k\to\infty} \sigma(F_k), \qquad (3.8)$$

where τ is the $*$-weak or the norm topology in X^*.

This inclusion is hard to obtain. In fact, it is not difficult to construct sequences F_k of linear operators such that $\sigma(F_k)$ is empty and the spectrum of the limit is nonempty. In the infinite-dimensional setting, a simple sequence of this type is given in Example 3.1. In the finite dimension case a simple example of this type is given by the matrix sequence

$$\mathbf{A_k} = \begin{pmatrix} 0 & \frac{1}{k} \\ -\frac{1}{k} & 0 \end{pmatrix}.$$

Despite of this negative comment, there are interesting particular cases where inclusion (3.8) holds. The next proposition and examples support this fact.

PROPOSITION 3.10. *If* $(\tau - \|\cdot\| \liminf_{k\to\infty} F_k)(x) \subset F_k(x)\ \forall x \in X,\ \forall k \in \mathbb{N}$, *then*

$$\sigma\Big(\tau - \|\cdot\| \liminf_{k\to\infty} F_k\Big) \subset \liminf_{k\to\infty} \sigma(F_k). \tag{3.9}$$

Proof. Trivial. □

EXAMPLE 3.3. The following sequences of operators satisfy the hypothesis of Proposition 3.10.

(i) $F_k = \partial_{\varepsilon_k} f \supset \partial f$ (ε_k-subdifferential of f, defined in [23]) with $\varepsilon_k \to 0^+$ where f is a proper closed convex function defined from X to $\overline{\mathbb{R}}$.
(ii) $F_k = F + \varepsilon_k \mathbb{B}_*$ with $\varepsilon_k \to 0^+$, $\mathbb{B}_* = \{x^* \in X^* : \|x^*\|_* \leqslant 1\}$ where F is any operator from X to X^*.
(iii) $F_k = T^{\varepsilon_k}$ with $\varepsilon_k \to 0^+$, where T is a maximal monotone operator and T^ε is the ε-enlargement of T (see [9]), i.e.,

$$T^\varepsilon(x) = \{x^* \in X^* : \langle x^* - y^*, x - y\rangle \leqslant -\varepsilon \quad \forall y \in X, \forall y^* \in T(x)\}.$$

A natural idea to repair this difficulty is to try to enlarge the spectrum. With this purpose, we introduce an extension, to the frame of Banach spaces, of the enlargement presented in [11] in the frame of Hilbert spaces. This idea was originally introduced in [14, 15] for linear operators. Interesting characterization and applications can be found in [12, 22, 27].

In order to define the enlargement $\sigma_\varepsilon(\cdot)$ let us introduce the concept of ε-eigenvalue of an operator $F\colon X \rightrightarrows X^*$ when X is a Banach space.

DEFINITION 3.4. Given $F\colon X \rightrightarrows X^*$, a real λ is an ε-eigenvalue of F if there exist $x \neq 0$, $x^* \in I(x)$ and $y^* \in F(x)$ such that $\|\lambda x^* - y^*\|_* \leqslant \varepsilon\|x\|$. We denote by $\sigma_\varepsilon(F)$ the set of all ε-eigenvalues of F.

The next proposition and corollary give the characterization of the ε point spectrum $\sigma_\varepsilon(\cdot)$ for some classes of operators.

PROPOSITION 3.11. *Let X be a Hilbert space and* $A\colon X \to X$ *be a selfadjoint compact injective linear operator. Then*

$$\sigma_\varepsilon(A) = \sigma(A) + \varepsilon[-1, 1] \quad \forall\varepsilon \geqslant 0. \tag{3.10}$$

Moreover, if X is finite-dimensional, the linear operator A may be not injective.

Proof. Let be $\lambda \in \sigma_\varepsilon(A)$. Since X is a Hilbert space, there exist x, with $\|x\| = 1$, such that

$$\|Ax - \lambda x\| \leqslant \varepsilon.$$

If A is a selfadjoint compact injective operator, there exists an orthonormal base $\{v_j\}_{j\in\mathbb{N}}$ of X with v_j eigenvector of A, and then there exist real numbers α_i such that

$$x = \sum_i \alpha_i v_i \quad \text{and} \quad \|x\|^2 = \sum_i \alpha_i^2.$$

If we denote by μ_i the eigenvalue associated with the eigenvector v_i, we can write:

$$\begin{aligned}\varepsilon^2 &\geqslant \|Ax - \lambda x\|^2 \\ &= \Big\| \sum_i \alpha_i(\mu_i - \lambda)v_i \Big\|^2 \\ &= \sum_i \alpha_i^2(\mu_i - \lambda)^2 \geqslant \Big(\inf_{\mu\in\sigma(A)} |\mu - \lambda|\Big)^2 \sum_i \alpha_i^2 = \Big(\inf_{\mu\in\sigma(A)} |\mu - \lambda|\Big)^2.\end{aligned}$$

Thus, $\inf_{\mu\in\sigma(A)} |\mu - \lambda| \leqslant \varepsilon$ which is equivalent to

$$\lambda \in \sigma(A) + \varepsilon[-1, 1].$$

On the other hand, if $\lambda \in \sigma(A) + \varepsilon[-1, 1]$, there exist $\mu \in \sigma(A)$ and an eigenvector x, associated with μ, with $\|x\| = 1$, such that

$$\|Ax - \lambda x\| = \|\mu x - \lambda x\| = |\mu - \lambda| \leqslant \varepsilon$$

which means that $\lambda \in \sigma_\varepsilon(A)$.

In finite-dimensional spaces the injectivity of the operator is not necessary for the existence of a base of eigenvectors. □

In the next corollary we define an important operator used in Fan theory [13].

COROLLARY 3.1. *Let X be a Hilbert space and $F\colon X \rightrightarrows X$ defined by*

$$F(x) := \{Ax : A \in \mathcal{A}\},$$

where $\mathcal{A}$ is a nonempty set of continuous linear operators defined from X to X as in the above proposition. Then

$$\sigma_\varepsilon(F) = \sigma(F) + \varepsilon[-1, 1] \quad \forall \varepsilon \geqslant 0.$$

PROPOSITION 3.12. *Let $F_k\colon X \rightrightarrows X^*$ be a sequence of operators. If the norm in the Banach space X is Fréchet differentiable, then for each $\varepsilon > 0$*

$$\sigma\Big(\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k\Big) \subset \liminf_{k\to\infty} \sigma_\varepsilon(F_k). \tag{3.11}$$

Proof. Let $\lambda \in \sigma(\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k)$ and $x \neq 0$ be such that

$$\lambda I(x) \in \Big(\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k\Big)(x),$$

that is, there exist sequences

$$x_k \longrightarrow x, \qquad x_k^* \longrightarrow \lambda I(x) \quad \text{with } x_k^* \in F_k(x_k).$$

By Proposition 2.4 for $\varepsilon > 0$ we can choose $k_0 \in \mathbb{N}$ such that

$$\left\| \frac{\lambda I(x_k)}{\|x_k\|} - \frac{\lambda I(x)}{\|x\|} \right\|_* \leqslant \frac{\varepsilon}{2} \quad \forall k \geqslant k_0,$$
$$\left\| \frac{x_k^*}{\|x_k\|} - \frac{\lambda I(x)}{\|x\|} \right\|_* \leqslant \frac{\varepsilon}{2} \quad \forall k \geqslant k_0,$$

which implies that

$$\|\lambda I(x_k) - x_k^*\|_* \leqslant \varepsilon \|x_k\| \quad \forall k \geqslant k_0,$$

in other words, $\lambda \in \sigma_\varepsilon(F_k)$ and hence,

$$\lambda \in \liminf_{k\to\infty} \sigma_\varepsilon(F_k). \qquad \square$$

COROLLARY 3.2. *The equality*

$$\bigcap_{\varepsilon>0} \liminf_{k\to\infty} \sigma_\varepsilon(F_k) = \liminf_{k\to\infty} \sigma(F_k), \tag{3.12}$$

is a sufficient condition for the lower semicontinuity inclusion

$$\sigma\Big(\|\cdot\|_* - \|\cdot\| \liminf_{k\to\infty} F_k\Big) \subset \liminf_{k\to\infty} \sigma(F_k). \tag{3.13}$$

In particular, if there exists a nondecreasing function $g\colon \mathbb{R} \to \mathbb{R}$ *with* $\lim_{\beta\to 0^+} g(\beta) = 0$ *such that*

$$\sigma_\varepsilon(F_k) \subset \sigma(F_k) + g(\varepsilon)[-1, 1] \quad \forall k \in \mathbb{N}, \tag{3.14}$$

then (3.12) *holds.*

COROLLARY 3.3. *Let X be a Hilbert space. Any sequence $A_k\colon X \to X$ of selfadjoint compact injective linear operators satisfies inclusion* (3.13). *More generally, the same holds for any sequence of operators $F_k\colon X \rightrightarrows X$ defined by $F_k(x) := \{Ax : A \in \mathcal{A}_k\}$ where $\mathcal{A}_k$ are sets of selfadjoint compact injective linear operators.*

Proof. From Propositions 3.11 and Corollary 3.1, we know that these sequences of operators satisfy (3.14) with $g(\varepsilon) = \varepsilon$. We conclude from the above corollary. $\square$

OBSERVATION 3.4. *It is interesting to check that if in the right-hand side of* (3.11) *we consider the intersection for all* $\varepsilon > 0$*, we obtain an equality for the sequences*

$$F_k(x)(t) = \frac{t}{k}x(t) \quad x \in L^2[0,1]$$

and

$$\mathbf{A_k} = \begin{pmatrix} 0 & \frac{1}{k} \\ -\frac{1}{k} & 0 \end{pmatrix}$$

mentioned at the beginning of this section as sequences for which the inclusion (3.13) *fails. To see this, it is enough to verify that*

$$\sigma_\varepsilon(F_k) = \left(-\varepsilon, \frac{1}{k} + \varepsilon\right) \quad \textit{and} \quad \sigma_\varepsilon(A_k) = \left[-\sqrt{\varepsilon^2 - \frac{1}{k^2}}, \sqrt{\varepsilon^2 - \frac{1}{k^2}}\right].$$

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