Spanning Trees in Graphs of High Minimum Degree with a Universal Vertex I: An Asymptotic Result

Bruce Reed* Maya Stein†

Abstract

In this paper and a companion paper, we prove that, if m is sufficiently large, every graph on $m+1$ vertices that has a universal vertex and minimum degree at least $\lfloor \frac{2m}{3} \rfloor$ contains each tree T with m edges as a subgraph. Our result confirms, for large m, an important special case of a recent conjecture by Havet, Reed, Stein, and Wood. The present paper already contains an approximate version of the result.

1 Introduction

A recurring topic in extremal graph theory is the use of degree conditions (such as minimum/average degree bounds) on a graph to prove that it contains certain subgraphs. One of the easiest classes of subgraphs for which this question is not yet properly understood are trees. This is the focus of the present paper.

Clearly, any graph of minimum degree exceeding $m-1$ contains a copy of each tree with m edges: Just embed the root of the tree anywhere in the host graph, and greedily continue, always embedding vertices whose parents have already been embedded. The bound on the minimum degree is sharp (see below).

*School of Computer Science McGill University. Research supported by NSERC.

†Department of Mathematical Engineering and Centro de Modelamiento Matemático, Universidad de Chile, UMI 2807 CNRS. Research supported by CONICYT + PIA/Apoyo a centros científicos y tecnológicos de excelencia con financiamiento Basal, Código AFB170001, and by Fondecyt Regular Grant 1183080.

Our paper is one of a large number which discuss possible strengthenings of the above observation by replacing the minimum degree condition with a different condition on the degrees of the host graph. One of these is the Loebl-Komlós-Sós conjecture from 1995 (see [EFLS95]), which replaces the minimum degree with the median degree. This conjecture has attracted a fair amount of attention over the last decades, and has been settled asymptotically [HKP$^+$a, HKP$^+$b, HKP$^+$c, HKP$^+$d]. More famously, Erdős and Sós conjectured in 1963 that every graph of average degree exceeding $m-1$ contains each tree with m edges as a subgraph. This conjecture would be best possible, since no $(m-1)$-regular graph contains the star $K_{1,m}$ as a subgraph. Alternatively, consider a graph that consists of several disjoint copies of the complete graph K_m; this graph has no connected $(m+1)$-vertex subgraph at all. Note that for these examples it does not matter whether we considered the average degree (as in the Erdős–Sós conjecture) or the minimum degree (as in the observation above).

The Erdős–Sós conjecture poses an extremely interesting question. It is trivial for stars, and it holds for paths by an old theorem of Erdős and Gallai [EG59]. It also holds when some additional assumptions on the host graph are made, see for instance [BD96, Hax01, SW97]. In the early 1990's, Ajtai, Komlós, Simonovits and Szemerédi announced a proof of the Erdős–Sós conjecture for sufficiently large m.

It is well-known that every graph of average degree $> m$ has a subgraph of minimum degree $> \frac{m}{2}$. So, if it were true that every graph of minimum degree exceeding $\frac{m}{2}$ contained each tree on m edges, then the Erdős–Sós conjecture would immediately follow. Of course, the statement from the previous sentence is not true: It suffices to consider the examples given above. Still, for bounded degree spanning trees an approximate version of the statement does hold. Komlós, Sarközy and Szemerédi show in [KSS01] that every large enough $(m+1)$-vertex graph of minimum degree at least $(1+\delta)\frac{m}{2}$ contains each tree with m edges whose maximum degree is bounded by $\frac{cn}{\log n}$, where the constant c depends on δ. Variations of the bounds and the size of the tree are given in [BPS18, CLNS10]. However, the result is essentially best possible in the sense that (even if the minimum degree of the host graph is raised) it does not hold for trees of significantly larger maximum degree [KSS01].

So, if we wish to find a condition that guarantees we can find *all* trees of a given size as subgraphs, only bounding the minimum degree is not enough. Nevertheless, there can be at most one vertex of degree at least $\frac{m}{2}$ in any tree on $m+1$ vertices, and so, we might not need many vertices of large degree in the host graph. Therefore, it seems natural to try to pose a condition on both the minimum and the maximum degree of the host graph.

The first conjecture of this type has been put forward recently by Havet, Reed, Stein, and Wood [HRSW16]. They believe that a maximum degree of at least m and a minimum degree of at least $\lfloor \frac{2m}{3} \rfloor$ is enough to embed all m-edge trees.

Conjecture 1.1 (Havet, Reed, Stein, and Wood [HRSW16])**.** *Let $m \in \mathbb{N}$. If a graph has maximum degree at least m and minimum degree at least $\lfloor \frac{2m}{3} \rfloor$ then it contains every tree with m edges as a subgraph.*

The conjecture holds if the minimum degree condition is replaced by $(1-\gamma)m$, for a tiny but explicit[1] constant γ, and it also holds if the maximum degree condition is replaced by a large function[2] in m [HRSW16]. An approximate version of the conjecture holds for bounded degree trees and dense host graphs [BPS18].

As further evidence we shall prove, in this paper and its companion paper [RS19b], that Conjecture 1.1 holds for sufficiently large m, under the additional assumption that the graph has $m+1$ vertices, i.e., when we are looking for a spanning tree. That is, building on the results from the present paper, we will show the following theorem in [RS19b].

Theorem 1.2. **[RS19b]** *There is an $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ every graph on $m+1$ vertices which has minimum degree at least $\lfloor \frac{2m}{3} \rfloor$ and a universal vertex contains every tree T with m edges as a subgraph.*

Observe that Theorem 1.2 is easy if T has a vertex t that is adjacent to a set L of at least $\lceil \frac{m}{3} \rceil$ leaves. We root T at t, embed t in the universal vertex v^* of G, greedily embed $T-L$, and then embed L in neighbours of v^*. This is possible since v^* is universal. It turns out that this approach can be extended if, for a small positive number δ, the tree T contains a vertex adjacent to at least δn leaves. Although the greedy argument no longer works, we will be able to prove a result, namely Lemma 1.3 below, which achieves the embedding of any tree T as above. This lemma will be crucial for the proof of Theorem 1.2 in our companion paper [RS19b].

Lemma 1.3. *For every $\delta > 0$, there is an m_δ such that for any $m \geq m_\delta$ the following holds for every graph G on $m+1$ vertices which has minimum*

[1]Namely, $\gamma = 200^{-30}$.
[2]Namely, $f(m) = (m+1)^{2m+6} + 1$.

degree at least $\lfloor \frac{2m}{3} \rfloor$ *and a universal vertex. If* T *is a tree with* m *edges, and some vertex of* T *is adjacent to at least* δm *leaves, then* G *contains* T*.*

Also, the results from the present paper already imply an asymptotic version of Theorem 1.2.

Theorem 1.4. *For every* $\delta > 0$*, there is an* m_δ *such that for every* $m \geq m_\delta$*, every graph* G *on* $m+1$ *vertices having minimum degree at least* $\lfloor \frac{2m}{3} \rfloor$ *and a universal vertex contains every tree* T *having at most* $(1-\delta)m$ *edges.*

Both Theorem 1.4 and Lemma 1.3 follow from Lemma 2.1, which is stated in Section 2, and whose proof occupies almost all the remainder of this paper. In the companion paper [RS19b], we will prove the full Theorem 1.2, building on Lemma 1.3 and another auxiliary result, namely Lemma 7.3, which is to be stated and proved in the last section of the present paper (Section 7).

Let us end the introduction with a very short overview of our methods of proof. A more detailed overview can be found in Section 3.

Given a tree T we wish to embed in the host graph G, we first cut T into a constant number of connecting vertices, and a large number of very small subtrees. Applying regularity to G, we can ensure that all small trees that are not just leaves can be embedded into matching structures we find in the reduced graph. This is more complicated than in earlier work on tree embeddings using the regularity approach, as our assumptions are too weak to force one matching structure we can work with throughout the whole embedding. Instead, we have to employ ad-hoc matchings, plus some auxiliary structures, one for each of the connecting vertices. Finally, we have to deal with the leaves adjacent to connecting vertices. These are more difficult to embed than the other small trees, because an embedded vertex might only see two thirds of the graph, and there is no way to reach the remaining third of the graph in only one step. For this reason, we have to come up with a delicate strategy on where we place the connecting vertices, in order to ensure that at the very end of the embedding process we will be in a position to embed all these leaves at once with a Hall-type argument.

2 The Proof of Theorem 1.4

The lemma behind our two results from the introduction (Lemma 1.3 and Theorem 1.4) is the following.

Lemma 2.1. *For every $\delta > 0$, there is an m_δ such that for any $m \geq m_\delta$ and α with $\delta \leq \alpha \leq 1$ the following holds.*
Let G be an $(m+1)$-vertex graph of minimum degree at least $\lfloor \frac{2m}{3} \rfloor$, and let $w \in V(G)$. Let T be a tree with at most $(1-\alpha)m$ edges, let $t \in V(T)$, and assume that no vertex of T is adjacent to more than αm leaves. Then one can embed T in G, mapping t to w.

Let us see how Lemma 2.1 implies the results from the introduction. Here is the proof of the lemma that we will need in the companion paper [RS19b].

Proof of Lemma 1.3. Let m_δ be given by Lemma 2.1 for input δ. Given G and T as in Lemma 1.3, we let t be a vertex of T having the maximum number of leaf neighbours. Let L be the set of its leaf neighbours and set $\alpha := \frac{|L|}{m}$. By assumption, $\delta \leq \alpha \leq 1$, so we may apply Lemma 2.1 to obtain an embedding of $T - L$ in G with t embedded in the universal vertex of G. We can then embed the vertices of L into the remaining vertices of G. □

Here is the proof of the approximate result.

Proof of Theorem 1.4. Let m_δ be the maximum of the numbers m_δ given by Lemma 2.1 and by Lemma 1.3 for input δ. Given G and T as in the theorem, consider a vertex of T with the maximum number of leaf neighbours, say these are βm leaf neighbours. If $\beta \leq \delta$, we are done by Lemma 2.1. If $\beta > \delta$, we are done by Lemma 1.3. □

3 A Sketch of the Proof of Lemma 2.1

The purpose of this section is to give some more detailed insight into the proof of Lemma 2.1, going a little more below the surface than in the Introduction. We remark that for the understanding of the rest of the paper, it is not necessary to read this section (but we hope it will be helpful).

The number m_δ is chosen in dependence of the output of the regularity lemma for some constant depending on δ. Now, given the approximation constant α, the tree T and the host graph G, we prepare each of T and G separately for the embedding.

Similar as in earlier tree embedding proofs [AKS95, HKP$^+$d, PS12], we cut T into a set W of *seeds* (connecting vertices), such that W has constant size, and a large set $\mathcal{T}$ of very small subtrees. The trees in $\mathcal{T}$ are only connected through W, and they each have size $< \beta m$, where β is a small constant (smaller than all other constants in this paper). Differently from earlier approaches to tree cutting, we now categorise the small trees contained in $\mathcal{T}$. They fall into three categories: trees consisting of a leaf of T, trees that are smaller than a (huge) constant, and trees that are larger than this constant. We name the categories L, F_1 and F_2. The last category is further subdivided into two sets, F_2' and $F_2 \setminus F_2'$, according to whether the small tree is adjacent to one or more of the seeds. Each seed from W may have trees of any (or all) of these categories hanging from it (and there may also be seeds hanging from it). The details of this cut-up of T is explained in Section 5.1.

Next, in Section 5.2, we order and group the seeds obtained from this decomposition. Our strategy of ordering the seeds takes into account their position in a natural embedding order, but also the number of leaves hanging from them. We will come back to this point at a later stage during this outline, and will then explain the why and how of the ordering.

Independently, in Section 6.1.1, we regularize the host graph G, with parameter ε, such that $\beta \ll \varepsilon \ll \alpha$. (For an introduction to regularity, see Section 4.3.) Furthermore, we partition each of its clusters C arbitrarily into subsets $C_W, C_L, C_{F_1}, C_{F_2}, C_{\tilde{V}}$ of appropriate sizes into which we aim to embed the different parts of the tree, namely, W, L, F_1, and F_2, while the last subset, $C_{\tilde{V}}$, is reserved for neighbours of seeds in trees of F_2'. The set of these neighbours will be denoted by $\tilde{V}$.

We fix a matching M_{F_2} in the reduced graph R_G. This matching will be used when we embed the trees from F_2. More precisely, we will embed each tree $\bar{T} \in F_2 \cup F_2'$ into $C_{F_2} \cup D_{F_2}$ for a suitable (i.e. sufficiently unoccupied) edge $CD \in M_{F_2}$, except for the root $r_{\bar{T}}$ of $\bar{T}$. The root $r_{\bar{T}}$ will go to one of the subsets $C'_{\tilde{V}}$, for a suitable cluster C' that connects CD with the cluster containing the seed adjacent to $\bar{T}$. In case $\bar{T} \in F_2'$, which means that $\bar{T}$ contains a second vertex $\tilde{v}$ from $\tilde{V}$, we embed $\tilde{v}$ into one of the subsets $C''_{\tilde{V}}$, for a suitable cluster C''. Throughout the embedding process, we will keep each of the edges of M_{F_2} as balanced a possible. That is, the sets of used vertices in the corresponding slices C_{F_2} or D_{F_2} on either side of such an edge never differ by more than βm.

Since we do not have enough space in the slices $C_{\tilde{V}}$ for all roots of trees in F_1 (because we have no control over the number of trees in F_1), we need to proceed differently with the small trees from F_1. For embedding these trees, we use a family of matchings M_s, one for each embedded seed s. Since these matchings M_s are possibly different for each s, we now will have to keep the set of *all* slices C_{F_1} balanced. This is not easy but possible since the trees from F_1 have constant size, and we choose M_s so that it intersects the neighbourhood of the image of s in a nice way. In addition to M_s, we also employ two auxiliary matchings which we combine with M_s to obtain a partition of almost all of $V(R_G)$ with short paths. We call these structures *good path partitions* and, together with the matchings M_s, they will be defined and proved to exist in Subsection 4.2.

The actual embedding of the tree will be performed as follows. In Sections 6.2 and 6.3, we go through the seeds in a connected way, and embed each seed s together with all the trees from $F_1 \cup F_2$ hanging at s in the corresponding slices in the way we discussed above. We leave out any leaves from L, as we will deal with them in the final phase of the embedding. Our precautions from above ensure that we can embed all of $T - L$ without a problem, never running out of space. However, if we do not take care where exactly we embed the seeds, we may run into problems in the final phase when we need to embed the leaves. For instance, we need to avoid embedding all parents of L into vertices having the same neighbourhood in G, as then, the leaves might not fit.

For this reason, we take some extra care when choosing the target clusters and the images for the seeds (this happens in Section 5.2). As already shortly hinted at above, we order the seeds into a system of groups according to the number of leaves hanging from them, and also according to the order the seeds appear in our planned embedding order. Moreover, each seed s will be assigned a *relevant* set X_s of seeds that come before it. In the actual embedding, in Subsection 6.2, we choose the image $\varphi(s)$ of a given seed s in a way that $\varphi(s)$ has many neighbours outside the union of the neighbourhoods of $\varphi(X_s)$. (We remark that is is crucial here that no vertex of T is adjacent to more than αm leaves.) This precaution will ensure that for each subset of seeds, their images have enough neighbours in $Z := \bigcup C_L$. We will then be able to embed all the leaves in L at once by using Hall's theorem. The whole procedure will be explained in detail in Subsection 6.4.

The last section of this paper, Section 7, is devoted to the proof of Lemma 7.3, which will need Lemma 7.3 in our companion paper [RS19b]. The lemma deals with a similar situation as the one treated in Lemma 2.1, the main difference being that now, a small part of the tree is already embedded (and thus possibly blocking valuable neighbourhoods), but, on the positive side, throughout [RS19b], we will be able to assume that no seed is adjacent to many leaves, and so we can assume this as well in Lemma 7.3.

4 Preliminaries

4.1 An edge-double-counting lemma

We will need the following easy lemma.

Lemma 4.1. *Let G be a graph on n vertices, let $0 < \psi < \frac{1}{3}$, and let $S \subseteq V(G)$. If each vertex in S has degree at least $(\frac{2}{3} - \psi)n$, then there are at least $(\frac{1}{3} + \frac{\sqrt{\psi}}{10})n$ vertices in G that each have at least $(\frac{1}{2} - \sqrt{\psi})|S|$ neighbours in S.*

Proof. Let $A \subseteq V(G)$ denote the set of all vertices having at least $(\frac{1}{2} - \sqrt{\psi})|S|$ neighbours in S. Writing $e(S, V(G))$ for the number of all edges touching S, where edges inside S are counted twice, we calculate that

$$\begin{aligned}(\frac{2}{3} - \psi)n \cdot |S| &\leq e(S, V(G)) \\ &\leq |V(G) \setminus A| \cdot (\frac{1}{2} - \sqrt{\psi})|S| \ + \ |A| \cdot |S| \\ &\leq n \cdot (\frac{1}{2} - \sqrt{\psi})|S| \ + \ |A| \cdot (\frac{1}{2} + \sqrt{\psi})|S|,\end{aligned}$$

and conclude that

$$|A| \ \geq \ \frac{\frac{1}{6} + \frac{\sqrt{\psi}}{2}}{\frac{1}{2} + \sqrt{\psi}} \cdot n \ \geq \ (\frac{1}{3} + \frac{\sqrt{\psi}}{10}) \cdot n,$$

as desired. □

4.2 Matchings and good path partitions

The purpose of this subsection is to find some matchings in a graph H (which will later be the reduced graph R_G of our host graph G), and combinations of some of these matchings to covers of H with short paths. These structures will be used for the embedding of T in the proof of Lemma 2.1, more specifically in Subsection 6.3. The important result of this section is Lemma 4.3.

We need a quick definition before we start. For any graph H, and any $N \subseteq V(H)$, an N-*good* matching is one whose edges each have at most one vertex outside N.

Lemma 4.2. *Let $0 < \xi < \frac{1}{20}$ and let H be a p-vertex graph of minimum degree at least $(\frac{2}{3} - \xi)p$. Let $N \subseteq V(H)$ be such that $|N| = \lceil (\frac{2}{3} - 2\xi)p \rceil$. Then $H - N$ contains a set Y of size at most $3\xi p + 1$ such that $H - Y$ has an N-good perfect matching.*

Proof. First of all, note that we can greedily match all but a set X of at most $3\xi p$ vertices from $V(H) \setminus N$ to N, simply because of the condition on the minimum degree. If necessary, add one more vertex to X, in order to have that $H - X$ is even. So $|X| \leq 3\xi p + 1$. Now, take any maximal N-good matching M in the graph H that covers all vertices of $V(H) \setminus (N \cup X)$. We would like to see that M covers all of $H - X$, so for contradiction assume that $N \setminus V(M)$ contains at least two vertices.

By the maximality of M, we know that $N \setminus V(M)$ is an independent set, and no two vertices $D, D' \in N \setminus V(M)$ can be adjacent to different endpoints of an edge in M. So, for each edge $EF \in M$, we know that either one of the endvertices, say E, sees no vertex in $N \setminus V(M)$, or E and F each see only one vertex in $N \setminus V(M)$ (and that is the same vertex). Therefore, at least one of the vertices in $N \setminus V(M)$ sees at most half of the vertices in $V(M)$, and thus less than half of the vertices in $H - X$, a contradiction to our condition on the minimum degree. □

Before we state the second lemma of this section, we need another definition. An *N-out-good path partition* of a graph H, with $N \subseteq V(H)$, is a set $\mathcal{P}$ of disjoint paths, together covering all the vertices of H, such that for each $P \in \mathcal{P}$ one of the following holds:

- $P = AB$, with $A, B \in N$;
- $P = ABCD$, with $B, C \in N$; or
- $P = ABCDEF$, with $B, C, D, E \in N$.

(Note that if P has four vertices, then there is no restriction on the whereabouts of A and D, and similar for six-vertex paths P.)

An *N-in-good path partition* of a graph H, with $N \subseteq V(H)$, is a set $\mathcal{P}$ of disjoint paths, together covering all the vertices of H, such that for each $P \in \mathcal{P}$ one of the following holds:

- $P = AB$, with $A, B \in N$;
- $P = ABCD$, with $A, D \in N$; or
- $P = ABCDEF$, with $A, C, D, F \in N$.

Now we are ready to state the main result of this section. Note that the first item is a direct consequence of the previous lemma, and all structures exist independently of each other.

Lemma 4.3. *Let* $0 < \xi < \frac{1}{20}$*, and let* H *be a* p*-vertex graph of minimum degree at least* $(\frac{2}{3} - \xi)p$*. Let* $N \subseteq V(H)$ *be any set with* $|N| = \lceil (\frac{2}{3} - 2\xi)p \rceil$*. Then* H *contains a set* X *of at most* $\lfloor 15\xi p \rfloor + 1$ *vertices, such that*

- $H - X$ *has an* $(N \setminus X)$*-good perfect matching;*
- $H - X$ *has an* $(N \setminus X)$*-in-good path partition; and*
- $H - X$ *has an* $(N \setminus X)$*-out-good path partition.*

Proof. Lemma 4.2 provides us with a set Y and an N-good perfect matching M of $H - Y$. Note that M would be as desired for the first item, if X was chosen as Y. Now, set $Q := V(M) \setminus N$, and let R be the set of all vertices from H that are matched by M to a vertex from Q. Since M is N-good, we know that $R \subseteq N$.

We take a maximal matching $\tilde{M}^Q$ inside $H[Q]$. Let $\tilde{Q}$ be the set of vertices in Q not covered by $\tilde{M}^Q$. We augment $\tilde{M}^Q$ to a matching M^Q by matching as many vertices of $\tilde{Q}$ as possible to vertices of $N \setminus R$. Because of the minimum degree condition of the lemma, because $|N \setminus R| \geq (\frac{1}{3} - 2\xi)p$ and because $\tilde{Q}$ is an independent set, we can ensure that there are at most $3\xi p$ vertices in $\tilde{Q}$ not covered by M^Q. Let Z denote the set of these vertices, and let $Z' \subseteq R$ be the set of all vertices matched to vertices of Z by M.

We define a second auxiliary matching M^R in a very similar way. Matching M^R consists of edges with both ends in $R \setminus Z'$, and a matching of almost all the remaining vertices of $R \setminus Z'$ to a set R' of vertices from $N \setminus R$. Indeed, we can ensure that M^R covers all but at most $3\xi p$ vertices of $R \setminus Z'$. Let Z'' denote the set of all such vertices and their partners in M, and set $X := Y \cup Z \cup Z' \cup Z''$. Note that $|X| \leq 3\xi p + 1 + 4 \cdot 3\xi p = 15\xi p + 1$.

Now, discarding any edge that touches X from M, M^Q and M^R, we find that the union of the edges in $M \cup M^Q$ gives an $(N \setminus X)$-in-good path partition of $H - X$. Indeed, for each of the edges $AB \in M$ with $B \in Q \setminus X$ there is an edge $BC \in M^Q$ such that $C \in N \setminus R$ or $C \in Q$. Let D be the partner of C in M. If D is not matched in M^Q, then $ABCD$ is as desired for the path partition. If D is matched in M^Q, then necessarily $D \in N \setminus R$, and so, D is matched to a vertex $E \in Q \setminus X$, and $ABCDEF$ is as desired for the path partition, where F is the partner of E in M.

Similarly, the union of the edges in $M \cup M^R$ gives the desired N-out-good path partition. Finally, M is an $(N \setminus X)$-good matching of $H - X$. □

4.3 Regularity

We need to quickly discuss Szemerédi's regularity lemma and a couple of other preliminaries regarding regularity. Readers familiar with this topic are invited to skip this subsection.

The *density* of a pair (A, B) of disjoint subsets $A, B \subseteq V(G)$ is $d(A, B) = \frac{|E(A,B)|}{|A|\cdot|B|}$. A pair (A, B) of disjoint subsets $A, B \subseteq V(G)$ is called *ε-regular* if

$$|d(A, B) - d(A', B')| < \varepsilon$$

for all $A' \subseteq A,\ B' \subseteq B$ with $|A'| \geq \varepsilon|A|,\ |B'| \geq \varepsilon|B|$. In many ways, regular pairs behave like random bipartite graphs with the same edge density.

If (A, B) is an ε-regular pair, then we call a subset A' of A *ε-significant* (or simply significant, if ε is clear from the context) if $|A'| \geq \varepsilon|A|$. We call a vertex from A *ε-typical* (or simply typical, if ε is clear from the context) with respect to a set $B' \subseteq B$ if it has degree at least $(1-\varepsilon)d(A, B)|B'|$ to B'.

The following well known and easy-to-prove facts (see for instance [KSS02]) state that in a regular pair almost every vertex is typical to any given significant set, and also that regularity is inherited by subpairs. More precisely, if (A, B) is an ε-regular pair with density d, then

(R1) for any ε-significant $B' \subset B$, all but at most $\varepsilon|A|$ vertices from A are ε-typical to B'; and

(R2) for each $\delta \geq 0$, and for any subsets $A' \subseteq A$, $B' \subseteq B$, with $|A'| \geq \delta|A|$ and $|B'| \geq \delta|B|$, the pair (A', B') is $\frac{2\varepsilon}{\delta}$-regular with density between $d - \varepsilon$ and $d + \varepsilon$.

Szemerédi's regularity lemma states that every large enough graph has a partition of its vertex set into one 'trash' set of bounded size, and a bounded number of sets of equal sizes, such that almost all pairs of these sets are ε-regular.

Lemma 4.4 (Szemerédi's regularity lemma)**.** *For each $\varepsilon > 0$ and $M_0 \in \mathbb{N}$ there are $M_1, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, every n-vertex graph G has a partition $V_0 \cup V_1 \cup \ldots \cup V_p$ of $V(G)$ into $p+1$ partition classes (or* clusters*) such that*

(a) $M_0 \leq p \leq M_1$;

(b) $|V_1| = |V_2| = \ldots = |V_p|$ *and* $|V_0| < \varepsilon n$;

(c) *apart from at most* $\varepsilon\binom{p}{2}$ *exceptional pairs, the pairs* (V_i, V_j) *are* ε*-regular, for* $i, j > 0$ *with* $i \neq j$.

As usual, we define the *reduced graph* R_G corresponding to this decomposition of G as follows. The vertices of R_G are all clusters V_i ($i = 1, \dots, p$), and R_G has a edge between V_i and V_j if the pair (V_i, V_j) is ε-regular, and has density at least $10\sqrt{\varepsilon}$. By standard calculations (see for instance [KSS02]), and assuming we take $M_0 \geq \lceil \frac{1}{\varepsilon} \rceil$, it follows that

$$\delta_w(R_G) \geq (1 - 12\sqrt{\varepsilon}) \cdot \frac{p}{|V(G)|} \cdot \delta(G), \tag{1}$$

where $\delta_w(R_G)$ is the weighted minimum degree. (That is, the densities of the pairs of clusters provide weights on the edges of R_G, and the weighted degree of a vertex is the sum of the corresponding edge-weights. The weighted minimum degree is the minimum of these weighted degrees. Observe that $\delta_w(R_G) \leq \delta(R_G)$ since weights do not exceed 1.)

Almost all vertices of any cluster $C \in V(R_G)$ are typical to almost all significant sets, in the following sense. If $\mathcal{Y}$ is a set of significant subsets of clusters in $V(R_G)$, then

$$\begin{aligned}&\text{all but at most } \sqrt{\varepsilon}|C| \text{ vertices } v \in C \text{ are typical with respect to}\\ &\text{all but at most } \sqrt{\varepsilon}|\mathcal{Y}| \text{ sets in } \mathcal{Y}.\end{aligned} \tag{2}$$

To see this well-known observation, assume that the set $C' \subseteq C$ of vertices not satisfying (2) is larger than $\sqrt{\varepsilon}|C|$. Then

$$\begin{aligned}\sum_{Y \in \mathcal{Y}} |\{v \in C : \ v \text{ is not typical to } Y\}| &\geq \sum_{v \in C'} |\{Y \in \mathcal{Y} : \ v \text{ is not typical to } Y\}|\\ &\geq |C'|\sqrt{\varepsilon}|\mathcal{Y}|\\ &> \varepsilon|C| \cdot |\mathcal{Y}|.\end{aligned}$$

Thus there is a $Y \in \mathcal{Y}$ such that more than $\varepsilon|C|$ vertices in C are not typical to Y, a contradiction.

Regularity will help us when embedding small trees into a pair of adjacent clusters of R_G.

Lemma 4.5. *Let* CD *be an edge of* R_G*, and let* $U \subseteq G$ *with* $|C \setminus U|, |D \setminus U| \geq \sqrt{\varepsilon}|C|$*. Let* $\bar{T}$ *be a tree of size* $\leq \varepsilon|C|$ *with root* $r_{\bar{T}}$.

Then $\bar{T}$ can be embedded into G, with $\bar{T} - r_{\bar{T}}$ going to $(C \cup D) \setminus U$, and with $r_{\bar{T}}$ going to any prescribed set of $\geq 2\varepsilon|C|$ vertices of C, or to any prescribed set of $\geq 2\varepsilon|C|$ vertices of C', where C' is any other cluster of R_G that is adjacent to D.

Proof. We construct the embedding $\bar{T}$ levelwise, starting with the root, which is embedded into a typical vertex of $(C \cup D) \setminus U$. At each step i we ensure that all vertices of level i are embedded into vertices of $C \setminus U$ (or $D \setminus U$) that are typical with respect to the unoccupied vertices of $D \setminus U$ (or $C \setminus U$). This is possible, because at each step i, and for each vertex v of level i, by (R1), the degree of a typical vertex into the unoccupied vertices on the other side is at least $4\varepsilon|C|$, and there are at most $\varepsilon|C|$ nontypical vertices and at most $|\bar{T}| \leq \varepsilon|C|$ already occupied vertices. □

5 Preparing the tree

5.1 Cutting a tree

In this section, we will show how any tree T can be cut up into small subtrees and few connecting vertices. The ideas is that later, we can use regular pairs to embed many tiny trees.

We will make use of a procedure which in a very similar shape has already appeared in [AKS95, HKP$^+$d, PS12], although there, no distinctions between the trees from L, F_1, F_2, were made. The resulting cut-up is given in the following statement.

Lemma 5.1. *For any $m \in \mathbb{N}$, for any tree T on $m+1$ vertices, for any $r \in V(T)$, and for any $\beta > 0$, there is a set $W \subseteq V(T)$, and a partition $\mathcal{T} = L \cup F_1 \cup F_2$ of the family $\mathcal{T}$ of components of $T - W$, distinguishing a subset $F_2' \subseteq F_2$, such that*

(a) $r \in W$;

(b) $|W| \leq \frac{2}{\beta^2}$;

(c) if $\beta m > 1$ *then each* $w \in W$ *has a child in* T;

(d) $|V(\bar{T})| = 1$ *for every tree* $\bar{T} \in L$;

(e) $1 < |V(\bar{T})| \leq \frac{1}{\beta}$ *for every tree* $\bar{T} \in F_1$;

(f) $\frac{1}{\beta} < |V(\bar{T})| \leq \beta m$ *for every tree* $\bar{T} \in F_2$*;*

(g) each $\bar{T} \in L \cup F_1 \cup (F_2 \setminus F_2')$ *has exactly one neighbour in* W*;*

(h) each $\bar{T} \in F_2'$ *has exactly two neighbours in* W*; and*

(i) $|\tilde{V}| < 2\beta m$*, where* $\tilde{V}$ *is the set of all neighbours of vertices of* W *in* $\bigcup F_2'$*.*

The vertices in W *will be called the* seeds *of* T*.*

Proof. In a sequence of at most $\frac{1}{\beta}$ steps i, we define vertices w_i and trees T_i as follows. Set $T_0 := T$. Now, for each $i > 0$, let $w_i \in V(T_{i-1})$ be a vertex at maximal distance from r (the root of T) such that the components of $T_i - w_i$ that do not contain r each have size at most βm. Delete w_i and all of these components from T_{i-1} to obtain T_i. We stop when we reach r, which will be the last vertex w_i to be defined.

Let W_0 be the union of all w_i, and let $\mathcal{T}_0$ be the set of all components of $T - W_0$. These two sets already fulfill items (a), (b) and (c). (To see (b), note that at each step i, we cut off βm vertices. Hence we actually have that $|W_0| \leq \frac{1}{\beta}$.)

In order to obtain sets W, $\mathcal{T}$ that also fulfill items (g) and (h), we add some vertices to W_0 as follows. For each $\bar{T} \in \bigcup \mathcal{T}_0$ that has $\ell > 2$ neighbours $v_1, v_2, \ldots, v_\ell$ in W_0, we add to W_0 a set of at most $\ell - 1$ vertices w_j' from $V(\bar{T})$ that separate all v_i's from each other. Note that these are at most $\frac{1}{\beta}$ vertices in total (counting over all affected $\bar{T}$), since each of the newly added vertices w_j' can be associated to one of the 'old' vertices v_j from W_0 such that w_j' lies between v_j and r. So, letting $\mathcal{T}_1$ be the family of the trees in $T - W_1$, the new sets W_1, $\mathcal{T}_1$ still fulfill (a), (b) and (c) (actually, we have that $|W_1| \leq \frac{2}{\beta}$). They furthermore have the property that each of the trees in $\bigcup \mathcal{T}_1$ has at most two neighbours in W_1.

We modify our sets once more to ensure that only the large trees can have two seed neighbours. We proceed as follows. For each $\bar{T} \in \bigcup \mathcal{T}_1$ that has at most $\frac{1}{\beta}$ vertices and is adjacent to two seeds $w_1, w_2 \in W_1$, we add to W_1 all vertices in $V(\bar{T})$. In total, these are at most $\frac{1}{\beta} \cdot |W_1| \leq \frac{2}{\beta^2}$ vertices. Call the new set of seeds W.

Defining $\mathcal{T}$ as the family of the trees in $T - W$, and adequately dividing $\mathcal{T}$ into three sets L, F_1, F_2, and letting F_2' be the appropriate subset of F_2, we obtain sets that fulfill all properties of the claim (where (i) follows directly from (f), (h) and the observation that $|\tilde{V}| \leq 2|F_2'| < \frac{2|T-r|}{\frac{1}{\beta}} = 2\beta m$). $\square$

5.2 Ordering the seeds of a tree

In order to be able to choose well the clusters of $V(R_G)$ into which we will embed the seeds other than r later on, we will define a convenient ordering of the seeds of a tree T with a cut-up as in Section 5.1. Together with this ordering we will define a set of relevant seeds X_s for each seed s of the tree, and ensure that the seeds in X_s come before s in the ordering.

The purpose of this ordering and the definition of the sets X_s is that later, when we embed the trees from L, F_1 and F_2 in G, it will turn out that the smaller the trees, the harder they are to embed, with the most difficult ones being the trees from L, i.e., the leaves of T adjacent to seeds. An embedded seed has only degree $\frac{2}{3}m$ in G, of which a large part might already be used, so we have to plan ahead in order to avoid getting stuck when embedding the leaves. For this reason we have to choose very well into which clusters the seeds go, and the sets X_s will help us with this.

The reader might wish to skip the remainder of this unfortunately rather technical section at a first reading, because everything we do here is only necessary for the embedding of L. Even the embedding of L can be followed with only a vague understanding of the definitions of the present section if the reader takes the 'Degrees of the embedded seeds into Z' as stated in Subsection 6.2.4 for granted.

5.2.1 Grouping and ordering

Let us start with the ordering. Assume we are given a tree T which has been treated by Lemma 5.1 for some $\beta > 0$, let W denote the set of seeds we obtained. Throughout the rest of this section we will assume that

$$|W| = 47 \cdot 2^{j^*}, \text{ where } j^* = \lceil \log \frac{2}{\beta^2} \rceil. \qquad (3)$$

(This can be assumed by adding some new vertices to the tree T, and declaring all of them seeds. We will explicitly discuss why this can be done in Subsection 6.1.2.)

We will order the seeds in two different ways, before we get to the third and final order. The first order is determined by the number of leaves hanging from each seed, the second order is determined by the position of the seeds in the tree T, and the third order is a mixture of both. We explain the orderings in detail in the following.

We start by ordering the seeds in a way that the number of leaf children of the seeds is decreasing, and we call this the *size order* σ on the seeds. Now, we will define ordered sets of seeds, which we will call *groups* of seeds. First we will define the *large groups*: loosely speaking, the largest of these groups consists of all seeds, then we define two groups consisting of the first and the second half of the seeds respectively, then each of these groups is divided into two new groups, and so on, until the size is down to 47. More precisely, for each $j = 0, \ldots, j^*$, we partition the set of all seeds into 2^{j^*-j} consecutive groups of size $2^j \cdot 47$, under the size order σ, and we call these the *large groups*. Clearly, each large group of size exceeding 47 is the union of two large groups half its size.

We break up each group B of size 47 into twelve consecutive groups (consecutive under σ) of the following sizes:

$$4, \mathbf{4}, 4, 4, 5, 4, \mathbf{4}, 4, 4, 5, 4, 1. \tag{4}$$

We call these the *small groups*. (So the small subgroup of size 1 of B consists of the very last seed of B in the size order σ.) We say the second and sixth group of size four are of type 1 (they are marked in boldface in (4)). The remaining groups of size four (i.e. the first, third, fourth, fifth, seventh, eighth and ninth group of size four) will be called type 2.

It would be difficult to embed the seeds in the size order σ, as this enumaration might not be suitable for embedding the tree in a connected way. For this reason, we employ a second order τ, which we call the *transversal order*, obtained by performing a preorder tranversal on T, starting with the root r, and then restricting this order to W. (The transversal order is the actual order the seeds will be embedded in.)

The third order, which we call the *rearranged order* ρ, is obtained by reordering the order σ. First, we reorder the seeds in each small group so that each small group is ordered according to τ. Next, for every large group B of size 47, we reorder all its subgroups so that their first seeds form an increasing sequence in the transversal order τ. Finally, for every large group B of size > 47 (in successive steps according to the group size), we reorder the two subgroups within B so that the first subgroup contains the first seed in B under the transversal order τ (i.e. we reorder them such that the first seed of B under τ becomes the first seed of B). This finishes the definition of the rearranged order ρ.

We note that ρ maintains the structure given by breaking down the set of seeds into large and small groups. That is, if we partition the set of all seeds into 2^{j^*-j} consecutive groups under ρ of sizes $2^j \cdot 47$, we obtain the same groups as above for σ. Further, each group B of size 47 breaks down into twelve small groups as above, although these are no longer ordered as in the sequence from (4). (For instance, in ρ, the small group of size 1 from B could become the first group in B, or be at any other position.)

We will embed according to τ but momentarily work with ρ. We write $s <_\rho s'$ to denote that s comes before s' in order ρ (and similar for τ).

5.2.2 Sequences

In this subsection, we will follow the rearranged order ρ. We define for each large group B two sequences

$$(x_i^B)_{i=1\dots,j+6} \text{ and } (y_i^B)_{i=1\dots,j+7},$$

where j is such that $|B| = 2^j \cdot 47$, and vertices $x_i^B, y_i^B \in B$ are as specified in what follows.

We construct our sequences inductively. For $j = 0$, we have 2^{j^*} large groups of size 47. For each such large group B, we take $x_1^B = y_1^B$ as the first seed of the group. The first seed of the second, third, fourth, fifth and sixth small subgroup of B is chosen as $x_2^B, x_3^B, x_4^B, x_5^B, x_6^B$, respectively. The first seed of the seventh, eigth, ninth, tenth, eleventh and twelfth small subgroup of B is chosen as $y_2^B, y_3^B, y_4^B, y_5^B, y_6^B, y_7^B$, respectively. (We always work under ρ, both when talking about the 'first seed of a group' and when talking about the 'ith subgroup'.)

For $j \geq 1$, we have to deal with all large groups of size $2^j \cdot 47$. For each such group B, do the following. By construction B is made up of two large subgroups of size $47 \cdot 2^{j-1}$, say these are B' and B'' (in this order, under ρ). We set

$$x_i^B := y_i^{B'} \text{ for all } 1 \leq i \leq j+6,$$

and

$$y_1^B := x_1^B = y_1^{B'}, \text{ and } y_i^B := y_{i-1}^{B''} \text{ for all } 2 \leq i \leq j+7.$$

This finishes the definition of the sequences. We remark that we will only use the sequences (x_i) in what follows (the sequences (y_i) were only used to make the definition of (x_i) more convenient).

Observe that for all blocks B, and for all $i < j$, we have that $x_i^B <_\tau x_j^B$.

5.2.3 Relevant seeds

In order to be later able to choose well the clusters we embed the seeds into (which in turn will enable us to embed the leaves at an even later stage), we need to define, for each seed s, a set X_s of relevant seeds for s, as follows.

Definition 5.2 (Relevant seeds for s)**.**
Let s be a seed of T, and let B be the small group s belongs to.

(a) If B is a group of four of type 2, and s is the last seed of B, then we set

$$X_s := \{x \ : \ x \text{ is the third seed in } B \text{ (under } \rho)\}.$$

(b) If s is not the first seed of B, and, in case B is a group of four of type 2, s is not its last seed, then we set

$$X_s := \{x \ : \ x \in B,\ x <_\rho s\}.$$

(c) If s is the first seed of B, then we set

$$X_s := \{x \ : \ \exists \tilde{B}, i, i' \text{ such that } i' < i,\ s = x_i^{\tilde{B}} \text{ and } x = x_{i'}^{\tilde{B}}\}.$$

Observe that $\tilde{B}$, if it exists, is unique, and that if s only appears as a first vertex in the sequences $(x_i^{\tilde{B}})$, then $X_s = \emptyset$.

Let us make a quick observation which follows directly from the definition of the order ρ, of the sequences (x_i) and of the sets X_s.

Observation 5.3. *Let s be a seed. Then for all $x \in X_s$ it holds that $x <_\tau s$.*

6 The Proof of Lemma 2.1

6.1 Preparations

6.1.1 Setting up the constants

First of all, given δ, we choose

$$\varepsilon \leq \frac{\delta^4}{10^{18}}, \tag{5}$$

and apply Lemma 4.4 (the regularity lemma) with input ε^2 and $M_0 := \frac{1}{\varepsilon^2}$. This yields numbers M_1 and n_0. We then set

$$\beta := \frac{\varepsilon}{100M_1}.$$

Finally, we choose

$$m_\delta := (n_0 + 1) \cdot \frac{400M_0}{\beta^{10} \cdot \varepsilon \cdot \delta} \tag{6}$$

for the output of Lemma 2.1. So, given the approximation constant α, satisfying $1 \geq \alpha \geq \delta$, we will have that

$$0 < \frac{1}{m_\delta} \ll \beta \ll \varepsilon \ll \delta \leq \alpha, \tag{7}$$

with the explicit dependencies given above. Now, given $m \geq m_\delta$, and given an $(m+1)$-vertex graph G of minimum degree at least $\lfloor \frac{2m}{3} \rfloor$, and a tree T with at most $m - \alpha m$ edges, rooted at r, we will prepare both T and G for the embedding.

6.1.2 Preparing T for the embedding

We apply Lemma 5.1 to obtain a partition of T into a set W of seeds and a set $\mathcal{T}$ of small trees. The small trees divide into L, F_1 and F_2, with two-seeded trees $F_2' \subseteq F_2$, and the lemma also gives us a set $\tilde{V}$. Set

$$f_1 := \sum_{\bar{T} \in F_1} |V(\bar{T})| \quad \text{and} \quad f_2 := \sum_{\bar{T} \in F_2} |V(\bar{T})|.$$

Next, add a set W' of vertices to T, each adjacent to r, such that, setting $\tilde{W} := W \cup W'$, we have

$$|\tilde{W}| = 47 \cdot 2^{j^*}$$

for

$$j^* := \lceil \log \frac{2}{\beta^2} \rceil. \tag{8}$$

The only reason for this is that we plan to apply the grouping and ordering of seeds from Subsection 5.2, that is, we would like to see (3) fulfilled. We are going to embed $T \cup W'$ instead of T. Since the number of vertices in W' is a constant, space is not a problem. Indeed, clearly,

$$|\tilde{W}| + |L| + f_1 + f_2 \; = \; |V(T)| + |\tilde{W} \setminus W| \; \leq \; m - \alpha m + \frac{200}{\beta^2}. \tag{9}$$

6.1.3 Preparing G for the embedding

As a preparation of G for the embedding, we take an ε^2-regular partition of G as given by Lemma 4.4 (the regularity lemma), into p clusters, for some p with $M_0 < p < M_1$. Consider the reduced graph R_G of G with respect to this partition, as defined below Lemma 4.4.

Because of the minimum degree of G and by (1), we have that

$$\delta_w(R_G) \geq (\frac{2}{3} - 13\varepsilon)p. \tag{10}$$

Let us now partition the clusters of R_G further. We will divide each clusters into several slices, into which we plan to embed the distinct parts of the tree T which we identified above.

First of all, we choose a set Z of vertices into which we plan to embed L. More precisely, we arbitrarily choose a set $Z \subseteq V(G)$ of size

$$|Z| = |L| + \lceil (\alpha - \frac{\alpha^4}{10^6})m \rceil, \tag{11}$$

choosing the same number of vertices in each part of the partition (plus/minus one vertex). Now, we will split up the remainder $C \setminus Z$ of each cluster $C \in V(R_G)$ arbitrarily into four sets $C_{\tilde{V}}$, C_W, C_{F_1}, C_{F_2}, and a leftover set $C \setminus (Z \cup C_{\tilde{V}} \cup C_W \cup C_{F_1} \cup C_{F_2})$ which will not be used. The sets are chosen having the following sizes:

$$|C_{\tilde{V}}| = |C_W| = \lceil \frac{\frac{\alpha^4}{5}m}{p} \rceil; \tag{12}$$

$$|C_{F_1}| = \lceil \frac{f_1 + \frac{\alpha^4}{5}m}{p} \rceil; \tag{13}$$

and

$$|C_{F_2}| = \lceil \frac{f_2 + \frac{\alpha^4}{5}m}{p} \rceil. \tag{14}$$

This is possible because of (7) and (9). As we mentioned above, the idea behind this slicing up is that we are planning to put each part X of the tree ($X \in \{W, \tilde{V}, F_1, F_2, L\}$) into the parts C_X of the clusters of R_G, or into Z, respectively. We reserve a bit more than is actually needed for the embedding, in order to always be able to choose well-behaved (typical) vertices, and also in order to account for slightly unbalanced use of the regular pairs when embedding the trees from $\mathcal{T}$. Since the sets C_X are large enough, regularity properties will be preserved between these sets (cf. Section 4.3).

Let us remark that it is not really necessary to slice the clusters C up as much as we do: the vertices destined to go into slices $C_{\tilde{V}}$ and C_W are actually so few that they could go to any other slice without a problem. But we think the exposition might be clearer if everything is well-controlled.

Finally, we fix a perfect matching M_{F_2} of R_G which exists because of (10). This matching will be used for embedding the trees from F_2.

6.1.4 The plan

For convenience, for each seed $s \in \tilde{W}$, let $\mathcal{T}_s$ denote the set of all trees from $\mathcal{T} \setminus L$ that hanging from s. We are going to traverse the seeds in the transversal order τ, placing each seed s into a suitable cluster $S(s)$ (we will determine this cluster right before embedding s into it). We then embed $\bigcup \mathcal{T}_s$ before embedding any other seed. After having embedded all seeds $s \in \tilde{W}$ and all corresponding trees from $\bigcup \mathcal{T}_s$, we embed all of L in one step at the very end of the embedding process. So, if the seeds are ordered as $s_1, s_2, s_3, \ldots, s_{|\tilde{W}|}$ in τ, then we embed in the order

$$s_1,\ \bigcup \mathcal{T}_{s_1},\ s_2,\ \bigcup \mathcal{T}_{s_2},\ s_3,\ \bigcup \mathcal{T}_{s_3},\ \ldots s_{|\tilde{W}|},\ \bigcup \mathcal{T}_{s_{|\tilde{W}|}},\ L,$$

and at every point in time, the embedded parts of the tree will form a connected set in T.

Each of the three different embedding procedures will be described in detail in one of the following subsections, namely, in Subsection 6.2 (embedding a seed s), in Subsection 6.3 (embedding $\bigcup \mathcal{T}_s$) and in Subsection 6.4 (embedding L).

6.2 Embedding the seeds

6.2.1 Preliminaries

Assume we are about to embed some seed s. Denote by U the set of vertices that, up to this point, have been used for embedding seeds and small trees. So $U \cap Z = \emptyset$ (we will ensure that this will always remain so), and every cluster $C \in V(R_G)$ divides into six sets: $C \cap U$, $C \cap Z$, $C_W \setminus U$, $C_{\tilde{V}} \setminus U$, $C_{F_1} \setminus U$, and $C_{F_2} \setminus U$.

Apart from U, it will be useful to have a set $U' \subseteq \bigcup_{C \in V(R_G)}(C_{F_1} \setminus U)$ of vertices for which at some point we decided that they will never be used for the embedding. The main purpose of this set U' is that after embedding certain trees from $\mathcal{T}_s \cap F_1$ for some seed s, we can just make all sets C_{F_1} of clusters C equally 'occupied' by discarding some of the vertices of the emptier sets C_{F_1} by putting them into U'. This will be the only time we add vertices to U'. We will make sure that for each seed s the number u'_s of vertices we add to U' while, or directly after, embedding $\mathcal{T}_s$ is bounded by

$$u'_s \leq \frac{3}{\beta^{10}} + 600\varepsilon \cdot |\mathcal{T}_s \cap F_1|. \tag{15}$$

Since there at most $\frac{2}{\beta}$ (original) seeds in the tree, this means that at any time,

$$|U'| \leq \frac{6}{\beta^{11}} + 600\varepsilon \cdot \sum_{s \in \tilde{W}} |\mathcal{T}_s \cap F_1| \leq 601\varepsilon m.$$

In other words, the set U' will always stay so small that we can ignore it while embedding.

Throughout the embedding, we will ensure that for each parent u of a seed (the parent u might be a seed, or a vertex from $\tilde{V}$) the following holds. If u was embedded in vertex $\varphi(u)$, then we have that

$$\varphi(u) \text{ is typical to slice } C_W \text{ for all but at most } \varepsilon p \text{ clusters } C \text{ of } R_G. \tag{16}$$

Note that by Observation 5.3, by the time we reach a seed s, the 'relevant' seeds in X_s have already been embedded into a set $\varphi(X_s)$. Let $N_Z(X_s)$ denote the set of all neighbours of vertices from X_s in Z, i.e. $N_Z(X_s) := N(X_s) \cap Z$. Let $\mathcal{N}_s$ be the set of the corresponding subsets of the clusters of R_G (i.e., $\bigcup \mathcal{N}_s = N_Z(X_s)$).

6.2.2 Finding the target cluster $S(s)$ for s

Before actually choosing the vertex $\varphi(s)$ we will embed s into, we will determine the target cluster $S(s)$ for a seed s.

Observe that by Lemma 4.1, with $\psi := 13\varepsilon$, we know that at least $(\frac{1}{3} + \varepsilon^{\frac{1}{3}})m$ of the vertices of G see a $(\frac{1}{2} - \varepsilon^{\frac{1}{3}})$-portion of the vertices in $Z \setminus N_Z(X_s)$. So, for significantly more than a third of the clusters of R_G we have that a significant portion of their vertices see at least $(\frac{1}{2} - \varepsilon^{\frac{1}{3}}) \cdot |Z \setminus N_Z(X_s)|$ vertices in $Z \setminus N_Z(X_s)$. Because of regularity, and because of (2), this means that for any such cluster C, all but at most an ε-fraction of the vertices in C_W has at least $(\frac{1}{2} - 3\varepsilon^{\frac{1}{3}}) \cdot |Z \setminus N_Z(X_s)|$ neighbours in $Z \setminus N_Z(X_s)$.

Choose $S(s)$ as any one of the clusters as above, i.e., such that

(α) all but at most $\varepsilon|S(s)_{\tilde{W}}|$ vertices of the set $S(s)_{\tilde{W}}$ have degree at least $(\frac{1}{2} - 3\varepsilon^{\frac{1}{3}}) \cdot |Z \setminus N_Z(X_s)|$ into $Z \setminus N_Z(X_s)$;

and such that in addition (unless $s = r$, in which case the following two conditions are void),

(β) $S(s)$ is adjacent to $S(p(s))$;

(γ) $\varphi(p(s))$ is typical with respect to $S(s)_{\tilde{W}}$,

where $S(p(s))$ denotes the cluster the parent $p(s)$ of s was embedded into. Such a choice of $S(s)$ is possible since by (10), cluster $S(p(s))$ has degree almost $\frac{2p}{3}$ in R_G, and because of (16).

6.2.3 Embedding seed s into target cluster $S(s)$

We place s in a vertex $\varphi(s)$ from $S(s)_{\tilde{W} \setminus U}$ such that

(A) $\varphi(s)$ is a neighbour of $\varphi(p(s))$ (where $p(s)$ is the parent of s, and if $s = r$ this restriction is empty);

(B) $\varphi(s)$ is typical to $C_{\tilde{W}}$ for all but at most εp clusters $C \in V(R_G) \setminus S(s)$;

(C) $\varphi(s)$ is typical to $C_{\tilde{V}}$ for all but at most εp clusters $C \in V(R_G) \setminus S(s)$;

(D) $\varphi(s)$ is typical to $C_{F_1} \setminus (U \cup U')$ for all but at most εp clusters $C \in V(R_G) \setminus S(s)$; and

(E) $\varphi(s)$ is typical to C_L for all but at most εp clusters $C \in V(R_G) \setminus S(s)$.

Such a choice is possible since by (2), almost all vertices in any given cluster are typical with respect to any fixed significant subsets of almost all other clusters. Note that in particular, (E) implies that

$$deg_Z(\varphi(s)) \geq (\frac{2}{3} - \varepsilon^{\frac{1}{3}})|Z|. \tag{17}$$

6.2.4 Degrees of the embedded seeds into Z

The reason for our choice of $S(s)$ as a cluster fulfilling property (α) from Subsection 6.2.2 is that it allows us to accumulate degree into Z. More precisely, if we consider a seed s together with its relevant seeds X_s, then we know that the union of their neighbourhoods in Z is significantly larger than the neighbourhood of s alone. Better still, the more vertices X_s contains, the larger becomes our bound on this neighbourhood.

We make this informal observation more precise in the following claim.

Claim 6.1. *Let B be a group of seeds.*

(i) If B has size five, then

$$|N(\varphi(B)) \cap Z| \geq (\frac{47}{48} - \varepsilon^{\frac{1}{4}}) \cdot |Z|.$$

(ii) If B has size four and is of type 1, then

$$|N(\varphi(B)) \cap Z| \geq (\frac{23}{24} - \varepsilon^{\frac{1}{4}}) \cdot |Z|.$$

(iii) If $B = \{b_1, b_2, b_3, b_4\}$ (with the seeds b_i appearing in this order in σ) is of type 2, then

$$|N(\varphi(B)) \cap Z| \geq (\frac{11}{12} - \varepsilon^{\frac{1}{4}}) \cdot |Z|,$$

and

$$\min\big\{|N(\varphi(\{b_1, b_2\})) \cap Z|, |N(\varphi(\{b_3, b_4\})) \cap Z|\big\} \geq (\frac{5}{6} - \varepsilon^{\frac{1}{4}}) \cdot |Z|.$$

(iv) If B is large, say of size $47 \cdot 2^j$, then

$$|N(\varphi(\{x_i^B : i = 1 \ldots, j+6\})) \cap Z| \geq (1 - \frac{1}{96 \cdot 2^j} - \varepsilon^{\frac{1}{4}}) \cdot |Z|,$$

where $(x_i^B)_{i=1\ldots,j+6}$ is the sequence defined in Subsection 5.2.2.

Proof. This follows rather directly from (α) and (E) (from Subsections 6.2.2 and 6.2.3, respectively), from (17), and from the definition of the set X_s of relevant seeds (Definition 5.2). For instance, we can calculate the bound in item (i) by using (17), (α), (E), and Definition 5.2 (b) to see that

$$|N(\varphi(B)) \cap Z| \geq (\frac{2}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \frac{1}{48} - 5 \cdot 3\varepsilon^{\frac{1}{3}}) \cdot |Z| \geq (\frac{47}{48} - \varepsilon^{\frac{1}{4}}) \cdot |Z|.$$

For item (iv), we need to take slightly more care with the calculation. Note that the degree into Z of the image of the first seed is off $\frac{2}{3}|Z|$ by at most $3\varepsilon^{\frac{1}{3}}|Z|$. The degree of second seed's image is only off $\frac{1}{2}|Z \setminus N_Z(x_1^B)|$ by less than $3\varepsilon^{\frac{1}{3}}\frac{|Z|}{2}$. For the third seed we are only off $\frac{1}{2}|Z \setminus (N_Z(x_1^B) \cup N_Z(x_2^B))|$ by $3\varepsilon^{\frac{1}{3}}\frac{|Z|}{4}$, and so on, which means we can bound the error in our estimate for the size of the joint neighbourhod in (iv) by $(1 + \frac{1}{2} + \frac{1}{4} + \ldots) \cdot 3\varepsilon^{\frac{1}{3}}|Z| \leq 2 \cdot 3\varepsilon^{\frac{1}{3}}|Z| \leq \varepsilon^{\frac{1}{4}} \cdot |Z|$. □

6.3 Embedding the small trees

Assume we have sucessfully embedded a seed s, and are now, before we proceed to the next seed, about to embed all small trees from $\mathcal{T}_s$ (while still leaving any leaves from L adjacent to s unembedded).

Our plan is to embed those trees of $\mathcal{T}_s$ that belong to F_1 into $\bigcup_{C \in V(R_G)} C_{F_1}$, and those trees of $\mathcal{T}_s$ that belong to F_2 into $\bigcup_{C \in V(R_G)} C_{F_2}$. We first explain how we deal with the larger trees, i.e. those in $F_2 \setminus F_2'$, and those in F_2'. After that we explain how we deal with the constant sized trees, i.e. those in F_1. Note that actually, it does not matter in which order we deal with the sets $F_1, F_2 \setminus F_2', F_2'$.

6.3.1 Embedding the trees from $F_2 \setminus F_2'$

For each $\bar{T} \in \mathcal{T}_s \cap (F_2 \setminus F_2')$, let $r_{\bar{T}}$ denote its root. We plan to put $r_{\bar{T}}$ into $C'_{\tilde{V}} \setminus U$ for some suitable cluster C'. (We will explain below how exactly we do that.) For the rest of $V(\bar{T})$, we proceed as follows.

Recall that we defined a perfect matching of R_G near the end of Section 6.1.1. Choose an edge CD of M_{F_2} that contains at least $3\varepsilon \cdot \frac{m}{p}$ unused vertices in each of C_{F_2}, D_{F_2}. If $|C_{F_2} \setminus U| \geq |D_{F_2} \setminus U|$, we will aim at putting the larger colour class of $\bar{T} - r_{\bar{T}}$ into $C_{F_2} \setminus U$, and otherwise, we aim at putting it into $D_{F_2} \setminus U$. Observe that if we manage to do this for every tree $\bar{T}$ we embed, we can ensure that throughout the process (even when embeeding trees from $\mathcal{T}_{s'}$, for some $s' \neq s$), the edges from M_{F_2} keep their free space in a more or less balanced way, that is, for all edges $C'D'$ in M_{F_2},

$$|C'_{F_2} \setminus U| \text{ and } |D'_{F_2} \setminus U| \text{ differ by at most } \beta \frac{m}{p}. \tag{18}$$

Let us now explain how we manage to embed $\bar{T}$ in this way. Assume our aim is to embed the children of $r_{\bar{T}}$ in to $C_{F_2} \setminus U$, the grandchildren into $D_{F_2} \setminus U$, the grand-grandchildren into $C_{F_2} \setminus U$, and so on. The embedding of $\bar{T} - r_{\bar{T}}$ will be easy using Lemma 4.5 once we found a vertex $\varphi(r_{\bar{T}})$ to embed $r_{\bar{T}}$ into, that is, a vertex that is both a neighbour of $\varphi(s)$ and typical with respect to $C_{F_2} \setminus U$.

So we only need to find a suitable vertex for $\varphi(r_{\bar{T}})$, the root of $\bar{T}$ (which belongs to $\tilde{V}$). In order to do so, we first determine a cluster C' that is adjacent to both C and $S(s)$, and that fulfills $d(S(s), C') \geq \frac{1}{4}$. At least nearly a third of the clusters in R_G qualify for this, because of (10). Now, by (C) in the choice of $\varphi(s)$ (in Subsection 6.2.3), we know that $\varphi(s)$ has typical degree into the set $C'_{\tilde{V}}$ for all but very few clusters C'. Typical degree means that $\varphi(s)$ has at least $(\frac{1}{4} - \varepsilon^2) \cdot |C'_{\tilde{V}}|$ neighbours in $C'_{\tilde{V}}$, and by Lemma 5.1 (i), at most $2\beta m$ vertices have been used for earlier vertices from $\tilde{V}$. So, by (7), we can choose a suitable C' such that $\varphi(s)$ has a large enough neighbourhood in $C'_{\tilde{V}}$ to ensure it contains a vertex $\varphi(r_{\bar{T}})$ that is typical with respect to $C_{F_2} \setminus U$, i.e. such that $\varphi(r_{\bar{T}})$ has at least $(10\sqrt{\varepsilon} - \varepsilon^2)|C_{F_2} \setminus U| \geq \beta m$ neighbours in $C_{F_2} \setminus U$, where the inequality follows from (14) and (18).

6.3.2 Embedding the trees from F'_2

For each $\bar{T} \in \mathcal{T}_s \cap F'_2$, we proceed exactly as in the preceding paragraph, except that now, we have to make a small adjustment when we are close to embedding $\tilde{v}$, the second vertex from $\tilde{V}$ contained in $V(\bar{T})$.

Suppose s' is the seed which is adjacent to $\tilde{v}$ in T. Because we embed the seeds following the transversal order, we know that s' is not yet embedded by the time we deal with $\bar{T}$. We take care to embed $\tilde{v}$ into a vertex that is typical with respect to almost all the sets $C_{\tilde{W}}$. That is, the image of $\tilde{v}$ will be chosen such that (16) holds.

Finally, observe that (14) and (18) ensure that the space we had assigned to F_2 is enough for embedding first all of $F_2 \setminus F'_2$, and now, all of F'_2.

6.3.3 Embedding the trees from F_1

We now explain how we embed the trees from $\mathcal{T}_s \cap F_1$. Note that because of (13) and (15), we have enough space to embed all of $\mathcal{T}_s \cap F_1$. Furthermore, because the trees from F_1 are small, and because of regularity, we have no problem with the actual embedding of them into the regular pairs of G. The only thing we need to make sure is that the roots of the trees from $\mathcal{T}_s \cap F_1$ are embedded into neighbours of $\varphi(s)$, and that we maintain the unused parts of the cluster slices C_{F_1} balanced at all times.

Since there is no matching like M_{F_2} that can be used throughout the whole embedding (i.e., for all seeds), we will have to simultaneously keep *all of the clusters* reasonably balanced. This will be possible because of the rather delicate embedding strategy we employ, and which we will start to explain now.

Preparing the slices C_{F_1}. Assume we are about to start the embedding process of the trees from $\mathcal{T}_s \cap F_1$. First of all, note that we can partition the free space $C_{F_1} \setminus (U \cup U')$ of the slices C_{F_1} of each of the clusters $C \in V(R_G) \setminus \{S(s)\}$ into sets $Q_0^C, \ldots, Q_r^C$ for some r, such that $|Q_0^C| < 2\lceil \varepsilon \frac{m}{p} \rceil$ and $|Q_i^C| = \lceil \varepsilon \frac{m}{p} \rceil$ for $i = 1, \ldots, r$, and such that for each $i = 1, \ldots, r$, either all or none of the vertices in Q_i^C are adjacent to $\varphi(s)$. The reason for doing this is that we now have total control over where exactly the neighbours of $\varphi(s)$ are (since the sets Q_0^C are small enough to be ignored during this step of the embedding). Observe that the sets Q_i^C, for $i = 1, \ldots, r$, are large enough to preserve regularity properties, although now we have to replace the regularity parameter ε^2 with $\frac{\varepsilon^2}{\varepsilon} = \varepsilon$.

Consider the graph H with vertex set $\{Q_i^C\}_{i=1,\ldots,r, C \in V(R_G)}$ and an edge for each ε-regular pair of sufficient density. Say H has p' vertices. By (10), the weighted minimum degree of R_G is bounded by $\delta_w(R_G) \geq (\frac{2}{3} - 13\varepsilon)p$, and therefore, the weighted minimum degree of H is bounded by $\delta_w(H) \geq (\frac{2}{3} - 17\varepsilon)p'$.

So, by our choice of $\varphi(s)$, in particular by (D) of Subsection 6.2.3, we know that $\varphi(s)$ has neighbours in at least $(\frac{2}{3} - 20\varepsilon)p'$ of the sets Q_i^C. Let N consist of a set of $\lceil (\frac{2}{3} - 20\varepsilon)p' \rceil$ sets Q_i^C that contain neighbours of $\varphi(s)$. We apply Lemma 4.3 with $\xi := 17\varepsilon$ to H to obtain a set X, of size

$$|X| \leq \lfloor 255\varepsilon p' \rfloor + 1 \leq 300\varepsilon p', \tag{19}$$

as well as an $(N \setminus X)$-good matching M $(= M_s)$, an $(N \setminus X)$-in-good path partition $\mathcal{P}_A$ and an $(N \setminus X)$-out-good path partition $\mathcal{P}_B$ of $H - X$.

Set $\mathcal{Q} := \bigcup_{C\in V(R_G)}\{Q_1^C, Q_2^C, \dots, Q_r^C\}\setminus X$. By (13), and by (15), we know that $\bigcup \mathcal{Q}$ is large enough to host all of $\bigcup \mathcal{T}_s \cap F_1$. In fact, if $\bigcup \mathcal{T}_s\cap F_1$ could be embedded absolutely balanced into the sets $Q\in\mathcal{Q}$, then there would even be a leftover space of more than $100\varepsilon\frac{m}{p}$ in each of the sets Q.

Recall that during the embedding of the trees from $\mathcal{T}_s \cap F_1$, we will add some vertices to a set U', for keeping better track of the balancing of the edges. We will keep U' small, that is, we will ensure that (15) holds.

Preparing $\mathcal{T}_s \cap F_1$. We now partition the set of trees from $\mathcal{T}_s \cap F_1$ into three sets[3]: the set T_{Bal} contains all the balanced trees, i.e. those trees whose color classes have the same size; the set $T_{NearBal}$ contains all trees having the property that their colour classes differ by exactly one, with the bigger class containing the root; and the set T_{Unbal} contains all the remaining trees, that is all unbalanced trees not belonging to $T_{NearBal}$.

Phase 1. In the first phase of our embedding, we embed all trees from T_{Bal}, using the matching M. We try to spread these trees as evenly as possible among the edges of M. It is not difficult to see that by Lemma 5.1 (e), it is possible to make the used part of the clusters differ by at most $\frac{1}{\beta}$ (but even the more obvious weaker bound $\frac{1}{\beta^2}$ is sufficient for our purposes). At the end of this phase of the embedding, we add to U' at most $\frac{1}{\beta}$ unused vertices from each of the clusters $Q\in V(M)$, and can thus make sure each of the clusters has exactly the same number of vertices in $Q_{F_1}\setminus(U\cup U')$.

Phase 2. In the second phase of our embedding, we embed all trees from T_{Unbal}. We group the trees from T_{Unbal} by their number of vertices, which is some number between 3 and $\frac{1}{\beta}$. Then we subdivide these groups according to the number of vertices belonging to the same colour class as their root. The final groups represent the *types* of trees. Since all trees we consider have order at most $\frac{1}{\beta}$,

$$\text{there are at most } \frac{1}{\beta^2} \text{ different types of trees in } T_{Unbal}. \tag{20}$$

For each of the types $\bar{T}$, say with t vertices, and colour classes of sizes t_1 and t_2, where the class of size t_1 contains the root, we proceed as follows.

[3] We remark that it is not really necessary to treat the trees from T_{Bal} separately (as they could be treated together with the trees from T_{Unbal} in Phase 2), but we believe that embedding $\bigcup T_{Bal}$ first (in Phase 1) is more instructive.

We go through the elements of our N-out-good path partition $\mathcal{P}_B$ in some fixed order, always embedding only a constant number of trees of type $\bar{T}$. In each round, we keep the clusters of $H - X$ perfectly balanced. Only when we run out of trees of type $\bar{T}$, we will (necessarily) have to make a last round, possibly not reaching all elements of $\mathcal{P}_B$, and thus unbalancing some of the clusters a bit (by at most $\frac{1}{\beta}$).

To make the above description more precise, recall that $\mathcal{P}_B$ consists of

(M1) single edges AB with both ends in N;

(M2) paths $ABCD$ with $B, C \in N$; and

(M3) paths $ABCDEF$ with $B, C, D, E \in N$.

Say there are m_1 paths AB as in (M1), m_2 paths $ABCD$ as in (M2), and m_3 paths $ABCDEF$ as in (M3). Let us now analyse how the sets $Q \in \mathcal{Q}$ lying in edges or paths from (M1)–(M3) fill up when we embed small trees of type $\bar{T}$ into them in the following specific ways.

First, the sets A, B of any edge as in (M1) will each get filled up with t vertices if we embed one tree of type $\bar{T}$ in one 'direction' and a second tree of type $\bar{T}$ in the other 'direction'. In other words, we can embed a total of $2m_1h$ trees of type $\bar{T}$ into the edges from (M1), filling each of the corresponding clusters Q with th vertices (where h is any not too large natural number).

The sets Q on paths $ABCD$ as in (M2) will get filled as follows. If we

- perform x rounds in which we embed one tree of the current type $\bar{T}$ in the edge AB, with the root going to B;
- perform x rounds in which we embed a tree of type $\bar{T}$ in the edge CD, with the root going to C;
- perform y rounds of embedding a tree of type $\bar{T}$ with the root going to C, but the rest of the tree going to AB; and
- perform y rounds of embedding a tree of type $\bar{T}$ with the root going to B, but the rest of the tree going to CD,

then after these $2x + 2y$ rounds, A and D each have received $xt_2 + y(t_1 - 1)$ vertices, while B and C each have received $xt_1 + y(t_2 + 1)$ vertices.

So, if $t_1 > t_2$ (observe that then actually $t_1 \geq t_2 + 2$, since $\bar{T} \notin T_{NearBal}$), we will have filled each of the four sets A, B, C, D with exactly $(t_1 - t_2 - 1)t$

vertices if we choose $x = t_1 - t_2 - 2$ and $y = t_1 - t_2$. If $t_2 \geq t_1$, we can fill each of the four sets A, B, C, D with exactly $(t_2 - t_1 + 1)t$ vertices by taking $x = t_2 - t_1 + 2$ and $y = t_2 - t_1$.

Resumingly, for any not too large natural number h', there is a way to embed a total of $|t_1 - t_2 - 1| \cdot 4m_2h'$ trees of type $\bar{T}$ into the edges from (M2), filling up each of the corresponding clusters Q with $|t_1 - t_2 - 1| \cdot t$ vertices. Even more, taking into account what we said above for edges from (M1), we conclude that for any not too large h', we are able to embed a total of

$$|t_1 - t_2 - 1| \cdot 2m_1h' + |t_1 - t_2 - 1| \cdot 4m_2h' = |t_1 - t_2 - 1| \cdot (2m_1 + 4m_2)h'$$

trees of type $\bar{T}$ into the edges from (M1) and (M2), filling up each of the corresponding clusters Q with $|t_1 - t_2 - 1| \cdot th'$ vertices.

For the paths $ABCDEF$ from (M3) we can calculate similarly: Say we do x rounds of embedding of a tree of type $\bar{T}$ in the edge AB and another x rounds embedding it into EF. We then do y rounds of embedding the tree into AB, but with the root of the tree going into C, and another y rounds putting it into EF, with the root going into D. Moreover, we perform $2z$ rounds where we embed the tree into CD, of which z rounds in each 'direction'. Then after these $2x + 2y + 2z$ rounds, we filled each of A and F with $xt_2 + y(t_1 - 1)$ vertices, each of B and E with $xt_1 + yt_2$ vertices, and each of C and D with $y + zt$ vertices.

So, if $t_2 \geq t_1$, then with $x = t \cdot (t_2 - t_1 + 1)$, $y = t \cdot (t_2 - t_1)$, and $z = (t - 1) \cdot (t_2 - t_1) + t_1$, we have filled each of the six sets A, B, C, D, E, F with exactly the same amount of vertices, namely with $t \cdot (t \cdot (t_2 - t_1) + t_1)$ vertices each. If $t_1 > t_2$, we choose $x = t \cdot (t_1 - t_2 - 1)$, $y = t \cdot (t_1 - t_2)$, and $z = (t - 1) \cdot (t_1 - t_2) - t_1$, and fill each of the six sets A, B, C, D, E, F with $t \cdot (t \cdot (t_1 - t_2) - t_1)$ vertices. So, adopting the convention that $\pm t_1$ means $+t_1$ if $t_2 \geq t_1$ and $-t_1$ otherwise, we can embed, for any not too large h'', a total of $6(t \cdot |t_2 - t_1| \pm t_1)m_3h''$ trees of type $\bar{T}$ into the edges from (M3), placing exactly $t \cdot (t \cdot |t_2 - t_1| \pm t_1)h''$ vertices into each of the corresponding clusters Q.

Recalling the earlier observations on embedding trees of type $\bar{T}$ into edges from (M1) and (M2), we conclude that we are able to embed

$$\begin{aligned} d_{\bar{T}} := {} & |t_1 - t_2 - 1| \cdot (2m_1 + 4m_2)(t \cdot |t_2 - t_1| \pm t_1) \\ & + 6(t \cdot |t_2 - t_1| \pm t_1)m_3 \cdot |t_1 - t_2 - 1| \\ = {} & |t_1 - t_2 - 1| \cdot (t \cdot |t_2 - t_1| \pm t_1) \cdot (2m_1 + 4m_2 + 6m_3) \end{aligned}$$

trees of the current type $\bar{T}$, using all sets $Q \in \mathcal{Q}$ in a completely balanced way (each receives exactly $t \cdot (t \cdot |t_2 - t_1| \pm t_1) \cdot |t_1 - t_2 - 1|$ vertices).

Now, we embed the first $d_{\bar{T}}$ trees of type $\bar{T}$ in this way, then proceed to embed the next $d_{\bar{T}}$ trees of this type, then the next $d_{\bar{T}}$ such trees, and so on. If at some point (this might happen in the first round already), there are less than $d_{\bar{T}}$ trees of type $\bar{T}$ left, then we perform a last round for embedding these trees, simultaneously blocking at most $d_{\bar{T}} \cdot t$ vertices which we add to U' (note that they have not been used for the embedding). In this way, we can finish the embedding of all the trees of the current type $\bar{T}$ while perfectly balancing $Q \cap (U \cup U')$ for all clusters $Q \in \mathcal{Q}$. In particular, each of the clusters has exactly the same number of vertices in $Q_{F_1} \setminus (U \cup U')$.

Note that, when working on one t-vertex tree $\bar{T} \in T_{Unbal}$, the number of vertices we add to U' although they are not actually used for the embedding is at most

$$d_{\bar{T}} t \leq 2t^4 \cdot p' \leq \frac{2}{\beta^4} \cdot \frac{p}{\varepsilon} \leq \frac{1}{50\beta^5} \leq \frac{1}{\beta^5},$$

where we used that each tree in $T_{Unbal} \subseteq F_1$ has at most $\frac{1}{\beta}$ vertices. So, the number of vertices we add to U' after working on all trees from T_{Unbal} is at most the number of types of trees multiplied by $\frac{1}{\beta^5}$, and thus, by (20), at most $\frac{1}{\beta^{10}}$.

Phase 3. In this phase, we embed the trees from $T_{NearBal}$. Each of these trees has (at least) one leaf in its heavier colour class. Instead of the root, as in phase 2, we will now put this leaf into a different cluster, and instead of the N-out-good path partition we will be using the N-in-good path partition $\mathcal{P}_A$.

Again we go through the different types $\bar{T}$ of trees, of which there are at most $\frac{1}{2\beta}$. Now say we are working on the trees of a fixed type $\bar{T}$, with t vertices. Easier considerations than in the previous case show that we can embed exactly t vertices into each of the slices Q_{F_1} of clusters $Q \in V(M)$ if we embed six trees of type $\bar{T}$ into the six clusters corresponding to a path of length 6, four trees of type $\bar{T}$ into the six clusters corresponding to a path of length 4, and two trees of type $\bar{T}$ into the six clusters corresponding to a path of length 2. So, putting at most $\frac{1}{2\beta} \cdot \frac{1}{\beta} \cdot p' \leq \frac{1}{\beta^3}$ unused vertices into U', we can finish the embedding of all the trees of $T_{NearBal}$ balancing all slices as desired. In particular, each of the clusters $Q \in \mathcal{Q}$ has exactly the same number of vertices in $Q_{F_1} \setminus (U \cup U')$.

After finishing Phase 3, we still put some more vertices into U', before declaring the embedding procedure of the trees in $\mathcal{T}_s \cap F_1$ finished. Namely, we put an appropiate number of vertices from the sets in X into U'. That is, the number of vertices from any of the sets of X we add to U' is the same as the number of vertices from any of the sets $Q \in \mathcal{Q}$ that went to U or to U' during the embedding of $\mathcal{T}_s \cap F_1$. This cleaning-up is only done because it will be nicer to be able to start the embedding of the trees at the next seed with all slices C_{F_1} perfectly balanced.

Observe that the number of vertices we added to U' while dealing with the trees from $\mathcal{T}_s \cap F_1$ is at most[4]

$$u'_s \ \leq \ \frac{1}{\beta} + \frac{1}{\beta^{10}} + \frac{6}{\beta^3} + |X| \cdot \frac{2|\mathcal{T}_s \cap F_1|}{p'} \ \leq \ \frac{3}{\beta^{10}} + 600\varepsilon \cdot |\mathcal{T}_s \cap F_1|,$$

where we used (6) for the last inequality. Hence the bound (15) we had claimed above is correct. This ensures we have enough space for all future trees from F_1.

6.4 Embedding the leaves

This section is devoted to the embedding of the leaves. That is, we are now at a stage where we have sucessfully embedded all seeds and all small trees, and all that is left to embed is L, the set of leaves adjacent to seeds. We will show we can embed all of L at once.

If we cannot embed L into Z, then by Hall's theorem[5], there is some subset $K \subseteq \tilde{W}$ such that

$$|N(\varphi(K)) \cap Z| < |L_K|, \tag{21}$$

where L_K is the set of leaves adjacent to elements of K, the set $\varphi(K)$ is the set of images of K, and $N(\varphi(K)) \cap Z$ is the union of the neighbours in Z of the elements of $\varphi(K)$.

Recall that by (E) from Subsection 6.2.3, we chose as the image of a seed s a vertex $\varphi(s)$ that is typical with respect to C_L for almost all clusters C of R_G. Because of (10), this means that

$$\text{each element of } \varphi(K) \text{ sees at least } (\frac{2}{3} - 20\varepsilon)|Z| \text{ vertices of } Z. \tag{22}$$

[4]Note that we did not add $\bigcup_{C \in V(R_G)} (Q_0^C \cup Q_1^C)$ to U'.

[5]Hall's theorem can be found in any standard textbook, it states that a bipartite graph with bipartition classes A and B either has a matching covering all of A, or there is an 'obstruction': a set $A' \subseteq A$ such that $|N(A')| < |A'|$.

In particular, by (11), each element of $\varphi(K)$ sees more than $\frac{5}{8}(|L| + \frac{9}{10}\alpha m)$ vertices of Z. Thus, we may assume that

$$|L| > \frac{3\alpha m}{2}, \tag{23}$$

as otherwise $\frac{5}{8}(|L| + \frac{9}{10}\alpha m) \geq |L|$, which means that we could have embedded L without a problem.

Our aim is to reach a contradiction to the assumption that the set K exists. We will reach this contradiction by proving in Claims 6.2–6.6 that K misses a vertex in each of the large groups and also in most of the small groups of seeds we defined in Subsection 5.2.1. In some of the small groups K actually misses more than one vertex. We will prove these claims by repeatedly using (21). This means that in total, K misses many vertices from $\tilde{W}$, and these vertices spread out among the blocks (and thus have a corresponding proportion of the leaves hanging from them). Therefore, we can conclude that $|L_K|$ is smaller than the bound for the neighbourhood of $\varphi(K)$ given in (22), and thus, L_K could have been embedded without a problem, which is a contradiction.

Let us make this outline more precise. We start by proving that each of the large groups has a vertex outside K.

Claim 6.2. *No large group is completely contained in K.*

Proof. Assume otherwise, and consider the largest j for which there is a group of size $47 \cdot 2^j$ completely contained in K. Then by Claim 6.1 (iv) from Subsection 6.2.4, we know that

$$|N(\varphi(K)) \cap Z| \geq (1 - \frac{1}{95 \cdot 2^j} - \varepsilon^{\frac{1}{4}})|Z|. \tag{24}$$

If $j \geq j^{\circledast} := \lceil \log \frac{1}{95 \cdot \frac{999}{1000}\alpha} \rceil$, then by (11), the bound from (24) exceeds $|L|$, which yields a contradiction to (21). So

$$j < j^{\circledast}. \tag{25}$$

In particular, because of (7) and (8), we know that $j < j^*$, and hence, there exists a large group of size $47 \cdot 2^{j+1} = 94 \cdot 2^j$. For any such group B, we know that, by the choice of j, there is a vertex $v_B \in B$ that is not in K. Let L_B be the set of leaves adjacent to seeds in B, and let $\ell_{\max}(B)$ and

$\ell_{\min}(B)$ be the number of leaves adjacent to the first and the last seed in B, respectively, under the size order σ. Thus, every seed $b \in B$ is adjacent to a number ℓ_b of leaves, with $\ell_{\min}(B) \le \ell_b \le \ell_{\max}(B)$.

Set

$$\mathit{dif}(B) := \ell_{\max}(B) - \ell_{\min}(B).$$

Then

$$\begin{aligned}\ell_{v_B} \ge \ell_{\min}(B) &\ge \frac{|L_B|}{|B|} - \mathit{dif}(B) \cdot \frac{|B|-1}{|B|} \\ &= \frac{|L_B|}{94 \cdot 2^j} - \mathit{dif}(B) \cdot (1 - \frac{1}{94 \cdot 2^j}).\end{aligned}$$

Since the groups are consecutive in the size order σ, and since no seed has more than αm leaves adjacent to it, we know that

$$\sum_{B\colon\ |B| = 94 \cdot 2^j} \mathit{dif}(B) \le \alpha m.$$

So, the number of leaves adjacent to seeds that are not in K can be bounded by calculating

$$\begin{aligned}|L| - |L_K| &\ge \sum_{B\colon\ |B| = 94 \cdot 2^j} \ell_{v_B} \\ &\ge \frac{|L|}{94 \cdot 2^j} - \sum_{B\colon\ |B| = 94 \cdot 2^j} \mathit{dif}(B) \cdot (1 - \frac{1}{94 \cdot 2^j}) \\ &\ge \frac{|L|}{94 \cdot 2^j} - \alpha m \cdot (1 - \frac{1}{94 \cdot 2^j}).\end{aligned}$$

Therefore,

$$\begin{aligned}|L_K| &\le (1 - \frac{1}{94 \cdot 2^j}) \cdot (|L| + \alpha m) \\ &\le (1 - \frac{1}{94 \cdot 2^j}) \cdot (|Z| + \frac{\alpha^4}{10^6} m) \\ &\le (1 - \frac{1}{95 \cdot 2^j} - \varepsilon^{\frac{1}{4}})|Z| \\ &\le |N(\varphi(K)) \cap Z|, \end{aligned} \tag{26}$$

where the second and last inequalities follow from (11) and (24), respectively, and the third inequality follows from the observation that

$$\frac{\alpha^4}{10^6}m \;\le\; \frac{2}{3}\cdot\frac{\alpha^3}{10^6}|Z| \;\le\; \Big(\frac{1}{94\cdot 95\cdot 2^j} - \varepsilon^{\frac{1}{4}}\Big)|Z|,$$

where for the first inequality we used that $|Z| \ge |L| > \frac{3\alpha m}{2}$ (by (23)), and the second inequality follows from the facts that $\varepsilon \le \frac{\alpha^4}{10^{18}}$ (by (5)) and $j \le j^{\circledast}$ (by (25)).

Now, inequality (26) gives a contradiction to (21). This proves Claim 6.2. □

Next, we show a similar fact for all small groups of size five.

Claim 6.3. *No small group of size* 5 *is completely contained in* K*.*

Proof. Indeed, otherwise, because of Claim 6.1 (i), we know that

$$|N(\varphi(K)) \cap Z| \ge \Big(\frac{47}{48} - \varepsilon^{\frac{1}{4}}\Big)|Z|. \tag{27}$$

Moreover, Claim 6.2 implies that every large group B of size 47 has a vertex v_B which is not in K, and thus we can calculate, similar as above for Claim 6.2, that

$$|L| - |L_K| \;\ge \sum_{B:\ |B|=47} \ell_{v_B} \;\ge\; \frac{|L|}{47} - \alpha m\cdot\Big(1-\frac{1}{47}\Big),$$

and thus, employing (11), we find that

$$|L_K| \le \Big(1-\frac{1}{47}\Big)\cdot\Big(|Z| + \frac{\alpha^4}{10^6}m\Big),$$

which, with the help of (23) and (27), and using the fact that $\alpha \gg \varepsilon$, yields a contradiction to (21). This proves Claim 6.3. □

Next, we turn to the groups of size four, separating the treatment of these into two cases depending on their type.

Claim 6.4. *No small group of size* 4 *and of type 1 is completely contained in* K*.*

Proof. Otherwise, because of Claim 6.1 (ii), we know that

$$|N(\varphi(K)) \cap Z| \geq (\frac{23}{24} - \varepsilon^{\frac{1}{4}})|Z|. \tag{28}$$

By Claim 6.3 we know that every group of size five has a vertex which is not in K. Hence, every large group B of size 47 contains at least two vertices v_B^1 and v_B^2 that are not in K. Moreover, by the definition of the small groups, the first of these vertices, v_B^1, is one of the first 23 vertices of B under the size order, and the second vertex v_B^2, is one of the next 23 vertices of B under the size order.

So, we can split the group B minus its last vertex into two groups B_1, B_2 containing the first 23 and the next 23 consecutive elements in the size order, respectively, with $v_B^1 \in B_1$ and $v_B^2 \in B_2$. Defining $\mathit{dif}(B_1)$, $\mathit{dif}(B_2)$ as in Claim 6.2 for each of these two subgroups B_1, B_2 of the group B of size 47, and letting $\ell_{v_B^i}$ denote the number of leaves at v_B^i, for $i = 1, 2$, we can calculate that

$$\ell_{v_B^1} \geq \frac{|L_{B_1}|}{23} - \frac{22}{23} \cdot \mathit{dif}(B_1),$$

and

$$\ell_{v_B^2} \geq \frac{|L_{B_2}|}{23} - \frac{22}{23} \cdot \mathit{dif}(B_2),$$

where L_{B_1} and L_{B_2} denote the sets of leaves adjacent to vertices from B_1 and B_2, repectively. Thus, letting L_{B_3} denote the set of leaves adjacent to the very last vertex of the group B (of size 47), and noticing that $|L_{B_3}| \leq \frac{|L_B|}{47}$, we can calculate that

$$\begin{aligned} |L| - |L_K| &\geq \sum_{B\colon\ |B|=47} (\ell_{v_B^1} + \ell_{v_B^2}) \\ &\geq \frac{|L| - \sum_{B\colon\ |B|=47} |L_{B_3}|}{23} - \frac{22}{23} \cdot \sum_{B\colon\ |B|=47} (\mathit{dif}(B_1) + \mathit{dif}(B_2)) \\ &\geq \frac{|L|}{23} - \frac{|L|}{23 \cdot 47} - \frac{22}{23}\alpha m. \end{aligned}$$

Therefore, using (11) and (28), we obtain that

$$\begin{aligned}|L_K| &\le \frac{22}{23}(|L| + \alpha m) \cdot (1 + \frac{1}{22 \cdot 47}) \\ &\le \frac{22}{23}(|Z| + \frac{\alpha^4}{10^6} m) \cdot (1 + \frac{1}{22 \cdot 47}) \\ &\le (\frac{23}{24} - \varepsilon^{\frac{1}{4}})|Z| \\ &\le |N(\varphi(K)) \cap Z|,\end{aligned}$$

a contradiction to (21). This proves Claim 6.4. □

Claim 6.5. *No small group of size* 4 *and of type 2 is completely contained in* K*.*

Proof. Otherwise, because of Claim 6.1 (iii), we know that

$$|N(\varphi(K)) \cap Z| \ge (\frac{11}{12} - \varepsilon^{\frac{1}{4}})|Z|. \tag{29}$$

However, by Claims 6.3 and 6.4, we know that every small group of size 5 and every small group of size 4 and of type 1 has a vertex which is not in K. So, we can split every large group B of size 47 into five subgroups B_1, B_2, B_3, B_4, B_5, each consecutive in the size order, and with $|B_i| = 11$ for $i = 1, 2, 3, 4$ and $|B_5| = 3$, such that each of B_1, B_2, B_3, B_4 contains a vertex $v_B^1, v_B^2, v_B^3, v_B^4 \notin K$.

Similar as above, we can calculate that

$$\begin{aligned}|L| - |L_K| &\ge \sum_{B\colon\ |B| = 47} (\ell_{v_B^1} + \ell_{v_B^2} + \ell_{v_B^3} + \ell_{v_B^4}) \\ &\ge \frac{|L|}{11} - \frac{3|L|}{11 \cdot 47} - \frac{10}{11}\alpha m,\end{aligned}$$

where numbers $\ell_{v_B^i}$ are defined as in the previous claim. Now, using (11) and (29), we obtain that

$$\begin{aligned}|L_K| &\le \frac{10}{11}(|L| + \alpha m) \cdot (1 + \frac{1}{10 \cdot 15}) \\ &\le (\frac{11}{12} - \varepsilon^{\frac{1}{4}})|Z| \\ &\le |N(\varphi(K)) \cap Z|,\end{aligned}$$

a contradiction to (21). This proves Claim 6.5. □

Next, we will show that we can actually get some more out of the groups of type 2.

Claim 6.6. *No small group of size* 4 *and of type 2 has three or more vertices in* K.

Proof. Indeed, otherwise, because of the second part of Claim 6.1 (iii), we know that

$$|N(\varphi(K)) \cap Z| \geq (\frac{5}{6} - \varepsilon^{\frac{1}{4}})|Z|. \tag{30}$$

By Claims 6.3, 6.4 and 6.5, every small group B, except possibly those of size one, has a vertex v_B which is not in K. So, similar as in the previous claims, but now going over all groups of sizes 4 and 5, and temporarily considering the groups of size 1 to form part of the previous group (which had size 4, and so now has size 5), we calculate that

$$\begin{aligned} |L| - |L_K| \geq \sum_{B:\ 4 \leq |B| \leq 5} \ell_{v_b} &\geq \sum_{B:\ 4 \leq |B| \leq 5} (\frac{|L_B|}{|B|} - \mathit{dif}(B) \cdot \frac{|B|-1}{|B|}) \\ &\geq \min\{\frac{1}{4}, \frac{1}{5}\} \cdot |L| - \max\{\frac{3}{4}, \frac{4}{5}\} \cdot \alpha m \\ &\geq \frac{|L|}{5} - \frac{4}{5}\alpha m, \end{aligned}$$

where L_B is the set of leaves at B, and $\mathit{dif}(B)$ is the difference between the biggest number of leaves at a vertex in B and the smallest such number. We then use (11) and (30), to obtain that

$$|L_K| \leq \frac{4}{5}(|L| + \alpha m) \leq (\frac{5}{6} - \varepsilon^{\frac{1}{4}})|Z| \leq |N(\varphi(K)) \cap Z|,$$

a contradiction to (21). This proves Claim 6.6. □

Resumingly, Claims 6.3–6.6 tell us that K misses at least one vertex of each small group of size four or five, and misses at least two vertices from each small group of size four and type 2. Recalling our ordering $4, \mathbf{4}, 4, 4, 5, 4, \mathbf{4}, 4, 4, 5, 4, 1$ of the small groups inside each group B of size 47 as given in (4) in Subsection 5.2 (under ordering σ), we see that we can split B into five groups B_1, B_2, B_3, B_4, B_5 such that

- for $i = 1, 3$, the group B_i has 8 vertices, at least 3 of which are not in K;

- for $i = 2, 4$, the group B_i has 13 vertices, at least 5 of which are not in K; and
- B_5 has 5 vertices, at least two of which are not in K.

Therefore, similar calculations as for the previous claim give that

$$\begin{aligned} |L| - |L_K| &\geq \min\{\frac{3}{8}, \frac{5}{13}, \frac{2}{5}\} \cdot |L| - \max\{\frac{5}{8}, \frac{8}{13}, \frac{3}{5}\} \cdot \alpha m \\ &\geq \frac{3|L|}{8} - \frac{5}{8}\alpha m, \end{aligned}$$

and thus by (11), we get

$$|L_K| \leq \frac{5}{8}\big(|L| + \alpha m\big) < \big(\frac{2}{3} - 20\varepsilon\big)|Z|,$$

a contradiction to (22). This means the Hall-obstruction K cannot exist, and we can thus finish the embedding of T by embedding all leaves from L in one step. This finishes the proof of Lemma 2.1.

7 Extending a given embedding

For the companion paper [RS19b], which contains the proof of the exact version of Theorem 1.4, we will not only need Lemma 2.1, but also a second result, namely Lemma 7.3, the main result of this section. Both lemmas are very similar. The difference is that in the context of [RS19b], a small tree T^* is already embedded, except for a small set $Y \subseteq V(T^*)$. As images of neighbours of Y are well chosen, we will later be able to absorb Y, i.e., we choose a suitable set $S \subseteq V(G)$ somewhat smaller than Y, embed $T-(T^*-Y)$ into $G - S$, and then complete the embedding by using the leftover plus S for the embedding of Y. So, in Lemma 7.3, we wish to embed $T - T^*$, and just as in Lemma 2.1 we have some extra free space, but now we have to cope with the already embedded $T^* - Y$, which may block neighbourhoods. However, as we will see below, our proof of Lemma 2.1 can be adapted to the new setting, with two possible exceptions. First, if G is γ-special (see Definition 7.1 below), our embedding scheme will fail, because the matchings we need for the embedding of F_1 might not exist. In [RS19b] we show how to embed T in that case. Second, if T has a specific shape, and G is close to containing a complete tripartite graph, we may not be able to find the in-good path partition needed for embedding $T_{NearBal}$. This case is covered by Lemma 7.4 below, which shows that then we can embed all of T.

Another important difference to Lemma 2.1 is that in Lemma 7.3, we know that no seed of T has many leaves hanging from it. So, we can forget about all the extra work that was done in the proof of Lemma 2.1 to embed the set L of leaves hanging from seeds.

Let us now give the two definitions we need to state Lemma 7.3.

Definition 7.1. *We say a graph G on $m+1$ vertices is γ-special, for some $\gamma > 0$, if $V(G)$ consists of three mutually disjoint sets X_1, X_2, X_3 such that*

(i) $\frac{m}{3} - 3\gamma m \leq |X_i| \leq \frac{m}{3} + 3\gamma m$ *for each* $i = 1, 2, 3$*; and*

(ii) there are at most $\gamma^{10}|X_1| \cdot |X_2|$ edges between X_1 and X_2.

Definition 7.2. *Let T be a tree with m edges. Call a subtree T^* of T with root t^* a γ-nice subtree if $|T^*| < \gamma m$ and every component of $T - T^*$ is adjacent to t^*.*

We are now ready for the result we will need in [RS19b].

Lemma 7.3. *For all $\gamma < \frac{1}{10^6}$ there are $m_0 \in \mathbb{N}$ and $\lambda > 0$ such that the following holds for all $m \geq m_0$.*
Let G be an $(m+1)$-vertex graph of minimum degree at least $\lfloor \frac{2m}{3} \rfloor$ and with a universal vertex, such that G is not γ-special. Let T be a tree with m edges such that $T \not\subseteq G$ and no vertex in T is adjacent to more than λm leaves. Let T^* be a γ-nice subtree of T with root t^*, let $Y \subseteq V(T^*) \setminus \{t^*\}$, and let $S \subseteq V(G)$ with $|S| \leq |Y| - (\frac{\gamma}{2})^4 m$.
If, for any $W \subseteq V(G) - S$ with $|W| \geq \gamma m$, there is an embedding of $T^* - Y$ into $G - S$, with t^* embedded in W, then there is an embedding of $T - Y$ into $G - S$ extending the given embedding of $T^* - Y$.

In the proof of Lemma 7.3, we will need the following lemma. Call a rooted tree *bad* if it is a three-vertex path whose root is not the middle vertex.

Lemma 7.4. *Let H be a graph on $m+1$ vertices with universal vertex w. Let $H_1 \cup \ldots \cup H_5$ be a partition of $V(H)$, with $|H_1| = |H_2| = |H_3| \geq \frac{33}{100}m$ and $|H_5| \leq \frac{1}{2}|H_4|$. Assume that for all $i, j \in \{1, 2, 3\}$ with $i \neq j$, each vertex of H_i is adjacent to at least $\frac{99}{100}$ of the vertices in H_j, and for $i = 4, 5$, each vertex of H_i is adjacent to at least two fifths of the vertices of H_{i-2}.*
Let T be a tree with m edges, and let W, L, F_1, F_2 be sets as in Lemma 5.1

for some β with $\beta^2 m > 250$, except that we do not require the upper bound in Lemma 5.1 (f). *If F_1 contains at least $\frac{33}{100}m$ bad trees, then $T \subseteq G$.*

We leave the proof of Lemma 7.4 to the end of the section, and first prove Lemma 7.3, mainly following the lines of the proof of Lemma 2.1.

Proof of Lemma 7.3. We structure the proof according to the main steps.

Setting the constants. Given γ, we set $\alpha := (\frac{\gamma}{2})^4$; this will be our approximation factor for the embedding of $T - Y$. We choose ε such that

$$\varepsilon \leq \gamma^{20}. \tag{31}$$

Apply Lemma 4.4 (the regularity lemma) to ε^2 and $M_0 := \frac{1}{\varepsilon^2}$ obtaining numbers M_1 and n_0. We choose $\beta \ll \varepsilon$, and $\lambda \leq \frac{\beta^2 \cdot \varepsilon}{3000}$. Finally, choose a sufficiently large number m_0 for the output of Lemma 7.3. Resumingly, we have

$$\frac{1}{m_0} \ll \lambda \ll \beta \ll \varepsilon \ll \alpha \ll \gamma. \tag{32}$$

Holes. Call a subset of $V(G)$ containing at least $\frac{m}{3} - 7\gamma m$ vertices a *hole* if it induces less than $100\varepsilon m^2$ edges. Let V_{Bad} be the set of all vertices in $V(G)$ whose non-neighbourhood contains a hole. For now, suppose that

$$|V_{Bad}| \leq \frac{m}{2}. \tag{33}$$

(The other case will be treated at the end.) Now, assume we are given a graph G as in the lemma, a set S, a tree T and a subtree T^* of T, with

$$|V(T^*)| < \gamma m, \tag{34}$$

a vertex $t^* \in V(T^*)$, a set $Y \subseteq V(T^*) \setminus \{t^*\}$, and an embedding φ of $T^* - Y$ into $G - S$. By (33), we may assume that

$$\varphi(t^*) \notin V_{Bad}. \tag{35}$$

Regularising the host graph. We take an ε^2-regular partition of $G' := G \setminus (\varphi(V(T^* - Y) \cup S)$, with a reduced graph $R_{G'}$ on $M_0 \leq p' \leq M_1$ vertices. We wish to extend the embedding of $T^* - Y$ to an embedding of all of $T - Y$ into $G - S$. Note that the minimum degree of $R_{G'}$ is no longer bounded from below by $(\frac{2}{3} - 13\varepsilon)p$, as in the proof of Lemma 2.1, because of the possible degree into the set $\varphi(V(T^* - Y))$. But we can guarantee the following bound:

$$\delta_w(R_{G'}) \geq (\frac{2}{3} - \gamma + (\frac{\gamma}{2})^4 - 13\varepsilon)p' \geq (\frac{2}{3} - \gamma + \frac{\gamma^4}{20})p'. \tag{36}$$

Cutting the tree. We use Lemma 5.1 to cut up the tree induced by $V(T - T^*) \cup \{t^*\}$, making t^* a seed. Add all neighbours of t^* belonging to a tree from F_2 to the set W of seeds. Use Lemma 5.1 (f), (g) and (h) to see that there are at most $2\beta(m - |V(T^*)|)$ new seeds (with exactly the same argument as the one used to prove Lemma 5.1 (i)). Note that the new seeds transform the partition of the tree a little, as any new seed cuts the tree from F_2 it belonged to. We just add the newly formed small trees to L, F_1, $F_2 \setminus F_2'$ or F_2', as appropriate, and, slightly abusing notation, continue to call these sets L, F_1, $F_2 \setminus F_2'$ or F_2'. Let W^* be the set of all seeds adajcent to t^*. It will not be necessary to add any more extra seeds, as we did in the proof of Lemma 2.1, so the total number of seeds is bounded by $\frac{3}{\beta^2}$.

Embedding leaves at t^*, and reserving for W^*. First embed the leaves at t^*, into any cluster, using the minimum degree of G. As t^* has at most $\lambda m \ll \varepsilon m$ leaves hanging from it, this will not disturb the rest of the embedding process. Let L' denote the set of the remaining leaves from L.

Next, we choose a cluster C^* such that at least a third of its vertices are neighbours of $\varphi(t^*)$. We reserve a set $C^*_W \subseteq C^*$ of size $\varepsilon^{\frac{1}{3}} m$ consisting of neighbours of $\varphi(t^*)$ in C^*. This reservation ensures that we will not block the neighbourhood of $\varphi(t^*)$ before embedding W^*.

Embedding the trees from F_1^*. Now we embed the trees from $F_1^* := F_1 \cap \mathcal{T}_{t^*}$. We provisionally slice up each of the clusters of $R_{G'}$ into $\frac{1}{\varepsilon}$ smaller sets (slices) of equal sizes (plus a very small garbage set), in a way that at most one of the new slices contains both neighbours and non-neighbours of $\varphi(t^*)$. Let $\mathcal{S}$ be the set of all these slices except for the garbage set and the mixed slice, and let $R'_{G'}$ be the reduced graph on $\mathcal{S}$. Say $|\mathcal{S}| = p''$. Note that $R'_{G'}$ is still regular (with a slightly worse approximation), and in $R'_{G'}$, the minimum degree bound from (36) becomes

$$\delta_w(R'_{G'}) \geq (\frac{2}{3} - \gamma)p''. \tag{37}$$

Consider a set $N \subseteq \mathcal{S}$ of size $\lfloor (\frac{2}{3} - \gamma)p'' \rfloor$ such that $\varphi(t^*)$ is adjacent to all vertices in all clusters of N. We will now find a matching M^* and path partitions $\mathcal{P}^*_A$, $\mathcal{P}^*_B$, or slight variations thereof, as in the proof of Lemma 2.1, where we employed Lemmas 4.2 and 4.3.

Let us start with the matching M^* from Lemma 4.2. While the conditions of the lemma are still satisfied if we take ξ of the order of γ, we would like to have an outcome with ξ having the order of ε. The only possible reason that could prevent us from finding a set Y of order around $500\varepsilon p''$ and an N-good perfect matching M^* of $R'_{G'} - Y$ as in Lemma 4.2 is that the first line of the proof of Lemma 4.2, where we greedily match using the minimum degree condition, fails in our new circumstances (for the rest of the argument in that proof we only need a much weaker minimum degree condition). So, if we cannot find M^*, then each matching from $V(R'_{G'}) \setminus N$ to N leaves more than $\lfloor 500\varepsilon p'' \rfloor$ vertices from $V(R'_{G'}) \setminus N$ uncovered. Thus there is a Hall-obstruction, i.e., a set $X'_1 \subseteq V(R'_{G'}) \setminus N$ with less than $|X'_1| - \lfloor 500\varepsilon p'' \rfloor$ neighbours in N. Now, apply (37) to any vertex from X'_1, and set $X_3 := N(X'_1) \cap N$ and $X_1 := V(R'_{G'}) \setminus N$ to see that

$$|X_1 \cup X_3| \geq \left(\frac{2}{3} - \gamma\right) p''. \tag{38}$$

Moreover, as $|X_1| = p'' - |N|$, we see that

$$\left(\frac{1}{3} + \gamma\right) p'' \geq |X_1| \geq |X'_1| \geq \left(\frac{1}{3} - \frac{\gamma}{2} + 250\varepsilon\right) p'' \geq \left(\frac{1}{3} - \frac{\gamma}{2}\right) p'', \tag{39}$$

where for the third inequality we use (38) and the fact that $|X_3| < |X'_1| - \lfloor 500\varepsilon p'' \rfloor$. Thus, we can bound the size of X_3 as follows:

$$\left(\frac{1}{3} + \gamma\right) p'' \geq |X_1| > |X_3| = |X_1 \cup X_3| - |X_1| \geq \left(\frac{1}{3} - 2\gamma\right) p''. \tag{40}$$

Letting X_2 denote the non-neighbours of X'_1 in N, we obtain from (37) in a similar way as for X_1 that $|X_2 \cup X_3| \geq (\frac{2}{3} - \gamma)p''$, and therefore, using (40) for a bound on $|X_3|$, we obtain that

$$|X_2| = |X_2 \cup X_3| - |X_3| \geq \left(\frac{1}{3} - 2\gamma\right) p''. \tag{41}$$

Also, by (38), we have that

$$|X_2| \leq \left(\frac{1}{3} + \gamma\right) p''. \tag{42}$$

But then G is γ-special. Indeed, (34), (39), (40), (41) and (42) imply that Definition 7.1 (i) holds for $\bigcup X_1$, $\bigcup X_2$, and $\bigcup X_3 \cup \varphi(V(T^*) - Q) \cup S \cup$

$(V(G') \setminus \bigcup_{S\in\mathcal{S}} V(S))$, while Definition 7.1 (ii) holds by the definition of X_2, and since by (31), any non-edge in $R'_{G'}$ corresponds to a very sparse pair of clusters in G. However, G being γ-special is against the assumptions of Lemma 7.3. So, no Hall-type obstruction can exist, and we find the matching M^* as desired.

Next, we turn to the good path partitions $\mathcal{P}^*_Q$ and $\mathcal{P}^*_R$ from Lemma 4.3. Let us start with $\mathcal{P}^*_Q$. In the proof of Lemma 4.3, we constructed $\mathcal{P}^*_Q$ using M^* and an auxiliary matching M^Q, which, in turn, was obtained as the union of two matchings, the first of which is $\tilde{M}^Q$, a maximum matching inside $V(M^*)\setminus N$. We define $\tilde{M}^Q$ in the same way here. Note that $|V(\tilde{M}^Q)| \geq 6\gamma p''$, as otherwise, $(Q \setminus V(\tilde{M}^Q))$ is a hole, which is impossible by (35). In the proof of Lemma 4.3, $\tilde{M}^Q$ is completed to M^Q by matching clusters from $Q := V(M^*) \setminus (N \cup V(\tilde{M}^Q))$ to clusters in $N \setminus R$, where R are the clusters matched to Q by M^*. By the maximality of $\tilde{M}^Q$, Q is independent, and all but at most one of the vertices in Q see at most one of the endvertices of any edge in $\tilde{M}^Q$. So, these clusters see at least $|N \setminus R| - \frac{|V(\tilde{M}^Q)|}{2} \geq |Q \setminus V(\tilde{M}^Q)|$ clusters in $N \setminus R$, implying that we can match all but at most one vertex of Q in M^Q, and thus find an N-in-good path partition $\mathcal{P}^*_Q$, as desired.

Let us now turn to the N-out-good path partition $\mathcal{P}^*_R$ from Lemma 4.3. To find $\mathcal{P}^*_R$ we employed a maximum matching $\tilde{M}^R$ inside R. If $|V(\tilde{M}^R)| > 6\gamma p''$, we can proceed as above to find $\mathcal{P}^*_R$. Otherwise, it is not hard to see that we can match all except at most $6\gamma p''$ clusters of R to $N \setminus R$, which leads to almost all thus obtained paths having 6 vertices. We can then match the remaining $6\gamma p''$ clusters of R to the innermost vertices of these paths. In this way, we find a partition into (vertex-disjoint copies of) the three types of paths allowed in an N-out-good path partition, plus the following type of graph on 8 vertices: its vertices are $A, A', B, B', C, D, E, E', F, F'$, with $B, B', C, D, E, E' \in N$, and its edges are AB, $A'B'$, BC, $B'C$, CD, DE, DE', EF, $E'F'$. We will call this partition $\mathcal{P}^*_{mod}$.

We embed $\bigcup F_1^*$ using M^*, $\mathcal{P}^*_Q$ and $\mathcal{P}^*_{mod}$, all the time avoiding C^*_W. Note that when using $\mathcal{P}^*_{mod}$ we will have to adapt our strategy from Section 6.3.2, but this is not hard[6]. In order to leave all clusters balanced during the

[6]For any graph of the new type, consider embedding x trees of type $\bar{T}$ in the edge AB, putting the bipartition class $\bar{T}_1$ that contains the root $r_{\bar{T}}$ into B and the other class $\bar{T}_2$ into A. Then embed x trees similarly into each of $A'B'$, EF and $E'F'$. Next, do y rounds of embedding $\bar{T}$ with $r_{\bar{T}}$ going into C, $\bar{T}_2$ going to B, and $\bar{T}_1 - r_{\bar{T}}$ going to A, and y analogous rounds for each of the edges $A'B'$, EF, and $E'F$. Finally, embed $\bar{T}$ $2z$ times into CD, of which z rounds in each 'direction'. After these $4x + 4y + 2z$ rounds, we filled

embedding of $\bigcup F_1^*$, we again use a small set U' of pseudo-used vertices.

Slicing up the clusters. We now go back to work in $R_{G'}$. We slice up the yet unused parts of the clusters as before (but avoiding C_W^*), into sets C_L, C_W, $C_{\tilde{V}}$, $C_{F_1 \setminus F_1^*}$, and C_{F_2}. The slices C_X reflect the sizes of the corresponding sets X, but we leave sufficient buffer space in each.

We now go through the subtree induced by W and the non-trivial trees hanging from them in a connected way, starting with the root t^*. As before, we embed each seed together with all small trees from $F_1 \cup F_2$ hanging from it. We always avoid the set C_W^*, unless we are embedding a seed from W^*.

Embedding a seed s. We embed each seed s in a neighbour $\varphi(s)$ of the image $\varphi(p)$ of its parent p, with $\varphi(s) \notin V_{Bad}$ (this is possible by (33)), and such that $\varphi(s)$ is typical with respect to the slices C_L, C_W, $C_{\tilde{V}}$ and $C_{F_1 \setminus F_1^*}$. Usually, seeds go to C_W, but seeds from W^* go to their reserved space C_W^*. Note that we did not need to group and order our seeds as in the proof of Lemma 2.1, and we also do not need to choose the target clusters as carefully.

Embedding the trees from $F_2 \cup F_1$ at s. For the trees from F_2, we can find the matching M_{F_2} just as before, as the minimum degree bound (37) is sufficient. Again, we make the connections through the slices $C_{\tilde{V}}$, using the fact that the corresponding seeds are embedded in typical vertices (with respect to almost all slices $C_{\tilde{V}}$).

For the trees from F_1, we proceed as above for F_1^*, i.e we slice up the clusters, so that almost all of the obtained slices behave uniformly with repect to being adjacent to $\varphi(s)$. Let $R''_{G'}$ be the graph on these slices, after momentarily discarding the mixed slice and the garbage slice. Since G is not γ-special, and since we embedded s outside any hole, we can proceed exactly as above (when we embedded the trees from F_1^*) to find a matching M, an in-good path partition and a modified out-good path partition in $R''_{G'}$.

---

each of A, A', F, F' with $xt_2 + y(t_1 - 1)$ vertices, each of B, B', E, E' with $xt_1 + yt_2$ vertices, and each of C, D with $2y + zt$ vertices, where $t_i = |\bar{T}_i|$ for $i = 1, 2$. So, taking $x = (t_1 + t_2) \cdot (t_2 - t_1 + 1)$, $y = (t_1 + t_2) \cdot (t_2 - t_1)$, and $z = (t_1 + t_2 - 2) \cdot (t_2 - t_1) + t_1$, we fill each of the eight clusters with exactly the same amount of vertices. Similarly as in Section 6.3.2, we can calculate how to fill all types of graphs from $\mathcal{P}^*_{mod}$ simultaneously with the same amount of vertices.

Embedding the leaves from L'. In (32), we chose λ such that

$$|L'| \leq \frac{3}{\beta^2} \cdot \lambda m \leq \frac{1}{1000}\varepsilon m.$$

Since there are about p' slices C_L, each much larger than $\varepsilon\frac{m}{p'}$, and since the embedded seeds (except possibly t^*) see about two thirds of almost all slices C_L, we can embed the leaves from L' greedily into $\bigcup_{C\in V(R_{G'})} C_L$. This finishes the proof of Lemma 7.3 for the case that (33) holds.

More holes. Now assume that (33) does not hold. We can delete a few vertices from a hole so that inside the remaining set, no vertex has degree more than $\sqrt{\varepsilon}m$. By deleting a few more vertices, we arrive at a hole of size exactly $\lceil(\frac{1}{3}-8\gamma)m\rceil$. Let $\mathcal{H}$ be the set of all holes obtained in this way. As G has minimum degree at least $\lfloor\frac{2m}{3}\rfloor$, for each $H \in \mathcal{H}$ and $x \in V(H)$ we have

$$\deg(x, V(G) \setminus V(H)) \geq |V(G) \setminus V(H)| - 9\gamma m. \tag{43}$$

In particular, since (33) is not true, $\mathcal{H}$ contains at least two holes, H_1 and H_2, and their intersection is empty.

By (43) it is easy to calculate that for $i = 1, 2$, all but at most $3\sqrt{\gamma}m$ vertices of $V(G) \setminus (H_1 \cup H_2)$ have degree at least $|H_i| - \sqrt{\gamma}m$ into H_i. We choose a set $H_3 \subseteq V(G) \setminus (H_1 \cup H_2)$ with $|H_3| = |H_1| = |H_2|$ and such that for each $x \in H_3$, and each $i \in \{1, 2\}$, $\deg(x, H_i) \geq |H_i| - \sqrt{\gamma}m$.

Set $H_0 := V(G) \setminus (H_1 \cup H_2 \cup H_3)$. As $\delta(G) \geq \lfloor\frac{2m}{3}\rfloor$, each vertex in H_0 sees at least two fifths the vertices of at least two of the sets H_1, H_2, H_3. So, there is an index $i \in \{1, 2, 3\}$, without loss of generality let us assume $i = 2$, such that at least two thirds of the vertices in H_0 are adjacent to at least two fifths of the vertices in H_2. In other words, we can split H_0 into two sets H_4, H_5 as in Lemma 7.4.

Let T_{Bad} be the set of all bad trees in F_1. If $|T_{Bad}| \geq \frac{33}{100}m$, we add the trees from T^*-t^* to L, T_{Bad}, $F_1 \setminus T_{Bad}$, and F_2, where we now allow that trees in F_2 have more than βm vertices. Forgetting about the embedding φ of T^*, we apply Lemma 7.4 to G, finding that $T \subseteq G$, contrary to the assumptions of Lemma 7.3. So,

$$|T_{Bad}| \leq \frac{33}{100}m.$$

Now, we let t^* be embedded into the hole H_1. We regularise $G - \varphi(T^* - Y)$, respecting the prepartition given by $H_1, H_2, V(G) \setminus (H_1 \cup H_2)$. In the reduced graph R_G on p clusters, we use (43) to find an edge X_1X_2 with $X_i \subseteq H_i$ for $i = 1, 2$, and $|X_2 \cap N(\varphi(t^*))| \geq \frac{|X_2|}{2}$. Using (43) again, we find a set $\mathcal{D}$ of disjoint triangles ABC with $A \subseteq H_1$, $B \subseteq H_2$ and $C \subseteq H_3$, such that for each $i = 1, 2$, two of the pairs (A, X_i), (B, X_i), (C, X_i) have large density (larger than $\frac{2}{3}$), and such that moreover, $\varphi(t^*)$ sees at least $\frac{2}{3}$ of the vertices in B and in C. We can choose $\mathcal{D}$ such that $|\mathcal{D}| = \lceil (\frac{1}{3} - 100\gamma)p \rceil$.

We now split the clusters from $V(R_G) \setminus \{X_1, X_2\}$ into slices. Each cluster belonging to a triangle from $\mathcal{D}$ is split into one slice D_1 of size $\frac{|T_{Bad}|}{|\mathcal{D}|} + 100\varepsilon m$, and one slice D_2 containing the remaining vertices. Doing this, we put as many neighbours of $\varphi(t^*)$ as possible into D_1. Let $\mathcal{S}$ be the set of all slices of type D_1 and note that together they are large enough to accommodate all trees from T_{Bad} (there is even some buffer space). Let $\mathcal{D}'$ denote the set of triangles in the reduced graph on $\mathcal{S}$ corresponding to triangles from $\mathcal{D}$. Now, take the set of all slices of type D_2, and all clusters in $V(R_G) \setminus (V(\mathcal{D}) \cup \{X_1, X_2\})$, and split each of these elements into slices of size $\varepsilon \frac{m}{p}$ (plus possibly one garbage set). Let $\mathcal{S}'$ be the set of all these slices, and denote by R'_G the reduced graph on $\mathcal{S}'$. By our bound on $|T_{Bad}|$, we know that

$$\delta(R'_G) \geq (\frac{2}{3} - 500\gamma)p', \tag{44}$$

where $p' = |\mathcal{S}'|$. Now, we embed the leaves at t^* as before, and embed the bad trees at t^* into $\bigcup \mathcal{S}$, using regularity inside the triangles, and filling the clusters from $\mathcal{S}'$ as evenly as possible. For this, consider a triangle $A'B'C' \in \mathcal{D}'$ and note that $\varphi(t^*)$ sees almost $\frac{2}{3}$ of the clusters B' and C'. So, if necessary we can fill almost all $A' \cup B' \cup C'$ with trees from T_{Bad}, by distributing their first vertices among B' and C'.

We then embed the trees from $F_1 \setminus T_{Bad}$ adjacent to t^*. For this, we temporarily splice up the clusters from $\mathcal{S}'$ into about $\frac{1}{\varepsilon}$ new slices, so that almost all new slices contain either only neighbours or only non-neighbours of $\varphi(t^*)$, disregarding the possible garbage slice and the possible mixed slice, as before. Call this new set of slices $\mathcal{S}'_{t^*}$. In the reduced graph on $\mathcal{S}'_{t^*}$, we find a matching M^* (for this, we use (44) and the fact that most of the vertices in the non-neighbourhood of $\varphi(t^*)$ lie in H_1, which means that they see almost all of $V(G) \setminus H_1$). We then find a modified out-path partition $\mathcal{P}_{mod}$ as before. We may be unable to find a in-good path partition $\mathcal{P}_Q$ as before (as now $\varphi(t^*)$ lies in a hole). But, in a similar way as we found $\mathcal{P}_{mod}$, we can find a modified partition $\mathcal{P}'_{mod}$, which allows for graphs of the three types from the definition of the in-good path partition, plus a new type of graph on eight vertices $A, A', B, B', C, D, E, E', F, F'$, with $A, A', C, D, F, F' \in N$, and edges AB, $A'B'$, BC, $B'C$, CD, DE, DE', EF, $E'F'$. Recall that the in-good path partition was only used for the trees from $T_{NearBal}$. It is not hard to see that all trees in $T_{NearBal} \setminus T_{Bad}$ can be embedded into the modified in-good path partition $\mathcal{P}'_{mod}$.[7] We fill all clusters from $\bigcup \mathcal{S}''$ evenly (again using a set U').

Next, we slice up the yet unused parts of the clusters as before into sets C_L, C_W, $C_{\tilde{V}}$, $C_{F_1 \setminus F_1^*}$, C_{F_2}, each of the size needed, plus buffer space. We go through the seeds s and the trees from $F_1 \cup F_2$ hanging from s. We embed s into a vertex $\varphi(s) \in X_i$, for some $i = 1, 2$, such that $\varphi(s)$ is typical with respect to all clusters from triangles $A'B'C' \in \mathcal{D}'$, with respect to all unused parts of these clusters, with respect to all unused parts of slices of clusters from $\mathcal{S}'$, and with respect to X_{3-i}. We also require that s is embedded into a neighbour of the image of its parent, which is not a problem, since all such parents are embedded into vertices having sufficient degree into $X_1 \cup X_2$.

We embed all trees from T_{Bad} hanging from s into $\bigcup \mathcal{S}$, using regularity and filling all clusters from $\mathcal{S}$ almost evenly. Note that $\varphi(s)$ is typical with respect to the unused part of at least two of the clusters of any triangle $A'B'C' \in \mathcal{D}'$, i.e. $\varphi(s)$ sees almost $\frac{2}{3}$ of these unused parts. So, as above for $\varphi(t^*)$, we can distribute the first vertices of the trees in T_{Bad} at s among the two clusters and thus fill up the triangle if necessary.

We then embed the trees from $F_1 \setminus T_{Bad}$ at s into $\bigcup \mathcal{S}'$, using the same strategy as before, i.e. temporarily splicing up the clusters from $\mathcal{S}'$ into about $\frac{1}{\varepsilon}$ new slices, so that almost all contain either only neighbours or only non-neighbours of $\varphi(s)$. Finding a matching M^* and modified path partitions as above, we can fill all clusters from $\bigcup \mathcal{S}''$ evenly. For the trees in F_2 hanging from s, we use a matching M_{F_2}, as before. If $\bar{T} \in F_2$ contains a parent p of a seed, we embed p into a suitable vertex $v \in C_{\tilde{V}}$ with many neighbours in $X_1 \cup X_2$. We finish by embedding the leaves at seeds as before. □

It remains to prove Lemma 7.4. For its proof it will be convenient to have the following auxiliary result at hand.

[7] In order to show this, let us just prove that any graph of the new type can be equally filled with a fixed type of tree $\bar{T} \in T_{NearBal} \setminus T_{Bad}$. For this, let t_1 denote the size of the larger partition class of $\bar{T}$. Note that $t_1 \geq 3$ and the other class has size $t_1 - 1$. Let $r_{\bar{T}}$ be the root of $\bar{T}$. Embed $2t_1 - 1$ times $r_{\bar{T}}$ into C, the smaller bipartition class into B, and the rest into A. Do the same three more times, replacing the triple C, B, A with C, B', A', with D, E, F, and with D, E', F'. Embed $t_1 - 3$ times the larger bipartition class into C, and the smaller class into D. Also embed $t_1 - 3$ copies of $\bar{T}$ the other way around into D and C. Then each of the eight clusters is filled with exactly $(2t_1 - 1)(t_1 - 1)$ vertices.

Lemma 7.5. *Let H be a graph with universal vertex w. Let $H_1, H_2, H_3 \subseteq V(H)$ be disjoint, with $|H_i| \geq \frac{3}{10}m$ and such that each vertex of H_i is adjacent to at least $\frac{9}{10}$ of the vertices in H_j, for $i = 1, 2, 3$ and $j \in \{1, 2, 3\} \setminus \{i\}$. Let T be a tree on $m+1$ vertices, with sets W, L, F_1, F_2 as in Lemma 5.1 for some β with $\beta^2 m > 600$, except that we do not require the upper bound in Lemma 5.1* (f). *If F_1 contains at least $\frac{33}{100}m$ bad trees, then there is a tree $T' \subseteq T$ with $|V(T')| \leq \frac{m}{50}$ and an embedding φ of T' into H such that*

(i) each component of $T - T'$ is a bad tree from F_1, and for at least $\frac{33}{400}m$ of these trees, their neighbour in $\varphi(T')$ is embedded in $H_2 \cup \{w\}$; and

(ii) $|\varphi(V(T')) \cap H_2| \geq |\varphi(V(T')) \cap H_i|$ for $i = 1, 3$.

Proof. Let T_{Bad} be the set of all bad trees in F_1 and let T'' be the union of W and all trees from $(L \cup F_1 \cup F_2) \setminus T_{Bad}$. Consider the bipartition $W_0 \cup W_1$ of W induced by the bipartition of T and say W_0 is adjacent to at least as many trees from T_{Bad} as W_1. So, there are at least $\frac{33}{200}m$ bad trees at W_0. Partition W_0 into three sets W_0', W_2', W_3' such that W_0' contains all seeds from W_0 not adjacent to any leaves, and such that $|(\ell_{W_3'}+|W_2'|)-(\ell_{W_2'}+|W_3'|)|$ is minimised, where we define ℓ_X as the number of leaves adjacent to the vertices of X, for any $X \subseteq W$. Say W_2' is adjacent to at least as many bad trees as W_3'. If $\ell_{W_3'} + |W_2'| \geq \ell_{W_2'} + |W_3'|$, choose s^* arbitrarily in W_3', and if $\ell_{W_3'} + |W_2'| < \ell_{W_2'} + |W_3'|$, there is at least one vertex in W_2' adjacent to more than one leaf, let s^* be such a vertex. Then, setting $W_2 := (W_0' \cup W_2') \setminus \{s^*\}$ and $W_3 := W_3' \setminus \{s^*\}$,

$$\text{there are at least } \frac{33}{400}m \text{ bad trees hanging from } W_2, \tag{45}$$

$$\text{every seed in } W_3 \text{ is adjacent to at least one leaf, and} \tag{46}$$

$$\ell_{W_3} + \ell_{\{s^*\}} + |W_2| \geq \ell_{W_2} + |W_3|. \tag{47}$$

(Indeed, (47) is obvious if $s^* \in W_3'$, and if $s^* \in W_2'$, note that by our choice of W_2' and W_3', we have $\ell_{W_3'} + \ell_{\{s^*\}} + |W_2' \setminus \{s^*\}| > \ell_{W_2'} - \ell_{\{s^*\}} + |W_3' \cup \{s^*\}|$ which implies (47).) Finally, for each $s \in W_1$ that is adjacent to a tree from T_{Bad}, choose one such tree, and let T_{W_1} be the set of all these trees. Let T' be the tree induced by $T'' \cup T_{W_1}$.

Now, proceed as follows for each $s \in W$, starting with any seed and then always choosing a seed whose parent is already embedded. If $s \in W_i$, then embed s into H_i, and embed s^* into w. In what follows we will often use the minimum degree between the sets H_i without explicitly mentioning it.

Let $\bar{T} \in T_{W_1} \cup (F_1 \setminus (T_{Bad}) \cup F_2$ be a tree from $T' - W$ hanging from s, with root $r_{\bar{T}}$. Our aim is to show that we can embed $\bar{T}$ such that

$$|\varphi(V(\bar{T})) \cap H_2| \geq \max\{|\varphi(V(\bar{T})) \cap H_3|, |\varphi(V(\bar{T}) \cup s) \cap H_1|\}. \tag{48}$$

This is easy if $\bar{T} \in T_{W_1}$, because then we can embed the first and the last vertex of $\bar{T}$ into H_2, and the middle vertex into H_1. So assume $\bar{T} \notin T_{W_1}$. Let $\bar{T}_1, \bar{T}_2$ be the partition of $\bar{T} - r_{\bar{T}}$ induced by the bipartition classes of $\bar{T}$, with $|\bar{T}_2| \geq |\bar{T}_1|$. If $|\bar{T}_2| = |\bar{T}_1|$, suppose $\bar{T}_1$ is the class not containing neighbours of $r_{\bar{T}}$, and let $v \in \bar{T}_1 - r_{\bar{T}}$ (such a vertex exists as in this case $|\bar{T}_1| \geq 2$ since $\bar{T} \notin T_{Bad} \cup L$). If $\bar{T}$ contains a parent p of a seed, we can choose $v \neq p$.

Let us first find an adequate set H_i for $r_{\bar{T}}$. For this, if $s \in W_j$ for $j \in \{1, 3\}$, then set $i := 4 - j$, and if $s \in W_2$, then choose any $i \in \{1, 3\}$. If $\bar{T}$ contains no parents of seeds, we can embed $r_{\bar{T}}$ into H_i, embed $\bar{T}_2$ into H_2, and embed $\bar{T}_1$ into H_{4-i}, unless $|\bar{T}_1| = |\bar{T}_2|$ and $s \in W_1$, in which case we embed $r_{\bar{T}}$ and v into H_3, embed $\bar{T}_2$ into H_2, and embed $\bar{T}_1 - v$ into H_1. In this way we ensure (48) for all $\bar{T}$ not containing parents of seeds.

Now assume $\bar{T}$ contains a parent p of a seed s', say $s' \in W_j$. If $p = r_{\bar{T}}$, then s and s' belong to the same bipartition class of T, i.e. both are in W_1 or both are in $W_2 \cup W_3$. So we can proceed exactly as above (note that if $s \in W_2$ we are at liberty to choose $i = 1$), as $r_{\bar{T}}$ will always be embedded outside H_j. The same is true if $p \neq r_{\bar{T}}$ and $j = i$, and because of the liberty we have for choosing i if $s \in W_2$, we can from now on assume that

$$p \neq r_{\bar{T}},\ j \neq i \text{ and if } s \in W_2, \text{ then } j = 2. \tag{49}$$

If $r_{\bar{T}}p \notin E(T)$, we embed $r_{\bar{T}}$ and p into H_i, embed the larger bipartition class of $\bar{T} - r_{\bar{T}} - p$ (as induced by the bipartition of T) into H_2, and embed the rest into H_{4-i}, unless both classes $\bar{T}_1'$, $\bar{T}_2'$ have the same size and $s \in W_1$, in which case we wish to embed an additional vertex v' into H_i. Note that it does not matter which partition class v' comes from (as we can embed the other one into H_2), so if there is no suitable v' then all vertices of $\bar{T} - r_{\bar{T}} - p$ are adjacent to either $r_{\bar{T}}$ or p. Hence, as $|\bar{T}_1'| = |\bar{T}_2'| \geq \frac{1}{\beta} > 2$ by Lemma 5.1 (e), $r_{\bar{T}}$ and p lie in different partition classes of T, and since $r_{\bar{T}}p \notin E(T)$, there is a four-vertex path $r_{\bar{T}}x_1x_2p$ in $\bar{T}$. We can embed this path with $\varphi(r_{\bar{T}}), \varphi(p) \notin H_2$ and $\varphi(p) \notin H_j$. We then embed all other vertices of $\bar{T}$ into H_2, thus ensurig (48).

Finally, assume $r_{\bar{T}}p \in E(T)$. Then s and s' lie in distinct bipartition classes of T. So by (49), $s \notin W_2$, and moreover, if $s \in W_3$, then $s' \in W_1$, i.e. $j = 1 = i$, a contradiction. So $s \in W_1$, and thus $s' \in W_2 \cup W_3$, more precisely, since $j \neq i$, we have $s' \in W_2$. We embed $r_{\bar{T}}$ into H_3 and p into H_1, and consider the two components $\bar{T}_r$, $\bar{T}_p$ of $\bar{T} - r_{\bar{T}}p$. We embed the larger bipartition class of each of $\bar{T}_r - r_{\bar{T}}$, $\bar{T}_p - p$ into H_2, and embed the other class into H_1 and H_3, respectively. This results in an embedding as needed for (48), unless we are in one the following two situations. First, it could be that $T_r - r_{\bar{T}} = \emptyset$ while the two bipartition classes of $T_p - p$ have the same size (and are thus large). In this case we reembed a vertex from H_3 to H_1, similarly as before. Seondly, it could be that $T_p - p$ has at most two vertices and the sizes of the bipartition classes of $T_r - r$ differ by at most one, in which case we reembed up to two vertices from H_1 to H_3. We thus proved (48).

Now consider any $\bar{T} \in L$ at s. If $s \in W_1 \cup W_3 \cup \{s^*\}$, we embed $\bar{T}$ into H_2, and if $s \in W_2$, we embed $\bar{T}$ into H_3. Note that (47) ensures that we embedded at least as much of $W \cup L$ into H_2 as into H_3. If we can also show that we embedded at least as much of $W \cup L$ into H_2 as into H_1, then, by (45) and (48), and since $|V(T_{W_1})| \leq \frac{6}{\beta^2} < \frac{m}{100}$ by Lemma 5.1 (b), we embedded T' in a way that (i) and (ii) hold.

To see this, first note that no $\bar{T} \in L$ was embedded into H_1. Consider a seed $s \in W_1$. If there is a leaf v at s, then v was embedded into H_2, so s is accounted for. If there is a tree $\bar{T} \in F_1 \cup F_2$ hanging from s, then by the choice of T_{W_1}, we can assume $\bar{T} \in T_{W_1} \cup (F_1 \setminus (T_{Bad}) \cup F_2$, and by (48), s is accounted for. Otherwise, there are no trees hanging at s, so by Lemma 5.1 (c), s has a child $s' \in W$. As $s \in W_1$, we know that $s' \in W_2 \cup W_3$. If $s' \in W_2$, then $\varphi(s') \in H_2$, which accounts for s, and if $s' \in W_3$, then by (46), s' has a leaf, which was embedded in H_2 and which accounts for s. □

We are now ready to prove the final ingredient for the proof, Lemma 7.4.

Proof of Lemma 7.4. We start by applying Lemma 7.5 to find a tree T' with $|V(T')| \leq \frac{m}{50}$, and an embedding φ of it into $H_1 \cup H_2 \cup H_3 \cup \{w\}$ with the properties given in the lemma. In particular, there are at least $\frac{33}{400}m$ unembedded bad trees whose seeds were embedded into H_2. We extend φ by embedding some of these trees, in a way that we fill all of $H_4 \cup H_5$. By the assumptions of Lemma 7.4, (and using the minimum degree between the sets H_i, and the fact that both T' and $H_4 \cup H_5$ are small,) we know that for any $x \in V(H_5)$ we can embed such a $\bar{T}$ mapping its first vertex into H_1, its second vertex into an appropriate vertex of H_3, and its third vertex into x. For any $x \in V(H_4)$, we can put the first vertex of $\bar{T}$ into either H_1 or H_3, the second vertex into H_2, and the third vertex into x. Alternating H_1 and H_3 when filling H_4, and recalling Lemma 7.5 (ii), we can ensure that for $i = 1, 3$,

$$|\varphi(T) \cap H_i| \leq |\varphi(T) \cap H_2| \leq |V(T')| + |H_4 \cup H_5| \leq \frac{3}{100}m, \text{ and} \tag{50}$$

$$|T_{H_2}| \geq \frac{29}{400}m, \tag{51}$$

where T_{H_2} is the set of all yet unembedded trees whose seed is embedded in H_2. We now embed some more trees from $\bar{T} \in T_{H_2}$, in two phases. This time our aim is to balance the sizes of the used parts of the three sets H_1, H_2, H_3. Let H_i be the set containing the least number of embedded vertices. By (50), we know that $i \neq 2$. In the first phase, if H_i has strictly less used vertices than H_2, we embed trees from T_{H_2} by putting their first vertex into H_i, their second vertex into H_2, and their third vertex into H_i. We do this until the used parts of H_1 and H_3 differ by at most one vertex. In the second phase, we embed trees from T_{H_2}, only using H_1 and H_3, and keeping their used parts almost perfectly balanced, until the one of them has exactly as many used vertices as H_2. Then, since the unused part of G is divisible by three (as the unembedded part of T is), all three of H_1, H_2, H_3 have exactly the same number of used vertices. By (50), and since the used part of H_2 has augmented by a factor of at most $\frac{3}{2}$, this number is at most $\frac{m}{20}$.

Moreover, we used at most $\frac{3}{200}m$ trees in the first phase, and at most $\frac{3}{200}m$ trees in the second phase. So by (51), we still have at least $\frac{m}{25} + 2$ yet unembedded trees in T_{H_2}. We embed up to two more of these trees, keeping the sets H_i balanced, so that the number of unembedded trees in T_{H_2} is now divisible by 3.

We next embed all yet unembedded bad trees from $T_{Bad} \backslash T_{H_2}$, distributing their three vertices equally among the three sets H_i. Note that the minimum degree between the sets H_i is large enough so that it does not matter that in this step, all but $\frac{m}{25}$ vertices of each H_i may become occupied.

In order to embed the remaining trees, we will still use the minimum degree between the sets H_i to embed their first and second vertices, and then a Hall-type argument to embed their last vertex. Let us make this more precise. We always embed three paths at a time. The first vertex of the first path goes to H_3, and the first vertex of each of the other two paths goes to H_1. The second vertex of the second path goes to H_3, and the second vertex of the other two paths goes to H_2. Doing this for all remaining trees, the unused space U_i in each H_i has the same size, which is at least $\frac{m}{75}$.

Our plan is to embed the third vertex v_P of each path P as follows: v_P goes into H_1 if P was a 'first path', into H_2 if P was a 'second path' and into H_3 if P was a 'third path'. Let A_i be the set of all images of second vertices on paths whose third vertex is scheduled to go to H_i. Using Hall's theorem we see that if we cannot embed all third vertices as planned, then there is an obstruction, that is, there is an index $i \in \{1, 2, 3\}$ and a set $A' \subseteq A_i$ such that $|N(A') \cap U_i| < |A'|$. In particular, there is a vertex $v \in U_i \setminus N(A')$, and by our minimum degree condition, v is adjacent to all but at most $\frac{m}{250}$ vertices of A_i. Hence $|A'| \leq \frac{m}{250} \leq \frac{|A_i|}{2}$. On the other hand, any vertex $a \in A'$ is adjacent to all but at most $\frac{m}{250}$ vertices of U_i, and therefore $|A'| > |N(A') \cap U_i| \geq \frac{m}{75} - \frac{m}{250} \geq \frac{|A_i|}{2}$. This finishes the embedding of T. □

References

[AKS95] M. Ajtai, J. Komlós, and E. Szemerédi. On a conjecture of Loebl. In *Graph theory, combinatorics, and algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992)*, 1135–1146. Wiley, New York, 1995.

[BPS18] G. Besomi, M. Pavez-Signé, and M. Stein. Degree conditions for embedding trees. *SIAM Journal on Discrete Mathematics*, 33(3): 1521–1555, 2019.

[BD96] S. Brandt and E. Dobson. The Erdős–Sós conjecture for graphs of girth 5. *Discr. Math.*, 150:411–414, 1996.

[CLNS10] B. Csaba, I. Levitt, J. Nagy-György, and E. Szemerédi. Tight bounds for embedding bounded degree trees. In *Katona G.O.H., Schrijver A., Szenyi T., Sági G. (eds) Fête of Combinatorics and Computer Science*, vol. 20, 2010.

[EFLS95] P. Erdős, Z. Füredi, M. Loebl, and V. T. Sós. Discrepancy of trees. *Studia Sci. Math. Hungar.*, 30(1-2):47–57, 1995.

[EG59] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Mathematica Academiae Scientiarum Hungarica* 10(3): 337– 356, 1959.

[HRSW16] F. Havet, B. Reed, M. Stein, and D. Wood. A Variant of the Erdős-Sós Conjecture. *J. Graph Theory*, 94(1): 131–158, 2020.

[Hax01] P. E. Haxell. Tree embeddings. *J. Graph Theory*, 36(3):121–130, 2001.

[HKP$^+$a] J. Hladký, J. Komlós, D. Piguet, M. Simonovits, M. Stein, and E. Szemerédi. The approximate Loebl–Komlós–Sós Conjecture I: The sparse decomposition. *SIAM Journal on Discrete Mathematics* 31-2: 945-982, 2017.

[HKP$^+$b] J. Hladký, J. Komlós, D. Piguet, M. Simonovits, M. Stein, and E. Szemerédi. The approximate Loebl–Komlós–Sós Conjecture II: The rough structure of LKS graphs. *SIAM Journal on Discrete Mathematics* 31-2: 983-1016, 2017.

[HKP$^+$c] J. Hladký, J. Komlós, D. Piguet, M. Simonovits, M. Stein, and E. Szemerédi. The approximate Loebl–Komlós–Sós Conjecture III: The finer structure of LKS graphs. *SIAM Journal on Discrete Mathematics* 31-2: 1017-1071, 2017.

[HKP$^+$d] J. Hladký, J. Komlós, D. Piguet, M. Simonovits, M. Stein, and E. Szemerédi. The approximate Loebl–Komlós–Sós Conjecture IV: Embedding techniques and the proof of the main result. *SIAM Journal on Discrete Mathematics* 31-2: 1072-1148, 2017.

[KSS01] J. Komlós, G. Sárközy, and E. Szemerédi. Spanning Trees in Dense Graphs. *Combinatorics, Probability and Computing,* 5: 397–416, 2001.

[KSS95] J. Komlós, G. Sárközy, and E. Szemerédi. Proof of a Packing Conjecture of Bollobás. *Combinatorics, Probability and Computing,* 4 (3): 241–255, 1995.

[KSS02] J. Komlós, A. Shokoufandeh, M. Simonovits, and E. Szemerédi. The regularity lemma and its applications in graph theory. *Theoretical aspects of computer science,* Lecture Notes in Comput. Sci., 2292: 84–112, 2002.

[PS12] D. Piguet and M. J. Stein. An approximate version of the Loebl-Komlós-Sós conjecture. *J. Combin. Theory Ser. B*, 102(1):102–125, 2012.

[RS19b] B. Reed and M. Stein. Embedding Spanning Trees in Graphs of High Minimum Degree with a Universal Vertex II: A Tight Result. Accepted for publication in *J. Graph Theory.*

[SW97] J.-F. Saclé and M. Woźniak. A note on the Erdős–Sós conjecture for graphs without C_4. *J. Combin. Theory (Series B)*, 70(2):229–234, 1997.