On claw-free t-perfect graphs

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Abstract

A graph is called t-perfect, if its stable set polytope is defined by non-negativity, edge and odd-cycle inequalities. We characterise the class of all claw-free t-perfect graphs by forbidden t-minors. Moreover, we show that claw-free t-perfect graphs are 3-colourable. Such a colouring can be obtained in polynomial time.

1 Introduction

Perfect graphs can be determined by the structure of their stable set polytope. The *stable set polytope*, or SSP for short, is the convex hull of the characteristic vectors of independent vertex sets, the stable sets. In the case of a perfect graph, the SSP is fully described by non-negativity and clique inequalities. Vice versa, if the SSP of some graph is given by these types of inequalities then the graph is perfect.

In analogy to the relationship between perfect graphs and the SSP, Chvátal [8] proposed to investigate a class of graphs now called t-perfect: the class of graphs whose SSP is determined by non-negativity, edge and odd-cycle inequalities. (For precise definitions see next section.) The class of t-perfect graphs includes the series-parallel graphs (Boulala and Uhry [2]) and the almost bipartite graphs, i.e. those graphs that become bipartite upon deletion of a single vertex (Fonlupt and Uhry [12]). Gerards and Shepherd [18] characterise the graphs with all subgraphs t-perfect. A prime example of graph that is not t-perfect is the complete graph on four vertices, the K_4. Indeed, it will play an important role in what follows.

In this paper, we prove two theorems for t-perfect graphs that are, in addition, claw-free. We show that these graphs can be 3-coloured and we characterise them in terms of forbidden substructures.

Standard polyhedral methods assert that the fractional chromatic number of a t-perfect graph is at most 3. Shepherd suggested that t-perfect graphs might always be k-colourable for some fixed small k. As Laurent and Seymour found a t-perfect graph with $\chi = 4$ (see [26, p. 1207]), this number k has to be at least 4.

Conjecture 1. *Every t-perfect graph is 4-colourable.*

We prove that if the graphs are additionally claw-free then three colours suffice.

Theorem 2. *Every claw-free t-perfect graph is 3-colourable.*

Moreover, such a 3-colouring can be computed in polynomial time (Corollary 15).

We remark that compared to a result of Chudnovsky and Ovetsky [5] our Theorem 2 yields an improvement of 1. Indeed, Chudnovsky and Ovetsky show that the chromatic number of a quasi-line graph G is bounded by $\frac{3}{2}\omega(G)$. As no t-perfect graph can contain a clique of at least four vertices and, furthermore, as a claw-free t-perfect graph is quasi-line, Chudnovsky and Ovetsky's bound is applicable and yields $\chi \leq 4$ for all claw-free t-perfect graphs.

The celebrated strong perfect graph theorem of Chudnovsky, Robertson, Seymour and Thomas [7] characterises perfect graphs in terms of forbidden induced subgraphs: a graph is perfect if and only if it does not contain odd holes or anti-holes. We prove an analogous, although much more modest, result for claw-free t-perfect graphs. While, in order to describe perfect graphs, induced subgraphs are suitable as forbidden substructures, for t-perfect graphs a more general type of substructure, called a t-minor, is more appropriate. Briefly, a t-minor is any graph obtained from the original graph by two kinds of operations, both of which preserve t-perfection: vertex deletions and simultaneous contraction of all the edges incident with a vertex whose neighbourhood forms an independent set. With this notion our second result is as follows.

Theorem 3. *A claw-free graph is t-perfect if and only if it does not contain any of K_4, W_5, C_7^2 and C_{10}^2 as a t-minor.*

Here, K_4 denotes the complete graph on four vertices, W_5 is the 5-wheel, and for $n \in \mathbb{N}$ we denote by C_n^2 the square of the cycle C_n on n vertices, see Figure 1. (The square of a graph is obtained by adding edges between any two vertices of distance 2.)

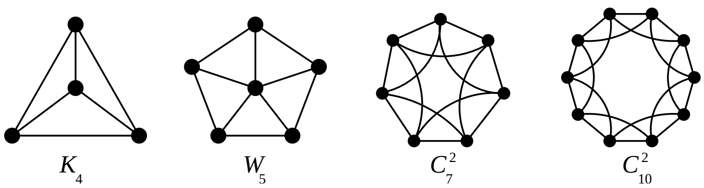

Figure 1: The forbidden t-minors.

The graphs from Theorem 3 already appear implicitly in Galluccio and Sassano [15]. They showed that every rank facet in a claw-free graph comes from a combination of inequalities describing cliques, line graphs of 2-connected factor-critical graphs, and circulant graphs $C_{\alpha\omega+1}^{\omega-1}$. However, as a claw-free graph may have non-rank facets we will not be able to make use of these results.

Ben Rebea's conjecture describes the structure of the stable set polytope of quasi-line graphs. As the conjecture has been solved (see Eisenbrand et al [11] and Chudnovsky and Seymour [6]), and as claw-free t-perfect graphs are quasi-line, it seems conceivable to use Ben Rebea's conjecture to prove Theorem 3. We have not pursued this approach for three reasons. First, Theorem 3 does not appear to be a direct consequence of the conjecture. Second, the solution of the conjecture rests on Chudnovsky and Seymour's characterisation of claw-free graphs, which is far from trivial. Finally, our proof of Theorem 3 (with a little extra effort) yields a 3-colouring of claw-free t-perfect graphs.

Let us give a brief outline of the paper. In the next section, we give a more formal definition of t-perfect graphs and introduce some of their properties. In Section 3, we prove our main results for the special case of line graphs. We will make use of these results later on. After a necessary (and short) detour in Section 4, in which we determine which squares of cycles are t-perfect, we prove our main lemma in Section 5. This lemma, Lemma 14, will be essential in the proofs of both our main results. It implies that every claw-free t-perfect 3-connected graph is a line graph or one of three exceptional graphs. We prove Theorem 2 in Section 6 and Theorem 3 in Section 7. We conclude the paper by proving, in Section 8, that C_7^2 and C_{10}^2 are minimally strongly t-imperfect and by posing three open questions in the last section.

2 Definition of t-perfect graphs

All our graphs are simple and finite. For general graph-theoretic concepts and notation we refer the reader to Diestel [9], for more on t-perfect and claw-free graphs to Schrijver [26, Chapters 68 and 69].

Let $G = (V, E)$ be a graph. The *stable set polytope* $\mathrm{SSP}(G) \subseteq \mathbb{R}^V$ of G is defined as the convex hull of the characteristic vectors of stable, i.e. independent, subsets of V. We define a second polytope $\mathrm{TSTAB}(G) \subseteq \mathbb{R}^V$ for G, given by

$$\begin{aligned} & x \geq 0, \\ & x_u + x_v \leq 1 \text{ for every edge } uv \in E, \\ & x(C) \leq \lfloor |C|/2 \rfloor \text{ for every induced odd cycle } C \text{ in } G. \end{aligned} \tag{1}$$

These inequalities are respectively known as non-negativity, edge and odd-cycle inequalities. Clearly, it holds that $\mathrm{SSP}(G) \subseteq \mathrm{TSTAB}(G)$.

We say that the graph G is *t-perfect* if $\mathrm{SSP}(G)$ and $\mathrm{TSTAB}(G)$ coincide. Equivalently, G is t-perfect if and only if $\mathrm{TSTAB}(G)$ is an integral polytope, i.e. if all its vertices are integral vectors.

Neither the complete graph on four vertices K_4 nor the 5-wheel W_5 are t-perfect. Indeed, for K_4 the vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ lies in TSTAB but not in the SSP of K_4 as the sum of over all entries is larger than $\alpha(K_4) = 1$. The vector that assigns a value of $\frac{2}{5}$ to each vertex on the rim and a value of $\frac{1}{5}$ to the centre shows that 5-wheel is t-imperfect. Again, the vector lies in TSTAB but the sum of all entries is larger than $\alpha(W_5) = 2$.

The following fact is well-known:

bipartite graphs are t-perfect. (2)

In fact, the SSP of a bipartite graph is fully described by just non-negativity and edge inequalities.

It is easy to check that vertex deletion preserves t-perfection (edge deletion, however, does not). A second operation that maintains t-perfection is described in Gerards and Shepherd [18]:

for a vertex v for which $N(v)$ is a stable set contract all edges in $E(v)$. $(*)$

We call this operation a *t-contraction at v*. Let us say that H is a *t-minor* of G if it is obtained from G by repeated vertex-deletion and t-contraction. Then, if G

is t-perfect, so is H. We call a graph *minimally t-imperfect* if it is not t-perfect but every proper t-minor of it is t-perfect. Obviously, in order to characterise t-perfect graphs in terms of forbidden t-minors it suffices to find all minimally t-imperfect graphs.

The following simple lemma ensures that we stay within the class of claw-free graphs when taking t-minors. (For a proof, observe that a claw in a t-minor can only arise from an induced subdivided claw in the original graph.)

Lemma 4. *Every t-minor of a claw-free graph is claw-free.*

3 t-perfect line graphs

We begin by proving our main results for line graphs (Lemma 7 and Lemma 8). Cao and Nemhauser [3], among other results, already characterise t-perfect line graphs in terms of forbidden subgraphs. Unfortunately, their characterisation appears erroneous. While we therefore cannot make use of their theorem, we will pursue an approach that is inspired by their work. In particular, we take advantage of Edmonds [10] celebrated theorem on the matching polytope.

For a graph G, define the *matching polytope* $M(G) \subseteq \mathbb{R}^{E(G)}$ to be the convex hull of the characteristic vectors of matchings. Recall that a graph G is *factor-critical* if $G - v$ has a perfect matching for every vertex v.

Theorem 5 (Edmonds [10], Pulleyblank and Edmonds [23])**.** *Let G be a graph and $x \in \mathbb{R}^{E(G)}$. Then $x \in M(G)$ if and only if*

$$x \geq 0 \tag{3}$$

$$\sum_{e \in E(v)} x_e \leq 1 \qquad \text{for each } v \in V(G) \tag{4}$$

$$\sum_{e \in E(F)} x_e \leq \lfloor \frac{|V(F)|}{2} \rfloor \qquad \text{for each 2-connected factor-critical } F \subseteq G. \tag{5}$$

We say that G has a *proper odd ear decomposition* if there is a sequence $G_0, G_1, \ldots, G_n$ so that G_0 is an odd cycle, $G_n = G$ and G_k is obtained from G_{k-1} for $k = 1, \ldots, n$ by adding an odd path between two (distinct) vertices of G_{k-1} whose interior vertices are disjoint from G_{k-1}.

Theorem 6 (Lovász [20])**.** *A graph is 2-connected and factor-critical if and only if it has a proper odd ear-decomposition.*

For the proof of the next two lemmas, we define C_5^+ to be the 5-cycle plus an added chord, and a *totally odd subdivision of* C_5^+ to be a subdivision of C_5^+ in which every edge is replaced by a path of odd length.

Lemma 7. *Let H be a line graph (of a simple graph). Then H is t-perfect if and only if H does not contain K_4 as a t-minor.*

Proof. One direction is clear, so assume that H does not contain K_4 as a t-minor, and let G be such that $L(G) = H$. Since $M(G)$ =SSP(H), all we have to show is that TSTAB(H) is a subset of the polytope described by (3), (4), and (5) from Theorem 5. That is, we have to check that the inequalities from Theorem 5 are valid for TSTAB(H).

Condition (3) is clear, and for (4), pick a (non-isolated) vertex v of G. If v has degree 2 then (4) follows from an edge inequality in H, and if $d(v) = 3$ then (4) follows from an odd-cycle inequality for a triangle. This shows (4), since clearly, $\Delta(G) \leq 3$ as otherwise H contains K_4 as a subgraph.

For (5), suppose that G contains a 2-connected factor-critical subgraph F, which, by Theorem 6, has an odd ear-decomposition. So either F is an odd cycle, or F contains a totally odd subdivision X of C_5^+. But in the latter case, $L(X)$ is an induced subgraph of H, from which we obtain a K_4 as t-minor by performing t-contractions at vertices of degree 2, a contradiction.

Hence F is an odd cycle, and (5) follows from some odd-cycle inequality in H. Thus, we have shown that SSP(H) coincides with TSTAB(H), as desired. □

Let G be a claw-free graph with an edge-colouring, and let i, j be two colours. Denote the subgraph consisting of the edges coloured i or j together with their incident vertices by $G_{i,j}$. Note that the components of $G_{i,j}$ are paths or cycles.

Lemma 8. *Let H be a t-perfect line graph of a graph. Then $\chi(H) \leq 3$.*

Proof. Let G be a graph such that $H = L(G)$. We do induction on $|E(G)|$. Pick an edge $e = uv$. Then clearly, we may apply the induction hypothesis to the t-perfect line graph $L(G - e)$ to deduce that the edges of $G - e$ can be coloured with three colours.

So, let c be a colouring of the edges of $G - e$ with colours $\{1, 2, 3\}$. If there is a colour that is not used by the edges adjacent to e, then we can colour e with that colour and we are done. Thus, assume that all colours $\{1, 2, 3\}$ are used by edges adjacent to e. Since H does not contain K_4 as a subgraph we know that $\Delta(G) \leq 3$. We may therefore assume that u is incident with two edges f_1, f_2 with $c(f_i) = i$ and that v is incident either with one edge g_3, or with two edges g_1, g_3, so that $c(g_i) = i$. We suppose that $E(G)$ cannot be coloured with three colours, which will lead to a contradiction.

Let P' be the component of $G_{2,3}$ containing f_2. If g_3 does not lie in P', then we can swap colours along P', such that e is no longer incident with any edge coloured 2, a contradiction. Thus, $g_3 \in E(P')$, and the subpath $P := uP'v$ has even length. Hence $P + e$ is an odd cycle.

Next, let Q_1 be the component of $G_{1,3}$ containing f_1. In fact, Q_1 is a path. Suppose Q_1 meets P outside u, and let w be the first vertex after u in Q_1 that also lies in P. Then the last edge of $Q_1 w$ is coloured 1, and $Q_1 w$ therefore of odd length. We see that $(P + e) \cup Q_1 w$ is a totally odd subdivision of C_5^+, which in H induces K_4 as a t-minor, which is impossible as H is t-perfect. Therefore, $V(Q_1 \cap P) = \{u\}$.

We swap colours along Q_1 and denote the resulting colouring by c'. Note that $c'(f_1) = 3$ and that P' is still coloured with $\{2, 3\}$. Now, if g_1 does not exist, then we can colour e with 1. On the other hand, if g_1 exists, then in the same way as before for Q_1, we deduce that the component Q_2 of $G_{1,2}$ (with respect to c') containing g_1 meets P only in v. In particular, by recolouring along Q_2 we obtain a colouring of $E(G - e)$ where no edge incident with e is coloured with 1, yielding a 3-edge-colouring of G. □

The proof of the lemma can easily be turned into an algorithm with running time $O(n^3)$, where n is the number of vertices. (We are cheating here a bit. The proof supposes that we know the graph G of which H is the line graph. However, with minor complications, the same induction can be performed directly in H.)

Corollary 9. *A t-perfect line graph can be coloured with three colours in polynomial time.*

4 Squares of cycles

As a preparation for our main lemma we show in this section that most squares of cycles are t-imperfect. In fact, the only t-perfect squares of cycles are C_3^2, which is a triangle, and C_6^2, the line graph of K_4.

Recall that C_n^2 denotes the square of a cycle of order n. We shall always assume that $V(C_n^2) = \{v_1, \ldots, v_n\}$ where the vertices are labelled in cyclic order.

Lemma 10. *Let $n \geq 4$, and let $n \notin \{6, 7, 10\}$. Then K_4 is a t-minor of C_n^2. Moreover, for $n \geq 8$ the K_4-t-minor is already contained in $C_n^2 - v_5$.*

Proof. Since $C_4^2 = K_4$ we only need to concern ourselves with C_n^2 for $n \geq 5$. Depending on n mod 4 we perform vertex-deletions and then t-contractions as

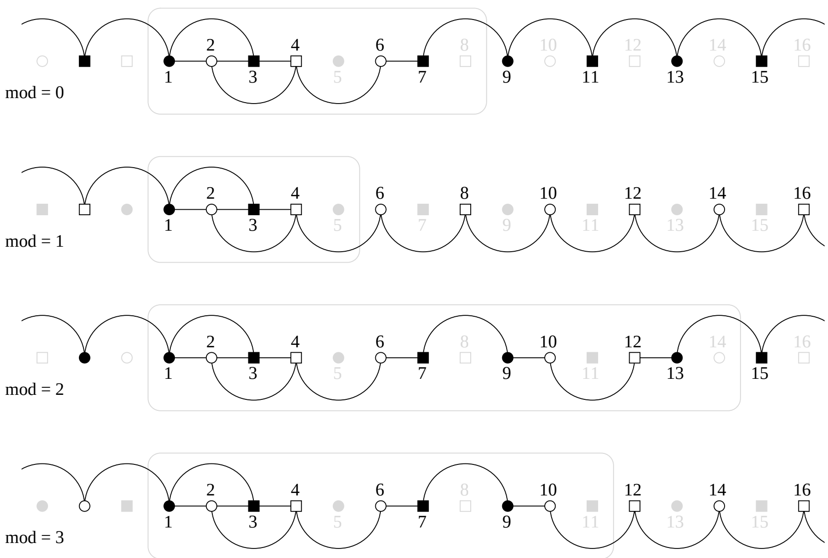

Figure 2: K_4-t-minors in C_n^2 depending on n mod 4.

indicated in Figure 2 until the only vertices left are $v_1, \ldots, v_4$. In particular, we delete the grey vertices in the initial segment (marked by a dashed box). Outside this segment we delete every other vertex until we reach the first vertex v_1 again. Finally, we contract the odd path between v_4 and v_1 to a single edge.

The length of the initial segment poses a constraint on the minimal size of the graph. For $n \equiv 0 \pmod 4$ the construction is possible for $n \geq 8$, for $n \equiv 1$ we need $n \geq 5$, for $n \equiv 2$ we need $n \geq 14$, and $n \geq 11$ is necessary for $n \equiv 3$. So the only cases we have not dealt with are $n = 6, 7, 10$, which are precisely the exceptions. The second part of the assertion follows directly from the construction of the subdivision of K_4. □

Lemma 11. *No square of a cycle of length at least 7 is t-perfect.*

Proof. By Lemma 10, we only need to check C_7^2 and C_{10}^2. However, neither C_7^2 nor C_{10}^2 is t-perfect. Indeed, the vector $x \in \mathbb{R}^{V(C_7^2)}$ defined by $x_v = 1/3$ for each $v \in V(C_7^2)$ clearly lies in TSTAB(C_7^2) but not in SSP(C_7^2) as $\mathbf{1}^T x = 7/3 > 2 = \alpha(C_7^2)$. We get a similar contradiction by assigning a value of 1/3 to every vertex in C_{10}^2. □

5 The main lemma

Before we prove our main lemma, which will play an essential part in the proof of both Theorem 2 and Theorem 3, we quickly note two facts.

Lemma 12. *Let G be a claw-free graph. If $\Delta(G) \geq 5$ then G contains K_4 or W_5 as an induced subgraph.*

Proof. Consider a vertex v of G. If v has at least six neighbours, then, by Ramsey theory, $N(v)$ contains a triangle or three independent vertices. The former leads to a K_4, and the latter to a claw, which is impossible. If $|N(v)| = 5$ then $G[v \cup N(v)]$ is a 5-wheel, or contains a K_4, since the 5-cycle is the only triangle-free graph on five vertices with $\alpha \leq 2$. □

We call a triangle T *odd* if there is a vertex v outside T that is adjacent to an odd number of the vertices in T. We need the following theorem.

Theorem 13 (Harary [19, Theorem 8.4])**.** *Let G be a claw-free graph. Then G is a line graph if and only if every pair of odd triangles that shares exactly one edge induces a K_4.*

Let us now state the main lemma. It shows that the structure of a claw-free t-perfect graph is rather restricted, provided the graph is 3-connected.

Lemma 14 (Main lemma)**.** *Let G be a 3-connected claw-free graph with $\Delta(G) \leq 4$. If G does not contain K_4 as t-minor then one of the following statements holds true:*

(i) *G is a line graph;*

(ii) $G \in \{C_6^2 - v_1v_6, C_7^2 - v_7, C_{10}^2 - v_{10}, C_7^2, C_{10}^2\}$.

Proof. We shall repeatedly make use of the following argument. Assume that in the neighbourhood of a vertex u we find a path xyz, and assume that u has a fourth neighbour $v \notin \{x, y, z\}$. As K_4 is not a subgraph of G we know that $xz \notin E(G)$. Then, because G is claw-free, v has to be adjacent to x or to z or to both.

First of all, we shall show that

$$P_6^2 \text{ is a subgraph of } G. \tag{6}$$

Recall that P_k denotes a path on k vertices.

Indeed, as we may assume that G is not a line graph, there exist by Theorem 13 two odd triangles that share exactly one edge, say $u_1u_2u_3$ and $u_2u_3u_4$. As G is 3-connected, $\{u_1, u_4\}$ does not separate G, and thus one of u_2 and u_3 has a neighbour $u_5 \notin \{u_1, u_2, u_3, u_4\}$. By symmetry, we may assume that $u_3u_5 \in E(G)$ and by the argument outlined at the beginning of this proof, we deduce from $u_1u_2u_4 \subseteq G[N(u_3)]$ that u_5 is adjacent to u_1 or to u_4 (or to both). Symmetry, again, allows us to assume that u_5 is adjacent to u_4.

As K_4 is not a subgraph of G, u_1 and u_5 each send exactly two edges to the triangle $u_2u_3u_4$. That triangle, however, is odd. Thus there exists a vertex $u_6 \notin \{u_1, \ldots, u_5\}$ that is adjacent to an odd number of vertices of the triangle. Since u_3 has four neighbours already among the u_i, it follows that u_6 is either adjacent to u_2 or to u_4. By forgetting that $u_1u_2u_3$ is odd, we obtain again a symmetric graph on $u_1, \ldots, u_5$, which means that we may, without loss of generality, assume that $u_6u_4 \in E(G)$, and that $u_6u_2 \notin E(G)$. The path $u_2u_3u_5$ that is contained in the neighbourhood of u_4 together with $u_6u_2 \notin E(G)$ ensures that u_6 is adjacent to u_5. This proves (6).

Next, we prove that

if $k \geq 6$ so that $P_k^2 \subseteq G$, then either $P_{k+1}^2 \subseteq G$ as well, or $V(G) = V(P_k)$. (7)

Assume that G has a vertex outside $P_k = v_1 \ldots v_k$. Because G is 3-connected and $\Delta(G) \leq 4$, one of v_2 and v_{k-1}, let us say the latter, has a neighbour $v_{k+1} \notin V(P_k)$; if not then v_1 and v_k would separate $V(P_k)$ from the rest of the graph. From the fact that the path $v_{k-3}v_{k-2}v_k$ is contained in the neighbourhood of v_{k-1} we deduce that v_{k+1} is adjacent to v_{k-3} or to v_k. However, v_{k-3} is already adjacent to four vertices, namely to $v_{k-5}, v_{k-4}, v_{k-2}, v_{k-1}$ (recall that $k \geq 6$). Thus, $\Delta(G) \leq 4$ implies that v_{k+1} is in fact adjacent to v_k. Thus $P_{k+1}^2 \subseteq G$ and we have proved (7).

Now, by repeated application of (7) we arrive at a path $P_n = v_1 \ldots v_n$, for some $n = |V(G)| \geq 6$, whose square is a subgraph of G. Observe that in the square of P_n every vertex has degree 4, except v_2 and v_{n-1}, which have degree 3, and except v_1 and v_n, which have degree 2. Since $\Delta(G) \leq 4$, the square of P_n and G may only differ in the presence or absence of the edges v_1v_{n-1}, v_1v_n, v_2v_{n-1} and v_2v_n in G. As G is 3-connected, each of v_1 and v_n is incident with at least one of these edges.

First, assume that $v_1v_n \notin E(G)$, which immediately entails that $v_1v_{n-1} \in E(G)$ and $v_2v_n \in E(G)$, and hence, as $\Delta(G) \leq 4$, that $v_2v_{n-1} \notin E(G)$. Since $v_1v_3v_4$ is a path in the neighbourhood of v_2, the fourth neighbour v_n of v_2 must be adjacent to v_4. This is only possible if $n = 6$, and we find that then $G = C_6^2 - v_1v_6$, which is as desired.

So, from now on, let us assume that

$$v_1v_n \in E(G). \tag{8}$$

Next, suppose that v_2v_{n-1} is an edge of G. Then $n > 6$ as otherwise $v_2, v_3, v_4, v_5 = v_{n-1}$ span a K_4. On the other hand, we find the path $v_{n-3}v_{n-2}v_n$ in the neighbourhood of v_{n-1}, which implies that v_2 is adjacent to v_{n-3} or to v_n. Since v_2 already has already four neighbours, namely v_1, v_3, v_4 and v_{n-1}, and since $n > 6$ it follows that $v_{n-3} = v_4$ and $n = 7$.

Consequently, G is isomorphic to $\tilde{C}_7^2$, which we define as the square of P_7 plus the edges v_1v_7 and v_2v_6. However, Figure 3 A shows that $\tilde{C}_7^2$ contains K_4 as a t-minor, a contradiction. (Alternatively, we might have argued that $\tilde{C}_7^2$ is the line graph of the graph obtained from K_4 by subdividing one edge.)

Thus,

$$v_2v_{n-1} \notin E(G). \tag{9}$$

So, by (8) and (9), G is isomorphic to one of the following graphs: $G = C_n^2$, $C_n^2 - v_1v_{n-1}$, and $C_n^2 - v_1v_{n-1} - v_2v_n$. Let us check these cases seperately.

First, assume $G = C_n^2$. Since $C_6^2 = L(K_4)$ and since by Lemma 10, for $n \geq 7$ every C_n^2 except C_7^2 and C_{10}^2 contains K_4 as a t-minor, we find that $G = C_7^2$ or $G = C_{10}^2$, which are two of the allowed outcomes of Lemma 14.

Next, assume that $G = C_n^2 - v_1v_{n-1}$. Observe that $(C_n^2 - v_1v_{n-1}) - v_1$ is isomorphic to $C_n^2 - v_5$. Hence, unless $n \in \{6, 7, 10\}$, Lemma 10 asserts that G contains K_4 as a t-minor. For $n = 7$ and $n = 10$, Figure 3 B and C indicate K_4-t-minors of G. So, $n = 6$, that is, $G = C_6^2 - v_1v_5$ which is isomorphic to $C_6^2 - v_1v_6$, and thus one of the allowed outcomes of the lemma.

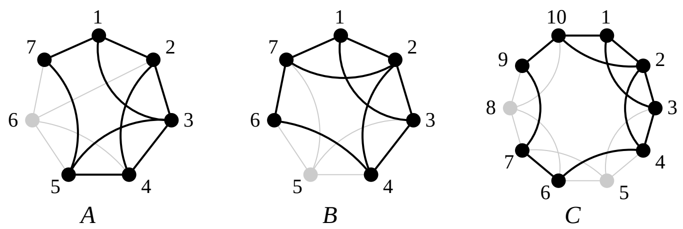

Figure 3: K_4 as a t-minor of $\tilde{C}_7^2$, $C_7^2 - v_1v_6$, $C_{10}^2 - v_1v_9$.

Finally, we treat the case when $G = C_n^2 - v_1v_{n-1} - v_2v_n$. Observe that then G is isomorphic to $C_{n+1}^2 - v_{n+1}$, and thus we may employ Lemma 10 again to deduce that $n + 1 \in \{6, 7, 10\}$. Of these cases, $n + 1 = 6$ is impossible as $n \geq 6$ by (6). Therefore, either $G = C_7^2 - v_7$ or $G = C_{10}^2 - v_{10}$, which is as desired. $\square$

6 Colouring claw-free t-perfect graphs

We now prove the first of our two main results, which we restate.

Theorem 2. *Every claw-free t-perfect graph is* 3*-colourable.*

Proof. Let G be claw-free and t-perfect. As every colouring of all the blocks yields a colouring of G, we may assume that G is 2-connected. We proceed by induction on $|V(G)|$.

Observe that we can 3-colour G if G is 3-connected. Indeed, by Lemma 12, we can then apply Lemma 14, which implies that either G is a line graph, or $G \in \{C_7^2, C_{10}^2\}$ or G is a subgraph of C_6^2 (recall that $C_7^2 - v_5$ is isomorphic to $C_6^2 - v_1v_5 - v_2v_6$), or $G = C_{10}^2 - v_{10}$. But $G \in \{C_7^2, C_{10}^2\}$ is impossible, by Lemma 11. If G is a line graph then it it follows from Lemma 8 that G is 3-colourable. Finally, C_6^2 as well as $C_{10}^2 - v_{10}$ are easily seen to be 3-colourable.

If G has at most three vertices, we clearly have $\chi(G) \leq 3$, too. So, let us assume now that G is not 3-connected and has at least four vertices. Then there are distinct vertices u, v, and induced proper subgraphs L and R of G so that $V(L) \cap V(R) = \{u, v\}$ and $L \cup R = G$. As $|V(L)| < |V(G)|$ there is, by induction, a 3-colouring c_L of L. Permuting colours, if necessary, we may assume that $c_L(u) = 1$ and $c_L(v) \in \{1, 2\}$.

Define $\tilde{R}$ to be the graph we obtain from R by identifying u and v. Observe that at least one of $\tilde{R}$ and $R + uv$ is a proper t-minor of G, and thus has a 3-colouring by induction. This colouring can be extended to a 3-colouring c_R

of R (with $c_R(u) = c_R(v)$, or with $c_R(u) \neq c_R(v)$, depending on the t-minor we found). Now, we can combine c_L and c_R, if necessary swapping colours on one side, to a 3-colouring of G. The only two situations where this is not possible is when $c_L(v) = 1$ and $\tilde{R}$ is not a t-minor of G, or when $c_L(v) = 2$ and $R + uv$ is not a t-minor of G.

In the former case, this means that there is no induced even u–v path in L. In particular, the Kempe-chain K at u in colours $1, 2$ does not contain v. We can thus recolour along K to obtain a colouring c'_L of L with $c'_L(u) \neq c'_L(v)$ that combines with c_R to a 3-colouring of G. In the latter case, i.e. when $c_L(v) = 2$ and $R + uv$ is not a t-minor of G, we proceed similarly, recolouring along a Kempe-chain at u to obtain a colouring c'_L with $c'_L(u) = c'_L(v)$ that combines with c_R to a 3-colouring of G □

Corollary 15. *Every claw-free t-perfect graph on n vertices can be coloured with three colours in polynomial time in n.*

Proof. Let us sketch how the proof of Theorem 2 can be turned into an algorithm. First observe that, by Lemma 12, Lemma 14 and Corollary 9 we can 3-colour any 3-connected claw-free t-perfect graph in polynomial time.

Now, starting with $L^0 := G$ we construct a sequence of graphs L^i and R^i. Indeed, if L^{i-1} fails to be 3-connected and has at least four vertices, we split L^{i-1} into two proper induced subgraphs L^i and R^i as in the proof above (with L^{i-1} in the role of G). Among all choices for L_i and R_i, we choose R^i so that $|V(R^i)|$ is minimal. (This can be accomplished by considering all of the $O(n^2)$ vertex sets of cardinality at most 2.)

We recursively compute a 3-colouring c_{L^i} of L^i and in order to check which of the graphs $R^i + uv$ and $\tilde{R}^i$ is a t-minor of G, we pick an induced u-v path P^i in L^i (for instance a shortest path). As we chose R_i minimal, the t-minor $R^i + uv$, respectively $\tilde{R}^i$, is 3-connected. Hence, we can compute its 3-colouring directly with the help of Lemma 14 and Corollary 9. If necessary, we then recolour L_i along a Kempe-chain at u. Since $L^{i-1} - L^i \neq \emptyset$, the procedure stops after at most n steps. □

Let us now turn for a moment to h-perfect graphs, to which our result on colourings easily carries over. Sbihi and Uhry [24] introduced h-perfect graphs as a common generalisation of perfect and t-perfect graphs. For the definition of h-perfect graphs we use the same inequalities as for t-perfect graphs, only that the edge inequalities are replaced with clique inequalities. So, a graph is called *h-perfect* if the SSP is determined by

$$\begin{aligned} & x \geq 0 \\ & x(K) \leq 1 \text{ for every clique } K \\ & x(C) \leq \lfloor |V(C)|/2 \rfloor \text{ for every induced odd cycle } C \end{aligned}$$

The proof of the following corollary is due to Sebő [22]. As it has not been published but contains a nice and useful technique we present it here.

Corollary 16. *Let G be a claw-free h-perfect graph. Then*

(i) $\chi(G) = \lceil \chi^*(G) \rceil$*; and*

(ii) $\chi(G) = \omega(G)$ *if* $\omega(G) \geq 3$.

Here, χ^* denotes the fractional chromatic number. More formally, if $\mathcal{S}$ denotes the set of all stable sets:

$$\begin{array}{rcl} \chi^*(G) & = & \min \mathbf{1}^T y,\ y \in \mathbb{R}^{\mathcal{S}} \\ \text{subject to} & & y \geq 0 \quad \text{and} \displaystyle\sum_{S \in \mathcal{S},\, v \in S} y_S \geq 1 \text{ for all } v \in V \end{array} \tag{10}$$

Proof of Corollary 16. Define the polytope

$$P = \{x \in \mathbb{R}^V : x(S) \leq 1 \text{ for each stable set } S,\ x \geq 0\}.$$

Observe that $\max_{x \in P} \mathbf{1}^T x$ is the dual program of (10), so that we get $\chi^*(G) = \max_{x \in P} \mathbf{1}^T x$. Moreover, it is not hard to check that the anti-blocking polytope of P coincides with $\mathrm{SSP}(G)$. As G is h-perfect, Theorem 2.1 in Fulkerson [14] (see also [13]) yields therefore that every vertex $\neq \mathbf{0}$ of P is either the characteristic vector χ_K of a clique K of G or the vertex is of the form $\frac{2}{|C|-1}\chi_C$ for an odd cycle C.

First, assume that $\omega(G) \geq 3$. We show that

there is a stable set S which intersects every clique of size $\omega(G)$. (11)

Since $\omega(G) \geq 3 > \mathbf{1}^T(\frac{2}{|C|-1}\chi_C)$ for every odd cycle C of length ≥ 5, we see that $\max_{x \in P} \mathbf{1}^T x = \omega(G)$ is attained in every clique of size $\omega(G)$. Consider an optimal solution y of (10) and a clique K of size $\omega(G)$. Then

$$\omega(G) = \mathbf{1}^T \chi_K \leq \sum_S y_S\, \chi_S^T \chi_K = \sum_S y_S |S \cap K| \leq \sum_S y_S = \omega(G).$$

Thus, each stable set S with $y_S > 0$ must meet each such clique K, which proves (11).

Next, we find with (11) stable sets $S_1, \ldots, S_k$ where $k = \omega(G) - 3$ such that $G' := G - S_1 - \ldots - S_k$ has no clique of size 4. Thus, G' is t-perfect and therefore, by Theorem 2, colourable with three stable sets, $S_{k+1}, S_{k+2}, S_{k+3}$ say. Now, we can colour G with $S_1, \ldots, S_{\omega(G)}$. This proves assertion (ii), and (i), too, for $\omega(G) \geq 3$ as $\omega(G)$ is a lower bound for $\chi^*(G)$.

Finally, assume $\omega(G) < 3$. If G is not bipartite, in which case we are done, then $\chi^*(G) = \max_{x \in P} \mathbf{1}^T x$ is attained in $\frac{2}{|C|-1}\chi_C$ for some odd cycle C. Thus, $\chi^*(G) > 2$. On the other hand, G is t-perfect, and we can consequently, by Theorem 2, colour it with three colours. □

We remark that Sebő developed the arguments above to show that Conjecture 1 on the 4-colourability of t-perfect graphs is implied by the following claim.

Conjecture 17 (Sebő [22])**.** *Every triangle-free t-perfect graph is* 3*-colourable.*

7 Characterising claw-free t-perfect graphs

Lemma 14 together with Lemma 7 provides already a full characterisation of claw-free t-perfect graphs if, in addition, the graph is 3-connected. The task of the next few lemmas is to show that minimally t-imperfect claw-free graphs are, in fact, 3-connected.

The first of these lemmas is quite similar to Lemma 12 in Gerards and Shepherd [18]. As that lemma, however, is assembled from results of various authors, its proof is not easily verified. We therefore give a direct proof that draws on only two fairly simple facts.

Lemma 18. *Let G be a minimally t-imperfect graph, and assume $u, v \in V(G)$ to separate G. Then $G - \{u, v\}$ has exactly two components, one of which is a (possibly trivial) path. Moreover, $uv \notin E(G)$.*

Proof. Let $G = G_1 \cup G_2$ so that $\{u, v\}$ separates $G_1 - \{u, v\}$ from $G_2 - \{u, v\}$. Suppose that neither of G_1 and G_2 is a path. Let z be a non-integral vertex of TSTAB(G), denote by $\mathcal{I}$ the set of those non-negativity, edge and odd-cycle inequalities that are satisfied with equality by z. We define z^1 resp. z^2 to be the restriction of z to G_1 resp. G_2.

As in the proof of Theorem 1 in Gerards and Shepherd [18] we can deduce that

$$0 < z_w < 1 \textit{ for all } w \in V(G) \tag{12}$$

and

$$\textit{every odd cycle whose inequality is in } \mathcal{I} \textit{ fails to separate } G. \tag{13}$$

The last fact implies, in particular, that each odd cycle in $\mathcal{I}$ lies either completely in G_1 or in G_2 (recall that neither of G_1 and G_2 is a path). Thus, we can partition $\mathcal{I}$ in $(\mathcal{I}_1, \mathcal{I}_2)$ so that $\mathcal{I}_k$ pertains only to G_k. Now, if there is a $j \in \{1, 2\}$ so that $\dim \mathcal{I}_j = |V(G_j)|$ then z^j is a vertex of TSTAB(G_j) =SSP(G_j). Since z^j is non-integral we obtain a contradiction.

Therefore, we have $\dim \mathcal{I}_k = |V(G_k)| - 1$ for $k = 1, 2$, which means that $\mathcal{I}_k$ describes an edge of TSTAB(G_k). Denote the endvertices of this edge by s^k and t^k, i.e. $z^k = \lambda_k s^k + (1 - \lambda_k)t^k$ for some $0 \le \lambda_k \le 1$. As TSTAB(G_k) =SSP(G_k) by assumption, it follows that s^k is the characteristic vector of a stable set S_k of G_k; the same holds for t^k and a stable set T_k.

By (12), $z^1_u = z^2_u > 0$ and thus for each $k = 1, 2$ one of S_k and T_k needs to contain u. By renaming if necessary we may assume that $u \in S_1$ and $u \in S_2$. Then $u \notin T_k$ for $k = 1, 2$ as otherwise we obtain $z^k_u = \lambda_k + (1 - \lambda_k) = 1$ in contradiction to (12). This implies that

$$\lambda_1 = z^1_u = z_u = z^2_u = \lambda_2. \tag{14}$$

If $S_1 \cap \{v\} = S_2 \cap \{v\}$ then also $T_1 \cap \{v\} = T_2 \cap \{v\}$ as (12) implies as above that $v \in S_k$ if and only if $v \notin T_k$. In this case, $S := S_1 \cup S_2$ and $T := T_1 \cup T_2$ are stable sets of G and we obtain $z = \lambda_1 \chi_S + (1 - \lambda_1)\chi_T$, contradicting the choice of z as a non-integral vertex of TSTAB(G).

So, let us assume that S_1 and S_2 differ on $\{v\}$. Without loss of generality, let $v \in S_1$ but $v \notin S_2$. Then

$$\begin{aligned} S_1 \cap \{u, v\} &= \{u, v\}, & T_1 \cap \{u, v\} &= \emptyset, \\ S_2 \cap \{u, v\} &= \{u\} \quad \textit{and} & T_2 \cap \{u, v\} &= \{v\}. \end{aligned}$$

So, $\lambda_1 = z^1_v = z^2_v = 1 - \lambda_2$, and hence, by (14), $\lambda_1 = \lambda_2 = 1/2$. In particular, it follows with (12) again that $z_w = 1/2$ for all $w \in V(G)$.

Now, since bipartite graphs are t-perfect by (2), G contains an odd cycle of length $2k+1$, say. However, adding up z along the cycle yields $k+1/2$, contradicting the odd-cycle inequalities. $\square$

Next, let us prove that a minimally t-imperfect claw-free graph has minimum degree at least three. We start with a lemma that is a variant of Theorem 2.5 in Barahona and Mahjoub [1], and can be proved in a very similar way

Lemma 19 (Barahona and Mahjoub [1])**.** *Let G be a graph, and let uvw be a path in G so that $\deg(v)=2$ and $uw \notin E(G)$. Furthermore, let $a^Tx \le \alpha$ be a facet-defining inequality of SSP(G) so that $a_u = a_v = a_w$. Denote by G' the graph obtained from G by contracting uv and vw, and let $\tilde{v}$ be the resulting vertex, i.e. $V(G') \setminus V(G) = \{\tilde{v}\}$. If $a' \in \mathbb{R}^{V(G')}$ is defined by $a'_p = a_p$ for $p \in V(G'-\tilde{v})$ and $a_{\tilde{v}} = a_v$ then $a'^Tx \le \alpha - a_v$ is a facet-defining inequality of SSP(G').*

The following lemma serves to guarantee that $a_u = a_v = a_w$ as in Lemma 19.

Lemma 20. *Let G be a graph and assume that for $a \in \mathbb{R}^{V(G)}$, $a > 0$ the inequality $a^Tx \le \alpha$ is facet-defining in SSP(G), and that it is not a multiple of an edge inequality or of an odd-cycle inequality.*

(i) *If G contains a path uvw so that $\deg(v)=2$ then $a_v \le a_w$.*

(ii) *If G contains a triangle wpq and a neighbour $v \notin \{p,q\}$ of w so that $\deg(w)=3$ then $a_v \ge a_w$.*

Assertion (i) appears in Mahjoub [21].

Proof. For both cases, observe that as the SSP is full-dimensional there exists a set $\mathcal{S}$ of $|V(G)|$ affinely independent stable sets that satisfy $a^Tx \le \alpha$ with equality. Since $a > 0$ it follows that $\alpha \ne 0$, which, in turn, implies that the characteristic vectors of the stable sets in $\mathcal{S}$ are even linearly independent. In particular, any inequality satisfied with equality by all $S \in \mathcal{S}$ is a multiple of $a^Tx \le \alpha$.

(i) Since $a^Tx \le \alpha$ is not a multiple of the edge inequality $x_u + x_v \le 1$ there must exist an $S_0 \in \mathcal{S}$ so that $u \notin S_0$ and $v \notin S_0$. As $a > 0$ this implies that $w \in S_0$. Clearly, $S'_0 := S_0 \setminus \{w\} \cup \{v\}$ is a stable set and thus $a^T\chi_{S'_0} \le \alpha = a^T\chi_{S_0}$. Hence $a_v \le a_w$.

(ii) Since $a^Tx \le \alpha$ is not a multiple of the triangle inequality $x_w + x_p + x_q \le 1$ there must exist an $S_1 \in \mathcal{S}$ so that $\{w,p,q\} \cap S_1 = \emptyset$. Then, as $a > 0$ and $N(w) = \{v,p,q\}$, we have that $v \in S_1$ and that $S'_1 := S_1 \setminus \{v\} \cup \{w\}$ is stable. Again, we obtain $a^T\chi_{S'_1} \le \alpha = a^T\chi_{S_1}$ and therefore $a_w \le a_v$. $\square$

Lemma 21. *Let G be a minimally t-imperfect claw-free graph. Then G has minimum degree ≥ 3.*

Proof. It is easy to see that no vertex can have degree 1. Indeed, such a vertex would lead to a violation as in (12). So suppose there is a path $P = w_1 \ldots w_k$ with $k \ge 3$ so that all internal vertices have degree 2 in G but w_1 and w_k have degree > 2. Since G is claw-free and does not properly contain a K_4 we deduce that $\deg(w_1) = \deg(w_k) = 3$, and in fact there are neighbours p_1, q_1 of w_1 and p_k, q_k of w_k so that $w_1p_1q_1$ and $w_kp_kq_k$ are triangles in G.

As G is t-imperfect there exists a facet-defining inequality $a^T x \leq \alpha$ of SSP(G) with $a \geq 0$ that is not a multiple of a non-negativity, edge or odd-cycle inequality. Since G is minimally t-imperfect under vertex deletion it follows furthermore that $a > 0$.

Now, applying (i) of Lemma 20 we get that $a_{w_2} = \ldots = a_{w_{k-1}} \leq \min\{w_1, w_k\}$. Then, (ii) yields that $a_{w_1} = a_{w_2} = \ldots = a_{w_k}$.

Denote by G' the graph obtained from G by performing a t-contraction at w_2, and let $\tilde{w}$ be the resulting new vertex. Define $a'_u = a_u$ for $u \in V(G' - \tilde{w})$ and $a'_{\tilde{w}} = a_{w_2}$. Then, by Lemma 19, $a'^T x \leq \alpha - a_{w_2}$ is facet-defining for SSP(G'). However, as $a' > 0$ and as G' is t-perfect it follows that G' consists of a single vertex, a single edge or of a single odd cycle. Then G is such a graph, too, and thus t-perfect, a contradiction. □

In Section 8 we will show in Lemma 23 that $C_7^2 - v_7$ as well as $C_{10}^2 - v_{10}$ are (strongly t-perfect and thus) t-perfect. Considering Figure 4 we see that $C_6^2 - v_1v_6$ is a t-minor of $C_{10}^2 - v_{10}$. Hence, (assuming Lemma 23) the following lemma holds:

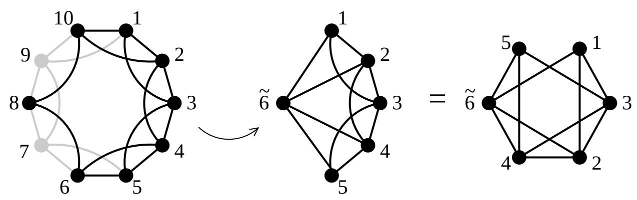

Figure 4: $C_6^2 - v_1v_6$ is a t-minor of $C_{10}^2 - v_{10}$.

Lemma 22. *$C_7^2 - v_7$, $C_{10}^2 - v_{10}$ and $C_6^2 - v_1v_6$ are t-perfect.*

We now restate and then prove our second main result.

Theorem 3. *A claw-free graph is t-perfect if and only if it does not contain any of K_4, W_5, C_7^2 and C_{10}^2 as a t-minor.*

Proof. As neither of K_4, W_5, C_7^2 and C_{10}^2 is t-perfect (note Lemma 11), necessity is obvious. To prove sufficiency, consider a claw-free and minimally t-imperfect graph G.

Lemmas 18 and 21 ensure that G is 3-connected. Moreover, as we are done if G contains K_4 or W_5 as a t-minor, we obtain with Lemma 12 that $\Delta(G) \leq 4$. Consequently, all preconditions of Lemma 14 are satisfied, and we may assume that G is of type (i) or (ii) as listed in Lemma 14.

Now, if G is a line graph then Lemma 7 forces G to contain K_4 as a t-minor, as desired. It remains to check the types listed in (ii). As by Lemma 22, the graphs $C_6^2 - v_1v_6$, $C_7^2 - v_7$ and $C_{10}^2 - v_{10}$ are t-perfect, it follows that $G \in \{C_7^2, C_{10}^2\}$, as desired. □

8 C_7^2 and C_{10}^2 are minimally strongly-t-imperfect

To conclude the proof of Theorem 3 it still remains to prove one last lemma (since we needed for the proof of Lemma 22 that $C_7^2 - v_7$ and $C_{10}^2 - v_{10}$ are t-perfect). We take this opportunity to show something slightly stronger, namely that C_7^2 and C_{10}^2 are minimally t-imperfect, and, moreover, minimally strongly t-imperfect.

In order to define strong t-perfection, consider a graph G and $w \in \mathbb{Z}^{V(G)}$, and denote by $\alpha_w(G)$ the maximum $w(S)$ over all stable sets S in G. Call a family $\mathcal{K}$ of edges and odd cycles a *w-cover of G* if every vertex v lies in at least $w(v)$ members of $\mathcal{K}$. If $\mathcal{K}$ consists of the subfamily of edges $\mathcal{E}$ and the subfamily of cycles $\mathcal{C}$ then it has *cost*

$$|\mathcal{E}| + \sum_{C \in \mathcal{C}} \frac{|C|-1}{2}.$$

We say that G is *strongly t-perfect* if for every $w \in \mathbb{Z}^{V(G)}$ there is a w-cover of cost at most $\alpha_w(G)$. (Clearly, any w-cover has cost at least $\alpha_w(G)$.) One may alternatively define strongly t-perfect graphs by requiring that (1) is totally dual integral.

Observe that it suffices to check the existence of the desired cover for all non-negative vectors w. Moreover, one can show that vertex deletion as well as t-contraction maintain strong t-perfection.

Strongly t-perfect graphs have been studied by Gerards [16] and Schrijver [25]; see also Schrijver [26, Chapter 68]. They show that bad resp. odd K_4-free graphs are strongly t-perfect. It is not known whether a t-perfect graph is necessarily strongly t-perfect, but the converse is true. So, the t-perfectness of $C_7^2 - v_7$ and $C_{10}^2 - v_{10}$ follows from the following lemma.

Lemma 23. *For $j \in \{7, 10\}$, the graph $C_j^2 - v_j$ is strongly t-perfect.*

The proof of this lemma is a bit involved and given below. Let us first get to the main result of this section:

Proposition 24. *The graphs C_7^2 and C_{10}^2 are minimally t-imperfect as well as minimally strongly-t-imperfect.*

Proof. Lemma 11 yields that C_7^2 and C_{10}^2 are t-imperfect, and by Lemma 23, we know that the deletion of one vertex makes these graphs strongly t-perfect. Hence, as strong t-perfection implies t-perfection, our proposition follows. □

Proof of Lemma 23. In both cases, $j = 7$ and $j = 10$, we proceed by induction on the total weight $w(V)$, where $V := V(C_j^2 - v_j)$ and w is the given non-negative vector in $\mathbb{Z}^V$ for which we have to find a w-cover. So, let $G \in \{C_7^2 - v_7, C_{10}^2 - v_{10}\}$. As the case when $w(V) = 0$ is trivial we will assume that w is given with $w(V) > 0$, and that the desired cover exists for all w' with $w'(V) < w(V)$.

Let $\{v_1, \ldots, v_{j-1}\}$ be the vertices of $V(G)$ in circular order, so that v_1, v_2, v_{j-2} and v_{j-1} have degree 3. Denote by $\mathcal{S}$ the set of all stable sets of weight $\alpha_w := \alpha_w(G)$, and write w_i for $w(v_i)$.

First of all, if there is triangle T so that every $S \in \mathcal{S}$ meets T, then we define $w'(v) := w(v) - 1$ for $v \in T^+ := V(T) \cap \bigcup_{S \in \mathcal{S}} S$ and $w'_i = w_i$ otherwise. As each $v \in T^+$ has positive weight $w(v)$—otherwise $S \setminus \{v\}$ would be in $\mathcal{S}$ and miss T—we conclude that w' is non-negative. Since $T^+ \neq \emptyset$ by assumption, the total weight $w'(V)$ is smaller than $w(V)$. Hence, by induction there is a w'-cover $\mathcal{K}'$ of cost $\alpha_{w'}$. Since $\alpha_{w'} = \alpha_w - 1$ the family $\mathcal{K}' \cup T$ is a w-cover of cost α_w, as desired.

We can argue similarly if every $S \in \mathcal{S}$ meets the edge $v_1 v_{j-1}$. So, let us assume from now on that for each triangle T in G there is a $S_T \in \mathcal{S}$ avoiding T, and that there exists a $S_{v_1 v_j}$ that is disjoint from $\{v_1, v_{j-1}\}$.

In the case when $G = C_7^2 - v_7$, the stable set $S_{v_4 v_5 v_6}$ of weight α_w needs to consist of a single vertex v_k with $k \in \{1, 2, 3\}$ as $v_1 v_2 v_3$ forms a triangle in G. Hence, $w_k = \alpha_w$. In the same way, we get that for some $l \in \{4, 5, 6\}$ the vertex v_l has weight α_w, too. Moreover, v_k and v_l have to be adjacent. If $(k, l) = (1, 6)$, then all other vertices have weight 0, and α_w times the edge $v_1 v_6$ is a w-cover of G. On the other hand, if $k \in \{2, 3\}$ and $l \in \{4, 5\}$, then $w_1 = w_6 = 0$. Furthermore, as $\{v_2\} = S_{v_3 v_4 v_5}$ and $\{v_5\} = S_{v_2 v_3 v_4}$ have weight α_w, the stable set $\{v_2, v_5\}$ has weight $2\alpha_w$, a contradiction.

Now, let us consider the case of $G = C_{10}^2 - v_{10}$. Let K be a triangle in G, or let K be the subgraph consisting of the edge $v_1 v_9$. Suppose that $k \in V(K)$.

If $w(k) > 0$ and k has only one neighbour s outside K then, as $w(S_K) = \alpha_w$, S_K contains s, since otherwise we could increase the weight of S_K by including k. Since $S_K \setminus \{s\} \cup \{k\}$ is stable, it follows that $w(k) \le w(s)$. Observe that this inequality trivially holds too, if $w(k) = 0$. We use this rule to obtain a number of inequalities that are listed in the table below.

K		
$v_1 v_2 v_3$	(a) $w_1 \le w_9$	(b) $w_2 \le w_4$
$v_7 v_8 v_9$	(c) $w_9 \le w_1$	(d) $w_8 \le w_6$
$v_2 v_3 v_4$	(e) $w_2 \le w_1$	
$v_6 v_7 v_8$	(f) $w_8 \le w_9$	

Now assume that the vertex $k \in V(K)$ has two adjacent neighbours s and t outside K (and then no other neighbours outside K). Because S_K can only contain one of s and t, we deduce as above that $w(k) \le \max\{w(s), w(t)\}$. Using this argumentation, we obtain

K	
$v_3 v_4 v_5$	(g) $w_3 \le \max\{w_1, w_2\}$
$v_5 v_6 v_7$	(h) $w_7 \le \max\{w_8, w_9\}$
$v_1 v_9$	(i) $w_1 \le \max\{w_2, w_3\}$
$v_1 v_9$	(j) $w_9 \le \max\{w_7, w_8\}$

From (a) and (c), we get that $w_1 = w_9$, and (g) together with (e) yields $w_3 \le w_1$. Symmetrically, we obtain $w_7 \le w_9$, and with (e), (f), (i) and (j) this results in

$$\max\{w_2, w_3\} = w_1 = w_9 = \max\{w_7, w_8\}. \tag{15}$$

Now, take two stable sets $S, S' \in \mathcal{S}$ of cardinality 2 that avoid $v_4 v_5 v_6$ (such sets exist, as we may, for example, take $S_{v_4 v_5 v_6}$, after adding a vertex, if necessary). Observe that by (15), and since a stable set may meet each of the triangles $v_1 v_2 v_3$ and $v_7 v_8 v_9$ at most once, we may choose S and S' so that $S = \{v_1, s\}$ for some $s \in \{v_7, v_8\}$ and so that $S' = \{v_9, s'\}$ for some $s' \in \{v_2, v_3\}$.

Comparing the stable set $\{v_1, s, v_4\}$ to S we get $w_1 + w(s) + w_4 \le w(S) = w_1 + w(s)$ and thus $w_4 = 0$. Hence, $w_2 = 0$ too, by (b), and $w_3 = w_1$, by (15). Symmetrically, comparing $\{v_9, s', v_6\}$ to S', we get that $w_6 = w_8 = 0$.

To sum up, we have discovered that $w_1 = w_3 = w_7 = w_9$ and that $w_2 = w_4 = w_6 = w_8 = 0$. Furthermore, $\alpha_w = w(S) = 2w_1$.

Finally, as $\{v_1, v_5\}$ is stable, it follows that $w_5 \leq w_1$. We conclude the proof by choosing a w-cover consisting of w_1 times the 5-cycle $v_1v_3v_5v_7v_9$ at a cost of $2w_1$. □

9 Open questions

Let us conclude the paper with three open problems. Since odd holes and antiholes in a graph can be detected in polynomial time, see Chudnovsky et al [4], the strong perfect graph theorem implies that perfect graphs can be recognised in polynomial time. A similar result for t-perfect graphs would be desirable:

Question 25. *Is there an algorithm with polynomial running time to test whether a given (claw-free) graph is t-perfect?*

A closer inspection of our proofs reveals that for such an algorithm it would be enough to detect K_4-t-minors in polynomial time. We note that Gerards [17] describes an algorithm with polynomial running time that tests whether a graph contains an odd-K_4 subdivision as a subgraph. (An odd-K_4 subdivision is a subdivision of K_4 in which each of the four triangles has become an odd cycle.)

In Theorem 3 we have determined all minimally t-imperfect graphs that are claw-free. To find all minimally t-imperfect graphs, with or without claws, seems a daunting task. Indeed, the class of these graphs already includes two infinite families, namely the odd wheels and the even Möbius ladders, see Shepherd [27]. Nevertheless, the examples that we know suggest two properties that all minimally t-imperfect graphs might share:

Question 26. *Are all minimally t-imperfect graphs* 3*-connected?*

In the light of Lemma 18, for an affirmative answer to this question, it would suffice to prove that all minimally t-imperfect graphs have minimum degree 3.

Question 27. *For a minimally t-imperfect graph G, does TSTAB(G) have precisely one non-integral vertex?*

We note that this is false if G is only minimal subject to vertex deletion. An example for this is the K_4 with one edge replaced by a path of length 3.

Acknowledgment

The first author would like to express his gratitude towards András Sebő and Paul Wollan for very inspiring discussions. Both authors would like to thank András Sebő for his donation of Corollary 16 to the paper.

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Version Jan 2009

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