

Partitioning two-coloured complete multipartite graphs into monochromatic paths and cycles

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Abstract

We show that any complete k -partite graph G on n vertices, with $k \geq 3$, whose edges are two-coloured, can be covered with two vertex-disjoint monochromatic paths of distinct colours, given that the largest partition class of G contains at most $n/2$ vertices. This extends known results for complete and complete bipartite graphs.

Secondly, we show that in the same situation, all but $o(n)$ vertices of the graph can be covered with two vertex-disjoint monochromatic cycles of distinct colours, if colourings close to a split colouring are excluded. From this we derive that the whole graph, if large enough, may be covered with 14 vertex-disjoint monochromatic cycles.

keywords: monochromatic path partition, monochromatic cycle partition, two-coloured graph

MSC: 05C38, 05C55.

1 Introduction

Gerencsér and Gyárfás [5] observed the vertex set of any complete graph whose edges are coloured red and blue¹ can be partitioned into a red and a blue path. This is fairly easy: just take a maximal set S of vertices that span two paths P_1, P_2 , one in each colour, which only meet in one of their endvertices, call this

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¹Note that a colouring is never meant to be a proper colouring in this paper: any assignment of colours red and blue to the edges will do.

vertex x . One quickly checks that any vertex $v \notin S$ can be used to augment S : we can add the edge xv to the path P_i of the same colour, and then go from v on reversely through P_{3-i} . It is a long-standing conjecture that this phenomenon carries over to arbitrarily many colours.

Conjecture 1.1 (Gyárfás [7]). *Let G be a complete graph whose edges are coloured with r colours. Then G can be partitioned into r monochromatic paths.*

A stronger conjecture, replacing paths by cycles, had been put forward by Erdős, Gyárfás and Pyber [4], but was recently disproved by Pokrovskiy [18] for $r \geq 3$. (Here, and throughout the paper, a cycle is allowed to consist of a single vertex or an edge, or to be totally empty.) In the case $r = 2$, however, the stronger result with cycles does hold (even with cycles of distinct colours). This used to be known as Lehel’s conjecture, and was shown for all n by Bessy and Thomassé [2], after having been proved for large values of n by Łuczak, Rödl and Szemerédi [16] and by Allen [1].

Theorem 1.2 (Bessy and Thomassé [2]). *Let G be a complete graph whose edges are coloured red and blue. Then G can be partitioned into a red and a blue cycle.*

Together with Conlon, the second author showed in [3] that Theorem 1.2 literally extends to 2-local colourings: those are colourings with any number of colours, where each vertex is incident with at most two colours.

For arbitrary r , the best known bound on the number of vertex-disjoint cycles needed to cover the r -coloured complete graph K_n is $100r \log r$, if n is large, this bound is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [9]. For $r = 3$, the same authors show in [11] that there is a partition of all but $o(n)$ vertices of K_n into 3 or less monochromatic cycles. From this they deduce that 17 cycles partition the whole graph.

If one aims for similar results in complete bipartite graphs, it is reasonable to assume these are *balanced*, i.e. the two partition classes have the same size. As observed by several authors, an obstruction for partitions of two-coloured balanced complete bipartite graphs into two paths/cycles is a certain type of colouring, which we will now describe.

For any bipartite graph with partition classes U, V , call a red/blue colouring of $E(G)$ a *split colouring* if there are partitions $U = A \cup B$ and $V = C \cup D$ such that all edges in $E_G(A, C) \cup E_G(B, D)$ are blue, and all edges in $E_G(A, D) \cup E_G(B, C)$ are red. A split colouring is *proper* if $\min\{||A| - |C||, ||A| - |D||\} \geq 2$.

It is easy to see that a balanced complete bipartite graph with a proper split colouring cannot be partitioned into two monochromatic paths (even if the paths are allowed to have the same colour). That the converse is also true was shown by Pokrovskiy [18], improving an earlier result of Gyárfás and Lehel [6, 8] (they allowed one uncovered vertex).

Theorem 1.3 (Pokrovskiy [18]). *Let G be a balanced complete bipartite graph whose edges are coloured red and blue. If the colouring is not a proper split colouring, then G can be partitioned into a red and a blue path.*

It is not difficult to check that the vertices of any balanced complete bipartite graph with a proper split colouring can be partitioned into three monochromatic paths, or cycles.

Haxell [12] proved that any r -edge coloured balanced complete bipartite graph can be partitioned into $O((r \log r)^2)$ monochromatic cycles, and if $r = 3$, then 1695 monochromatic cycles suffice. In [14] the second author and Lang improve the bound for arbitrary r to $4r^2$, in [13] both authors and Lang improve the bound for $r = 3$ to 18 (both results are for large enough graphs only).

Here, for the first time to our best knowledge, monochromatic path or cycle partitions into complete multipartite graphs with more than two partition classes are investigated. We always assume that none of the partition classes is empty, e.g., a complete tripartite graph is assumed to be non bipartite. We call a multipartite graph *fair* if no partition class contains more than half of the vertices of the graph. Note that a complete bipartite graph is fair if and only if it is balanced, and, more generally, a complete multipartite graph is fair if and only if it admits a Hamilton cycle.

We show an extension of Theorem 1.3 to multipartite graphs with more than two partition classes under the necessary restriction of fairness. It is interesting that for these graphs, there is no analogue for the exceptional case of the split colouring.

Theorem 1.4. *Let G be a fair complete k -partite graph, with $k \geq 3$, whose edges are coloured red and blue. Then G can be partitioned into a red and a blue path.*

It seems plausible that Theorem 1.4 can be strengthened such that instead of two paths one can partition G into a path and a cycle. Indeed, we shall see this is true if we allow for at most one vertex to be uncovered.

Corollary 1.5. *Let G be a fair complete k -partite graph, with $k \geq 3$, whose edges are coloured red and blue. Then all but at most one vertex of G can be partitioned into a monochromatic path and a monochromatic cycle of distinct colours.*

We prove Theorem 1.4 and Corollary 1.5 in Section 2. The main tool for this proof is Lemma 2.2, which is shown in Section 3.

It is natural to ask whether Theorems 1.3 and 1.4 extend to cycle partitions instead of path partitions. Note that we need to exclude the situation that there is a proper split colouring between a partition class that contains half the vertices of the graph, and the rest of the graph. Clearly, in that case a partition into two monochromatic cycles cannot exist (while a partition into two monochromatic paths is possible, by Theorem 1.4).

We show an approximate result for the cycle partition problem in complete multipartite graphs, including the bipartite case. For this, we say that a colouring of the edges of a complete multipartite graph G with two colours is δ -close to a split colouring if by deleting at most $\delta|E(G)|$ edges we can make G bipartite and the colouring a split colouring.

Theorem 1.6. *For all $\delta > 0$ there is a n_0 such that the following holds for every fair complete k -partite graph G on $n > n_0$ vertices, with $k \geq 2$. If the edges of G are coloured red and blue, and the colouring is not δ -close to a split colouring, then there are two disjoint monochromatic cycles of distinct colours, which together cover all but at most δn vertices of G .*

It is easy to check that no colouring with two colours of a complete multipartite graph on n vertices, with at least three partition classes of size greater than $2\sqrt{\delta}n$, can be δ -close to a split colouring. So, for these graphs we can drop the condition on the colouring in Theorem 1.6. Also, notice that any complete multipartite graph on n vertices, with a colouring using two colours that is δ -close to a split colouring, contains three disjoint monochromatic cycles that together cover all but at most $8\sqrt{\delta}n$ vertices of the graph.²

We prove Theorem 1.6 in Section 5. The strategy uses the regularity method, and a well-known technique due to Łuczak for blowing up connected matchings of the reduced graph to cycles in the original graph. The existence of the connected matchings in the given circumstances is shown in Lemma 4.2 and Lemma 4.3 of Section 4.

Theorem 1.6 is probably not the best possible result. It might be possible to cover all but a constant number of vertices of our multipartite graph. For an open problem in this direction, and more discussion, see Section 7.

Using tools of Gyárfás et al. [10] and Haxell [12], we derive from Theorem 1.6 that a small finite number of monochromatic cycles is always sufficient to partition a multipartite graph.

Theorem 1.7. *Let G be a sufficiently large fair complete k -partite graph, whose edges are coloured red and blue. Then G can be partitioned into 14 monochromatic cycles. If $k = 2$, then G can be partitioned into 12 monochromatic cycles.*

We prove Theorem 1.7 in Section 6. We believe that the number of cycles can be dropped further, but the point of our result is rather that a reasonable finite number of cycles always suffices.

We end the introduction with a useful lemma, which tells us that for all our results for fair k -partite graphs with $k \geq 3$, we may restrict our attention to the tripartite case.

Lemma 1.8. *Every fair k -partite graph G with $k \geq 3$ has a spanning induced subgraph which is fair and tripartite.*

Proof. Assume $k \geq 4$ and delete all edges between the smallest two partition classes. If the resulting graph is not fair, then these two classes together contain more than $|V(G)|/2$ vertices. But then, also the third and the fourth smallest class together have more than $|V(G)|/2$ vertices, a contradiction. Inductively, the statement follows. \square

2 Proof of Theorem 1.4

Throughout this section, let $G = (V, E)$ be a fair complete k -partite graph on n vertices whose edges are coloured red and blue. Let V_1, \dots, V_k be the partition classes of G and let $n_i = |V_i|$ for $i = 1, \dots, k$, with $n_1 \geq \dots \geq n_k$. Our aim is to partition G into a red and a blue path. By Lemma 1.8, we may assume $k = 3$.

²In fact, in the graph induced by the edges of the split colouring, we can delete a balanced set of at most $2\sqrt{\delta}n$ vertices so that in the remaining graph H , each vertex has at most $\sqrt{\delta}n$ non-neighbours in the other partition class. Then we split H into three monochromatic balanced bipartite graphs, two in blue, and one in red. It is easy to check that each of these three graphs has a cycle covering all but at most $2\sqrt{\delta}n$ of its vertices.

In the following, we will aim to find a red path and a blue path sharing a common endvertex, but otherwise disjoint. Slightly abusing notation, we will find it convenient to call this a partition into a red path and a blue path sharing a common endvertex.

Lemma 2.1. *If $n_1 = n/2$, then G can be partitioned into a red and a blue path sharing a common endvertex.*

Proof. Consider the balanced complete bipartite graph $H = (V, E_G(V_1, V_2 \cup V_3))$ whose partition classes are V_1 and $V_2 \cup V_3$. By Theorem 1.3, H and thus G can be partitioned into a red and a blue path if the edges of H are not split-coloured. So, we may assume that the edges of H are coloured with a split colouring, that is, there are disjoint non-empty sets $A, B \subseteq V_1$ and $C, D \subseteq V_2 \cup V_3$ with $A \cup B = V_1$ and $C \cup D = V_2 \cup V_3$, such that $E_G(A, C) \cup E_G(B, D)$ is entirely coloured blue, and $E_G(A, D) \cup E_G(B, C)$ is entirely coloured red.

Since $V_2, V_3 \neq \emptyset$ we have that $E_G(C, D) \neq \emptyset$. Let $uv \in E_G(C, D)$, say uv is blue. Let P_u (P_v) be a blue path in H , of maximum even length starting in u (in v). Let P be the blue path $P_u uv P_v$. If $|A| \leq |C|$, then $|D| \leq |B|$, and thus $A \cup D \subseteq V(P)$. Otherwise, $B \cup C \subseteq V(P)$. In both cases $H[V \setminus V(P)]$ is a balanced complete bipartite graph with red edges only. Hence, G can be partitioned into a red and a blue path.

It is straightforward that there is an edge joining an endvertex of the red path to an endvertex of the blue path. \square

The following is the main tool for our proof of Theorem 1.4.

Lemma 2.2. *Assume that $n_1 \leq n_2 + n_3 - 2$, or that $n_1 = n_2 + n_3 - 1$ and $n_3 > 1$. Then G can be partitioned into a red and a blue path sharing a common endvertex.*

The proof of Lemma 2.2 is the subject of Section 3.

Lemma 2.3. *Assume that $n_1 = n_2$ and $n_3 = 1$. Then G can be partitioned into a red and a blue path sharing a common endvertex.*

Proof. Say $V_3 = \{z\}$. Consider the balanced complete bipartite graph H induced by the vertex set $V_1 \cup V_2$. First, assume that the edges of H are not properly split-coloured. Then by Theorem 1.3, H can be partitioned into a red path R and a blue path B . Clearly, as H is balanced, there are an endvertex r of R and an endvertex b of B which lie in distinct partition classes. We may assume the edge rb to be blue, the other case is analogous. If the edge rz is red, we may extend the red path to include z . Together with the edge bz , this gives the desired partition. Otherwise, if rz is blue, we may extend the blue path to include z . Together with the edge $r'z$, where r' is the second to last vertex on R , this gives the desired partition.

So we may assume that H is properly split coloured. That is, there are disjoint non-empty sets $A, B \subseteq V_1$ and $C, D \subseteq V_2$ with $A \cup B = V_1$ and $C \cup D = V_2$, such that $E_G(A, C) \cup E_G(B, D)$ is entirely coloured blue, and $E_G(A, D) \cup E_G(B, C)$ is entirely coloured red. Now, there are two colour-components, either $A \cup D$ and $B \cup C$, or $A \cup C$ and $B \cup D$ which are connected in G via z . We treat the case that there are vertices Let $a \in A, b \in B$ such that the edges az, bz are red, all other cases can be treated analogously.

Choose a longest balanced red path X starting in a , and a longest balanced red path Y starting in b (where balanced means the path should have an even number of vertices). Take a longest blue path Z covering $G - (V(X) \cup V(Y) \cup \{z\})$. The latter choice is possible since $A \cup C \subseteq V(X) \cup V(Y)$ or $B \cup D \subseteq V(X) \cup V(Y)$. Thus, G can be partitioned into the red path XzY and the blue path Z . \square

We are now ready for the proof of our first main theorem.

Proof of Theorem 1.4. By Lemma 1.8, it suffices to prove our result for $k = 3$. If either $n_1 \leq n_2 + n_3 - 2$, or $n_1 = n_2 + n_3 - 1$ and $n_3 > 1$, we may apply Lemma 2.2 and are done, so assume otherwise. Then either $n_1 = n_2$ and $n_3 = 1$, in which case we may apply Lemma 2.3 and are done, or $n_1 \geq n_2 + n_3$, which we assume from now on. Since G is fair, we actually have $n_1 = n_2 + n_3$, and are thus in conditions to apply Lemma 2.1 to obtain the desired partition. \square

We end this section with the proof of Corollary 1.5.

Proof of Corollary 1.5. By Lemma 1.8, we may assume G is tripartite, with partition classes V_1, V_2, V_3 . We observe that the lemmas we use to prove Theorem 1.4, Lemmas 2.1, 2.2, and 2.3, yield a partition of G into a red and a blue path, say R and B , that share their last vertex x . Say among all such partitions, R is chosen of maximum length. W.l.o.g. assume $x \in V_1$.

If any of the two paths R, B is trivial, or has only one edge, we are done. Note that we may assume all edges between the first two and the last two vertices of R to be blue, and all edges between the first two and the last two vertices of B to be red, as otherwise we are done. In particular, by maximality of R , this implies that the first vertex v_1 on B lies in V_1 .

First assume the first vertex w_1 on R does not lie in V_1 . Then w_1x is blue, and by maximality of R , we know that w_1v_1 is blue, too. Thus we are done.

So assume $w_1 \in V_1$. By maximality of R , we know w_1 sends a blue edge to the second last vertex on B . Let v_2 be the second vertex on B . Now if v_2w_1 is red we find a red cycle and a blue path covering all but one vertex, and if v_2w_1 is blue we find a blue cycle and a red path covering all but one vertex. \square

3 Proof of Lemma 2.2

Before turning to Lemma 2.2 we shall state the following simple lemma for later use. Since the proof is straightforward, we omit it.

Lemma 3.1. *Assume that $n_i \leq n_{3-i} + n_3 - 1$ for some $i \in \{1, 2\}$, and let P be any Hamilton path in G . Then either P contains an edge $uv \in E_G(V_{3-i}, V_3)$, or $n_i = n_{3-i} + n_3 - 1$ and both endvertices of P are in $V_{3-i} \cup V_3$.*

The remainder of this section is devoted to the proof of Lemma 2.2. For easier notation it will be convenient to now refrain from the assumption that $n_1 \geq n_2$. We still keep the convention that V_3 is the smallest of the three classes, that is, $n_3 \leq \min\{n_1, n_2\}$. We thus have to prove the following statement.

Assume that $n_i \geq n_3$ for $i \in \{1, 2\}$, and that either $n_i \leq n_{3-i} + n_3 - 2$, or $n_i = n_{3-i} + n_3 - 1$ and $n_3 > 1$. Then G can be partitioned into a red and a blue

path sharing a common endvertex.

The assumptions of the lemma imply that $\min\{n_1, n_2\} \geq 2$. Let $v_1 \in V_1$ and $v_2 \in V_2$ be arbitrary, let $H = G - \{v_1, v_2\}$, and let n'_1, n'_2, n'_3 be the sizes of the partition classes of H . If n'_1, n'_2, n'_3 satisfy the statement above, we may apply induction to see that H can be partitioned into a red path R and a blue path B that share a common endvertex.

If n'_1, n'_2, n'_3 violate the statement above, it must be that $\min\{n'_1, n'_2\} < n'_3$. As $n'_1 = n_1 - 1$ and $n'_2 = n_2 - 1$, we may w.l.o.g. assume that $n_2 = n_3$. So, $n_1 \geq n_2 = n_3$.

Let us first discuss the case $n_1 = n_2 = n_3$. It must be that $n_1 = 2$, for otherwise $n'_3 \leq n'_1 + n'_2 - 1$, $n'_1 \leq n'_2 + n'_3 - 1$ and $n'_2 > 1$, a contradiction to our assumption that n'_1, n'_2, n'_3 violate the statement. Thus $n'_3 = n'_1 + n'_2$ and so we may apply Lemma 2.1 to obtain a red path R and a blue path B sharing a common endvertex that partition H .

Now assume $n_1 > n_2 = n_3$. Hence, $n'_1 \geq n'_3 > n'_2 \geq 1$. Again we have $n'_1 \leq n'_2 + n'_3 - 1$, $n'_3 \leq n'_1 + n'_2 - 1$. Since n'_1, n'_2, n'_3 violate the statement, it must be that $n'_2 = 1$. As $n_2 \geq n_3$ and $n'_2 = n_2 - 1$, we have $n_2 = 2$ and $n'_3 = n_3 = 2$. Therefore $n_1 = 3$ and thus $n'_1 = 2$. So, we can apply Lemma 2.3 to H and obtain a red path R and a blue path B sharing a common endvertex that partition H .

Summing up, we may inductively assume Lemma 2.2 to hold for H . Hence, we know that H can be partitioned into a red path R and a blue path B , both possibly trivial, that share a common endvertex. Let $R = (r_1, \dots, r_s, x)$ and $B = (b_1, \dots, b_t, x)$ where R and B have only x in common.

Throughout the rest of the proof we suppose for contradiction that G cannot be partitioned into a red and a blue path that share a common endvertex. In several cases treated below we make use of the following simple fact.

Fact 3.2. *Assume we can partition G into a red path R' and a blue path B' such that one of these paths has its endvertices in distinct partition classes. Then one of R', B' can be extended such that the paths have exactly one vertex in common, namely an endvertex of both paths. The same holds if an endvertex of one path and an endvertex of the other path are in distinct partition classes.*

Below, we will construct path partitions in several different cases. In some cases we will explicitly check that the constructed paths have their endvertices in distinct partition classes (as in Fact 3.2), but we will omit this if it is straightforward.

For the remainder of the proof, we assume w.l.o.g. that v_1v_2 is red. At this point, we advise the reader to get their coloured pencils ready.

Claim 3.3. *Neither R nor B is trivial. (That is $R \neq (x) \neq B$.)*

Proof. We first suppose that the path R is trivial, that is, B covers H . W.l.o.g. assume $b_1 \notin V_1$. If the edge b_1v_1 is red, then G can be partitioned into the red path (v_2, v_1, b_1) and the blue path (b_1, \dots, b_t, x) , a contradiction. Otherwise, G can be partitioned into the red path (v_2, v_1) and the blue path $(v_1, b_1, \dots, b_t, x)$, another contradiction.

Now suppose that B is trivial, that is, R covers H . Again, we may w.l.o.g. assume that $r_1 \notin V_1$. The edge r_1v_1 is blue, since otherwise the red path $(v_2, v_1, r_1, \dots, r_s, x)$ covers G , a contradiction.

If $r_1 \in V_3$, then $r_1v_2 \in E$. In this case, however, G is covered by the red path $(v_1, v_2, r_1, \dots, r_s, x)$, if r_1v_2 is red. If r_1v_2 is blue, G can be partitioned into the red path (r_2, \dots, r_s, x) and the blue path (v_1, r_1, v_2) . As both cases are contradictory, $r_1 \notin V_3$ and so $r_1 \in V_2$. By symmetry, $x \notin V_3$.

First we suppose that $x \in V_1$. Then $xv_2, xr_1 \in E$. Like above, xv_2 must be blue. But then r_1x is red, for otherwise G can be partitioned into the red path (r_2, \dots, r_s) and the blue path (v_1, r_1, x, v_2) , a contradiction (because of the observation stated before the claim we are presently proving). We may pick $i \in \{2, \dots, s\}$ such that $r_i \in V_3$. If the edge v_1r_i is red, G can be partitioned into the red path $(v_2, v_1, r_i, \dots, r_s, x, r_1, \dots, r_{i-1})$, a contradiction. Thus, by symmetry, both v_1r_i and v_2r_i are blue. This means G can be partitioned into the red path $(r_{i+1}, \dots, r_s, x, r_1, \dots, r_{i-1})$ and the blue path (v_1, r_i, v_2) , a contradiction.

So, $x \in V_2$. Then $v_1x \in E$, and this edge must be blue, as it is interchangeable with v_1r_1 . Consider the edge v_2r_2 : it must be blue, else G can be partitioned into the red path $(v_1, v_2, r_2, \dots, r_s, x)$ and the blue path (v_1, r_1) . Thus, the edge r_2x is red, since otherwise G can be partitioned into the red path $(r_3, \dots, r_{s-1}, r_s)$ and the blue path (v_2, r_2, x, v_1, r_1) .

By Lemma 3.1, we may pick $i \in \{2, \dots, s\}$ such that $r_i \in V_3$ and $r_{i-1} \in V_1$ or $r_{i+1} \in V_1$. Note that we may apply Lemma 3.1 even if $n'_i = n'_{3-i} + n'_3 - 1$, for some $i \in \{1, 2\}$, since $r_1 \in V_2$. Let us assume that $r_{i-1} \in V_1$, the other case is similar. If r_iv_1 is red, then G can be partitioned into the red path $(v_2, v_1, r_i, \dots, r_s, x, r_2, \dots, r_{i-1})$ and the blue path (r_1) , a contradiction. Similarly, r_iv_2 cannot be red. So, both r_iv_1 and r_iv_2 are blue. But now G can be partitioned into the red path $(r_{i+1}, \dots, r_s, x, r_2, \dots, r_{i-1})$ and the blue path (v_2, r_i, v_1, r_1) , which is contradictory. This completes the proof of Claim 3.3. \square

Over the next few claims, we deal with the case that $x \in V_3$.

Claim 3.4. *If $x \in V_3$, then $r_1 \notin V_3$.*

Proof. Suppose $x, r_1 \in V_3$. Consider the edges xv_1 and xv_2 . If any of these edges is red, then the respective path $(r_1, \dots, r_s, x, v_1, v_2)$ or $(r_1, \dots, r_s, x, v_2, v_1)$ is red, and together with the blue path (b_1, \dots, b_t) covers G . Thus xv_1 and xv_2 are blue.

If either of r_1v_1, r_1v_2 is red, we may simply extend R from r_1 to v_1 and v_2 . So these two edges are blue, too. Observe that one of the blue paths $(b_1, \dots, b_t, x, v_1, r_1, v_2)$, $(b_1, \dots, b_t, x, v_2, r_1, v_1)$ has its endpoints in different partition classes. Together with the red path (r_2, \dots, r_s) , this blue path covers G , a contradiction. \square

Claim 3.5. *Let $i \in \{1, 2\}$. If $x \in V_3$, then at most one of r_1 and r_s lie in V_i .*

Proof. Because of symmetry, it is enough to prove this claim for $i = 1$. So for contradiction suppose that $x \in V_3$ and $r_1, r_s \in V_1$.

As above, we see that xv_1 and r_1v_2 are blue. Also, r_sv_2 is blue, since otherwise G can be partitioned into the red path $(r_1, \dots, r_s, v_2, v_1)$ and the blue path $(v_1, x, b_t, \dots, b_1)$.

We claim that

$$b_1 \in V_1. \tag{1}$$

Suppose $b_1 \notin V_1$. Then xv_2 must be blue, since otherwise G can be partitioned into the red path $(r_1, \dots, r_s, x, v_2, v_1)$ and the blue path (b_1, \dots, b_t) . As $v_1b_1 \in$

E , this edge must be red, for otherwise G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(v_2, x, b_t, \dots, b_1, v_1)$. Thus, $r_1 b_1$ is blue, else G can be partitioned into the red path $(r_s, \dots, r_1, b_1, v_1, v_2)$ and the blue path $(v_2, x, b_t, \dots, b_2)$. Hence, G can be partitioned into the red path (r_2, \dots, r_s) and the blue path $(v_1, x, b_t, \dots, b_1, r_1, v_2)$, a contradiction. This proves (1).

Hence, the edge $b_1 v_2$ exists, and

$$b_1 v_2 \text{ is red,} \quad (2)$$

since otherwise G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(v_1, x, b_t, \dots, b_1, v_2)$. If $t = 1$, $x b_1$ is blue by definition. If $t \geq 2$ and $x b_1$ is red, G can be partitioned into the red path $(r_1, \dots, r_s, x, b_1, v_2, v_1)$ and the blue path (b_2, \dots, b_t) (note that by (1) we know that $b_2 \notin V_1$). Thus,

$$x b_1 \text{ is blue.} \quad (3)$$

We now show that

$$b_t \in V_1. \quad (4)$$

Indeed, suppose otherwise. Then the edge $b_t r_1$ exists. If $b_t r_1$ is blue, then G can be partitioned into the red path (r_2, \dots, r_s) and the blue path $(v_1, x, b_1, \dots, b_t, r_1, v_2)$. This is contradictory since $v_1 \in V_1$ and $v_2 \in V_2$.

So $b_t r_1$ is red. Since $x b_t \in E$, we have $b_t \in V_2$. Thus, $v_1 b_t \in E$, and this edge must be blue: otherwise G can be partitioned into the red path $(x, r_s, \dots, r_1, b_t, v_1, v_2)$ and the by (3) blue path (x, b_1, \dots, b_t) . Moreover, $x v_2$ must be blue, else G can be partitioned into the red path $(r_1, \dots, r_s, x, v_2, v_1)$ and the blue path (v_1, b_t, \dots, b_1) . But this means G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(v_2, x, v_1, b_t, \dots, b_1)$, a contradiction. This proves (4).

By (4), the edge $b_t v_2$ exists. However, this edge cannot be blue, as then G can be partitioned into the red path (r_s, \dots, r_1) and the blue path $(r_s, v_2, b_t, \dots, b_1, x, v_1)$. So, $b_t v_2$ is red.

We next show that

$$x r_1 \text{ is red.} \quad (5)$$

For contradiction, suppose that $x r_1$ is blue. Then, in particular, $s \geq 2$, since $x r_s$ is red. The edge $r_2 v_1$ must be blue, as otherwise G can be partitioned into the red path $(v_2, v_1, r_2, \dots, r_s)$ and the blue path $(v_2, r_1, x, b_t, \dots, b_1)$. Moreover, $b_t r_2$ is blue, as follows. Suppose $b_t r_2$ is red. If $t = 1$, G can be partitioned into the red path $(v_1, v_2, b_t, r_2, \dots, r_s, x)$ and the blue path (r_1, x) . Otherwise, b_{t-1} exists and does not belong to V_1 . Thus, G can be partitioned into the red path $(v_1, v_2, b_t, r_2, \dots, r_s)$ and the blue path $(r_1, x, b_1, \dots, b_{t-1})$. As both are contradictory, we see that $b_t r_2$ is blue. But now the red path (r_3, \dots, r_s) and the blue path $(r_s, v_2, r_1, x, v_1, r_2, b_t, \dots, b_1)$ partition G , a contradiction. This proves (5).

We now apply Lemma 3.1 to H , to see that there is an edge uv in R or B with $u \in V_3$ and $v \in V_2$. Note that we may apply Lemma 3.1 even if $n'_i = n'_{3-i} + n'_3 - 1$, for some $i \in \{1, 2\}$, since $r_1 \in V_1$ by assumption. Since $r_s, b_t \in V_1$, we know that $v \notin \{r_s, b_t\}$, and thus $u \neq x$.

We claim that

$$u, v \in V(B). \quad (6)$$

Indeed, suppose $u, v \in V(R)$. We assume v lies between u and x on R , say $u = r_i$ and $v = r_{i+1}$, the other case is similar. It must be that $r_i v_2$ is blue, since otherwise G can be partitioned into the red path $(v_1, v_2, r_i, r_{i-1}, \dots, r_1, x, r_s, \dots, r_{i+1})$ and the blue path (b_1, \dots, b_t) . Similarly, the edges $r_i v_1$ and $r_{i+1} v_1$ are blue.

Since $r_{i+1} \in V_2$ and $b_1 \in V_1$, $r_{i+1} b_1 \in E$. If $r_{i+1} b_1$ is red, then G can be partitioned into the red path $(v_1, v_2, b_1, r_{i+1}, \dots, r_s, x, r_1, \dots, r_i)$ and the blue path (b_2, \dots, b_t) (recall that $b_1 v_2$ is red by (2)). However, if $r_{i+1} b_1$ is blue, then G can be partitioned into the red path $(r_{i+2}, \dots, r_s, x, r_1, \dots, r_{i-1})$ and the blue path $(v_2, r_i, v_1, r_{i+1}, b_1, \dots, b_t)$. This proves (6).

Say v lies between u and x on B , say $u = b_i$ and $v = b_{i-1}$, the other case is similar. We now show that

$$r_1 b_i \text{ is blue.} \quad (7)$$

Indeed, suppose that the edge $r_1 b_i$ is red. Thus $b_i v_1$ must be blue: otherwise, G can be partitioned into the red path $(v_2, v_1, b_i, r_1, \dots, r_s)$ and the blue path $(b_{i-1}, \dots, b_1, x, b_t, \dots, b_{i+1})$. Similarly, $b_i v_2$ must be blue. Hence $b_{i-1} v_1$ must be red, as otherwise G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(v_2, b_i, v_1, b_{i-1}, \dots, b_1, x, b_t, \dots, b_{i+1})$. Supposing $b_{i-1} r_1$ is red, G can be partitioned into the red path $(v_2, v_1, b_{i-1}, r_1, \dots, r_s)$ and the blue path $(b_i, \dots, b_t, x, b_1, \dots, b_{i-2})$. Thus, the edge $b_{i-1} r_1$ is blue, and so G can be partitioned into the red path (r_2, \dots, r_s) and the blue path $(v_1, b_i, v_2, r_1, b_{i-1}, \dots, b_1, x, b_t, \dots, b_{i+1})$. This proves (7).

Now we see that $v_1 b_{i-1}$ must be red: otherwise, G can be partitioned into the red path (r_2, \dots, r_s) and the blue path $(v_2, r_1, b_i, \dots, b_t, x, b_1, \dots, b_{i-1}, v_1)$. Also, the edge $r_1 b_{i-1}$ is blue, as otherwise we can cover G with the red path $(v_2, v_1, b_{i-1}, r_1, \dots, r_s)$ and the blue path $(b_i, \dots, b_t, x, b_1, \dots, b_{i-2})$. Therefore, $b_i v_1$ is red, as otherwise G can be partitioned into the red path (r_2, \dots, r_s) together with the blue path $(v_2, r_1, b_{i-1}, \dots, b_1, x, b_t, \dots, b_i, v_1)$.

This implies that $r_s b_{i-1}$ is red, else G could be covered by the red path (r_2, \dots, r_{s-1}) and the blue path $(v_1, x, b_t, \dots, b_i, r_1, v_2, r_s, b_{i-1}, \dots, b_1)$. But then we can cover G with the red path $(v_1, b_{i-1}, r_s, \dots, r_2)$ and the blue path $(b_{i-2}, \dots, b_1, x, b_t, \dots, b_i, r_1, v_2)$ (here we use (3) and (7)), giving the final contradiction. \square

Claim 3.6. *Let $i \in \{1, 2\}$. If $x \in V_3$, then at most one of $r_1 \in V_i$ and $r_s \in V_{3-i}$ hold.*

Proof. For symmetry, we only need to treat the case $i = 1$. So suppose $r_1 \in V_1$ and $r_s \in V_2$. As before one can easily see that the edges $v_2 r_1$ and $v_1 x$ must be blue. Moreover,

$$r_s v_1 \text{ is blue,} \quad (8)$$

since otherwise G can be partitioned into the red path $(r_1, \dots, r_s, v_1, v_2)$ and the blue path (b_1, \dots, b_t, x) . Thus,

$$r_1 r_s \text{ is red,} \quad (9)$$

since otherwise G can be partitioned into the red path (r_2, \dots, r_{s-1}) and the blue path $(v_2, r_1, r_s, v_1, x, b_t, \dots, b_1)$.

Let us show that

$$b_1 \in V_1. \quad (10)$$

Suppose otherwise. Then $r_1b_1 \in E$, and this edge must be red: if it was blue, G could be covered by the red path (r_2, \dots, r_s) together with the blue path $(r_s, v_1, x, b_t, \dots, b_1, r_1, v_2)$. Moreover, $v_1b_1 \in E$, and this edge must be blue, since otherwise G can be partitioned into the red path $(x, r_s, \dots, r_1, b_1, v_1, v_2)$ and the blue path (x, b_t, \dots, b_2) . Also, xv_2 is blue, else G can be partitioned into the red path $(r_1, \dots, r_s, x, v_2, v_1)$ and the blue path (b_1, \dots, b_t) . Now, however, G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(v_2, x, b_t, \dots, b_1, v_1)$, a contradiction. This proves (10).

By (10), we know that $v_2b_1 \in E$, and this edge must be red, as otherwise G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(r_s, v_1, x, b_t, \dots, b_1, v_2)$ (recall that r_sv_1 is blue by (8)). Thus, r_sb_1 is blue, else G can be partitioned into the red path $(r_1, \dots, r_s, b_1, v_2, v_1)$ and the blue path $(v_1, x, b_t, \dots, b_1)$.

Let $j \in \{1, 2\}$ such that $n_j \geq n_{2-j}$. Hence $n_j \leq n_{2-j} + n_3 - 2$, so $n_3 \geq 2$. Thus, there is a vertex $y \in V_3 \setminus \{x\}$. We show that

$$y \in V(B). \quad (11)$$

Suppose otherwise, say $y = r_i$. Then, r_iv_2 is blue, as else G can be partitioned into the red path $(v_1, v_2, r_i, \dots, r_2, r_1, r_s, \dots, r_{i+1})$ (recall that r_1r_s is red by (9)) and the blue path (b_1, \dots, b_t, x) . Similarly, r_iv_1 is blue. But now G can be partitioned into the red path $(r_{i-1}, \dots, r_1, r_s, \dots, r_{i+1})$ and the blue path $(v_2, r_i, v_1, x, b_t, \dots, b_1)$. This proves (11).

So say $y = b_i$. Thus, v_2b_i is red, otherwise G can be partitioned into the red path (r_1, \dots, r_{s-1}) and the blue path $(v_2, b_i, \dots, b_t, x, v_1, r_s, b_1, \dots, b_{i-1})$. The edge r_1b_i is red, too, else G can be partitioned into the red path (r_2, \dots, r_{s-1}) and the blue path $(v_2, r_1, b_i, \dots, b_t, x, v_1, r_s, b_1, \dots, b_{i-1})$. (For this, note that $r_{s-1} \notin V_2$.) But now G can be partitioned into the red path $(v_2, b_i, r_1, \dots, r_{s-1})$ and the blue path $(b_{i+1}, \dots, b_t, x, v_1, r_s, b_1, \dots, b_{i-1})$, a contradiction. This finishes the proof of the claim. \square

Putting Claims 3.4, 3.5 and 3.6 together, and noting that if $x \in V_3$, then $r_s \notin V_3$, we obtain the following assertion.

Claim 3.7. $x \notin V_3$.

We now turn to the case that $x \notin V_3$. We first show an auxiliary claim.

Claim 3.8. *Let $i \in \{1, 2\}$. If $x \in V_i$ then the edges xv_{3-i} , r_sv_i are blue, and $r_1 \notin V_3$. Furthermore, if $r_1 \in V_j$ for some $j \in \{1, 2\}$, then r_1v_{3-j} is blue.*

Proof. For symmetry, it is enough to show this claim for $i = 1$. So, assume $x \in V_1$. Then, the edge xv_2 is present in G . If xv_2 is red, then the red path $(v_1, v_2, x, r_s, \dots, r_1)$ and the blue path (b_1, \dots, b_t) together cover G . So we know that xv_2 is blue.

The edge r_sv_1 is present in G as $r_s \notin V_1$. If this edge is red, G can be partitioned into the red path $(r_1, \dots, r_s, v_1, v_2)$ and the blue path (x, b_1, \dots, b_t) , a contradiction. Thus, r_sv_1 is blue.

Similarly, neither of the edges r_1v_1 and r_1v_2 , if present, can be red. On the other hand, at most one of r_1v_1 and r_1v_2 can be present and blue, else G can be partitioned into the red path (r_2, \dots, r_s) and the blue path $(v_1, r_1, v_2, x, b_t, \dots, b_1)$. Thus, either r_1v_1 or r_1v_2 is absent from G , which implies that $r_1 \notin V_3$. This proves the claim. \square

Claim 3.9. *Let $i \in \{1, 2\}$. If $x \in V_i$ then $r_1 \notin V_{3-i}$.*

Proof. For symmetry, it is enough to show this claim for $i = 1$. So, for contradiction, assume $x \in V_1$ and $r_1 \in V_2$. By Claim 3.8, the edges xv_2 , $r_s v_1$ and $r_1 v_1$ are blue.

We show first that

$$b_1 \notin V_1. \quad (12)$$

Suppose otherwise. Then clearly $r_1 b_1$ is red, since otherwise G can be partitioned into the red path (r_2, \dots, r_s) and the blue path $(v_1, r_1, b_1, \dots, b_t, x, v_2)$. Thus the edge $v_2 b_1$ is blue, otherwise G can be partitioned into the red path $(v_1, v_2, b_1, r_1, \dots, r_s)$ and the blue path (x, b_t, \dots, b_2) . A similar argument shows that $r_1 x$ is red. Moreover, $r_s b_1$ is red, for otherwise G can be partitioned into the red path (r_2, \dots, r_{s-1}) and the blue path $(r_1, v_1, r_s, b_1, \dots, b_t, x, v_2)$. Also, the edge $b_t v_1$ is red, as otherwise G can be partitioned into the red path (r_2, \dots, r_{s-1}) and the blue path $(r_1, v_1, b_t, \dots, b_1, v_2, x)$. Hence, $b_1 b_t$ is blue, else G can be partitioned into the red path $(v_2, v_1, b_t, b_1, r_1, \dots, r_s, x)$ and the blue path (b_2, \dots, b_{t-1}) .

Let $y \in V_3$. First suppose that $y = r_i$ for some $1 \leq i \leq s$. Thus both $v_1 r_i$ and $v_2 r_i$ are blue: if $v_1 r_i$ is red, say, G can be partitioned into the red path $(v_2, v_1, r_i, \dots, r_1, x, r_s, \dots, r_{i-1})$ and the blue path (b_1, \dots, b_t) , a contradiction. (Note that $b_1 \in V_1$ and $b_t \notin V_1$.) But this means G can be partitioned into the red path $(r_{i+1}, \dots, r_s, x, r_1, \dots, r_{i-1})$ and the blue path $(v_1, r_i, v_2, b_1, \dots, b_t)$.

Thus $y = b_i$ for some $1 \leq i \leq t$. Hence, $v_1 b_i$ is red, as otherwise G can be partitioned into the red path (r_2, \dots, r_s) and the blue path $(r_1, v_1, b_i, \dots, b_1, v_2, x, b_t, \dots, b_{i+1})$. So, the edge $r_1 b_i$ must be blue, for otherwise G can be partitioned into the red path $(v_1, b_i, r_1, \dots, r_s)$ and the blue path $(b_{i+1}, \dots, b_t, x, v_2, b_1, \dots, b_{i-1})$. Consequently, G can be partitioned into the red path (r_2, \dots, r_s) and the blue path $(v_1, r_1, b_i, b_{i+1}, \dots, b_t, x, v_2, b_1, \dots, b_{i-1})$. This finishes the proof of (12).

Next, we show that

$$b_1 \notin V_3. \quad (13)$$

Suppose otherwise. Then $r_1 b_1 \in E$ and this edge must be red, since otherwise G can be partitioned into the red path (r_2, \dots, r_s) and the blue path $(v_1, r_1, b_1, \dots, b_t, x, v_2)$. In the case that $v_1 b_1$ is red, G can be partitioned into the red path $(v_2, v_1, b_1, r_1, \dots, r_s)$ and the blue path (x, b_t, \dots, b_1) . So $v_1 b_1$ is blue, thus G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(v_2, x, b_t, \dots, b_1, v_1)$. This finishes the proof of (13).

Putting (12) and (13) together, we see that

$$b_1 \in V_2.$$

Then $b_1 v_1$ must be red, as otherwise G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(v_2, x, b_t, \dots, b_1, v_1)$. Hence $x b_1$ is blue, since otherwise G can be partitioned into the red path $(r_1, \dots, r_s, x, b_1, v_1, v_2)$ and the blue path (b_2, \dots, b_t) . Moreover, $b_t v_1$ is red, otherwise G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(v_2, x, b_1, \dots, b_t, v_1)$.

We claim that

$$r_s, b_t \in V_2. \quad (14)$$

For this, first assume the edge $r_s b_t$ does exist. If $r_s b_t$ is red, G can be partitioned into the red path $(r_1, \dots, r_s, b_t, v_1, v_2)$ and the blue path (x, b_1, \dots, b_t) . If $r_s b_t$ is blue, however, G can be partitioned into the red path (r_1, \dots, r_{s-1}) and the blue path $(v_1, r_s, b_t, \dots, b_1, x, v_2)$. This shows that $r_s b_t$ does not exist. Since $x \in V_1$, we get that $r_s, b_t \in V_i$ for some $i \in \{2, 3\}$.

Now, if $r_s, b_t \in V_3$, then $r_s v_2 \in E$. If $r_s v_2$ is red, G can be partitioned into the red path $(r_1, \dots, r_s, v_2, v_1)$ and the blue path (x, b_t, \dots, b_1) . But if $r_s v_2$ is blue, G can be partitioned into the red path (r_1, \dots, r_{s-1}) and the blue path $(v_1, r_s, v_2, x, b_t, \dots, b_1)$. This proves (14).

Since $r_1 \in V_2$, we have that $r_2 b_t \in E$. We show that

$$r_2 b_t \text{ is blue.} \quad (15)$$

Suppose otherwise. Then also $r_1 b_{t-1}$ is red, as else G can be partitioned into the red path $(v_2, v_1, b_t, r_2, \dots, r_s, x)$ and the blue path $(b_1, \dots, b_{t-1}, r_1)$. Hence, the edge $v_2 b_{t-1}$ is blue, for otherwise G can be partitioned into the red path $(b_t, v_1, v_2, b_{t-1}, r_1, \dots, r_s, x)$ and the blue path (b_1, \dots, b_{t-2}) . So, $r_s b_{t-1}$ is blue, else G can be partitioned into the red path $(v_2, v_1, b_t, r_2, \dots, r_s, b_{t-1}, r_1)$ and the blue path (x, b_1, \dots, b_{t-2}) . Now, however, G can be partitioned into the red path $(b_t, r_2, \dots, r_{s-1})$ and the blue path $(r_1, v_1, r_s, b_{t-1}, v_2, x, b_1, \dots, b_{t-2})$. This proves (15).

Observe that

$$r_2 r_s \text{ is red,} \quad (16)$$

for otherwise G can be partitioned into the red path (r_{s-1}, \dots, r_3) and the blue path $(r_1, v_1, r_s, r_2, b_t, \dots, b_1, x, v_2)$.

We now apply Lemma 3.1 to H , to see that there is an edge uv on R or on B with $u \in V_3$ and $v \in V_1$. We may apply Lemma 3.1 even if $n'_i = n'_{3-i} + n'_3 - 1$, for some $i \in \{1, 2\}$, since $r_1 \in V_2$.

We claim that

$$u \in V(B). \quad (17)$$

Suppose otherwise, i.e. assume $u = r_i$ for some $2 \leq i \leq t$. We discuss the case $v = r_{i+1}$ only, the other case is similar. Observe that not both $v_1 r_i$ and $v_2 r_i$ can be blue, for otherwise the red path $(r_{i-1}, \dots, r_2, r_s, \dots, r_{i+1})$ and the blue path $(r_1, v_1, r_i, v_2, x, b_t, \dots, b_1)$ cover G . Say $v_1 r_i$ is red, the other case can be resolved similarly. If $r_1 x$ is blue, then we can cover G with the red path $(v_2, v_1, r_i, \dots, r_1, r_s, r_{s-1}, \dots, r_{i+1})$ and the blue path $(r_1, x, b_t, \dots, b_1)$. Thus, the edge $r_1 x$ is red, and therefore, G can be partitioned into the red path $(v_2 v_1, r_i, r_{i-1}, \dots, r_1, x, r_s, \dots, r_{i+1})$ and the blue path (b_1, \dots, b_t) . This proves (17).

Say $u = b_i$ for some $2 \leq i \leq t$. We assume $v = b_{i-1}$, as the other case is similar. If $v_1 b_i$ is blue, $v_2 r_2$ must be red: otherwise, G can be partitioned into the red path (r_3, \dots, r_s) and the blue path $(r_1, v_1, b_i, \dots, b_t, r_2, v_2, x, b_1, \dots, b_{i-1})$. But then G can be partitioned into the red path (v_2, r_2, \dots, r_s) and the blue path $(r_1, v_1, b_i, \dots, b_t, x, b_1, \dots, b_{i-1})$.

Thus, $v_1 b_i$ must be red. Hence, $r_1 b_i$ is blue, for otherwise we can cover G with the red path $(v_2, v_1, b_i, r_1, \dots, r_s)$ and the blue path $(b_{i+1}, \dots, b_t, b_1, \dots, b_{i-1})$. So, the edge $r_2 v_2$ is blue, as otherwise G can be partitioned into the red path $(v_1, v_2, r_2, \dots, r_s)$ and the blue path $(r_1, b_i, \dots, b_t, x, b_1, \dots, b_{i-1})$. But then G can be partitioned into the red path (r_3, \dots, r_{s-1}) and the blue path $(r_s, v_1, r_1, b_i, \dots, b_t, r_2, v_2, x, b_1, \dots, b_{i-1})$, yielding the final contradiction. \square

Claim 3.10. *Let $i \in \{1, 2\}$. If $x \in V_i$ then $r_1 \notin V_i$.*

Proof. Because of symmetry, we only show the claim for $i = 1$. So suppose $x \in V_1$ and $r_1 \in V_1$. By Claim 3.8, the edges $r_s v_1$, $v_2 x$ and $r_1 v_2$ are blue.

We first show that

$$b_1 \notin V_1. \quad (18)$$

Suppose otherwise. Then the edge $r_s b_1$ is red, for otherwise G can be partitioned into the red path (r_2, \dots, r_{s-1}) and the blue path $(v_1, r_s, b_1, \dots, b_t, x, v_2, r_1)$. Thus, the edge $b_1 v_2$ is blue, else G can be partitioned into the red path $(v_1, v_2, b_1, r_s, \dots, r_1)$ and the blue path (b_2, \dots, b_t, x) . Moreover, $r_2 v_1$ is blue, since otherwise G can be partitioned into the red path (v_1, r_2, \dots, r_s) and the blue path $(r_1, v_2, b_1, \dots, b_t, x)$. Hence, $r_2 b_1$ must be red, as otherwise G can be partitioned into the red path (r_{s-1}, \dots, r_3) and the blue path $(r_s, v_1, r_2, b_1, \dots, b_t, x, v_2, r_1)$. Thus, $v_1 b_2$ must be red, as otherwise G can be partitioned into the red path $(r_{s-1}, \dots, r_2, b_1)$ and the blue path $(r_s, v_1, b_2, \dots, b_t, x, v_2, r_1)$.

Applying Lemma 3.1 to H , we see that there is an edge uv in R or B with $u \in V_3$ and $v \in V_2$. Note that we may apply Lemma 3.1 even if $n'_i = n'_{3-i} + n'_3 - 1$, for some $i \in \{1, 2\}$, since $r_1 \in V_1$ by assumption.

First assume $u \in V(R)$, i.e. $u = r_i$ for some $2 \leq i \leq t$. Say $v = r_{i-1}$, the other case is similar. If $r_i v_1$ is red, then G can be partitioned into the red path $(v_1, r_i, \dots, r_2, b_1, r_s, \dots, r_{i+1})$ and the blue path $(r_1, v_2, x, b_t, \dots, b_2)$. Thus $r_i v_1$ is blue and, similarly, $r_{i-1} v_1$ is blue. Either r_i or r_{i-1} is adjacent to b_2 , say $r_i b_2 \in E$ (the other case is analogous). If $r_i b_2$ is red, G can be partitioned into the red path $(v_1, b_2, r_i, \dots, r_s, b_1, r_2, \dots, r_{i-1})$ and the blue path $(r_1, v_2, x, b_t, \dots, b_3)$. Otherwise, G can be partitioned into the red path $(r_{i-1}, \dots, r_2, b_1, r_s, \dots, r_{i+1})$ and the blue path $(r_1, v_2, x, b_t, \dots, b_2, r_i, v_1)$. However, both is contradictory, and we may thus assume that $u \in V(B)$.

So, $u = b_i$ for some $2 \leq i \leq t$. We discuss the case $v = b_{i-1}$ only, as the case $v = b_{i+1}$ can be treated similarly. Now, the edge $b_i v_1$ is red, since otherwise G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(v_1, b_i, b_{i+1}, \dots, b_t, x, v_2, b_1, \dots, b_{i-1})$. Similarly, $b_{i-1} v_1$ is red. Clearly $r_1 b_i \in E$, and this edge must be blue. Otherwise, G can be partitioned into the red path $(v_1, b_i, r_1, \dots, r_s)$ and the blue path $(b_{i-1}, \dots, b_1, v_2, x, b_t, \dots, b_{i+1})$. Similarly, $r_1 b_{i-1}$ is blue.

It is clear that $r_2 b_i \in E$ or $r_2 b_{i-1} \in E$, say $r_2 b_i \in E$. If the edge $r_2 b_i$ is blue, then we can cover G with the red path (r_3, \dots, r_s) and the blue path $(v_1, r_2, b_i, r_1, b_{i-1}, \dots, b_1, v_2, x, b_t, \dots, b_{i+1})$. Otherwise, we can partition G into the red path $(v_1, b_i, r_2, \dots, r_s)$ and the blue path $(r_1, b_{i-1}, \dots, b_1, v_2, x, b_t, \dots, b_{i+1})$. This finishes the proof of (18).

Next, we show that

$$b_1 \notin V_3. \quad (19)$$

Suppose otherwise. Then $v_1 b_1 \in E$ and this edge must be red, otherwise G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(v_1, b_1, \dots, b_t, x, v_2)$.

Now assume $r_s \in V_2$. In that case, if $r_s b_1$ is red, we cover G with the red path $(v_2, v_1, b_1, r_s, \dots, r_1)$ and the blue path (b_2, \dots, b_t, x) . If $r_s b_1$ is blue, we cover G with the red path (r_{s-1}, \dots, r_1) and the blue path $(v_1, r_s, b_1, b_2, \dots, b_t, x, v_2)$. Thus $r_s \notin V_2$ and hence $r_s \in V_3$.

But now, the edge $r_s v_2$ cannot be red, because of the red path $(r_1, \dots, r_s, v_2, v_1)$ and the blue path $(b_1 \dots, b_t, x)$. It also cannot be blue, because of the red path (r_1, \dots, r_{s-1}) and the blue path $(b_1 \dots, b_t, x, v_2, r_s, v_1)$. This finishes the proof of (19).

By (18) and (19), we know that $b_1 \in V_2$. Thus $b_1 v_1 \in E$ and this edge must be red, as otherwise G can be partitioned into the red path (r_1, \dots, r_s) and the blue path $(v_1, b_1, \dots, b_t, x, v_2)$. So, $b_1 r_1$ is blue, for otherwise G can be partitioned into the red path $(v_2, v_1, b_1, r_1, \dots, r_s)$ and the blue path (x, b_t, \dots, b_1) . Thus, $b_t v_1$ is red, since otherwise G can be partitioned into the red path (r_2, \dots, r_s) and the blue path $(v_1, b_t, x, v_2, r_1, b_1, \dots, b_{t-1})$. This implies that the edge $b_t r_1$ is blue, as otherwise G can be partitioned into the red path $(v_2, v_1, b_t, r_1, \dots, r_s, x)$ and the blue path (b_{t-1}, \dots, b_1) . Moreover, $r_1 r_s$ is red, for otherwise G can be partitioned into the red path (r_2, \dots, r_{s-1}) and the blue path $(v_1, r_s, r_1, v_2, x, b_t, \dots, b_1)$. Also, $x b_1$ is blue, else G can be partitioned into the red path $(v_2, v_1, b_1, x, r_s, \dots, r_1)$ and the blue path (b_{t-1}, \dots, b_2) .

Applying Lemma 3.1 to H , we see that there is an edge uv in R or B with $u \in V_3$ and $v \in V_1$. Note that we may apply Lemma 3.1 even if $n'_i = n'_{3-i} + n'_3 - 1$, for some $i \in \{1, 2\}$, since we know that $b_1 \in V_2$.

First assume $u \in V(R)$, i.e. $u = r_i$ for some $2 \leq i \leq t$. We may suppose that $v = r_{i-1}$, the other case is similar. Then $r_i v_1$ is blue, for otherwise G can be partitioned into the red path $(v_2, v_1, r_i, \dots, r_s, r_1, \dots, r_{i-1})$ and the blue path (x, b_t, \dots, b_1) . Similarly, $r_i v_2$ is blue. But then G can be partitioned into the red path $(r_{i-1}, \dots, r_1, r_s, \dots, r_{i+1})$ and the blue path $(v_1, r_i, v_2, x, b_t, \dots, b_1)$, a contradiction.

Therefore, $u \in V(B)$. Say $u = b_i$ for some $2 \leq i \leq t$. Say $v = b_{i-1}$, the other case is similar. Thus, $b_i v_1$ is red, for otherwise G can be partitioned into the red path (r_2, \dots, r_s) and the blue path $(v_1, b_i, \dots, b_1, r_1, v_2, x, b_t, \dots, b_{i+1})$. Hence, the edge $r_1 b_i$ is blue, for otherwise G can be partitioned into the red path $(v_2, v_1, b_i, r_1, \dots, r_s)$ and the blue path $(b_{i-1}, \dots, b_1, x, b_t, \dots, b_{i+1})$.

If $r_s \in V_3$, then $r_s v_2 \in E$. But then $r_s v_2$ must be blue, since otherwise G can be partitioned into the red path $(r_1, \dots, r_s, v_2, v_1)$ and the blue path (x, b_t, \dots, b_1) . This implies that G can be partitioned into the red path (r_1, \dots, r_{s-1}) and the blue path $(v_1, r_s, v_2, x, b_t, \dots, b_1)$. So, $r_s \notin V_3$, and hence $r_s \in V_2$. Thus, the edge $r_s b_i$ exists and it must be red, for otherwise G can be partitioned into the red path (r_2, \dots, r_{s-1}) and the blue path $(v_1, r_s, b_i, \dots, b_t, x, v_2, r_1, b_1, \dots, b_{i-1})$, where $v_1 \in V_1$ and $r_2 \notin V_1$. Hence, G can be partitioned into the red path $(v_2, v_1, b_i, r_s, \dots, r_1)$ and the blue path $(b_{i+1}, \dots, b_t, x, b_1, \dots, b_{i-1})$, a contradiction. \square

Putting Claims 3.7, 3.9 and 3.10 together, we arrive at the final contradiction and thus complete the proof of Lemma 2.2.

4 Connected matchings

Let G be a graph whose edges are coloured red or blue. A monochromatic matching of G is called *connected* if it is contained in a connected component of the subgraph induced by the edges of the corresponding colour.

The following lemma, Lemma 4.1, says that any fair complete multipartite graph with at least three partition classes can be covered with two connected

matchings of distinct colours. This is a direct consequence of our Theorem 1.4, and thus there would be no reason to prove such a statement. But, as our aim is to apply the regularity method later, in order to pump up our paths/connected matchings to cycles that cover almost all vertices of the graph, we need a robust version of Lemma 4.1. (This version is given in Lemma 4.2 below.) It will be much easier to extend the proof of Lemma 4.1 for a robust version, than the one of Theorem 1.4. We prefer to give the proof of the exact version first, so that the idea becomes clear to the reader.

It will be convenient to formulate the two lemmas only for tripartite graphs. This is justified by Lemma 1.8.

Lemma 4.1. *Let G be a fair complete tripartite graph on an even number of vertices. If the edges of G are coloured red or blue, then there are two vertex-disjoint connected matchings of distinct colours that together cover all vertices of G .*

Proof. Say V_1 and V_2 are the largest two partition classes of G . Let $v_1 \in V_1$, $v_2 \in V_2$ and set $G' = G - \{v_1, v_2\}$. Note that G' is fair, and unless $|V_2| = 1$, we may apply induction to obtain two vertex-disjoint connected matchings of distinct colours that together cover all vertices of G' .

On the other hand, if $|V_2| = 1$, then $|V_3| = 1$, and thus by fairness, $|V_1| = 2$. In this case, any two disjoint edges e_1, e_2 cover all the vertices of G' . Clearly, it is either possible to choose e_1, e_2 of distinct colours, or in way that they give a monochromatic connected matching.

Resuming, in either case there are a red and a blue connected matching M_R, M_B that cover all of $V(G)$ except v_1, v_2 . Let V_R be the vertex set of the connected red component of G containing M_R , and let V_B be the analogue for blue. If one of the matchings is empty we just choose the vertex set of any component for V_R or V_B .

Also, V_R and V_B meet. In order to see this, observe that it is always possible to choose distinct $i, j \in \{1, 2, 3\}$ with $V_R \cap V_i \neq \emptyset \neq V_B \cap V_j$, say $x \in V_R \cap V_i$ and $y \in V_B \cap V_j$. Now, the edge xy is either red or blue, which implies that $V_R \cap V_B \neq \emptyset$.

Assume $v_1 v_2$ is red (the other case is analogous). Suppose that G cannot be covered with two connected matchings as desired. Thus, $\{v_1, v_2\} \cap V_R = \emptyset$, since otherwise we could add $v_1 v_2$ to M_R . This means that

$$\text{all edges going from } V_R \text{ to } \{v_1, v_2\} \text{ are blue.} \quad (20)$$

In particular, all edges going from $V_R \cap V_B$ to $\{v_1, v_2\}$ are blue. Hence, at least one of v_1, v_2 is contained in V_B , say $v_1 \in V_B$ (the other case is analogous), and moreover,

$$\text{if } v_2 \notin V_B \text{ then } V_R \cap V_B \subseteq V_2. \quad (21)$$

Now, as by (20), all edges from v_1 to V_R are blue, we know that $V_R \setminus V_1 \subseteq V_B$. So,

$$\text{if } v_2 \in V_B \text{ then } V_B \supseteq V_2 \cup V_3. \quad (22)$$

Otherwise $v_2 \notin V_B$. We claim that

$$\text{if } v_2 \notin V_B \text{ then } V_B \supseteq V_1 \cup V_3. \quad (23)$$

Indeed, if this was not the case, then $(V_1 \cap V_R) \setminus V_B \neq \emptyset$. As $V_3 \subseteq V_B$, the edges from $(V_1 \cap V_R) \setminus V_B$ to V_3 have to be red. But then $V_3 \subseteq V_R \cap V_B$, a contradiction to (21).

So, either by (22) or by (23), we find that V_B covers all but at most one partition class. Let M'_B be a largest blue matching in G such that $G'' := G - V(M'_B)$ is still fair. Since V_B covers all but at most one partition class, M'_B is a connected matching. If all edges in G'' are red, then G'' has a red connected perfect matching, and we are done.

So assume there is some blue edge $uv \in E(G'')$. By the choice of M'_B , the graph $G'' - \{u, v\}$ is not fair. This means neither of u, v lies in the largest partition class V_1'' of G'' , and, furthermore, $|V_1''| = |V(G'')|/2$. Thus all edges between V_1'' and the rest of G'' are red, and hence, we can cover G'' with a red connected matching. \square

We now give a robust version of Lemma 4.1. This is the result we need for applying regularity later.

Lemma 4.2. *Let G be a fair tripartite graph on n vertices, with partition classes V_1, V_2, V_3 , such that $|V_i| \geq 3\varepsilon n$, for some ε with $0 < \varepsilon < 1/5$. Suppose that*

$$d(v) \geq (1 - \varepsilon)(n - |V_i|)$$

for $i = 1, 2, 3$ and for each $v \in V_i$. If the edges of G are coloured red and blue, then there are two vertex-disjoint connected matchings of distinct colours that together cover all but at most $36\varepsilon n$ vertices of G .

Proof. Take a red connected matching M_R and a blue connected matching M_B , which together cover as much as possible of $V(G)$, while leaving $G' := G - V(M_R \cup M_B)$ fair. Let V_R, V_B be the vertex sets of the respective colour components, as above we may assume both are non-empty. We assume G' contains more than $36\varepsilon n$ vertices.

We claim that for all $i, j \in \{1, 2, 3\}$ with $i \neq j$ and $V_i \cap V_R \neq \emptyset$ and $|V_j \cap V_B| > \varepsilon n$, we have

$$(V_R \cap V_B) \cap (V_i \cup V_j) \neq \emptyset. \quad (24)$$

Indeed, by the degree condition of the lemma, there is an edge from $V_i \cap V_R$ to $V_j \cap V_B$, and clearly, one of its endpoints lies in $V_R \cap V_B$. This proves (24). Observe that (24) also holds if we interchange the roles of V_R and V_B .

Let

$$V_\varepsilon := V(G) \setminus (V_R \cup V_B).$$

Our next aim is to show that there are $\ell \in \{1, 2, 3\}$, $X \in \{R, B\}$ with

- (a) $V_X \supseteq V(G) \setminus (V_\ell \cup V_\varepsilon)$, and
- (b) $|V_\varepsilon \cap V_i| \leq \varepsilon n$ for $i \in \{1, 2, 3\}$, $i \neq \ell$.

First of all, since all edges are coloured, note that no vertex in V_ε may send an edge to $V_R \cap V_B$. So, by the degree condition of the lemma, we know that $|V_\varepsilon \cap V_\ell| \leq \varepsilon n$ for all ℓ with $(V_R \cap V_B) \setminus V_\ell \neq \emptyset$. In particular, if V_ℓ is such that $|V_\varepsilon \cap V_\ell| > \varepsilon n$, then $V_R \cap V_B \subseteq V_\ell$. This implies $|V_\varepsilon \cap V_i| \leq \varepsilon n$ and $|V_\varepsilon \cap V_j| \leq \varepsilon n$, where V_i, V_j are the other two partition classes. Moreover, since V_j has at least $3\varepsilon n$ vertices, we know that $V_j \setminus V_\varepsilon$ has at least εn vertices in either $V_R \setminus V_B$ or

in $V_B \setminus V_R$, say in $V_B \setminus V_R$. Then by (24), we have that $V_i \setminus V_\varepsilon$ is contained in $V_B \setminus V_R$. Again by (24) (interchanging the roles of i and j), we see that also $V_j \setminus V_\varepsilon$ is contained in $V_B \setminus V_R$. Thus, either (a) and (b) hold, or

$$|V_\varepsilon \cap V_\ell| \leq \varepsilon n \text{ for all } \ell \in \{1, 2, 3\}. \quad (25)$$

We now assume the latter assertion in order to show that (a) and (b) hold also in this case.

Since $|V(G')| > 36\varepsilon n$, and since G' is fair, the largest two partition classes V'_1, V'_2 of G' each contain at least $9\varepsilon n$ vertices. For notational ease, we assume $V'_i \subseteq V_i$ for $i = 1, 2$.

A well-known fact states that any graph H has a subgraph H' which has minimum degree at least half the average degree of H . Applying this fact to the bipartite graph spanned by V'_1 and V'_2 , in the majority colour, say this is red, we obtain sets $U_1 \subseteq V'_1$ and $U_2 \subseteq V'_2$ such that the minimum degree from U_i to U_{3-i} in red is greater than $2\varepsilon n$. In particular, $|U_i| > 2\varepsilon n$, for $i = 1, 2$.

By maximality of M_R , and since $G' - \{u_1, u_2\}$ is still fair for any red edge $u_1 u_2$ between U_1 and U_2 , we know that all edges between $U_1 \cup U_2$ and V_R are blue. In particular, the edges between $U_1 \cup U_2$ and some fixed $x \in V_R \cap V_B$ are blue. Without loss of generality, assume $x \notin V_1$. Since U_1 has size greater than $2\varepsilon n$, every vertex $y \in V_R \setminus V_1$ sends a blue edge to some blue neighbour of x in U_1 . Thus, by the definition of V_ε , we have

$$V(G) \setminus (V_\varepsilon \cup V_1) \subseteq V_B,$$

which is as desired for (a). Because of (25), also (b) holds. We have thus shown (a) and (b); let us assume they hold for $\ell = 1$ and $X = B$.

Set $V'_\varepsilon := V_\varepsilon \setminus V_1$. Now take a maximal blue matching M'_B in $G - V'_\varepsilon$ such that $G'' := G - V(M'_B)$ is still fair; by (a) we know that M'_B is connected in blue. Assume that $|V(G'')| > 36\varepsilon n$, as otherwise we are done.

By maximality of M'_B , for any blue edge in $E(G'' - V'_\varepsilon)$ we have that $G'' - e$ is not fair. Thus, similar as in the proof of Lemma 4.1, either all edges of $G'' - V'_\varepsilon$ are red, or G'' contains a spanning balanced bipartite graph H such that any blue edge in H is incident with V'_ε .

In either case, we can easily find a red connected matching in G'' covering almost all of G'' as follows. Take a maximal red connected matching M'_R in $G'' - V'_\varepsilon$, or in $H - V'_\varepsilon$, such that the remainder of the graph is still fair. Note that we may assume $M'_R \neq \emptyset$ as $|V'_\varepsilon| \leq 2\varepsilon n$ by (b). Let $xy \in M'_R$. The neighbourhoods of x and y in the uncovered part of $G'' - V'_\varepsilon$ or $H - V'_\varepsilon$ are large enough to span at least one red edge. Also, all edges between x and $G'' - V'_\varepsilon$ or $H - V'_\varepsilon$ are red, a contradiction to the maximality of M'_R . \square

We now give an analogue of Lemma 4.2 for bipartite graphs. With the obvious exclusion of the proper split colouring, we can cover all 2-edge-coloured balanced bipartite graphs that are almost complete bipartite, with two connected matchings of distinct colours.

Lemma 4.3. *Let G be a balanced bipartite graph on n vertices, with partition classes V_1, V_2 . Suppose G has minimum degree at least $(1 - \varepsilon)n/2$, for some ε with $0 < \varepsilon < 1/5$.*

If the edges of G are coloured red and blue, and this colouring is not a split

colouring, then there are two vertex-disjoint connected matchings of distinct colours that together cover all but at most $4\epsilon n$ vertices of G .

Proof. Take a red connected matching M_R and a blue connected matching M_B , together covering as much as possible of $V(G)$. Let V_R, V_B be the vertex sets of the respective colour components. We assume $G - V(M_R \cup M_B)$ contains more than $4\epsilon n$ vertices. As above we see that $V_R \cap V_B \neq \emptyset$, say there is a vertex $x \in V_R \cap V_B \cap V_2$.

Now, if there is an $X \in \{R, B\}$ and $i \in \{1, 2\}$ such that V_X covers all but a set V'_i of at most $2\epsilon n$ vertices of V_i , we can proceed as in the proof of Lemma 4.2. That is, we take a maximal matching M'_X in $G - V'_i$ in the colour corresponding to X , and note that M'_X is connected in this colour. Then all edges in $G - V'_i - V(M'_X)$ must have the other colour, and we can easily cover all but at most $4\epsilon n$ vertices of $G - V(M'_X)$ with a connected matching in this colour.

So, from now on, let us assume that there is no choice of $X \in \{R, B\}$ and $i \in \{1, 2\}$ as above. That is, we assume

$$|V_i \setminus V_X| > 2\epsilon n \text{ for all } X \in \{R, B\} \text{ and } i \in \{1, 2\}. \quad (26)$$

Set $V_\epsilon := V(G) \setminus (V_R \cup V_B)$. Since there can be no edges from x to $V_\epsilon \cap V_1$, we have that

$$|V_\epsilon \cap V_1| \leq \epsilon n. \quad (27)$$

Moreover,

$$\text{if } V_R \cap V_B \cap V_1 \neq \emptyset, \text{ then } |V_\epsilon \cap V_2| \leq \epsilon n. \quad (28)$$

Since there is no edge between $V_R \setminus V_B$ and $V_B \setminus V_R$, there are $j \in \{1, 2\}$ and $Z \in \{R, B\}$ such that $|V_j \cap (V_Z \setminus V_Y)| \leq \epsilon n$, where $Y \in \{R, B\}$ is such that $Y \neq Z$. In other words,

$$|V_j \setminus (V_\epsilon \cup V_Y)| \leq \epsilon n.$$

Together with (26), this implies that $|V_\epsilon \cap V_j| > \epsilon n$. So by (27), we have that $j = 2$, and by (28), we know that $V_R \cap V_B \cap V_1 = \emptyset$. Hence, $V_1 \setminus V_\epsilon$ is covered by the two disjoint sets $V_{1R} := V_1 \cap (V_R \setminus V_B)$ and $V_{1B} := V_1 \cap (V_B \setminus V_R)$.

By (26) and (27), we have that $|V_{1R}|, |V_{1B}| > \epsilon n$. Now, as no edge may exist between V_{1R} and $V_B \setminus V_R$, or between V_{1B} and $V_R \setminus V_B$, we see that $V_2 \setminus V_\epsilon = V_R \cap V_B$. Observe that thus, all edges between $V_2 \setminus V_\epsilon$ and V_{1R} are red, and all edges between $V_2 \setminus V_\epsilon$ and V_{1B} are blue.

Also, by definition of V_ϵ , all edges between $V_\epsilon \cap V_2$ and V_{1R} are blue, and all edges between $V_\epsilon \cap V_2$ and V_{1B} are red. Additionally, since there are no edges between $V_\epsilon \cap V_1$ and $V_2 \setminus V_\epsilon = V_R \cap V_B$, either $V_\epsilon \cap V_1$ is empty, or $V_2 \setminus V_\epsilon$ has less than ϵn vertices. In the first case, we have a split colouring, and in the latter case, we can easily cover all but at most $2\epsilon n$ vertices of G with two monochromatic connected matchings of distinct colours. \square

5 Covering almost everything with three cycles

5.1 Regularity preliminaries

In this subsection we introduce some very well-known concepts; the reader familiar with regularity is invited to skip this. We start giving the standard definition

of regularity. Given $\varepsilon > 0$ and disjoint subsets A, B of the vertex set of a graph G , we say that the pair (A, B) is ε -regular, and of density $d(A, B)$, if, for every pair (A', B') with $A' \subseteq A$, $|A'| \geq \varepsilon|A|$, $B' \subseteq B$, $|B'| \geq \varepsilon|B|$, we have

$$|d(A', B') - d(A, B)| < \varepsilon.$$

When there is no danger of confusion, we simply say that the pair (A, B) is ε -regular. It is well-known and easy to see that together with a pair (A, B) , its complement is ε -regular, and of density $1 - d(A, B)$.

For a graph G , we say a partition $V_0 \cup V_1 \cup \dots \cup V_t$ of its vertex set is ε -regular if the following hold:

- (i) $|V_1| = |V_2| = \dots = |V_t|$ and $|V_0| \leq \varepsilon|V(G)|$, and
- (ii) for each i , $1 \leq i \leq t$, all but at most εt^2 of the pairs (V_i, V_j) , $1 \leq j \leq t$, are ε -regular.

We state Szemerédi's regularity lemma [19] in its form with a prepartition. We say a partition $V_1 \cup V_2 \cup \dots \cup V_t$ refines another partition $W_1 \cup W_2 \cup \dots \cup W_s$ if for each i there is a j such that $V_i \subseteq W_j$.

Theorem 5.1 (Regularity lemma with prepartition). *For every $\varepsilon > 0$ and $m_0, s \in \mathbb{N}$, there is an $m_1 \in \mathbb{N}$ such that the following holds for each graph G on $n \geq m_1$ vertices, and with a partition W_1, \dots, W_s of its vertex set. There exists an ε -regular partition $V_0 \cup V_1 \cup \dots \cup V_t$ such that*

- $V_1 \cup V_2 \cup \dots \cup V_t$ refines $W_1 \cup W_2 \cup \dots \cup W_s$, and
- $m_0 \leq t \leq m_1$.

It is usual to define a reduced graph of a graph G , for a given ε -regular partition, and a threshold density ρ . This is the graph $R_G = ([t], E(R_G))$ which has an edge for each ε -regular pair of density at least ρ . (Note that in spite of the notation R_G , the reduced graph R_G depends on G , on ρ , and on the given partition.)

5.2 Proof of Theorem 1.6

Theorem 5.1 applied with parameters $\varepsilon \ll \delta$, $m_0 = 1/\varepsilon$ and $s = 2$ and $s = 3$, gives two values for m_1 of which we take the larger one. Let G be a fair complete multipartite graph on $n > m_1$ vertices whose edges are coloured red and blue. We assume the colouring is not δ -close to a split colouring.

Use Lemma 1.8 to obtain a spanning fair subgraph $G' \subseteq G$ that is complete tripartite or complete bipartite (possibly $G' = G$). In the case that the smallest partition class of G' has less than $\delta n/20$ vertices (and thus G' is tripartite), we delete this class, and the same number of vertices from the other classes, in a way that the obtained bipartite graph (which we still call G') is balanced. By the proof of Lemma 1.8, we know that $|E(G) \setminus E(G')| < \delta n^2/10$.

Now, Theorem 5.1 applied to the red subgraph of G' yields a partition of $V(G')$ refining the bi- or tripartition which is ε -regular in both colours. Using the majority colouring of each pair (that is, we use $\rho = 1/2$, and in case of a tie we give the edge any colour), we can work with a two-coloured reduced graph $R_{G'}$.

Note that the non-neighbours of a vertex $v \in V(R_{G'})$ correspond to irregular pairs containing v . So, there are at most $\sqrt{\varepsilon}t$ vertices v in $R_{G'}$ that have more than $\sqrt{\varepsilon}t$ non-neighbours in the other partition class(es). We discard these vertices from the reduced graph $R_{G'}$ to ensure that each vertex of the remaining graph $R'_{G'}$ has at most $\sqrt{\varepsilon}t$ non-neighbours in the other partition class(es).

Observe that the size of any class C_i of the bi- or tripartition lies between

$$\frac{|\hat{C}_i| - |V_0|}{|V_1|} - \sqrt{\varepsilon}t \geq \frac{|\hat{C}_i|}{|V_1|} - 2\sqrt{\varepsilon}t \quad \text{and} \quad \frac{|\hat{C}_i|}{|V_1|}, \quad (29)$$

where \hat{C}_i is the class of the bi-/tripartition of G' corresponding to C_i . So, since G' is fair, and since $|V_0| \leq \varepsilon n$, we have that $|C_1| \leq |C_2| + 3\sqrt{\varepsilon}t$, or $|C_1| \leq |C_2| + |C_3| + 2\sqrt{\varepsilon}t$, respectively, if C_1 is the largest partition class of $R'_{G'}$. If necessary, we discard at most $3\sqrt{\varepsilon}t$ vertices from C_1 to make the remaining graph $R''_{G'}$ fair.

Resuming, we have obtained a bi- or tripartite subgraph $R''_{G'}$ of $R_{G'}$ on at least $(1 - 4\sqrt{\varepsilon})t$ and at most t vertices. Let the partition classes of $R''_{G'}$ be denoted C'_1, C'_2 , and, if applicable, C'_3 . In $R''_{G'}$, every vertex of C'_i has at most $\sqrt{\varepsilon}t$ many non-neighbours outside of C'_i . Since $R''_{G'}$ is fair, $t - |C'_i| \geq t/2$, and so every vertex of C'_i has degree at least

$$t - |C'_i| - \sqrt{\varepsilon}t \geq (1 - 2\sqrt{\varepsilon})(t - |C'_i|),$$

where we sum the degree over both colours. We may view $R''_{G'}$ as a reduced graph for a subgraph G'' of G' , with $|V(G'')| \geq (1 - 5\sqrt{\varepsilon})n$ (allowing $\sqrt{\varepsilon}n \geq \varepsilon n$ for the exceptional set V_0). Note that

$$|E(G) \setminus E(G'')| \leq \delta n^2/10 + 5\sqrt{\varepsilon}n^2 \leq \delta n^2/5. \quad (30)$$

Observe that in the case that G' is bipartite, the colouring in the reduced graph $R''_{G'}$ might be a split colouring, even if the colouring in G is not a split colouring. But, in this case, note that if we change the colouring of any edge of $R''_{G'}$, we no longer have a split colouring. As all our arguments work the same whether we take the threshold for the colouring of $E(R''_{G'})$ to be $1/2$ or $\delta/2$, we may assume that we can either change the colouring of $R''_{G'}$ in a justified way so that the obtained colouring is not a split colouring, or that $R''_{G'}$ has a split colouring, and each red edge of $R''_{G'}$ corresponds to a pair in G'' which has blue density $< \delta/2$, and vice versa for the blue edges. So, deleting at most $\delta|E(G'')|/2$ edges from G'' we can make all pairs monochromatic. Thus, by (30), this means G is δ -close to a split colouring, a contradiction.

Therefore, we may assume the colouring in $R''_{G'}$ is not a split colouring. Since by (29), the classes of the bi- or tripartition of $R''_{G'}$ have size at least $\delta t/20 - 2\sqrt{\varepsilon}n$, we may apply Lemma 4.2 or Lemma 4.3 to $R''_{G'}$ to obtain two connected matchings, one in each colour, which together cover all but at most $72\sqrt{\varepsilon}t$ vertices. Now we use a crucial and well-known lemma that has appeared in similar forms before. In its original form it is due to Łuczak [15]. The version we use here is close to the one given in from Section 4 of [11].

Lemma 5.2. *Let R be the reduced two-coloured graph of a two-coloured graph H , for some γ -regular partition, where each edge of R corresponds to a γ -regular pair of density at least $\sqrt{\gamma}$.*

If all but at most $72\gamma|V(R)|$ vertices of R can be covered with two disjoint connected monochromatic matchings, one of each colour, then H has two disjoint monochromatic cycles, one of each colour, which together cover at least $(1 - 100\sqrt{\gamma})|V(H)|$ vertices of H .

For completeness, let us outline a proof of Lemma 5.2.

Sketch of a proof of Lemma 5.2. We first connect in H the pairs corresponding to matching edges with monochromatic paths, following their connections in R . We do this simultaneously for both colours. Note that in total, these paths consume only a constant number of vertices of H . Then we connect the monochromatic paths using the matching edges, blowing up the edges to long paths, where regularity ensures we can use all but a small fraction of the corresponding pairs. This gives the desired cycles. The above argumentation is also explained, rather detailed, in the proof of the main result of [10]. \square

Applying Lemma 5.2 to the graph R''_G , gives the desired two cycles which cover all but at most $100\sqrt{\epsilon}n < \delta n$ of our graph G . This finishes the proof of Theorem 1.6.

6 Covering all vertices with 14 cycles

6.1 Preliminaries

Call a balanced bipartite subgraph H of an $2n$ -vertex graph ϵ -hamiltonian, if any balanced bipartite subgraph of H with at least $(1 - \epsilon)n$ vertices in each partition class is hamiltonian. The next lemma is a straightforward combination of results of Haxell [12] and Peng, Rödl and Ruciński [17]. A proof of this lemma is given in our companion paper [13].

Lemma 6.1. [13] *For any $1 > \gamma > 0$, there is an $n_0 \in \mathbb{N}$ such that any balanced bipartite graph G on at least $2n \geq 2n_0$ vertices and of edge density at least γ has an $\gamma/4$ -hamiltonian subgraph of size at least $\gamma^{3024/\gamma}n/3$.*

In order to absorb vertices not covered with the cycles given by Theorem 1.6, we use the following result.

Lemma 6.2 (Gyárfás, Ruszinkó, Sárközy, and Szemerédi [10]). *There is an $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and $m \leq \frac{n}{(8r)^{8(r+1)}}$ and for any colouring of the edges of $K_{n,m}$ with r colours, there are $2r$ vertex-disjoint monochromatic cycles that cover the m vertices of the smaller side.*

6.2 Proof of Theorem 1.7

Let G be a fair complete k -partite graph on $n \geq n_0$ vertices, whose edges are 2-coloured. We assume n_0 to be large enough, its value may be extracted from the proof. By Lemma 1.8, we may assume that $2 \leq k \leq 3$. Let V_1, V_2 , and possibly V_3 , be the partition classes of G , where we assume that $|V_1| \geq |V_2| \geq |V_3|$. Set $\delta := 2^{-10^4}$.

For technical reasons, we split the argument into three cases. The explicit value of δ plays a role only in the last case of the proof.

We first discuss the case when

$$|V_1| \leq |V_2| + |V_3| - \delta n \text{ and } |V_3| \geq \delta n. \quad (31)$$

We pick disjoint subsets $U_1^1, U_1^2 \subseteq V_1$, $U_2^1, U_2^3 \subseteq V_2$, and $U_3^2, U_3^3 \subseteq V_3$ of size $\lfloor \delta n/4 \rfloor$ each. Due to (31), it holds that $G - \{U_i^j : i, j \in \{1, 2, 3\}\}$ is fair. Indeed, every graph H with $G - \{U_i^j : i, j \in \{1, 2, 3\}\} \subseteq H \subseteq G$ is fair, a fact that we will exploit later.

Let $G_1 := G[U_1^1, U_2^1]$, $G_2 := [U_1^2, U_2^3]$, and $G_3 := G[U_2^3, U_3^3]$. Assuming n_0 is large enough, we apply Lemma 6.1 with $\gamma := 1/2$ to G_i , $i = 1, 2, 3$, considering only the edges of the respective majority colour. This gives monochromatic $1/8$ -hamiltonian (and thus balanced) subgraphs $H_i = [W_1^i, W_2^i]$ of G_i , $i = 1, 2, 3$ with

$$|W_1^i| \geq \delta n/2^{6053}. \quad (32)$$

Now let $H := G - \bigcup_{i=1}^3 V(H_i)$ and recall that H is fair since $G - \{U_i^j : i, j \in \{1, 2, 3\}\} \subseteq H \subseteq G$. By (31) and by the choice of the sets U_j^i , each partition class of H has size at least $\delta n/2$. Let $\delta' := \delta/2^{6148}$. Assuming n_0 is large enough, we know H is not δ' -close to a split colouring, and Theorem 1.6 gives that all but at most $\delta' n$ vertices of H can be partitioned into two monochromatic cycles. If the set $X \subseteq V(G')$ of uncovered vertices has odd cardinality, we cover one of its vertices with a trivial cycle, and from now on assume $|X|$ is even.

Assume $|X \cap V_3| \geq |X \cap V_1|, |X \cap V_2|$ (the other cases are similar). Partition X into equal-sized sets X_1, X_2 with $X \cap V_i \subseteq X_i$ for $i = 1, 2$. Assuming n_0 is sufficiently large, the choice of δ' and (32) give that

$$|X_1| \leq \delta' n/2 \leq |W_2^1|/16^{24}. \quad (33)$$

So, Lemma 6.2 yields a set \mathcal{C} of eight vertex-disjoint monochromatic cycles in $[X_1, W_2^1]$ and $[X_2, W_1^1]$ that together cover $X_1 \cup X_2$. As we may assume that \mathcal{C} does not contain any trivial cycles, $V(\bigcup \mathcal{C}) \cap V(H_1)$ is balanced, and each side has at most $|W_2^1|/16^{24}$ vertices, by (33). Since H_1 is $(1/8)$ -hamiltonian, the monochromatic graph $H_1 - V(\bigcup \mathcal{C})$ thus has a hamilton cycle. Also, H_2 and H_3 each admit a hamilton cycle. In total, we covered $V(G)$ using at most $2 + 1 + 8 + 3 = 14$ vertex-disjoint monochromatic cycles.

We now discuss the case when $k = 2$. We use the same method as above. Observe that now, we only need to find one $(1/8)$ -hamiltonian subgraph of G (as opposed to the three graphs H_1, H_2, H_3 above). Thus in the last step we will use only one instead of three hamilton cycles. However, we might need three cycles to cover G almost entirely, since the colouring of G might be close to a split-colouring (a covering with 3 cycles does exist by the remark after Theorem 1.6). Since we may assume that none of these three cycles is trivial, we may assume the set X of vertices not covered with the three cycles to have even cardinality. In this way we avoid another cycle. This means we can cover G with $3 + 8 + 1 = 12$ vertex-disjoint monochromatic cycles in total.

To complete the proof of Theorem 1.7, we now consider the case when $k = 3$, but (31) is violated. Essentially, we proceed as in the case of $k = 2$. If n is odd, we pick a single vertex v from V_1 as a cycle. Since $G - v$ is fair, we may simply assume that n is even.

No matter whether we have $|V_3| < \delta n$, or we have $|V_3| \geq \delta n$ and $|V_1| > |V_2| + |V_3| - \delta n$, we proceed as follows. Delete a set W of $|V_2| + |V_3| - |V_1| < \delta n$ vertices from V_3 . Consider the bipartite graph G' spanned by V_1 and the remains of $V_2 \cup V_3$, which we call V'_2 . Note that G' is complete bipartite and balanced.

Let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ have size $\lceil n/4 \rceil$ each. (Note that clearly $|V_2| \geq n/4$.) Assuming n_0 is large enough, we may apply Lemma 6.1 with $\gamma = 1/2$ to the graph induced by the majority colour to find a $(1/8)$ -hamiltonian subgraph $H = [U'_1, U'_2]$ of $G'[U_1 \cup U_2]$, with $|U'_1| = |U'_2| \geq n/2^{6052}$. Assuming n_0 is large enough, the remark after Theorem 1.6 gives three monochromatic vertex-disjoint cycles that cover all but a set W' of at most δn many vertices of $G' - V(H)$. We may assume that none of these cycles is trivial, and thus $|W' \cap V_1| = |W' \cap V'_2|$.

Partition $W \cup W'$ into two equal-sized sets W_1 and W_2 such that for $i = 1, 2$, all edges between W_i and U'_i are present. As

$$16^{24}|W_1| < \delta n/2^{97} \leq n/2^{6052} \leq |U'_2|,$$

we may apply Lemma 6.2 to cover each of W_1, W_2 with four vertex-disjoint monochromatic cycles, which we again assume to be non-trivial. Since H is $(1/8)$ -hamiltonian, the yet unused vertices of H span a hamiltonian cycle. In total, we covered G with $1+3+8+1 = 13$ vertex-disjoint monochromatic cycles.

7 Conclusion

Our Theorem 1.4 and the earlier result of Pokrovskiy together cover all cases of monochromatic path covers in 2-edge-coloured multipartite graphs. It would be natural to investigate the same problem for more colours, in the spirit of Conjecture 1.1. For bipartite graphs whose edges are coloured with r colours, Pokrovskiy [18] conjectures that there is a partition into $2r - 1$ monochromatic paths, and he shows this number is best possible. For k -partite graphs with $k \geq 3$ a partition into less monochromatic paths, perhaps r paths, might be possible, in analogy to the case $r = 2$ we treated here.

For cycle partitions, as said in the introduction, we believe that Theorem 1.6 is not best possible. However, recalling the example given in the introduction, a split colouring between a partition class of size $n/2$ and the rest of the graph, it is not always possible to cover *all* vertices of any large fair multipartite graph with two vertex-disjoint monochromatic cycles. This remains true even if the colouring of this graph is far from a split colouring (the other classes might have size $n/4$ each).

Even if a partition class containing half of the vertices of the graph is forbidden, such constructions are possible. Indeed, given a properly split-coloured balanced complete bipartite graph one can add a third partition class consisting of a single vertex v_3 seeing only one colour. Then the obtained graph cannot be partitioned into two monochromatic cycles. One can even add a fourth partition class consisting of a single vertex v_4 seeing only the other colour, while giving v_3v_4 any colour. Still the obtained graph has no partition into two monochromatic cycles.

Problem 7.1. *For $k \geq 3$, under which conditions does a fair complete k -partite graph with a 2-colouring of the edges admit a partition into two monochromatic cycles?*

One candidate for a sufficient condition in Problem 7.1 could be balancedness, that is, having partition classes of equal size. In any case, we think that considering balanced multipartite graphs is a reasonable restriction that might be of its own interest.

The next natural step would be to extend Theorem 1.6 to more colours. In balanced complete bipartite graphs, the bound $O(r^2)$ from [14] is most likely not best possible. In analogy to Pokrovskiy’s conjecture for path covers [18] mentioned above, it might always be possible to cover almost all vertices of any large enough bipartite graph, whose edges are r -coloured, with $2r - 1$ disjoint monochromatic cycles (and perhaps the restrictions ‘almost’ and ‘large enough’ are not even necessary, or ‘almost’ might mean ‘all but a constant number’). In this direction, we showed in [13], together with Lang, that in large enough 3-edge coloured balanced complete bipartite graphs on n vertices, we can cover all but $o(n)$ vertices with five disjoint monochromatic cycles.

Turning to k -partite graphs with $k \geq 3$, note that an easy bound for *balanced* complete multipartite graphs can be obtained from the bound $4r^2$ for large r -coloured complete bipartite graphs [14]. Splitting up the multipartite graph into three suitable bipartite graphs if necessary, we deduce that any r -coloured balanced complete k -partite graph that is large enough can be covered by at most $4r^2$ disjoint monochromatic cycles if k is even, and at most $12r^2$ disjoint monochromatic cycles if k is odd.

It is not difficult to see that every *fair* complete multipartite graph can be split up into three balanced bipartite graphs, plus possibly one leftover vertex. So we can argue similarly as in the previous paragraph to see that any r -coloured fair complete multipartite graph can be covered by at most $12r^2 + 1$ disjoint monochromatic cycles.

All these numbers are most likely not best possible, and we feel the situation for fair complete multipartite graphs should not be worse than for balanced complete bipartite graphs. We suspect the number of cycles needed for fair complete multipartite graphs should be linear in r , and not more than what is needed for bipartite graphs. As evidence, we have seen here that three disjoint monochromatic cycles can cover all but $o(n)$ of the vertices of any large enough multipartite graph, whose edges are two-coloured, where three cycles are only needed if the colouring is very close to a split colouring. Perhaps the number of cycles needed to cover almost all vertices of any large enough r -coloured fair multipartite graph can even be dropped to r in k -partite graphs with $k \geq 3$, unless we are close to a specific problematic colouring, such as the split colouring. Maybe this is even possible in bipartite graphs.

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