Partitioning 2-coloured complete k-uniform hypergraphs into monochromatic ℓ-cycles

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Abstract

We show that for all ℓ, k, n with $\ell \leq k/2$ and $(k-\ell)$ dividing n the following hypergraph-variant of Lehel's conjecture is true. Every 2-edge-colouring of the k-uniform complete hypergraph $\mathcal{K}_n^{(k)}$ on n vertices has at most two disjoint monochromatic ℓ-cycles in different colours that together cover all but at most $4(k-\ell)$ vertices. If $\ell \leq k/3$, then at most two ℓ-cycles cover all but at most $2(k-\ell)$ vertices.
Furthermore, we can cover all vertices with at most 4 (3 if $\ell \leq k/3$) disjoint monochromatic ℓ-cycles.

1 Introduction

Cycle partitioning problems originated in Lehel's Conjecture [Aye79], which states that every 2-edge-colouring of the complete graph K_n contains two disjoint monochromatic cycles in different colours covering all vertices of K_n, where vertices and single edges count as (degenerate) cycles. Lehel's Conjecture was confirmed for large n in [All08, ŁRS98], and for all n by Bessy and Thomassé [BT10].

Not much is known for extensions of this question to k-uniform hypergraphs. Probably the most flexible concept for cycles in hypergraphs is the notion of *ℓ-cycles,* which are k-uniform hypergraphs with a cyclic ordering

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of their vertices such that every edge contains k consecutive vertices, and consecutive edges intersect in exactly ℓ vertices. We also consider vertex sets of size $k-\ell$ as *degenerate cycles*. Notice that 1-cycles and $(k-1)$-cycles often appear in the literature as *loose* or *tight* cycles, respectively.

It follows from work of Gyárfás and Sárközy [GS14] that the number of monochromatic 1-cycles needed to partition any 2-edge-coloured $\mathcal{K}_n^{(k)}$ is bounded by a function in k.[1] Replacing cycles with paths, the same authors conjectured [Gyá16, GS14] that any 2-edge-coloured $\mathcal{K}_n^{(k)}$ has two disjoint monochromatic 1-paths (an ℓ-path is obtained from an ℓ-cycle by deleting one edge), together covering all but at most $k-2$ vertices, and show this is best possible. This conjecture has recently been confirmed by Lu, Wang and Zhang [LWZ17].

The analogous problem for hypergraph cycles has not been resolved so far. In [BHS17], the present authors and Hán use hypergraph regularity to prove that all but $o(n)$ of the vertices of $\mathcal{K}_n^{(3)}$ can be covered by two monochromatic vertex-disjoint 2-cycles (tight cycles) of different colours, and an analogous statement for 1-cycles (loose cycles) also holds.

Here we show that for arbitrary k, and $\ell \leq k/2$ the bound on the number of uncovered vertices can be improved to a constant (depending on k), that is, all but a constant number of vertices can be covered by two disjoint monochromatic ℓ-cycles.

Theorem 1.1. *Let $\ell, k, n \in \mathbb{N}$ such that $0 < \ell \leq k/2$ and $k-\ell$ divides n. Let any 2-edge-colouring of $\mathcal{K}_n^{(k)}$ be given.*

(a) There are two vertex-disjoint monochromatic ℓ-cycles in different colours together covering all but at most $4(k-\ell)$ vertices.

(b) If $\ell \leq k/3$, the two ℓ-cycles cover all but at most $2(k-\ell)$ vertices.

Our proof does not use Bessy and Thomassé's theorem, and does not rely on hypergraph regularity.

We suspect that that a partition of all vertices into two cycles should always be possible. (It is not difficult to construct colourings which require at least two disjoint ℓ-cycles for covering all the vertices, so this would be best possible.)

[1] The actual result in [GS14] is on cycle partitioning in r-coloured complete hypergraphs (for arbitrary $r \geq 2$), following a recent surge of activity around cycle partitioning in r-coloured complete (and other) graphs. For an overview of the area, we recommend the surveys [FLM15] and [Gyá16].

Conjecture 1.2. *If $\ell, k, n \in \mathbb{N}$ with $n \equiv 0 \pmod{k-\ell}$, then every 2-edge-colouring of $\mathcal{K}_n^{(k)}$ contains two vertex-disjoint monochromatic ℓ-cycles in different colours covering all vertices.*

An easy argument shows that for $\ell = k/2$ the conjecture is true. In order to see this, take any partition $\mathcal{P}$ of the vertices of $\mathcal{K}_n^{(k)}$ into sets $S_i, i \in [2n/k]$, of size $k/2$. Consider an auxiliary 2-edge-colouring of the complete graph on $\mathcal{P}$, giving $\{S_i, S_j\}$ the colour of $S_i \cup S_j$ in $\mathcal{K}_n^{(k)}$. Bessy and Thomassé's theorem [BT10] yields two graph cycles, which correspond to two disjoint monochromatic ℓ-cycles in different colours in $\mathcal{K}_n^{(k)}$.

Also, we can obtain the following corollary from Theorem 1.1.

Corollary 1.3. *Let $\ell, k, n \in \mathbb{N}$ such that $0 < \ell \leq k/2$ and $k - \ell$ divides n. Then for any 2-edge-colouring of $\mathcal{K}_n^{(k)}$, one can cover all the vertices of $\mathcal{K}_n^{(k)}$ with four vertex-disjoint monochromatic ℓ-cycles, and if $\ell \leq k/3$, it can be done with three cycles instead of four.*

This follows directly from our main theorem together with the observation that the Ramsey number[2] of the k-uniform ℓ-cycle of length two is $2(k-\ell)$. This can be seen by observing that any 2-edge-colouring of $\mathcal{K} := \mathcal{K}_{2(k-\ell)}^{(k)}$ naturally defines a 2-edge-colouring of $\mathcal{K}^* := \mathcal{K}_{2(k-\ell)}^{(k-2\ell)}$ by giving any edge e^* in $\mathcal{K}^*$ the colour of $V(\mathcal{K}) \setminus e^*$ in $\mathcal{K}$. Then a monochromatic matching of size two in $\mathcal{K}^*$ corresponds to a monochromatic ℓ-cycle of length two in $\mathcal{K}$. Now, results of Alon, Frankl and Lovász [AFL86] imply that the Ramsey number of a 2-edge matching of uniformity r is at most $2r+1$, which, since $2(k-2\ell)+1 < 2(k-\ell)$, is enough for our purposes.

2 Partition into a path and a cycle

We will identify a hypergraph $\mathcal{H}$ with its edge set, so when we write $e \in \mathcal{H}$ it refers to the edge e of $\mathcal{H}$. Let us go through some necessary notation.

For an ℓ-path or ℓ-cycle $\mathcal{X}$, we order the k vertices of each edge in such a way that the last ℓ vertices of an edge e_i are the first ℓ vertices of the edge e_{i+1}. For an edge $e = \{v_1, \ldots, v_k\}$, we write $V_I(e)$ to denote the vertex set $\{v_i \in e : i \in I\}$, let e^- denote the vertex set $V_{[\ell]}(e)$, let e^+ denote the vertex set $V_{[k]\setminus[k-\ell]}(e)$, and use $\mathring{e}$ for the set $e \setminus (e^- \cup e^+)$.

[2] The two colour Ramsey number of a k-uniform hypergraph $\mathcal{H}$ is the least integer $R(\mathcal{H})$ for which every blue-red colouring of $\mathcal{K}_{R(\mathcal{H})}^{(k)}$ contains a monochromatic copy of $\mathcal{H}$.

An ℓ-path $\mathcal{P}$ of length m is a *blue-red ℓ-path* if there is $m_0 \in [m]$ such that the ℓ-paths $\{e_i : i \in [m_0]\}$ and $\{e_i : i \in [m] \setminus [m_0]\}$ are monochromatic and have different colours.

One of the first results in the field of monochromatic partitions was Gerenscer and Gyárfás' observation [GG67] that every two-edge-coloured complete graph has a spanning blue-red path. We extend this observation to ℓ-paths in hypergraphs in the following lemma.

Lemma 2.1. *Let $\ell, k, n \in \mathbb{N}$ such that $0 < \ell \leq k/2$ and $k-\ell$ divides n. Then every 2-edge-colouring of $\mathcal{K}_n^{(k)}$ contains a blue-red ℓ-path $\mathcal{P}$ with $|V(\mathcal{P})| = n - k + 2\ell$.*

Proof. Take a longest blue-red ℓ-path $\mathcal{P}$ in $\mathcal{K}_n^{(k)}$, with edges e_i for $i \in [m]$ and turning point m_0. Assume that all e_i with $i \in [m_0]$ are blue and all later edges on $\mathcal{P}$ are red.

If the set Z of all vertices not covered by $\mathcal{P}$ has size $k - 2\ell$, we are done. So assume otherwise; then Z contains at least $2k - 3\ell$ elements. Fix three disjoint sets $Z_0, Z_1, Z_2 \subset Z$ with $|Z_0| = \ell$ and $|Z_1| = |Z_2| = k - 2\ell$. Since $\mathcal{P}$ is maximal, we know that $e_R = e_1^- \cup Z_1 \cup Z_0$ is red, $e_B = e_m^+ \cup Z_1 \cup Z_0$ is blue, and $m_0 \neq m$.

By colour symmetry, we can assume the edge $e = e_{m_0}^+ \cup Z_2 \cup Z_0$ is red. Then $(\mathcal{P} \setminus \{e_{m_0}\}) \cup \{e, e_R\}$ is a blue-red ℓ-path longer than $\mathcal{P}$, which contradicts the maximality of $\mathcal{P}$. $\square$

Now we can show that there are a monochromatic ℓ-path and a monochromatic ℓ-cycle that together cover almost all the vertices.

Lemma 2.2. *Let $\ell, k, n \in \mathbb{N}$ such that $0 < \ell \leq k/2$ and $n = n_0(k - \ell)$ for some integer $n_0 \geq 3$. Then every 2-edge-colouring of $\mathcal{K}_n^{(k)}$ contains an ℓ-cycle $\mathcal{C}$ and an ℓ-path $\mathcal{P}$ with the following properties:*

1. *$\mathcal{C}$ and $\mathcal{P}$ are vertex-disjoint;*
2. *$\mathcal{C}$ and $\mathcal{P}$ are each monochromatic but use distinct colours;*
3. *$\mathcal{C}$ has at least two edges;*
4. *if $\mathcal{P} \neq \emptyset$, then $|V(\mathcal{C})| + |V(\mathcal{P})| \in \{n - k + 2\ell, n - 2k + 3\ell\}$; and*
5. *if $\mathcal{P} = \emptyset$, then $|V(\mathcal{C})| = n - k + \ell$.*

Proof. By Lemma 2.1, there is a blue-red ℓ-path with edges e_i for $i \in [m]$ and turning point m_0 that covers all but a set Z of $k - 2\ell$ vertices of $\mathcal{K}_n^{(k)}$.

Among such ℓ-paths, choose $\mathcal{P}_{\max}$ such that $\max\{m_0, m-m_0\}$ is as large as possible (i.e. $\mathcal{P}_{\max}$ maximises the length of a monochromatic sub-ℓ-path). By symmetry, we can assume that $\max\{m_0, m-m_0\} = m$, that $\mathcal{P}_B = \{e_i : i \in [m_0]\}$ is blue and that $\mathcal{P}_R = \{e_i : i \in [m]\backslash[m_0]\}$ is red. Since $n \geq 3(k-\ell)$, we know that $m \geq 2$. If $m_0 < m$ and the edge

$$e := e^+_{m_0} \cup Z \cup e^+_m$$

is blue, then $\big(\mathcal{P}_{\max}\backslash e_{m_0+1}\big) \cup \{e\}$ is a blue-red ℓ-path contradicting the choice of $\mathcal{P}_{\max}$. If $m_0 < m$ and the edge e is red, then the ℓ-cycle $\mathcal{C}_R = \mathcal{P}_R \cup e$ together with the ℓ-path $\mathcal{P}_B \setminus \{e_{m_0}\}$ are as desired.

So we can assume that $m_0 = m$, that is, $\mathcal{P}_{\max}$ is all blue. If also one of the two edges $e^+_1 \cup \mathring{e}_1 \cup e^+_m$, $e^+_1 \cup Z \cup e^+_m$ is blue, we can close $\mathcal{P}_{\max}$, forming an ℓ-cycle that covers all but $e^-_1 \cup Z$, or all but $e^-_1 \cup \mathring{e}_1$, respectively, which is as desired. So we can suppose both edges $e^+_1 \cup \mathring{e}_1 \cup e^+_m$, $e^+_1 \cup Z \cup e^+_m$ are red. They form an ℓ-cycle with two edges, which together with $\mathcal{P}_{\max}\backslash\{e_1, e_2, e_m\}$ covers all but $e^-_1 \cup \mathring{e}_2 \cup \mathring{e}_m$ (note that possibly $m = 2$, in which case e_2 coincides with e_m). So, we found an ℓ-cycle and an ℓ-path which are as required (in particular, either they cover $n - 2k + 3\ell$ vertices, or the ℓ-path is empty and the ℓ-cycle covers $n - k + \ell$ vertices). $\square$

3 Proof of Theorem 1.1 (a)

This section is devoted to the proof of Theorem 1.1 (a).

Consider a monochromatic ℓ-cycle $\mathcal{C}_B$ with at least two edges and a disjoint monochromatic ℓ-path $\mathcal{P}_R$ as given by Lemma 2.2. Note that if $\mathcal{P}_R$ has at most two edges, we are done, so assume otherwise. By deleting at most two edges from $\mathcal{P}_R$, if necessary, we can assume that

$$|V(\mathcal{C}_B \cup \mathcal{P}_R)| = n - 3k + 4\ell.$$

Among all such choices for $\mathcal{C}_B$ and $\mathcal{P}_R$ (including those where $\mathcal{P}_R$ is empty), assume we chose $\mathcal{C}_B$ and $\mathcal{P}_R$ such that

$$\mathcal{C}_B \text{ has as many edges as possible.} \tag{1}$$

By symmetry, we may assume that $\mathcal{C}_B$ is blue and $\mathcal{P}_R$ is red. Say $\mathcal{C}_B$ has edges $e_i, i \in [m_c]$ (and thus length $m_c \geq 2$), while $\mathcal{P}_R$ has edges $f_j, j \in [m_p]$ (and thus length $m_p \geq 0$).

Assuming that Theorem 1.1 (a) does not hold, we will reach a contradiction by analysing the connections from the first/last edge of $\mathcal{P}_R$ to $\mathcal{C}_B$.

If these cannot be used to close up $\mathcal{P}_R$ to an ℓ-cycle, we find a red ℓ-cycle on the same vertices as $\mathcal{C}_B$. In a last step, we will use this new red ℓ-cycle together with $\mathcal{P}_R$ to form one large red ℓ-cycle.

We start by making a couple of easy observations. First of all, note that

$$m_p \geq 2, \tag{2}$$

as otherwise $\mathcal{C}_B$ covers all but at most $4k - 4\ell$ vertices.

Let Z_1, Z_2, Z_3 be mutually disjoint subsets of vertices not covered by $\mathcal{C}_B \cup \mathcal{P}_R$ such that $|Z_1| = |Z_2| = |Z_3| = k - 2\ell$. Consider the edges

$$w^t := f_1^- \cup Z_t \cup f_{m_p}^+,$$

for $t = 1, 2, 3$. If any of the edges w^t is red, then $\mathcal{C}_B$ together with $\mathcal{P}_R \cup \{w^t\}$ are ℓ-cycles as in the theorem, covering all but $2k - 2\ell$ vertices. So,

$$w^t := f_1^- \cup Z_t \cup f_{m_p}^+ \text{ is blue, for } t = 1, 2, 3. \tag{3}$$

Consider the edges

$$v_i^t := f_{m_p}^+ \cup Z_t \cup e_i^-$$

for $i \in [m_c]$ and $t, t' \in \{1, 2, 3\}$. If for some triple i, t, t' with $t \neq t'$, both edges v_i^t, $v_{i+1}^{t'}$ are blue, then the ℓ-cycle $\mathcal{C}_B \cup \{v_i^t, v_{i+1}^{t'}\} \setminus \{e_i\}$ together with the ℓ-path $\mathcal{P}_R \setminus \{f_{m_p}\}$ contradicts (1). So, for each $i \in [m_c]$, and each pair and $t, t' \in \{1, 2, 3\}$ with $t \neq t'$,

$$\text{one of the edges } v_i^t,\ v_{i+1}^{t'} \text{ is red.} \tag{4}$$

Similary, for each $i \in [m_c]$, and each pair and $t, t' \in \{1, 2, 3\}$ with $t \neq t'$, setting

$$u_i^t := f_1^- \cup Z_t \cup e_i^-,$$

we observe that

$$\text{one of the edges } u_i^t, u_{i+1}^{t'} \text{ is red.} \tag{5}$$

We now establish that our ℓ-cycle is a bit longer than our ℓ-path.

Claim 1. *It holds that $m_c \geq m_p + 2$.*

Proof. Suppose to the contrary that $m_c < m_p + 2$. By (5), we can assume the edge u_1^1 is red. If the edge v_1^2 is red, too, then $\mathcal{P}_R \cup \{u_1^1, v_1^2\}$ and $\mathcal{C}_B \setminus \{e_1, e_{m_c}\}$ contradict the choice of $\mathcal{C}_B$ and $\mathcal{P}_R$ fulfilling (1). So the edge v_1^2 is blue, and thus by (4), the edge v_2^3 is red.

Now, if the edge u_2^1 is red, then either $\mathcal{P}_R \cup \{v_2^3, u_2^1\}$ is a red ℓ-cycle of length greater than m_c, which together with the path $\mathcal{C}_B \setminus \{e_1, e_2\}$ contradicts (1). Therefore, u_2^1 is blue. But now, since by (3), the edge w^3 is blue, we found a blue ℓ-cycle, namely $\mathcal{C}_B \cup \{u_2^1, v_1^2, w^3\} \setminus \{e_1\}$, which together with the red ℓ-path $\mathcal{P}_R \setminus \{f_1, f_{m_p}\}$ contradicts (1). $\square$

Note that Claim 1 together with (2) implies that

$$m_c \geq 4. \tag{6}$$

Let us now consider the edges

$$g_i := (e_i \setminus e_i^+) \cup e_{i+2}^+ \quad \text{and} \quad h_i := (e_i \setminus e_i^+) \cup e_{i+3}^+,$$

for $i \in [m_c]$ (considering all indices modulo m_c). The advantage of these edges is that on the one hand, each of these edges, if blue, provides a shortcut on $\mathcal{C}_B$ (and the vertices left out of $\mathcal{C}_B$ can be used for closing up $\mathcal{P}_R$). On the other hand, if all these edges are red, then they form new red ℓ-cycles on the vertex set of $\mathcal{C}_B$.

Let us first show why any of the edges g_i, h_i would be useful in blue.

Claim 2. *The edges g_i are red for all $i \in [m_c]$, and if $m_c > 4$, then the edges h_i are red for all $i \in [m_c]$.*

Proof. Suppose that one of these edges g_i or h_i is blue (the latter only in the case that $m_c > 4$). Then there is a blue cycle $\mathcal{C}_B'$ obtained from $\mathcal{C}_B$ by replacing the edges e_i, e_{i+1}, e_{i+2} with the edge g_i, or by replacing the edges $e_1, e_{i+1}, e_{i+2}, e_{i+3}$ with the edge h_i.

Consider the edges u_{i+1}^1 and v_{i+1}^2. If both of these edges are red then the theorem holds, since $\mathcal{C}_B'$ together with $\mathcal{P}_R \cup \{u_{i+1}^1, v_{i+1}^2\}$ either covers all but $3k - 3\ell$ vertices (if g_i is blue); or cover all but $4k - 4\ell$ vertices (if h_i is blue). So by symmetry, we can assume that u_{i+1}^1 is blue. Similarly, if the edges u_{i+2}^2 and v_{i+2}^3 are both red then the theorem holds, so at least one of them is blue.

Since u_{i+1}^1 is blue, (4) implies that u_{i+2}^2 is red, and thus v_{i+2}^3 is blue. Recall that by (3), the edge w^2 is blue, too, and so, the ℓ-cycle $\mathcal{C}_B \cup \{u_{i+1}^1, v_{i+2}^3, w^2\} \setminus \{e_{i+1}\}$ together with the ℓ-path $\mathcal{P}_R \setminus \{f_1, f_{m_p}\}$ contradicts (1). $\square$

Finally, consider the edge sets

$$R_j := \{g_i : i \equiv j \pmod 3\},$$

for $j = 0, 1, 2$. Notice that R_0, R_1, R_2 are three ℓ-cycles of length $\frac{m_c}{3}$ if $m_c \equiv 0 \pmod 3$, and together form one ℓ-cycle otherwise.

The remainder of the proof is split into several cases, depending on the value of m_c, and which of the edges u_i^t, v_i^t are red. Note that by (5) (and after possibly renaming the edges on $\mathcal{C}_B$, or the sets Z_t), we may assume that u_1^1 is red. Moreover, by (4), at least one of the edges v_4^2, v_5^3 is red (if $m_c = 4$, we take indices modulo 4, meaning that one of v_4^2, v_1^3 is red).

In all the cases considered below we use that $m_c \geq 4$ by (6).

Case 1. $m_c \not\equiv 0 \pmod 3$ *and* v_4^2 *is red.*

In this case, consider the red ℓ-cycle formed by $R_0 \cup R_1 \cup R_2$. We can substitute the edge g_1 from this ℓ-cycle with the path $\{u_1^1\} \cup \mathcal{P}_R \cup \{v_4^2\}$ to obtain one red ℓ-cycle which covers all but $2(k - \ell)$ vertices.

Case 2. $m_c \equiv 0 \pmod 3$ *and* v_4^2 *is red.*

Consider the auxiliary red ℓ-cycle formed by

$$(R_0 \setminus \{g_3\}) \cup (R_1 \setminus \{g_4\}) \cup (R_2 \setminus \{g_2, g_5\}) \cup \{h_2, h_3, h_4\}.$$

Similar as in the previous case, we can substitute the edge g_1 from the auxiliary ℓ-cycle with the path $\{u_1^1\} \cup \mathcal{P}_R \cup \{v_4^2\}$ to obtain one red ℓ-cycle which covers all but $3(k - \ell)$ vertices. (Note that this works fine even if $m_c = 6$.)

Case 3. $m_c \not\equiv 1 \pmod 3$ *and* v_5^3 *is red.*

Consider the auxiliary red ℓ-cycle formed by

$$(R_0 \setminus \{g_3\}) \cup (R_1 \setminus \{g_1\}) \cup (R_2 \setminus \{g_2\}) \cup \{h_1, h_2, h_3\},$$

and substitute its edge h_1 with $\{u_1^1\} \cup \mathcal{P}_R \cup \{v_5^3\}$ to obtain one red ℓ-cycle which covers all but $2(k - \ell)$ vertices. (Note that this works even if $m_c = 5$.)

Case 4. $m_c \equiv 1 \pmod 3$, $m_c \neq 4$, *and* v_5^3 *is red.*

If m_c is odd, then we can use the cycle spanned by all edges h_i except h_1, and the path $\mathcal{P}_R$, together with edges u_1^1 and v_5^3. This ℓ-cycle covers all but $2(k - \ell)$ vertices. Otherwise, since $m_c > 4$ and $m_c \equiv 1 \pmod 3$, we know that $m_c \geq 10$. So we can use a similar approach as above, using seven edges h_i instead of three. More precisely, the red ℓ-path formed by

$$(R_0 \setminus \{g_3, g_6, g_7\}) \cup (R_1 \setminus \{g_1, g_4\}) \cup (R_2 \setminus \{g_2, g_5\}) \cup \{h_2, h_3, h_4, h_5, h_6, h_7\}$$

and $\{u_1^1\} \cup \mathcal{P}_R \cup \{v_5^3\}$ covers all but $4(k - \ell)$ vertices.

Case 5. *$m_c = 4$, and v_1^3 is red while v_4^2 is blue.*

Then by (4), the edge v_3^1 is red, and we can close $\mathcal{P}_R$ using the edges v_1^3, v_3^1, g_1, g_4. We covered all but $3(k-\ell)$ vertices.

This finishes the proof of the theorem.

4 Proof of Theorem 1.1 (b)

In this section we give a proof of Theorem 1.1 (b), the first part of which follows very much the lines of the proof of Theorem 1.1 (a), while the last part is a bit different. In order to avoid repetition, we only sketch the first part, but give all details for the last part. We remark that much of the work can be avoided if we are only interested in two ℓ-cycles covering all but $3(k-\ell)$ vertices instead of the output of Theorem 1.1 (b) (see Footnote 1 for an explanation).

For the first part of the proof, the main difference is that now, we use Lemma 2.2 to find a monochromatic ℓ-cycle $\mathcal{C}_B$ of length $m_c \geq 2$ with edges $e_i, i \in [m_c]$, and a disjoint monochromatic ℓ-path $\mathcal{P}_R$ of length $m_p \geq 0$ with edges $f_j, j \in [m_p]$ such that

$$|V(\mathcal{C}_B \cup \mathcal{P}_R)| = n - 2k + 3\ell, \tag{7}$$

choosing $\mathcal{C}_B$ maximal under the these conditions. That is, now we leave only $2(k-\ell)+\ell$ vertices uncovered in the beginning. Instead of defining Z_1, Z_2, Z_3, we only define Z_1, Z_2 as two disjoint sets of vertices not covered by $\mathcal{C}_B \cup \mathcal{P}_R$ with $|Z_1| = |Z_2| = k - 2\ell$. The idea is that now, consecutive edges on $\mathcal{C}_B$ only intersect in at most $k/3$ vertices, which means that the interior of such an edge can be used in the same way as one of the sets Z_t. With this we can overcome the difficulty due to having only two sets Z_t to operate with.

Again we easily show that $m_p \geq 2$ (using (7)), and that edges w^t defined for $t = 1, 2$ and $i \in [m_c]$, have the same properties as in the proof of Theorem 1.1 (a). Now we define v_i^t and u_i^t as in that proof for $t = 1, 2$, and set

$$v_i^3 := f_{m_p}^+ \cup (e^i \setminus e_i^+) \text{ and } u_i^3 := f_1^- \cup (e^i \setminus e_i^+).$$

It is easy to see that for each $i \in [m_c]$, and each pair and $t, t' \in \{1, 2, 3\}$ with $t \neq t'$ and $t' \neq 3$, at least one of the edges v_i^t, $v_{i+1}^{t'}$ is red and at least one of the edges u_i^t, $u_{i+1}^{t'}$ is red.

For showing that $m_c \geq m_p + 2$ (and thus $m_c \geq 4$), observe that in the proof of Claim 1 in the proof of Theorem 1.1 (a), there is only one time

where we need that all three sets Z_t are present, and that is at the very end, when we form the blue ℓ-cycle $\mathcal{C}_B \cup \{u_2^1, v_1^2, w^3\} \setminus \{e_1\}$. Instead, we can use the ℓ-cycle $\mathcal{C}_B \cup \{u_2^1, v_1^3, w^2\} \setminus \{e_1\}$.

For the rest of the proof one might define the edges g_i, h_i as in the proof of Theorem 1.1 (a), show they are red, and then go through Cases 1-5.[3] However, for establishing that edges h_i are red, we would have to content ourselves with the outcome of two ℓ-cycles covering all but $3(k-\ell)$ vertices. We can do a slightly better than that by arguing as follows.

Consider the edges

$$a := f_1^- \cup \mathring{e}_3 \cup V_{[2\ell]\setminus[\ell]}(e_2)$$

and

$$a' := V_{[2\ell]\setminus[\ell]}(f_1) \cup \mathring{e}_4 \cup V_{[k-\ell]\setminus[k-2\ell]}(e_5)$$

(note that these edges are symmetric with respect to $e_3 \cap e_4$). If both a and a' are blue then we can replace the edges $e_3, e_4 \in \mathcal{C}_B$ with the edges a, a', w^1 and the blue[4] edge $V_{[2\ell]\setminus[\ell](f_1)} \cup Z_2 \cup f_{m_p}^+$ to obtain a blue cycle which together with the red path $\mathcal{P}_R \setminus \{f_1, f_m\}$ contradicts the maximality of $\mathcal{C}_B$. Therefore, we can assume that one of these edges is red, without loss of generality say

$$a \text{ is red.} \tag{8}$$

Next, consider the edges

$$q_i := V_{[2l]\setminus[\ell]}(e_i) \cup \mathring{e}_{i+1} \cup e_{i+3}^+,$$

for $i \in [m_c]$. It is easy to see that the edges q_i form an ℓ-cycle, which we will call $\mathcal{C}_R$.

Claim 3. *We may assume that q_i is red, for all $i \in [m_c]$.*

[3] To see this goes through we remark that first, near the end of the proof of Claim 2 of that theorem we used to occupy the set Z_3, by employing the edge v_{i+2}^3. With the new definition of this edge, this works here too.
Second, when going through Cases 1-5, we cannot use the edges u_1^1, v_4^2, v_5^3 as before. This problem is easily overcome by first finding out which of v_4^2, v_5^1 is red. Say this is v_5^1 (otherwise rename all edges). Now, if the edge u_2^2 is red, then we are in the same situation as in Case 1, 2 or 5 of the proof of Theorem 1.1 (a) (with indices augmented by one). Otherwise, the edge u_2^2 is blue, and thus the edge u_1^3 is red, as otherwise we could augment $\mathcal{C}_B$ using these two edges, and destroying one edge of $\mathcal{P}_R$. Now we are in a situation that is very similar to Cases 3 and 4 of Theorem 1.1 (a). As in these cases, neither of the edges g_1, h_1 was used, we have no problem finding our red ℓ-cycle using u_1^3 instead of u_1^1.

[4] This edge is blue for the same reason for which w^2 is blue.

Proof. Suppose one of these edges, say q_1, is blue. Obtain $\mathcal{C}'_B$ from $\mathcal{C}_B$ by replacing the edges e_2, e_3, e_4 with the edge q_1.

First assume u_3^3 is blue. Then u_4^1 is red, by our analogue of (5). Also, v_4^2 is red, as otherwise we can replace e_3 with the edges u_3^3, w^1 and v_4^2, and thus contradicting the maximality of $\mathcal{C}_B$. But now, $\mathcal{P}_R \cup \{u_4^1, v_4^2\}$ is a red ℓ-cycle, which, together with the blue ℓ-cycle $\mathcal{C}'_B$ is as desired for the theorem.

So from now on, assume that u_3^3 is red. Then, the edge v_3^1 is blue or we found ℓ-cycles $\mathcal{P}_R \cup \{u_3^3, v_3^1\}$ and $\mathcal{C}'_B$ which are as desired for the theorem. Now consider a set $Z'_2 \subseteq Z_2$ of size $k - 3\ell$. By the maximality of $\mathcal{C}_B$ and taking into account that v_3^1 is blue, we see that the edge

$$b := f_{m_p}^+ \cup Z'_2 \cup V_{[2\ell]\setminus[\ell]}(e_3) \cup e_3^+$$

has to be red. But then the ℓ-cycles $\mathcal{P}_R \cup \{u_3^3, b\}$ and $\mathcal{C}'_B$ give the desired output of the theorem. $\square$

We are now ready to prove Theorem 1.1 (b). For this, first assume that v_4^1 is red. Then, by (8) and by Claim 3, we know that $\big(\mathcal{C}_R \setminus \{q_1, q_2\}\big) \cup \mathcal{P}_R \cup \{a, v_4^1\}$ is a red ℓ-cycle, as desired for the theorem.

From now on assume v_4^1 is blue. Then $c := f_{m_p}^+ \cup \mathring{e}_4 \cup V_{[2\ell]\setminus[\ell]}(e_5)$ is red, as otherwise the cycle obtained by replacing $e_4 \in \mathcal{C}_B$ with the edges v_4^1, c yields a contradiction to the maximality of $\mathcal{C}_B$. So, by (8), and since we chose c so that it meets q_5 in exactly ℓ vertices,

$$\mathcal{C}'_R := \big(\mathcal{C}_R \setminus \{q_2, q_3, q_4\}\big) \cup \mathcal{P}_R \cup \{a, c\}$$

is a red ℓ-cycle covering all vertices, except the $3(k - \ell)$ vertices lying in

$$Z_1 \cup Z_2 \cup W \cup \{\mathring{e}_5 \setminus c\} \cup \{e_5^+\} \cup \{e_6^+\} \cup \{e_7^+\},$$

where W is a set of ℓ vertices outside $\mathcal{C}_B \cup \mathcal{P}_R$ disjoint from $Z_1 \cup Z_2$.

Consider $d := \{V_{[k-\ell]\setminus[k-2\ell]}(f_{m_p})\} \cup Z_2 \cup \{e_7^+\}$. Observe that either the ℓ-cycle

$$\big(\mathcal{C}_R \setminus \{q_2, q_3\}\big) \cup \mathcal{P}_R \cup \{a, d\}$$

is red, and then it covers all but $2(k-\ell)$ vertices, as desired for the theorem, or the edge d is blue, which we will assume from now on. Then by the maximality of $\mathcal{C}_B$, the edge $d' := \{V_{[k-\ell]\setminus[k-2\ell]}(f_{m_p})\} \cup Z_1 \cup \{e_6^+\}$ is red. Now consider the edge

$$e := \{e_6^+\} \cup Z_2 \cup \{e_7^+\}.$$

If e is blue, consider the blue path e and the red cycle $\mathcal{C}'_R$ to obtain a contradiction to the maximality of $\mathcal{C}_B$ in the choice of $\mathcal{C}_B$ and $\mathcal{P}_R$.

So e is red. Then the ℓ-cycle

$$\big(\mathcal{C}_R \setminus \{q_2, q_3\}\big) \cup \mathcal{P}_R \cup \{a, d', e\}$$

is red and covers all but $k - \ell$ vertices, as desired for the theorem.

This concludes the proof of Theorem 1.1 (b).

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