Degree conditions for trees in undirected and directed graphs

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Abstract

We survey some recent results on bounds on the minimum and maximum degree of graph that ensure it contains all trees of a fixed size, and corresponding results for digraphs. We also present a number of conjectures and open questions.

1 Introduction

At the core of extremal graph theory are subgraph containment problems, in particular those where the existence of a certain subgraph is connected to a condition on the minimum degree of the host graph. We focus on the case when the subgraph in question is a tree. One of the best-known results in this direction is the Komlós–Sárközy–Szemerédi theorem [12] which gives a lower bound on the minimum degree $\delta(G)$ of a graph G that ensures G contains any spanning tree of bounded degree. In recent years, some versions for smaller trees and corresponding lower bounds on $\delta(G)$ have been proposed. We will see below that in these versions, it is necessary to add an additional condition on the host graph, such as a lower bound on the maximum degree $\Delta(G)$ of G. We give a quick overview of the state of the art for graphs in Section 2.

A natural question is how to generalise problems of the above type to digraphs, following the increasingly popular trend of extending results from extremal graph theory to the digraph setting. As every graph can be viewed as a digraph, but the universe of digraphs is much richer than the world of graphs, this leads to fascinating

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new challenges. A few new results on using semidegree conditions to find oriented trees in digraphs are surveyed in Section 3.

For more context, we refer the interested reader to the surveys [18, 19], and remark that in particular, Section 3 here can be seen as a continuation of Section 6 of [19].

We remark that for better exposition, some of the results are not stated here in the same generality as they appear in the respective sources.

2 Trees in graphs

We start by noting that every graph G of minimum degree $\delta(G) \geq k$ contains all trees with k edges, as we can just embed the tree greedily into G. The bound on $\delta(G)$ could not be weaker: Consider the union of several disjoint copies of K_k which gives a graph of minimum degree $k-1$ that does not contain any tree with k edges. This example can be eliminated if we decide to add the constraint that G is connected and sufficiently large. Dirac and independently Erdős and Gallai observed the following.

Proposition 2.1 (See [4]). *If a graph G has minimum degree at least $k/2$ and a connected component on at least $k+1$ vertices then G contains a path with k edges.*

However, connectivity and large order alone are not helpful if we are looking for trees other than paths (see [18] for details).

Trying to overcome the example from above from a different angle, we may observe that the construction only works if $k < |V(G)|/2$. So for trees that are comparatively large with respect to the host graph G, a better bound on $\delta(G)$ would be possible. This has been confirmed for trees of bounded maximum degree in the folowing classical result:

Theorem 2.2 (Komlós, Sárközy and Szemerédi [12]). *For all $\delta > 0$, there are n_0 and c such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1+\delta)\frac{n}{2}$ contains each n-vertex tree T with $\Delta(T) \leq c\frac{n}{\log n}$.*

All bounds in Theorem 2.2 are tight up to the constant c. In particular, it is necessary to bound $\Delta(T)$. If we wish to omit the bound on $\Delta(T)$, that is, if we wish to find *all* spanning trees, then it will be necessary to require a vertex of large degree in G (because T could be a star). It turns out that having just one vertex of degree at least k is indeed sufficient, if we also elevate the bound on $\delta(G)$:

Theorem 2.3 (Reed and Stein [14, 15]). *There is n_0 such that for every $n \geq n_0$, every n-vertex graph with $\delta(G) \geq \lfloor 2(n-1)/3 \rfloor$ and $\Delta(G) \geq n-1$ contains each n-vertex tree.*

Theorem 2.3 solves a special case of the next conjecture, Conjecture 2.4, which considers smaller trees and corresponding weaker bounds on $\delta(G)$.

Conjecture 2.4 (Havet, Reed, Wood and Stein [5]). *Every graph of minimum degree at least $\lfloor 2k/3 \rfloor$ and maximum degree at least k contains each k-edge tree.*

Note that the bounds in Conjecture 2.4 are tight, which can be seen by considering the following example: Assume 3 divides k and consider the tree obtained from identifying the starting vertices of three distinct $\frac{k}{3}$-edge paths. This tree is not a subgraph of the graph obtained by taking two cliques of size $2k/3 - 1$ and adding a *universal vertex* (i.e., a vertex adjacent to all other vertices).

As a cross-over of Conjecture 2.4 and Theorem 2.2, it was conjectured in [2] that every graph of minimum degree at least $k/2$ and maximum degree at least $2k$ contains each k-edge tree. However, although an approximate version of this statement holds for large bounded degree trees [2], it is not true in general. Hyde and Reed [6] recently gave a counterexample. The following may still hold:

Conjecture 2.5. *Is it true that every graph with $\delta(G) > k/2$ and with $\Delta(G) \geq 2k$ contains each k-edge tree?*

Asymptotic versions of Conjectures 2.4 and 2.5 hold for bounded degree trees.

Theorem 2.6 (Besomi, Pavez-Signé and Stein [2]). *For every $\eta > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $k \geq \eta n$ every n-vertex graph G fulfilling*

(a) $\delta(G) \geq (1+\eta)k/2$ and $\Delta(G) \geq 2(1+\eta)k$; or

(b) $\delta(G) \geq (1+\eta)2k/3$ and $\Delta(G) \geq (1+\eta)k$

contains every k-edge tree T with $\Delta(T) \leq k^{1/67}$ as a subgraph.

Also, Hyde and Reed [6] recently proved the following for all host graphs.

Theorem 2.7 (Hyde and Reed [6]). *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph G with $\delta(G) \geq k/2$ and $\Delta(G) \geq f(k)$ contains each k-edge tree.*

Earlier it had been shown by Havet, Reed, Wood and the author [5] that there is an absolute constant $\gamma > 0$ such that every graph G with $\delta(G) \geq (1-\gamma)k$ and $\Delta(G) \geq k$ contains each k-edge tree.

If we wish to leave the minimum degree condition at $k/2$ but avoid requiring the existence of a vertex of very large degree, then a natural candidate for an alternative additional condition is to ask for some kind of expansion of G. In particular, forbidding short cycles should be helpful for finding trees. Note that it will still be necessary to require a vertex of degree at least k, because of the stars. The following result shows that it suffices to forbid 4-cycles.

Theorem 2.8 (Saclé and Woźniak (see Section 4 of [16])). *Every graph G having no 4-cycles and fulfilling $\delta(G) \geq k/2$ and $\Delta(G) \geq k$ contains any k-edge tree.*

Assuming G has large girth (where the *girth* of a graph G is the largest g such that G has no cycles of length $< g$), the bound on the minimum degree can be even smaller, as long as T has somewhat bounded degree.

Theorem 2.9 (Jiang [8]). *Every graph G of girth at least $2\ell + 1$ with $\delta(G) \geq \max\{k/\ell, \Delta(T)\}$ contains any k-edge tree.*

Note that the additional condition $\delta(G) \geq \Delta(T)$ is indeed necessary, which can be seen by considering a balanced double star and a suitable host graph.

3 Oriented trees in digraphs

We consider both *digraphs*, which may have up to one edge in each direction between any pair of vertices, and *oriented graphs* which have at most one edge between any pair of vertices. Loops are forbidden in both cases. Since the edges of a digraph have directions, there are several possibilities when it comes to translating the notion of the minimum degree to a digraph. The most popular one is the *minimum semidegree* $\delta^0(D)$, which is defined as the minimum over all the indegrees and all the outdegrees of the vertices of the digraph D.

In analogy to the graph case, every digraph of minimum semidegree at least k contains every oriented tree T with k edges. The bound cannot be weakened since T may be an oriented star. Even if T is an oriented path, a bound below k will not be sufficient: to see this, we can consider the disjoint union of *complete digraphs* (i.e. digraphs having all possible edges) of order k, which has minimum semidegree $k - 1$ but contains no oriented subgraph with k edges.

However, if we are only looking for oriented paths in oriented graphs, a generalisation of Proposition 2.1 may be possible. Indeed, the following was conjectured in [18].

Conjecture 3.1. **[18]** *Every oriented graph of minimum semidegree exceeding $k/2$ contains any oriented path of length k.*

Results of Kelly [10] imply that this is true if k is larger than $3n/4 + o(n)$, and an even earlier result of Jackson [7] shows that the conjecture holds for *directed paths*, i.e. oriented paths where each vertex has in- and outdegree at most one. Some more evidence can be found in [3, 11, 17].

One nice property of paths, apart from not having vertices of high degree, is that if they have an even number of vertices, they are *balanced*, i.e. their partition classes have the same size. It may be possible to extend Conjecture 3.1 to balanced oriented trees. In this direction, it was conjectured in [19] that every oriented n-vertex graph D with $\delta^0(D) > k/2$ contains each balanced antidirected k-edge tree with $\Delta(T) \leq o(n)$, where $\Delta(T)$ denotes *the maximum total degree of* T, i.e. the maximum degree of the underlying undirected tree, and a tree is *antidirected*, if each of its vertices has either outdegree or indegree 0. With a weaker bound on $\Delta(T)$, this conjecture is true if D is large, and k is large compared to the order of D:

Theorem 3.2 (Stein and Zárate-Guerén [21]). *For all $\eta > 0$, $c \in \mathbb{N}$ there is n_0 such that for all $n \geq n_0$ and $k \geq \eta n$, every oriented graph D on n vertices with $\delta^0(D) > (1+\eta)k/2$ contains every balanced antidirected tree T with k edges and with $\Delta(T) \leq (\log(n))^c$.*

We remark that the condition that T is balanced is necessary. To see this, assume k is even and consider the oriented graph D on sets V_1, V_2, V_3, each of size $k/2$, having all edges from V_i to V_{i+1} (mod 3). Then $\delta^0(D) = k/2$. Let T be an unbalanced antidirected tree. It is easy to see that T is not contained in D. In fact, the more unbalanced T is, the larger we could make the sets V_i, and D would still not contain T. The extreme case is when T is an antidirected star and each V_i has size $k-2$.

In the case that T and D have the same order and $\Delta(T)$ is bounded by a constant, we can omit the condition that T is antidirected in Theorem 3.2, and lower the bound on $\delta^0(D)$ to about $3n/8$, as we will see in the following new result. Interestingly, this bound on the minimum semidegree corresponds to the bound for oriented Hamilton cycles in oriented graphs due to Kelly [10].

Theorem 3.3 (Araújo, Santos and Stein [1]). *For every $\eta > 0$ and $\Delta \in \mathbb{N}$, there is an integer n_0 such that for all $n \geq n_0$ each oriented graph D on n vertices with $\delta^0(D) \geq (3/8+\eta)n$ contains each oriented tree T on n vertices with $\Delta(T) \leq \Delta$.*

Note that Theorem 3.3 can be viewed as a version of Theorem 2.2 for oriented graphs (albeit with a worse bound on $\Delta(T)$). Some years ago, Kathapurkar and Montgomery gave a generalisation of the same theorem to digraphs [9]. Again, all bounds are best possible up to the constant.

Theorem 3.4 (Kathapurkar and Montgomery [9]). *For each $\eta > 0$, there are $c > 0$ and $n_0 \in \mathbb{N}$ such that every n-vertex digraph on $n \geq n_0$ vertices and with $\delta^0(G) \geq (1/2+\eta)n$ contains each n-vertex oriented tree with $\Delta(T) \leq cn/\log n$.*

From the examples given above, we know that Theorem 3.4 becomes false if we change the size of the tree to arbitrary k (independent of n), and accordingly require $\delta^0(D) \geq (1/2+\eta)k$. However, this changes if we add the restriction that D has a vertex of large outdegree and a vertex of large indegree. Let us define $\Delta^{\pm}(D)$ as the largest ℓ such that D has a vertex of outdegree $\geq \ell$ and a vertex of indegree $\geq \ell$ (these two vertices may coincide).

In the spirit of Conjectures 2.4 and 2.5 we have the following new result for balanced antidirected trees.

Theorem 3.5 (Kontogeorgiou, Santos and Stein [13]). *For all $\eta > 0$ and $c \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $k \geq \eta n$, every digraph D on n vertices fulfilling*

(a) $\delta^0(D) \geq (\frac{3}{5}+\eta)k$ and $\Delta^{\pm}(D) \geq 2(1+\eta)k$ or

(b) $\delta^0(D) \geq (\frac{2}{3}+\eta)k$ and $\Delta^{\pm}(D) \geq (1+\eta)k$

contains every balanced antidirected tree T with k edges and with $\Delta(T) \leq (\log n)^c$.

Another new result is the generalisation of Theorem 2.8 to digraphs. We now forbid all orientations of the 4-cycle.

Theorem 3.6 (Stein and Trujillo-Negrete [20]). *Every digraph D having no oriented 4-cycles with $\delta^0(D) \geq k/2$ and $\Delta^{\pm}(D) \geq k$ contains all oriented trees T with k edges.*

It would be interesting to extend also Theorem 2.9 to digraphs. In this respect, we pose the following question.

Question 3.7. *Does every oriented graph D of girth at least $2\ell+1$ with $\delta^0(D) \geq \max\{k/\ell, \Delta(T)\}$ contain every orientation of each k-edge tree?*

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