MONOCHROMATIC CYCLE PARTITIONS IN LOCAL EDGE COLOURINGS

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Abstract. An edge colouring of a graph is said to be an r-local colouring if the edges incident to any vertex are coloured with at most r colours. Generalising a result of Bessy–Thomassé and others, we prove that the vertex set of any 2-locally coloured complete graph may be partitioned into two disjoint monochromatic cycles of different colours. Moreover, for any natural number r, we show that the vertex set of any r-locally coloured complete graph may be partitioned into $O(r^2 \log r)$ disjoint monochromatic cycles. This generalises a result of Erdős, Gyárfás and Pyber.

1. Introduction

A well-known result of Erdős, Gyárfás and Pyber [7] says that there exists a constant $c(r)$, depending only on r, such that if the edges of the complete graph K_n have been coloured with r colours, then the vertex set of K_n may be partitioned into at most $c(r)$ disjoint monochromatic cycles, where we allow single vertices and edges to be cycles. Moreover, they conjectured that this result should hold with $c(r) = r$.

The $r = 2$ case of this conjecture, attributed (see [2]) to Lehel, is slightly more specific, asking that the vertex set be partitioned into two disjoint cycles of different colours, where we now consider the empty set to be a cycle. This conjecture was proved, for n sufficiently large, by Łuczak, Rödl and Szemerédi [15] and later by Allen [1], though starting from a much smaller value of n. For all n, the conjecture was finally resolved by Bessy and Thomassé [3].

For $r \geq 3$, the conjecture was shown to be false by Pokrovskiy [16]. However, for the case $r = 3$, there are some partial results showing that the conjecture is still very close to being true [13, 16]. In general, the best known bound on $c(r)$, proved by Gyárfás, Ruszinkó, Sárközy and Szemerédi [12], is $O(r \log r)$ and, despite Pokrovskiy's counterexample, it seems likely that an approximate version of the original conjecture remains true.

We consider a generalisation of this monochromatic cycle partition question to graphs with locally bounded colourings. We say that an edge colouring of a graph is an *r-local colouring* if the edges incident to any vertex are coloured with at most r colours. Note that we do not restrict the total number of colours. Somewhat surprisingly, we prove that even for local colourings, a variant of the Erdős-Gyárfás-Pyber result holds.

Theorem 1.1. *The vertex set of any r-locally coloured complete graph may be partitioned into $O(r^2 \log r)$ disjoint monochromatic cycles.*

For $r = 2$, we have the following more precise theorem, which directly generalises the result of Bessy and Thomassé.

Theorem 1.2. *The vertex set of any 2-locally coloured complete graph may be partitioned into two disjoint monochromatic cycles of different colours.*

We will prove Theorem 1.1 in the next section by building on ideas of Erdős, Gyárfás and Pyber and using results on local Ramsey numbers. In Section 3, we will prove Theorem 1.2 using the result of Bessy and Thomassé as a black box. We conclude with some further remarks. In particular, we will discuss a further extension of Theorem 1.2 to 2-mean colourings.

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2. r-LOCAL COLOURINGS

In this section, we will prove Theorem 1.1 using a proof strategy similar to that of Erdős, Gyárfás and Pyber but with several elements added which are specific to local colourings. The first ingredient needed by Erdős, Gyárfás and Pyber is a result which says that if we have an unbalanced bipartite graph whose edges have been r-coloured then all of the vertices in the smaller set can be covered by a bounded number of cycles in terms of r. We will begin by proving an analogue of this result for local colourings.

We will need the following elementary lemma due to Pósa (see Problem 8.3 in [14]).

Lemma 2.1. *If a graph has independence number α, then the vertex set of the graph may be partitioned into at most α disjoint cycles.*

It will be useful to introduce some notation. Given a vertex x and a colour c, we let $N_c(x)$ be the neighbourhood of x in colour c. In the following proof, we will have a collection of colours $c_1, c_2, \ldots, c_r$ and we will simply write $N_i(x)$ for $N_{c_i}(x)$.

Lemma 2.2. *Suppose that A and B are vertex sets with $|B| \leq |A|/r^{r+3}$ and the edges of the complete bipartite graph between A and B are r-locally coloured. Then all vertices of B can be covered with at most r^2 disjoint monochromatic cycles.*

Proof. Fix a vertex b_1 in B. Since b_1 is incident to at most r colours, there exists a colour c_1 such that the neighbourhood $N_1(b_1)$ of b_1 in colour c_1 has size at least $|A|/r$. We let $A_1 = N_1(b_1)$.

Suppose now that $b_1, b_2, \ldots, b_i$ are distinct vertices in B, $c_1, c_2, \ldots, c_i$ are distinct colours and that $A_i = \cap_{j=1}^{i} N_j(b_j)$ satisfies $|A_i| \geq |A|/r^i$. If, for all $b \in B \setminus \{b_1, \ldots, b_i\}$, there exists a colour $c(b) \in \{c_1, c_2, \ldots, c_i\}$ such that

$$|A_i \cap N_{c(b)}(b)| \geq \frac{|A_i|}{r},$$

then, for each $j = 1, 2, \ldots, i$, we let $B_j = \{b : c(b) = c_j\}$ and $A' = A_i$. Otherwise, there exists a vertex $b_{i+1} \in B \setminus \{b_1, b_2, \ldots, b_i\}$ and a colour $c_{i+1} \notin \{c_1, c_2, \ldots, c_i\}$ such that $|A_i \cap N_{i+1}(b_{i+1})| \geq \frac{|A_i|}{r}$. Letting

$$A_{i+1} = A_i \cap N_{i+1}(b_{i+1}) = \cap_{j=1}^{i+1} N_j(b_j),$$

we see that $|A_{i+1}| \geq |A|/r^{i+1}$.

Suppose now that the process continues until we have defined a set $A_r = \cap_{j=1}^{r} N_j(b_j)$ with $|A_r| \geq |A|/r^r$. Since every vertex in A_r is adjacent to r different colours and the graph is r-locally coloured, we must have that all edges between A_r and B are coloured in $c_1, c_2, \ldots, c_r$. If this is the case, then, for each vertex $b \in B$, there exists a colour $c(b) \in \{c_1, c_2, \ldots, c_r\}$ such that $|A_r \cap N_{c(b)}(b)| \geq |A_r|/r$. For each $j = 1, 2, \ldots, r$, we let $B_j = \{b : c(b) = c_j\}$ and $A' = A_r$.

We may now assume that we have a subset A' of A with $|A'| \geq |A|/r^r$ and a partition of B into pieces $B_1, B_2, \ldots, B_r$ (some of which may be empty) such that, for each $j = 1, 2, \ldots, r$, every vertex $b \in B_j$ is adjacent to at least $|A'|/r$ vertices of A' in colour c_j. We define a graph G_j on vertex set B_j by joining $x, y \in B_j$ if and only if

$$|A' \cap N_j(x) \cap N_j(y)| \geq |A'|/r^3.$$

We claim that the graph G_j contains no independent set of order $r+1$. Suppose, for the sake of contradiction, that $x_1, x_2, \ldots, x_{r+1} \in B_j$ are the vertices of an independent set of order $r+1$. Then

$$\begin{aligned} |A'| &\geq \left|\cup_{k=1}^{r+1} \left(A' \cap N_j(x_k)\right)\right| \\ &\geq (r+1) \cdot \frac{|A'|}{r} - \sum_{1 \leq k < \ell \leq r+1} |A' \cap N_j(x_k) \cap N_j(x_\ell)| \\ &\geq |A'| \left(\frac{r+1}{r} - \frac{\binom{r+1}{2}}{r^3} \right) > |A'|, \end{aligned}$$

a contradiction. By Lemma 2.1, it follows that the vertex set B_j can be partitioned into at most r disjoint cycles from G_j. Using the definition of G_j and the fact that $|A' \cap N_j(x) \cap N_j(y)| \geq |A'|/r^3 \geq |B|$ for any two adjacent x, y from B_j, it is now easy to conclude that the vertices of B may be covered using at most r^2 disjoint monochromatic cycles. □

The *r-colour Ramsey number* of a graph H, denoted $R_r(H)$, is the smallest n such that in any r-colouring of the edges of K_n there is guaranteed to be a monochromatic copy of H. The local analogue of this concept, known as the *r-local Ramsey number* and denoted $R_{r-loc}(H)$, is the smallest n such that in any r-local colouring of the edges of K_n there is guaranteed to be a monochromatic copy of H. That the local Ramsey number exists was first proved by Gyárfás, Lehel, Schelp and Tuza [11]. We will need the following result of Truszczynski and Tuza [17], which says that for connected graphs the ratio of $R_r(H)$ and $R_{r-loc}(H)$ is bounded in terms of r.

Lemma 2.3. *For any connected graph H,*

$$\frac{R_{r-loc}(H)}{R_r(H)} \leq \frac{r^r}{r!} \leq e^r.$$

The following lemma generalises a result of Bollobás, Kostochka and Schelp [4]. For a class of graphs $\mathcal{H}$, the *r-local Ramsey number* $R_{r-loc}(\mathcal{H})$ is the smallest n such that in any r-local colouring of the edges of K_n there is guaranteed to be a monochromatic copy of some graph $H \in \mathcal{H}$.

Lemma 2.4. *Suppose that $\mathcal{H}$ is a class of graphs and c and ϵ are positive constants such that for all n any graph on n vertices with at least $cn^{2-\epsilon}$ edges contains a graph from $\mathcal{H}$. Then*

$$R_{r-loc}(\mathcal{H}) \leq (4cr)^{1/\epsilon}.$$

Proof. Suppose that the edges of K_n have been r-locally coloured with at most s colours, which we may assume are $\{1, 2, \ldots, s\}$. For each $i = 1, 2, \ldots, s$, let e_i be the number of edges in colour i and v_i the number of vertices which are incident with an edge of colour i. If there is no $H \in \mathcal{H}$ in colour i, then $e_i < cv_i^{2-\epsilon}$ for each $i = 1, 2, \ldots, s$. Since also $\sum_{i=1}^{s} v_i \leq rn$, we have

$$\frac{n^2}{4} \leq \binom{n}{2} = \sum_{i=1}^{s} e_i < c \sum_{i=1}^{s} v_i^{2-\epsilon} \leq c(\max_{1 \leq i \leq s} v_i^{1-\epsilon}) \sum_{i=1}^{s} v_i \leq crn^{2-\epsilon}.$$

This implies that $n < (4cr)^{1/\epsilon}$ and the result follows. □

We will only need the following corollary.

Corollary 2.5. *Let $\mathcal{C}_\ell$ be the collection of all cycles of length at least ℓ. Then*

$$R_{r-loc}(\mathcal{C}_\ell) \leq 2\ell r.$$

Proof. A classical result of Erdős and Gallai [6] shows that if a graph on n vertices contains at least $\ell n/2$ edges, then it contains a cycle of length at least ℓ. The result then follows from applying Lemma 2.4 with $\epsilon = 1$ and $c = \ell/2$. □

For any natural number k, we define the triangle cycle T_k to be the graph with vertex set $\{u_1, u_2, \ldots, u_k\} \cup \{v_1, v_2, \ldots, v_k\}$, where $u_1, u_2, \ldots, u_k$ form a cycle of length k and v_i is joined to u_i and u_{i+1} (with addition taken modulo k). That is, as the name suggests, we have a cycle formed from triangles. An important property of these graphs is that we can remove any subset of $\{v_1, v_2, \ldots, v_k\}$ and still find a cycle through all of the remaining vertices. The final ingredient we will need is a straightforward lemma of Erdős, Gyárfás and Pyber [7] about the Ramsey number of these triangle cycles.

Lemma 2.6.

$$R_r(T_k) = O(kr^{3r}).$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let K_n be a complete graph whose edges have been r-locally coloured. Combining Lemmas 2.3 and 2.6, we see that

$$R_{r-loc}(T_k) \leq e^r R_r(T_k) = O(kr^{4r}).$$

Therefore, there is a monochromatic triangle cycle T_k with $k = \Omega(n/r^{4r})$ in our colouring of K_n. We let A be the subset of this triangle cycle corresponding to the vertex set $\{v_1, v_2, \ldots, v_k\}$, that is, the collection of vertices which are not on the shortest cycle. We now restrict our attention to the complete graph on the vertex set $V(K_n) \setminus V(T_k)$.

By Corollary 2.5, any r-locally coloured K_t contains a monochromatic cycle of length at least $t/2r$. Removing the vertices of this cycle leaves an r-locally coloured complete graph with at most $t(1 - 1/2r)$ vertices. If we start with the vertex set $V(K_n) \setminus V(T_k)$ and apply this observation s times, we remove s disjoint monochromatic cycles and leave an r-locally coloured complete graph with at most

$$n\left(1 - \frac{1}{2r}\right)^s \leq ne^{-s/2r}$$

vertices. Therefore, for $s = cr^2 \log r$ and c sufficiently large, we see that the remaining set of vertices has size at most $|A|/r^{r+3}$, where we used that $|A| = k = \Omega(n/r^{4r})$. We let B be this remaining set.

We now apply Lemma 2.2 to the bipartite graph between A and B. This implies that there is a collection of at most r^2 disjoint monochromatic cycles which cover all vertices of B. Though we have deleted a vertex subset A' of A, the vertices $V(T_k) \setminus A'$ still contain a monochromatic cycle. Altogether, we have partitioned the vertex set of K_n using at most

$$cr^2 \log r + r^2 + 1 = O(r^2 \log r)$$

disjoint monochromatic cycles, completing the proof. □

3. 2-local colourings

In this section we will prove Theorem 1.2, that the vertex set of any 2-locally coloured complete graph may be partitioned into two disjoint monochromatic cycles of different colours. We begin with a simple corollary of Bessy and Thomassé's result that every 2-coloured complete graph may be partitioned into two disjoint monochromatic cycles of different colours. Throughout this section, we will say that a colour α *sees* a vertex v, and vice versa, if there is an edge of colour α incident with v.

Lemma 3.1. *Let K_n be 2-locally coloured such that there is a colour α which sees all vertices. Then there are two disjoint monochromatic cycles of different colours, one of these α, that together cover all of $V(K_n)$.*

Proof. Let β be the union of all colours other than α. Apply the result of Bessy and Thomassé to find two disjoint cycles covering all of K_n, one in colour α, the other in colour β. However, the cycle in colour β must be monochromatic in the original colouring, since every vertex sees at most one colour different from α. □

We next show a slight strengthening of Lemma 3.1 will only be used in the next section, in order to prove an extension of Theorem 1.2 to mean colourings. In Lemma 3.2, the colouring is not required to be a 2-local colouring, allowing a single vertex v to see more than two colours.

Lemma 3.2. *Let the edges of K_n be coloured in such a way that each vertex except possibly v sees α and at most one other colour. Then there are two disjoint monochromatic cycles of different colours, one of these α, that together cover all of $V(K_n)$.*

Proof. Let β be the union of all colours other than α. Apply the result of Bessy and Thomassé to find two disjoint cycles covering all of K_n, one in colour α, the other in colour β. If the cycle of colour α contains v, we are done as in the previous lemma. If the cycle of colour β contains v, then we may write the vertices of this cycle as $v_1, v_2, \dots, v_t$, where $v_1 := v$ and $v_i v_{i+1}$ is an edge for all $i = 1, 2, \dots, t$ (with addition taken modulo t). But then, since each vertex v_i with $2 \le i \le t$ sees at most one colour other than α, we must have that the edges $v_{i-1}v_i$ and $v_i v_{i+1}$ have the same colour for all $2 \le i \le t$. This implies that the cycle is monochromatic, completing the proof. □

Suppose now that we have a complete bipartite graph between sets A and B and each of A and B contain paths. The following simple lemma gives a condition under which these paths may be combined into a spanning cycle on $A \cup B$. Note that for a path P, we let $|P|$ be the number of vertices on P.

Lemma 3.3. *Let G be a graph and let A and B be disjoint subsets of $V(G)$ such that $G[A, B]$ is complete. Let P_A and P_B be paths in A and B, respectively, with $P_A, P_B \neq \emptyset$. If $|B - P_B| \le |A - P_A| \le |B| - 1$, then $A \cup B$ has a spanning cycle.*

Proof. We form a path by first following P_A and then alternating between B and A until we cover all of A. While doing so, we prefer vertices of B not covered by P_B but if we have to use vertices from P_B we use them in the order they lie on P_B. This gives a path P of order $2|A| - |P_A|$, starting and ending in A. Since $|A - P_A| \le |B| - 1$, we know P covers all of A, and since $|B - P_B| \le |A - P_A|$, the path P also covers all of $B - P_B$. Further, P possibly covers some, but not all, of P_B. We connect P with the remains of P_B to complete a cycle covering all of $A \cup B$. □

In the next lemma we generalize the situation above to three sets and two monochromatic cycles. See Figure 1 for an illustration.

Lemma 3.4. *Let G be a graph whose edges are coloured with colours 1 and 2 and let A_1, A_2 and B be disjoint subsets of $V(G)$. For $i = 1, 2$, let $G[A_i, B]$ be complete in colour i, let $P_{A_i} \neq \emptyset$ be a path of colour i in A_i and let P_B^i be a path of colour i in B. If*
(a) P_B^1 and P_B^2 partition B, and
(b) $|A_1 - P_{A_1}| + |A_2 - P_{A_2}| + 2 \le |B|$,
then $A_1 \cup A_2 \cup B$ has a partition into two monochromatic cycles, one of each colour.

Proof. First of all, note that we may assume that the paths P_B^i are non-empty. Otherwise, since $|P_B^1| + |P_B^2| = |B| \ge 2$ by (a) and (b), we can slightly modify our paths so that both are non-empty, without losing the conditions of the lemma.

Take a subpath P_i of P_B^i of order $\min\{|P_B^i|, |B| - |A_{3-i} - P_{A_{3-i}}| - 1\}$ for $i = 1, 2$. Note that $P_i \neq \emptyset$ since we assumed $P_B^i \neq \emptyset$ and also $|B| - |A_{3-i} - P_{A_{3-i}}| - 1 \ge 1$ by (b). Observe, by (a) and (b), that we have $P_i = P_B^i$ for at least one of $i = 1, 2$. Otherwise, we would have $|P_B^i| \ge |B| - |A_{3-i} - P_{A_{3-i}}|$

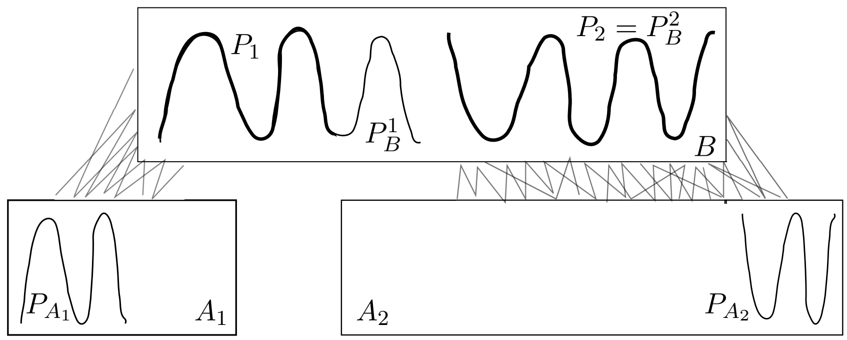

FIGURE 1. The paths P_i in the proof of Lemma 3.4. We choose P_1 in such a way that $B - P_1$ is large enough to form a cycle together with A_2.

for $i = 1, 2$ and thus $|B| = |P_B^1| + |P_B^2| \geq 2|B| - |A_1 - P_{A_1}| - |A_2 - P_{A_2}| \geq |B| + 2$, a contradiction. Without loss of generality, we may therefore assume that $P_2 = P_B^2$.

Clearly, either $|P_1| = |B| - |A_2 - P_{A_2}| - 1$ or $P_1 = P_B^1$ (or both). In the first case, we use (b) to see that $|P_1| \geq |A_1 - P_{A_1}| + 1$. In the second case, we see that $P_2 = P_B^2$ implies that $|P_B^2| \leq |B| - |A_1 - P_{A_1}| - 1$, by the choice of P_2. Thus, using (a), we get $|P_1| = |P_B^1| = |B| - |P_B^2| \geq |A_1 - P_{A_1}| + 1$. In either case, we obtain

$$|P_1| \geq |A_1 - P_{A_1}| + 1. \tag{1}$$

Let $B_1 := V(P_1)$ and $B_2 := B - B_1$. Note that $|B_i - P_i| \leq |A_i - P_{A_i}| \leq |B_i| - 1$ for $i = 1, 2$. The first inequality holds for $i = 1$ by the definition of B_1 and for $i = 2$ since $B_2 - P_2 = P_B^1 - P_1$, which is either empty or has size exactly $|P_B^1| - |B| + |A_2 - P_{A_2}| + 1 \leq |A_2 - P_{A_2}|$. The second inequality holds for $i = 1$ by the definition of B_1 and by (1). It holds for $i = 2$ since $B_2 = B - P_1$ and $|P_1| \leq |B| - |A_2 - P_{A_2}| - 1$ by the choice of P_1. We may therefore apply Lemma 3.3 separately to the pairs A_i, B_i with paths P_{A_i} and P_i (recall that $P_i \neq \emptyset$). This gives the desired partition. □

The proof of Theorem 1.2 splits into two cases, depending on whether or not there is a colour which sees every vertex. When there is a colour which sees every vertex, Lemma 3.1 gives the required result. When there is no such colour, it is easy to argue that the graph contains exactly 3 colours and looks like the configuration in Figure 2. The proof that configurations of this type may be coloured by two monochromatic cycles of different colours forms the core of our proof. While the proof of Lemma 3.1 relies crucially on the result of Bessy and Thomassé, we will only use the much simpler result of Gyárfás [9, 10] that every 2-coloured graph may be partitioned into two monochromatic paths of different colours to handle the remaining configurations.

Proof of Theorem 1.2. Suppose that the edges of K_n have been 2-locally coloured. Let S be the vertex set of a largest monochromatic connected subgraph, say in colour 1. If colour 1 sees all the vertices, then we are done by Lemma 3.1. So $V_{23} := K_n - S$ is not empty.

Let $x \in V_{23}$. Then, by the choice of S, we know that all edges between x and S receive at most two colours, both different from 1. Suppose these colours are 2 and 3. For $i = 2, 3$, let $V_{1i} = S \cap N_i(x)$. Note that these sets partition S and are non-empty (otherwise $\{x\} \cup S$ is a larger monochromatic connected component). Moreover, V_{1i} must contain only the colours 1 and i and the bipartite graph between V_{12} and V_{13} must be monochromatic in colour 1. Since every vertex in V_{1i} sees edges with colours 1 and i, we also see that the bipartite graph between V_{1i} and V_{23} must

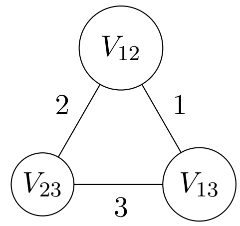

Figure 2. The special configuration in Theorem 1.2. All edges with an endpoint in $V_{i,j}$ must have colour i or colour j. In particular, all edges between V_{ij} and $V_{i\ell}$ have colour i.

be monochromatic in colour i. Finally, this implies that the set V_{23} only contains edges of colours 2 and 3. This situation is illustrated in Figure 2.

If $|V_{23}| = 1$, we first ignore the single vertex v in V_{23} and apply Lemma 3.1 to find a partition of $V_{12} \cup V_{13}$ into two monochromatic cycles, one of colour 1 and one of colour $i \in \{2, 3\}$. Since the second cycle will be contained entirely in V_{1i} and all edges between v and V_{1i} are coloured i, it is straightforward to extend this cycle to include v. We may therefore assume that $|V_{23}| \geq 2$ and, similarly, that $|V_{12}| \geq 2$ and $|V_{13}| \geq 2$.

By Gyárfás' observation, we know that each V_{ij} has a partition into two paths, one of colour i and order r^i_{ij} and one of colour j and order r^j_{ij} (the *order* of a path counts the vertices on it). We assume our graph does not admit a partition into two monochromatic cycles, of different colours, as otherwise we are done. So we can apply Lemma 3.4 to see that for any choice of paths as above (six paths in total), we have

$$\text{if } r^j_{ij}, r^\ell_{i\ell} > 0 \text{ then } r^i_{ij} + r^i_{i\ell} + 1 \geq |V_{j\ell}| \tag{2}$$

for any distinct $i, j, \ell \in \{1, 2, 3\}$. (In fact, we apply Lemma 3.4 three times, so that each of the sets V_{12}, V_{13}, V_{23} appears once in the role of B. Then (2) follows from the negation of condition (b) of Lemma 3.4.)

We now fix paths P^i_{ij} and P^j_{ij} partitioning V_{ij} for each distinct $i, j \in \{1, 2, 3\}$ and set $p^i_{ij} := |P^i_{ij}|$ and $p^j_{ij} := |P^j_{ij}|$. Since $|V_{ij}| \geq 2$ for all $i, j \in \{1, 2, 3\}$, we may assume that each of our paths has at least one vertex. Slightly abusing notation by setting $p^i_{ij} := p^i_{ji}$ whenever necessary, we clearly have

$$|V_{12}| + |V_{13}| + |V_{23}| = \sum_{i,j \in \{1,2,3\}, i<j} (p^i_{ij} + p^j_{ij}) = \sum_{i,j,\ell \in \{1,2,3\}, \ell \neq i < j \neq \ell} (p^\ell_{i\ell} + p^\ell_{j\ell}). \tag{3}$$

Hence, for at least two of the three sets V_{ij}, we have $|V_{ij}| \geq p^\ell_{i\ell} + p^\ell_{j\ell}$, where $i \neq \ell \neq j$ (if this inequality was false for two of the sets, then the third set would violate (2)). Without loss of generality, we will assume that

$$|V_{12}| \geq p^3_{13} + p^3_{23} \quad \text{and} \quad |V_{13}| \geq p^2_{12} + p^2_{23}. \tag{4}$$

Now consider the edges between the endpoints of the paths P^i_{ij} and P^j_{ij} for distinct $i, j \in \{1, 2, 3\}$. These edges have colours in $\{i, j\}$. Therefore, there is always one of the two paths which can be made shorter by 1 (possibly becoming empty), while augmenting the other path by 1. Even more is true: if one of the two paths, say P^i_{ij}, cannot be shortened[1] by 1, then the other one, P^j_{ij}, either can be shortened by at least 2 or is a one-vertex path. In the latter case, observe that P^i_{ij} can be extended to a spanning cycle of V_{ij}.

[1]A bit incorrectly, we say a path P^i_{ij} can/cannot be shortened by x if the other path, P^j_{ij}, can/cannot be augmented by x using the endpoints of P^i_{ij}.

Apply this reasoning to the paths P^i_{23}, $i = 2, 3$. By (2), neither of these paths may be shortened by $p^i_{1i} + p^i_{23} + 2 - |V_{1(5-i)}|$ (note that this is either 1 or 2). Otherwise, V_{23} would have a partition into two paths, with the path of colour i having order $q^i_{23} = |V_{1(5-i)}| - 2 - p^i_{1i}$. Since

$$p^i_{1i} + q^i_{23} = |V_{1(5-i)}| - 2,$$

this would contradict (2). If now each of the paths P^i_{23}, $i = 2, 3$ may be shortened by one, this implies that $p^i_{1i} + p^i_{23} + 2 - |V_{1(5-i)}| \geq 2$ and so $|V_{12}| = p^3_{13} + p^3_{23}$ and $|V_{13}| = p^2_{12} + p^2_{23}$, by (4). Next, assume that one of the paths cannot be shortened by 1. Hence, the other path, say P^2_{23}, can either be shortened by 2 or consists of only one vertex. The first case cannot hold since then $p^2_{12} + p^2_{23} + 2 - |V_{13}| \geq 3$ and so $|V_{13}| \leq p^2_{12} + p^2_{23} - 1$, contradicting (4). In the second case, since we can shorten P^2_{23} by one, we can again argue that $|V_{13}| = p^2_{12} + p^2_{23}$. Therefore, one of the following holds, after possibly swapping colours 2 and 3 for (b),

(a) each of the paths P^i_{23}, $i = 2, 3$, may be shortened by one and, thus, $|V_{12}| = p^3_{13} + p^3_{23}$ and $|V_{13}| = p^2_{12} + p^2_{23}$,
(b) $p^2_{23} = 1$, $|V_{13}| = p^2_{12} + p^2_{23}$ and V_{23} has a spanning cycle in colour 3.

In case (a), note that by (3), we also have $|V_{23}| = p^1_{12} + p^1_{13}$. In each of V_{1i}, $i = 2, 3$, one of the two paths P^1_{1i}, P^i_{1i} can be shortened by 1. However, by (2), not both P^1_{12} and P^1_{13} can be shortened by one. Thus, one of the P^i_{1i}, $i = 2, 3$, can be shortened by 1. Since, by (a), P^i_{23} can also be shortened by one, this contradicts (2).

So assume (b) holds. Then $|V_{13}| = p^2_{12} + p^2_{23} = p^2_{12} + 1$. We now wish to apply Lemma 3.3 with $A := V_{12}$, $B := V_{13}$, $P_A := P^1_{12}$ and P_B any one-vertex path in V_{13}. This is possible since both paths are nonempty and

$$|A - P_A| = |V_{12}| - p^1_{12} = p^2_{12} = |V_{13}| - 1 = |B| - 1.$$

Hence, there is a cycle in colour 1 that covers all of $V_{12} \cup V_{13}$. Together with the spanning cycle of V_{23} in colour 3, this gives the desired partition. □

4. Concluding remarks

Constructing counterexamples. The topic of this paper was originally motivated by an attempt to construct further counterexamples to the original conjecture. Concretely, suppose that one has an $(s - 1)$-locally coloured complete graph containing s colours in total and that at least s monochromatic cycles are necessary to cover all vertices. It will also be useful to assume that this property is somewhat robust. For our purposes, it will be sufficient to know that s monochromatic cycles are still needed to cover the graph whenever any vertex is deleted.[2]

We now add an additional vertex u to the graph. For each vertex v, we give uv a colour which did not appear at v in the original colouring. If we try to cover this new graph with monochromatic cycles, we see that u must appear either on its own or as part of a single edge. If it occurs on its own, it is clear that $s + 1$ monochromatic cycles are needed to cover the graph. If it appears as part of a single edge uv, we delete u and v. But then the robustness property tells us that the graph that remains requires at least s monochromatic cycles to cover all vertices, so we needed $s + 1$ cycles in total.

Unfortunately, since we found no $(s - 1)$-locally coloured complete graphs with the required properties, we could not use this technique to produce further counterexamples to the original conjecture. However, it may yet be a fruitful direction to consider.

The structure of r-local colourings. When proving Theorem 1.2, we saw that we may split our deduction into two cases, depending on whether or not there was a colour which was seen

[2]It is worth noting that Pokrovskiy's example [16] of a 3-coloured complete graph which requires four monochromatic cycles is not robust in this sense, since one may cover all but one vertex with three monochromatic cycles.

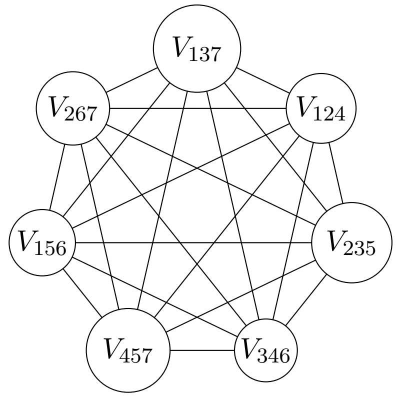

FIGURE 3. A 3-local colouring corresponding to the Fano plane. All edges with an endpoint in V_{ijk} must have colour i, j or k. In particular, all edges between V_{ijk} and $V_{i\ell m}$ have colour i.

by all vertices. The case where one colour sees all vertices followed easily as a corollary of the Bessy–Thomassé result, while the case where not all vertices are seen by one colour devolved into a special case which we had to study in depth.

As noted in [11], a similar methodology can be applied to r-local colourings. Suppose that we have an r-locally coloured complete graph using s colours in total, which we may assume to be $1, 2, \dots, s$. We form a hypergraph on the vertex set $\{1, 2, \dots, s\}$ by letting $\{c_1, c_2, \dots, c_i\}$ with $i \le r$ be an edge if and only if there exists a vertex v which sees precisely the colours $c_1, c_2, \dots, c_i$. Since any two vertices in our complete graph have an edge between them, any two edges in this hypergraph must intersect. A result of Erdős and Lovász [8] now implies that either there are $r - 1$ vertices such that every edge in the hypergraph contains at least one of these vertices or the hypergraph has at most r^r edges. Translated back to the original setting, either there are $r - 1$ colours such that every vertex sees at least one of these colours or every vertex sees one of at most r^r different colour combinations.

For $r = 2$, this result again reduces to saying that either there is a colour seen by every vertex or we have a configuration of the type illustrated in Figure 2. For $r = 3$, it tells us that there are two colours, say 1 and 2, such that either every vertex sees at least one of 1 and 2 or every vertex sees one of at most 27 different colour combinations (actually, as noted in [11], this may be reduced to 10). While this certainly gives us substantial extra information, a detailed analysis of monochromatic covers in 3-local colourings is likely to be unwieldy. For example, one would have to analyse colourings corresponding to the Fano plane illustrated in Figure 3.

It might also be interesting to restrict the total number of colours. To give an example, a result of Pokrovskiy [16] states that the vertex set of any 3-coloured complete graph may be partitioned into at most three monochromatic paths. Perhaps one can prove that this remains true in any 3-locally coloured complete graph containing at most 4 colours.

Mean colourings. An edge colouring of a graph is said to be an r-mean colouring if the average number of colours incident to any vertex is at most r. The r-mean Ramsey number of a graph H, denoted $R_{r-mean}(H)$, is the smallest n such that in any r-mean colouring of K_n there is guaranteed to be a monochromatic copy of H. This concept was introduced by Caro [5], who also proved that $R_{r-mean}(H)$ exists for all H. The proof is quite simple: we find a large subset on which the colouring is an $(r + 1)$-local colouring and then apply the existence of local Ramsey numbers.

Given the ease with which the concept of local Ramsey numbers generalises to mean Ramsey numbers, it is worth asking whether the vertex set of any r-mean coloured K_n can be partitioned into a finite number of disjoint monochromatic cycles. We have been unable to resolve this question in general. However, for $r = 2$, we can again show that two monochromatic cycles of different colours are sufficient to cover all vertices of K_n. We now sketch the proof.

To begin, note that the set of vertices V_1 seeing exactly one colour is at least as large as the set of vertices V_3 seeing three or more colours. In particular, if V_1 is empty, then so is V_3 and Theorem 1.2 applies. We may therefore assume that $V_1 \neq \emptyset$. Note that all vertices in V_1 must see the same colour, which we assume to be colour 1. It is now straightforward to choose a cycle in colour 1 which covers all vertices in $V_1 \cup V_3$. However, we will instead choose a cycle C_1 which covers all but one vertex v of V_3 and which uses an internal edge e in V_1 (unless $|V_1| = 1$, in which case the cycle is just a singleton). Consider now the set of vertices V_2 seeing exactly two colours. Since $V_1 \neq \emptyset$, every vertex in V_2 sees colour 1. We may therefore apply Lemma 3.2 to conclude that $V_2 \cup \{v\}$ may be covered by two monochromatic cycles of different colours, say C_1' and C_2, with C_1' of colour 1. Using the edge e (or the single vertex if $|C_1| = 1$), we may now combine C_1 and C_1' into a cycle, completing the proof.

Improving the bounds. We have proved that the vertex set of any r-locally coloured complete graph may be partitioned into $O(r^2 \log r)$ monochromatic cycles. It would be interesting to know whether these bounds can be substantially improved. While we cannot hope that r cycles are always enough, it seems plausible that $O(r)$ is. It would already be interesting to bring the bounds in line with the $O(r \log r)$ bound of Gyárfás, Ruszinkó, Sárközy and Szemerédi [12].

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