Kalai's conjecture in r-partite r-graphs

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Abstract

Kalai conjectured that every n-vertex r-uniform hypergraph with more than $\frac{t-1}{r}\binom{n}{r-1}$ edges contains all tight r-trees of some fixed size t. We prove Kalai's conjecture for r-partite r-uniform hypergraphs. Our result is asymptotically best possible up to replacing the term $\frac{t-1}{r}$ with the term $\frac{t-r+1}{r}$.

We apply our main result in graphs to show an upper bound for the Turán number of trees.

1 Introduction

For graphs, the well-known Erdős-Sós conjecture from 1963 states that any graph with more than $(t-1)\frac{n}{2}$ edges contains, as subgraphs, all trees with t edges. In 1984, Kalai introduced a natural generalisation of this conjecture to uniform hypergraphs. For simplicity, from now on r-uniform hypergraphs will be called *r-graphs*.

In order to be able to state Kalai's conjecture, we need to clarify the notion of a *tree* in an r-graph. The definition we will use relies on the following construction. Start with an r-edge e_1 and the r vertices it contains: This is T_1. Now, in every step i, we may add a new edge e_i. It is required that e_i

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contains precisely one new vertex v_i, and that $e_i \setminus \{v_i\}$ is a subset of some edge of T_{i-1}. The new hypergraph is T_i. Any hypergraph that can be constructed in this way will be called a *tight r-tree*.

Here is Kalai's generalisation of the Erdős-Sós conjecture.

Conjecture 1.1 (Kalai 1984, see [6]). *Let H be an r-graph on n vertices with more than $\frac{t-1}{r}\binom{n}{r-1}$ edges. Then H contains every tight r-tree T having t edges.*

As already noted in [6], it follows from constructions using a result of Rödl [14] (or alternatively, one can use designs whose existence is guaranteed by Keevash's work [12]) that this conjecture is tight as long as certain divisibility conditions are satisfied.

It has been observed (see e.g. [7]) that if the bound in Conjecture 1.1 is multiplied with a factor of r then the conjecture holds:

$$\text{Conjecture 1.1 holds if the bound is replaced by } (t-1)\binom{n}{r-1}. \tag{1}$$

The reason is that we can successively delete edges from the host r-graph until arriving at an r-graph H' having the property that each $(r-1)$-subset S of $V(H)$ either belongs to 0 or to at least t edges. Then we can embed the tree greedily into H', following the given ordering of the edges.

Not much is known on Kalai's conjecture in general, except for the case $r = 2$. We refer to [15] for an overview of known results in this case.

The known results for $r \geq 3$ all focus on specific types of tight r-trees. Frankl and Füredi [6] show that Conjecture 1.1 holds for all 'star-shaped' tight r-trees, that is, tight r-trees whose first edge intersects each other edge in $r-1$ vertices. Füredi, Jiang, Kostochka, Mubayi and Verstraëte [9, 10] show versions of Conjecture 1.1 for a broadened variant of the concept of 'star-shaped' (instead of the first edge, there is a constant number of first edges intersecting all other edges), and Füredi and Jiang [7] show the conjecture for special types of tight r-trees with many leaves.

For tight r-paths, bounds on the number of edges of the host r-graph below the bound $(t-1)\binom{n}{r-1}$ from (1) were established by Patkós [13] and by Füredi, Jiang, Kostochka, Mubayi and Verstraëte [8]. Namely, the bound in (1) can be replaced by $\frac{t-1}{2}\binom{n}{r-1}$ if r is even, and by a similar bound if r is odd. An asymptotic version of Kalai's conjecture for tight r-paths whose order is linear in the order n of the host r-graph has been confirmed by Allen, Böttcher, Cooley and Mycroft [1] for large n.

Also, the authors of [9] show that if one replaces the bound in Kalai's conjecture with $\frac{t-1}{r}|\partial H|$, an equivalent conjecture is obtained. (As it is usual, we define the *shadow* ∂H of an r-graph H as the set of all $(r-1)$-sets contained in edges of H.)

Our first contribution is a solution of Kalai's conjecture for r-partite r-graphs. We will actually show our result with the bound from [9], that is, the term $\binom{n}{r-1}$ from Kalai's conjecture will be replaced with the smaller term $|\partial H|$.

Theorem 1.2. *Let $r \geq 2$ and let H be an r-partite r-graph. If H has more than $\frac{t-1}{r}|\partial H|$ edges, then H contains every tight r-tree T having t edges.*

The proof of Theorem 1.2 relies on an auxiliary lemma, Lemma 2.1, which might be interesting in its own right. Lemma 2.1 is a variation of the earlier observation on subhypergraphs of convenient codegree that led to (1) (as usual, we will say $S \in \partial H$ has *codegree* c if S lies in c edges of H). The novelty in Lemma 2.1 is that it allows for a different minimum codegree into each of the partition classes. For instance, for bipartite graphs of average degree exceeding $t_1 + t_2 - 2$, the lemma yields a subgraph having minimum degree at least t_1 in one direction, and t_2 in the other direction, for any integers t_1, t_2 (see Corollary 2.2). For r-partite r-graphs, Lemma 2.1 gives an analogous statement based on codegrees.

Theorem 1.2 is not very far from best possible. We will see in Proposition 3.1 that there are r-partite r-graphs H not containing all tight r-trees with t edges fulfilling

$$|E(H)| \sim \frac{t-r+1}{r}|\partial H|. \tag{2}$$

For $r \geq 3$, this might indicate some room for a small improvement of the bound $\frac{t-1}{r}|\partial H|$ from Theorem 1.2, with other methods than the ones used here.

Theorem 1.2 also has an interesting application. Namely, for graphs, Theorem 1.2 can be used to obtain an upper bound for the *Turán number* $ex(n,T)$ of a tree T (this is the maximum number of edges a graph on n vertices can have without necessarily containing T as a subgraph).

Observe that for $r = 2$, the bound implied by (1) for the Turán number of a t-edge tree T is

$$ex(n,T) \leq (t-1)n, \tag{3}$$

which is a factor of 2 away from the bound $ex(n,T) \leq \frac{(t-1)n}{2}$ one can calculate from the Erdős–Sós conjecture. We will see in Proposition 4.2 that the bound (3) can be replaced by the slightly better bound

$$ex(n,T) \leq \frac{t}{t+1}(t-1)n. \tag{4}$$

Moreover, if t is even, the term $\frac{t}{t+1}$ can be replaced with the term $\frac{t-1}{t}$. We achieve the bound (4) by considering a maximum 2-*cut*, that is, a bipartition of the vertices of a graph G that maximises the number of edges crossing the bipartition. A result of Alon, Krivelevich and Sudakov [2] states that, if a fixed tree is excluded from G, then one can guarantee that substantially more than half of the edges of G cross some 2-cut. We then apply Theorem 1.2 to the bipartite graph spanned by the edges in the cut.

The paper is organised as follows. We will state and prove Lemma 2.1 and use it to prove Theorem 1.2 in Section 2. In Section 3 we will prove that Theorem 1.2 is not far from best possible in the above described sense, at least for balanced trees, by exhibiting r-graphs H fulfilling (2). Finally, in Section 4, we will prove (4), our Turán number bound for trees, in Proposition 4.2.

2 Proof of Theorem 1.2

The main ingredient for the proof of Theorem 1.2 is Lemma 2.1 below. For convenience, let us give a quick definition before we state the lemma.

If H is an r-partite r-graph, we say that $\delta_{(1,2,\ldots,r)}(H) \geq (t_1, t_2, \ldots, t_r)$ if there is a way of labeling the partition classes of H as $V_1, V_2, \ldots, V_r$ such that for each $i \in [r]$, every $S \in \partial H$ missing V_i is contained in at least t_i edges of H.

Lemma 2.1. *Let $r, t, t_1, t_2, \ldots, t_r \in \mathbb{N}$ such that*

$$t_1 + t_2 + \ldots + t_r = t + r - 1,$$

and let H be an r-partite r-graph with more than $\frac{t-1}{r}|\partial H|$ edges. Then there is a non-empty r-graph $H' \subseteq H$ such that

$$\delta_{(1,2,\ldots,r)}(H') \geq (t_1, t_2, \ldots, t_r).$$

Lemma 2.1 has the following corollary.

Corollary 2.2. *For all $t_1, t_2 \in \mathbb{N}$ every bipartite graph G with $d(G) > t_1 + t_2 - 2$ has a non-empty subgraph $G' = (V_1, V_2)$ such that each vertex in V_i has degree at least t_i in G', for $i = 1, 2$.*

Before proving Lemma 2.1, let us show how it implies Theorem 1.2. For this, we will need the following fact which is immediate from the definition of tight r-trees.

Fact 2.3. *Every tight r-tree has a unique r-partition.*

Now we are ready to prove our main result.

Proof of Theorem 1.2. Assume we are given an r-graph H, and a tight r-tree T with t edges. Consider the r-partition of T given by Fact 2.3, and let $t_1, t_2, \ldots, t_r$ be the sizes of the partition classes. Thus

$$t_1 + t_2 + \ldots + t_r = t + r - 1.$$

We apply Lemma 2.1 to see that there is an r-graph $H' \subseteq H$ with r-partition $V_1 \cup V_2 \cup \ldots \cup V_r$, such that for each $i \leq r$, any element of $\partial H'$ which avoids V_i is contained in at least t_i edges of H' (each containing a different vertex from V_i). We may therefore embed T following its natural order $v_1, v_2, \ldots, v_{t+r-1}$. At every step j, there is an unoccupied vertex we can choose as the image of v_j, because the total number of vertices from $V(T)$ we need to embed in any fixed class V_i is at most t_i. □

It only remains to prove Lemma 2.1.

Proof of Lemma 2.1. We may assume that $t_1 \leq t_2 \leq \ldots \leq t_r$. Set

$$\delta_i := t_i - \frac{t + r - 1}{r},$$

for all $i = 1, \ldots, r$. Clearly, we have

$$\delta_1 \leq \delta_2 \leq \ldots \leq \delta_r \quad \text{and} \quad \delta_1 \leq 0, \tag{5}$$

and moreover,

$$\sum_{i=1}^{r} \delta_i = 0. \tag{6}$$

We now turn to the r-graph H, with its r-partition $V_1 \cup V_2 \cup \ldots \cup V_r$. We let h_i denote the number of elements of ∂H that avoid V_i, for each $i \in [r]$. Clearly,

$$\sum_{i=1}^{r} h_i = |\partial H|, \tag{7}$$

and after possibly relabeling the partition classes of H, we may assume that

$$h_1 \geq h_2 \geq \ldots \geq h_r. \tag{8}$$

Let E_0 denote the set of all edges of H. For $j \geq 1$, we inductively define the set E_j as follows. If there is an $(r-1)$-set $S \subseteq V(H)$ missing V_i and contained in at least one, but less than t_i edges from E_{j-1}, then we set $E_j := \{e \in E_{j-1} : e \not\supseteq S\}$. If there is no set $S \subseteq V(H)$ as above, we terminate the process, and set $E := E_{j-1}$.

Observe that every $(r-1)$-subset S of $V(H)$ appears in at most one of the steps j as the reason for deleting edges, and in that step, we deleted at most $t_i - 1$ edges, where i is such that S misses V_i. Also, $(r-1)$-sets $S \notin \partial H$ never appear.

Therefore,

$$|E(H)| \ \leq \ |E| + \sum_{i=1}^{r} (t_i - 1) h_i.$$

We claim that

$$E \neq \emptyset. \tag{9}$$

Then, we can take H' to be the subhypergraph induced by the edges in E, and are done. So it only remains to prove (9).

In order to see (9), note that otherwise, by our assumption on the number of edges of H, we have that

$$\begin{aligned} \frac{t-1}{r}|\partial H| \ < \ |E(H)| \ \leq \ \sum_{i=1}^{r}(t_i - 1)h_i \ &\leq \ \sum_{i=1}^{r}\left(\frac{t+r-1}{r} - 1 + \delta_i\right)h_i \\ &\leq \ \frac{t-1}{r} \cdot \sum_{i=1}^{r} h_i + \sum_{i=1}^{r} \delta_i h_i, \end{aligned}$$

and so, using (7), we obtain that

$$\sum_{i=1}^{r} \delta_i h_i \ > \ 0. \tag{10}$$

By (5), we can choose an index $i \in [r]$ such that $\delta_i \leq 0$ for all $i \leq i^*$ and $\delta_i > 0$ for all $i > i^*$. This choice of i^*, together with (8) and (6), enables us to calculate that

$$\begin{aligned}\sum_{i=1}^{r} \delta_i h_i \;=\; \sum_{i=1}^{i^*} \delta_i h_i + \sum_{i=i^*+1}^{r} \delta_i h_i \;&\leq\; \sum_{i=1}^{i^*} \delta_i h_{i^*} + \sum_{i=i^*+1}^{r} \delta_i h_{i^*} \\ &\leq\; h_{i^*} \cdot \sum_{i=1}^{r} \delta_i \\ &= 0,\end{aligned}$$

a contradiction to (10). This proves (9), thus completing the proof of the lemma. □

3 Lower bounds for r-partite r-graphs

The bounds from Kalai's conjecture cannot be weakened much in r-partite r-graphs. This is asymptotically shown in Proposition 3.1 below. However, in this proposition, the term $\frac{t-1}{r}$ from Kalai's conjecture is replaced with the term $\frac{t-r+1}{r}$, which for $r \geq 3$ leaves us with a small gap. Possible finer scale improvements are discussed at the end of this section.

We call a tight r-tree *balanced* if all its partition classes have the same size.

Proposition 3.1. *For all $r \geq 2$, $t \geq 1$ such that $t+1$ is a multiple of r, and for all $\varepsilon > 0$, there exists an r-graph H with r-partition $V(H) = V_1 \cup V_2 \cup \ldots \cup V_r$ fulfilling*

$$|E(H)| \geq \big(1-\varepsilon\big)\frac{t-r+1}{r} \cdot \sum_{i=1}^{r} \prod_{\ell \neq i} |V_\ell|$$

such that H does not contain any balanced tight r-tree T with t edges.

Proof. Consider the sets V_i^j for $i \in [r]$ and $j = 1, 2$, where $|V_i^1| = \frac{t+1}{r} - 1$ and $|V_i^2| = \gamma^{-1}(\frac{t+1}{r} - 1)$, for $\gamma := (1-\varepsilon)^{\frac{1}{1-r}} - 1$. Let H be the r-partite r-graph with partition sets $V_i := V_i^1 \cup V_i^2$ (for $i \in [r]$) and all edges $\{v_1, \ldots, v_r\}$ having the property that $v_{i^*} \in V_{i^*}^1$ for exactly one index $i^* \in [r]$, and $v_i \in V_i^2$ for all $i \neq i^*$.

It is easy to see that no r-tight tree may contain vertices from both V_i^1 and V_i^2, for any $i \in [r]$. So, since $|V_i^1| < \frac{t+1}{r}$ for all i, we see that H does not contain any balanced tight r-tree T with t edges.

The number of edges of H is

$$|E(H)| = \sum_{i=1}^{r} |V_i^1| \prod_{\ell \neq i} |V_\ell^2| = r \cdot \gamma^{1-r} \big(\frac{t+1}{r} - 1\big)^r$$
$$= (1+\gamma)^{1-r} \big(\frac{t+1}{r} - 1\big) \cdot \sum_{i=1}^{r} \prod_{\ell \neq i} |V_\ell|,$$

giving the desired bound. □

In the example behind Proposition 3.1, the host r-graph is much larger than the r-tree we are looking for, and another error term hides behind the ε. On a finer scale, more improvements on Kalai's bounds might be possible for r-partite r-graphs. For $r = 2$, Gyárfás, Rousseau and Schelp [11] determine the extremal number of t-edge paths P_t in bipartite graphs with partition classes of sizes $n \geq m$ (that is, the maximum number $ex(n, m; P_t)$ of edges such a bipartite graph can have without necessarily containing P_t). In particular, if $t \leq m + 1$ is odd they obtain

$$ex(n, m; P_t) = \frac{t-1}{2}(n + m - t + 1). \tag{11}$$

Yuan and Zhang [16] conjecture similar results as (11) hold for all trees T (the exact bounds depend on how the bipartition sizes of T relate to n and m), and establish several special cases.

Analogous improvements might be possible for hypergraphs. Note that the quantity from (11) coincides with the number of edges of the r-graph from the proof of Proposition 3.1, for the case $r = 2$. For $r = 3$ one might add in all edges meeting all sets V_i^1, and the obtained r-graph still does not contain T. More generally, for $r \geq 4$ one might add in all edges meeting V_i^1 in an odd number of indices i.

4 A better Turán bound for all 2-graphs

We now discuss an implication of Theorem 1.2 for tree containment in graphs (i.e. 2-graphs) that are not necessarily bipartite. This will establish the bound (4.2) mentioned in the introduction.

We need some easy definitions first. We call a partition of the vertices of a graph into two sets a 2-*cut*. The *size* of a 2-cut is the number of edges that cross the cut (where an edge is said to *cross* the cut if it has one endvertex on either side).

It is well known and easy to prove that every m-edge graph has a 2-cut of size at least $\frac{m}{2}$. A random partition achieves this bound in expectation. But this is not best possible. A classical result of Edwards [4] states that instead of only $\frac{m}{2}$ edges we can actually guarantee a 2-cut of size at least $\frac{m}{2}+\Omega(\sqrt{m})$. Even better bounds can be achieved by excluding a fixed subgraph from G. Alon, Krivelevich and Sudakov [2] show that if we exclude any fixed tree T from the graph G, the maximum number of edges crossing some 2-cut can be bounded as follows.

Theorem 4.1 (Alon, Krivelevich and Sudakov [2]). *Let $t > 1$ and let T be a t-edge tree. Let G be a graph with m edges that does not contain T. Then G has a 2-cut of size at least $\frac{m}{2} + \frac{m}{2t}$ if t is odd, and of size at least $\frac{m}{2} + \frac{m}{2t-2}$ if t is even.*

We can use Theorem 4.1 to improve the bound from (1). The following proposition proves the bound (4) we mentioned in the introduction.

Proposition 4.2. *Let $t \in \mathbb{N}$ and let G be a graph on n vertices with more than $(1 - \frac{1}{t+1})(t-1)n$ edges if t is odd, and with more than $(1 - \frac{1}{t})(t-1)n$ edges if t is even. Then G contains every tree T having t edges.*

Proof. We may assume that $t > 1$. We only treat the case when t is odd, as the other case is very similar.

Given G and T, we use Theorem 4.1 to either find a copy of T in G, or to obtain a 2-cut of G having size greater than

$$\frac{\frac{t}{t+1}(t-1)n}{2} + \frac{\frac{t}{t+1}(t-1)n}{2t} = \frac{t-1}{2}n.$$

In the latter case, we apply Theorem 1.2 to the graph induced by this 2-cut to see that it contains T. □

Variants of Edwards' results for hypergraphs have been studied by Erdős and Kleitman [5]. They showed that in an m-edge r-graph, the expected size of an r-cut is $\frac{r!}{r^r}m$ (where an r-*cut* is a partition of the vertices into r sets, edges *cross* the cut if they have one vertex in each partition set, and the *size* of an r-cut is the number of crossing edges). Recently, Conlon, Fox, Kwan and Sudakov [3] improved this bound. In particular, they obtain that for $r \geq 3$, every m-edge r-graph has an r-cut of size at least $\frac{r!}{r^r}m + \Omega(m^{\frac{5}{9}})$. They conjecture the exponent in the second term can be improved to $\frac{2}{3}$. Unfortunately, we are not aware of any result in the spirit of Theorem 4.1 for hypergraphs and the bound from [3] alone does not seem to suffice to prove a meaningful version of Proposition 4.2 for r-graphs with $r \geq 3$.

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