DIRAC-TYPE CONDITIONS FOR SPANNING BOUNDED-DEGREE HYPERTREES

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ABSTRACT. We prove that for fixed k, every k-uniform hypergraph on n vertices and of minimum codegree at least $n/2 + o(n)$ contains every spanning tight k-tree of bounded vertex degree as a subgraph. This generalises a well-known result of Komlós, Sárközy and Szemerédi for graphs. Our result is asymptotically sharp. We also prove an extension of our result to hypergraphs that satisfy some weak quasirandomness conditions.

1. Introduction

Forcing spanning substructures with minimum degree conditions is a central topic in extremal graph theory. For instance, a classic result of Dirac [9] from 1952 states that any graph on $n \geqslant 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle. In the same spirit, Bollobás [5] conjectured in the 1970s that graphs on n vertices with minimum degree at least $n/2 + o(n)$ would contain every n-vertex tree of bounded maximum degree as a subgraph. Komlós, Sárközy and Szemerédi [21] proved this conjecture in 1995, introducing a prototype version of what is now known as the blow-up lemma.

In recent years many efforts have been made to extend Dirac's theorem to k-uniform hypergraphs, also called *k-graphs*, using various notions of degrees/cycles. Notably, Rödl, Ruciński and Szemerédi [26] proved that k-graphs of *minimum codegree* at least $n/2 + o(n)$ contain a *tight Hamilton cycle* (precise definitions below). More Dirac-type results for Hamilton cycles can be found in the survey of Simonovits and Szemerédi [27] and the references therein. In the present paper we prove an extension of the Komlós–Sárközy–Szemerédi theorem to k-graphs, to the best of our knowledge the first Dirac-type result for tightly connected spanning structures other than *tight paths* or *tight cycles*. As in [26], we will use a minimum codegree of at least $n/2 + o(n)$. The hypertrees in our result are *tight k-trees*, which we will introduce now.

For motivation, consider the following iterative definition of (non-trivial) trees in graphs. A single edge is a tree, and any graph obtained from a tree T by adding a new vertex v and a new edge vw, with $w \in V(T)$, is also a tree. Generalising the tree notion from the previous sentence, a *tight k-tree* is a k-graph whose edges can be ordered such that every edge e, except the first one, contains a vertex v which is not in any previous edge, and furthermore, $e \setminus \{v\}$ is contained in some previous edge. So, tight 2-trees are the usual trees in graphs, and the well-known k-uniform tight paths are tight k-trees. Since no other kinds of trees will be considered, we usually just write *k-tree* to refer to a tight k-tree.

Turán-type problems for k-trees have a long history. In 1984, Kalai conjectured (see [12, Conjecture 3.6]) that every k-graph on n vertices with more than $\frac{t-1}{k}\binom{n}{k-1}$ edges contains every k-tree with t edges. For all k, this conjecture is tight for infinitely many t and n. In general, Kalai's conjecture is open, but there are partial and asymptotic results for special families of k-trees [12–16], among these linear sized tight paths [3], for k-partite host k-graphs [28], and for the case $k = 2$ (see [29] for references), where the conjecture equals the famous Erdős–Sós conjecture.

Note that a global bound on the number of edges, as in Erdős-Sós conjecture, implies the existence of a subgraph of large minimum degree. Such a bound alone is, however, not sufficient

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in general to ensure all trees as subgraphs. This changes if the host graph and the tree we are looking for are of the same order. As mentioned above, this was shown in [21]:

Theorem 1.1 (Komlós, Sárközy and Szemerédi [21]). *For all $\gamma > 0$ and $\Delta \in \mathbb{N}$, there is n_0 such that every graph G on $n \geqslant n_0$ vertices with $\delta(G) \geqslant (1+\gamma)\frac{n}{2}$ contains each n-vertex tree T with $\Delta(T) \leqslant \Delta$.*

Our main result is a generalisation of Theorem 1.1 to hypergraphs. We need two definitions. For a k-graph H, the *maximum 1-degree* $\Delta_1(H)$ is the maximum number m such that some vertex of H is contained in m edges. The *minimum codegree* $\delta_{k-1}(H)$ is the largest m such that every set of $k-1$ vertices is contained in at least m edges of H.

Theorem 1.2. *For all $k, \Delta \geqslant 2$ and $\gamma > 0$ there is n_0 such that every k-graph H on $n \geqslant n_0$ vertices with $\delta_{k-1}(H) \geqslant (1/2+\gamma)n$ contains every k-tree T on n vertices with $\Delta_1(T) \leqslant \Delta$.*

The condition on $\delta_{k-1}(H)$ in Theorem 1.2 is best possible, up to the term γn and a term depending on k.

Proposition 1.3. *For every $k \geqslant 2$ and for every k-tree T on $n \geqslant k$ vertices, there is a k-graph H on n vertices not containing T, with $\delta_{k-1}(H) \geqslant \lfloor n/2 \rfloor - f(T)$, where $f(T) \leqslant 2^k + k - 1$. Moreover, there are k-trees T with $f(T) = k-1$.*

Theorem 1.2 generalises to host graphs that have certain quasirandom properties. For a k-graph H and a set F of distinct $(k-1)$-subsets of $V(H)$, define the *joint degree of F* as

$$\deg_H(F) = \big|\{v \in V(H) : f \cup \{v\} \in H \text{ for each } f \in F\}\big|. \tag{1.1}$$

We say H is $(\varrho, h, \varepsilon)$-*typical* if $\big|\deg_H(F) - \varrho^{|F|}n\big| \leqslant \varepsilon n$ for every family F of $(k-1)$-sets such that $|F| \leqslant h$. We show that for suitable choices of ϱ and ε, every large $(\varrho, 2, \varepsilon)$-typical k-graph contains every spanning k-tree with bounded degree.

Theorem 1.4. *For all $k, \Delta \geqslant 2$ and $\varrho > 0$ there are n_0 and ε_0 such that the following holds for all $\varepsilon \leqslant \varepsilon_0$. If H is a $(\varrho, 2, \varepsilon)$-typical k-graph on $n \geqslant n_0$ vertices, then H contains every k-tree T on n vertices with $\Delta_1(T) \leqslant \Delta$.*

Recently, Ehard and Joos [10] showed very general results for finding (almost perfect packings of) spanning bounded-degree hypergraphs in host hypergraphs satisfying certain strong quasirandom conditions. However, their results are incomparable with ours, as our quasirandomness conditions are much weaker.

We will deduce Theorem 1.4 from a slightly more general statement (Corollary 11.2). Weaker notions of quasirandomness and *minimum* 1-*degree* (defined in analogy to our definitions above) of order $\Theta(n^{k-1})$ are not enough to guarantee the existence of any spanning k-tree in dense k-graphs on n vertices. This follows from examples of Araújo, Piga and Schacht [4] for $k \geqslant 3$. See Section 11 for more details.

The paper is organised as follows. In Section 2 we introduce some notation and in Section 3 we show Proposition 1.3. In Section 4 we give an overview of the proof of Theorem 1.2 (which can be read independently of Sections 2 and 3). In Section 5 we investigate tight k-trees and their decomposition. In Section 6 we introduce some tools and preliminary results that will be used in the proof of our main result, Theorem 1.2. In Section 7 we show a connecting lemma that will allow us to prove a general embedding technique for bounded degree hypertrees in Section 8. In Section 9 we introduce the absorption method and in Section 10 we use the results form Sections 5-9 to prove Theorem 1.2. In Section 11 we extend Theorem 1.2 to hypergraphs satisfying certain quasirandomness conditions, in particular proving Theorem 1.4. Section 12 contains some concluding remarks and open questions.

2. Notation

We introduce some basic notation used throughout the paper here. More specific notions will be introduced where first needed. Throughout this section, let H be a k-graph, with $k \geqslant 2$.

Hypertrees. By definition, every k-tree on n vertices has $n-k+1 \geqslant 1$ edges, and hence $n \geqslant k$. Also by definition, every k-tree T on n vertices has orderings $e_1, ..., e_{n-k+1}$ of its edges, and $v_1, ..., v_n$ of its vertices such that $e_1 = \{v_1, ..., v_k\}$, and for all $i \in \{k+1, ..., n\}$,

(T1) $\{v_i\} = e_{i-k+1} \setminus \bigcup_{1 \leqslant j < i-k+1} e_j$, and
(T2) there exists $j \in [i-1]$ such that $e_{i-k+1} \setminus \{v_i\} \subseteq e_j$.

Clearly, an ordering of the edges implies an ordering of the vertices (and vice versa). Any ordering of $E(T)$ or $V(T)$ with properties (T1) and (T2) will be called a *valid ordering*. If $j \in [i-1]$ is the smallest index such that (T2) holds for e_i and v_i then we call e_j the *parent* of e_{i-k+1} and e_{i-k+1} a *child* of e_j. The *anchor of* v_i is $\alpha(v_i) := e_{i-k+1} \setminus \{v_i\}$.

A k-*subtree* of T is a k-tree T' such that $T' \subseteq T$. For instance, $e_1, ..., e_r$ induces a k-subtree of T, for any $1 \leqslant r \leqslant n-k+1$. Also, the tree $T - v_n$ obtained by removing v_n and e_{n-k+1} from T is k-subtree.

ℓ-partitions and blow-ups. We say H is ℓ-*partite* if there is a partition $\{V_1, ..., V_\ell\}$ of $V(H)$ such that $|e \cap V_i| = 1$ for each $e \in E(H)$ and $i \in [\ell]$. Using induction, it is easy to show that every k-tree is k-partite and, moreover, the k-partition of its vertices is unique.

If $V(H) = \{v_1, ..., v_\ell\}$, the M-*blowup of* H is the ℓ-partite k-graph with vertex classes $V_1, ..., V_\ell$ of size M each, such that for every edge $\{v_{i_1}, ..., v_{i_k}\} \in H$, all edges between the sets $V_{i_1}, V_{i_2} ...$ and V_{i_k} are present.

Homomorphisms and embeddings. If H_1, H_2 are hypergraphs, a *hypergraph homomorphism of* H_1 *in* H_2 is a function $\varphi : V(H_1) \to V(H_2)$ mapping edges to edges. If furthermore, $H_1 \subseteq H_1'$ and $\varphi' : V(H_1') \to V(H_2)$ is a hypergraph homomorphism such that φ' agrees with φ on $V(H_1)$, then we call φ' an *extension of* φ *to* H_1'.

Moreover, if φ is injective, we call φ a *embedding* of H_1 in H_2. An *extension of* φ is then an extension which is also an embedding.

Shadows and ordered shadows. The *shadow* $(k-1)$-*graph* ∂H of H is the $(k-1)$-graph on vertex set $V(H)$ whose edges are all the $(k-1)$-sets which are contained in some edge of H. We will sometimes need to view the elements of ∂H as ordered tuples. For this, we will write $\partial^o H$ for the *ordered shadow*, defined as the set of all tuples $(v_1, ..., v_{k-1})$ with $\{v_1, ..., v_{k-1}\} \in \partial H$. In particular, if $\varphi : \partial^o H \to \partial^o H'$ is a function, $f = (a_1, ..., a_\ell) \in \partial^o H$ and $f' = (b_1, ..., b_\ell) \in \partial^o H'$, then $\varphi(f) = f'$ means that $\varphi(a_i) = b_i$ for all $i \in \{1, ..., \ell\}$.

Neighbourhoods and degrees. For $S \subseteq V(H)$, the *neighbourhood of* S *in* H is defined as $N_H(S) = \{T \subseteq V(H) \setminus S : S \cup T \in H\}$. If $x_1, ..., x_\ell \in V(H)$, we will write $N_H(x_1, ..., x_\ell)$ instead of $N_H(\{x_1, ..., x_\ell\})$. The degree of $f \in \partial H$ is the number of edges of H which contain f and equals $|N_H(f)|$. Slightly abusing notation if $f = (x_1, ..., x_k) \in V(H)^k$ such that $\{x_1, ..., x_k\} \in H$, we will write $f \in H$.

Walks. An ordered sequence $(x_1, ..., x_n)$ of vertices from H is a *walk* if every k consecutive vertices form an edge of H. We will often just write a walk as the juxtaposition of its vertices, i.e. $x_1 \cdots x_n$. The *length* of a walk is the number of its edges, e.g. a walk on n vertices has length $n-k+1$. The order of the vertices is important: $x_1 x_2 \cdots x_n$ will generally be a different walk than $x_n \cdots x_2 x_1$ (even though they use the same vertices and edges). Note that a walk in which every vertex appears exactly once is a tight path in H. Slightly abusing notation, we will view a walk as a subgraph of H, so the $n-k+1$ edges of the walk constitute the walk.

Let $W = x_1 \cdots x_n$ be a walk in H. The *start of* W is $\mathrm{sta}(W) := (x_1, ..., x_{k-1})$, and the *end of* W is $\mathrm{ter}(W) := (x_{n-k+1}, ..., x_n)$. Note that both $\mathrm{sta}(W)$ and $\mathrm{ter}(W)$ belong to $\partial^o H$. If $\mathrm{sta}(W) = f$ and $\mathrm{ter}(W) = f'$, we also say that W *goes from* f *to* f'. The *interior* $\mathrm{int}(W)$ of W is the set $V(W) \setminus (\mathrm{sta}(W) \cup \mathrm{ter}(W))$. Thus $|\mathrm{int}(W)| \leqslant n - 2k - 2$, strict inequality being possible if W is not a path. If $|\mathrm{int}(W)| = q$, we also say that W *has* q *internal vertices.* If $\mathrm{int}(W) \cap S = \emptyset$, we call W *internally disjoint* from S.

Hierarchies. To simplify, we use hierarchies. We write $a \ll b$ to mean that for any $b > 0$, there exists $a_0 > 0$ such that for all $a \leqslant a_0$ the subsequent statements hold. Hierarchies with more constants are defined analogously, and should always be read from right to left. Implicitly, we assume that all constants appearing in a hierarchy are positive, and moreover if $1/m$ appears in a hierarchy then m is an integer, and a real number otherwise.

3. Extremal example

In this short section we prove Proposition 1.3. We will use the following family of k-graphs.

Definition 3.1. For disjoint sets A, B, and $0 \leqslant i \leqslant k$, let $H_i := \{e \subseteq A \cup B : |e| = k, |e \cap A| = i\}$, and $I := \{i \in \{0, ..., k\} : i \not\equiv \lfloor k/2 \rfloor \bmod 2\}$. Define $H(A,B) := \bigcup_{i \in I} H_i$.

Assuming that $|A \cup B| \geqslant k$, note that $\delta_{k-1}(H(A,B)) \geqslant \min\{|A|, |B|\} - k + 1$. There are not many ways to embed a k-tree into $H(A,B)$, as we will see now.

Lemma 3.2. *Let $k, n \in \mathbb{N}$, let $H(A,B)$ be as in Definition 3.1, with $|A \cup B| = n \geqslant k$. Let T be a k-partite k-tree, with a unique k-partition $V_1 \cup \cdots \cup V_k$, and with an embedding $\varphi : V(T) \to V(H(A,B))$. Then, for each $i \leqslant k$ either $\varphi(V_i) \subseteq A$ or $\varphi(V_i) \subseteq B$.*

Proof. We proceed by induction on $|E(T)|$, where the base case $|E(T)| \leqslant 1$ is trivial. Let v and e be the last vertex and edge of some valid ordering of T, let e' be the parent edge of e and let v' be the vertex in $e' \setminus e$. Note that there is $j \leqslant k$ such that $v, v' \in V_j$. Applying the induction hypothesis to $T - v$ and to φ restricted to $V(T')$, we see that we only need to show that $\varphi(v) \in A$ if and only if $\varphi(v') \in A$. This is true, as otherwise, $\big||A \cap e| - |A \cap e'|\big| = 1$, which contradicts the definition of $H(A,B)$. □

Now we are ready for the proof of Proposition 1.3.

Proof of Proposition 1.3. Given T, with unique k-partition $\{V_1, ..., V_k\}$ of $V(T)$, choose $a(T)$ as the largest integer such that $a(T) \leqslant n/2$ and $a(T) \neq |\bigcup_{j \in J} V_j|$ for all $J \subseteq [k]$. Note that $a(T) = \lfloor n/2 \rfloor$ for some trees (for instance star-like k-trees with $n \geqslant 2k$, or balanced k-trees for odd k), and that in general, $a(T) \geqslant \lfloor n/2 \rfloor - 2^k$. Set $f(T) = \lfloor n/2 \rfloor - a(T) + k - 1$.

Let A, B be disjoint sets such that $|A| = a(T)$ and $|A \cup B| = n$, and consider the k-graph $H(A,B)$ as in Definition 3.1. Then $\delta_{k-1}(H(A,B)) \geqslant a(T) - k + 1 = \lfloor n/2 \rfloor - f(T)$ (by the observation after Definition 3.1), and T does not embed in $H(A,B)$ because of Lemma 3.2. □

4. Overview of the proof of Theorem 1.2

Our proof of Theorem 1.2 relies on a decomposition method for hypertrees, combined with weak hypergraph regularity and absorption.

Given a k-graph H and a k-tree T, both on n vertices, the embedding of T into H will be performed in three steps. We first embed a small subtree T' of T into vertices of H that have some special properties. In the second step we embed the bulk of $T - T'$. In the third step we use absorption and the suitably embedded T' to embed the few missing vertices.

For the embedding of the bulk of $T - T'$, we start by separating $T - T'$ into a constant number of smaller subtrees, in a similar way as has been done for trees in graphs [2]. This is accomplished in Section 5, where in particular, we discuss *rooting* a k-tree at a $(k-1)$-set of its vertices, and we also develop the notion of *layerings* of hypertrees, which resemble BFS-layerings of rooted graphs. Using these notions, we show (cf. Proposition 5.14) that for any $\beta > 0$ one can partition the edges of any k-tree with n edges into at most β^{-1} parts, so that each of these parts spans a k-tree of size $O(\beta n)$. The parts can be ordered and each of them can be rooted so that the first ℓ parts, for any ℓ, form a connected subtree of $T - T'$, which contains the root of part $\ell + 1$.

We will then embed the parts successively, embedding in each step one part (except its root, which is already embedded). Each of the parts will be embedded into a suitable part of the host hypergraph using the regularity method. Fortunately, the *weak regularity lemma for hypergraphs* is sufficient for our purposes here. We apply this lemma to the host graph and then, in each step, we find a k-set of clusters with sufficient free space and of convenient density. Into this set we will embed almost all of the part that we wish to embed in the current step. That is, we embed all but the first layers, because these layers will be needed to make the connection between the already embedded root and the free cluster space.

In order to make these connections, we will use a part of H that we have separated earlier, before applying regularity, and that we will only use for the connections. This is the *reservoir*, a very small set $R \subseteq V(H)$ having (among others) the property that every $(k-1)$-set has several neighbours in R. The reservoir is found using a probabilistic argument (Lemma 6.7). A *connecting lemma* (Lemma 7.1) allows us to find many short walks between arbitrary pairs of ordered $(k-1)$-sets, whose internal vertices are all inside the reservoir. An enhanced version of this lemma (Lemma 8.1) allows us to embed not only walks or paths but instead bounded-size k-trees of bounded degree into the reservoir, joining given pairs of $(k-1)$-edges. This is what we need to finish the embedding described in the previous paragraph.

Finally, after embedding almost all of T, we use an *absorption technique*, inspired by [6, 7]. For this, we find in H, before embedding the bulk of the tree as described above, an *absorber*, which in our case is a carefully chosen sequence of constant-sized, vertex-disjoint and isomorphic k-trees in H. We embed T' into the absorber, and the bulk of $T - T'$ as described above. We can assume that the now embedded part T can be completed by adding leaves, one at a time (as in the definition of k-trees). We can add these leaves using the absorber. This mechanism is explained with detail in Section 9.

5. Hypertrees

In this section we establish some structural results about hypertrees. Most importantly, we show any large hypertree can be decomposed into smaller hypertrees (see Proposition 5.14).

5.1. **Link graph of a k-tree.** For a k-graph H and $v \in V(H)$, let $N_H(v)$ be the set of all vertices in $V(H) \setminus \{v\}$ in edges containing v. The *restricted link graph of v with respect to H*, denoted $H(v)$, is the $(k-1)$-graph whose vertex set is $N_H(v)$ and its edge set is $\{e \setminus \{v\} : v \in e\}$.

Lemma 5.1. *Let $k \geqslant 2$, let T be a k-tree, and let $v \in V(T)$. Then $T(v)$ is a $(k-1)$-tree on at most $\Delta_1(T) + k - 1$ vertices.*

Proof. Let $e_1, ..., e_m$ be a valid ordering of the edges of T, and let $I = \{i \in [m] : v \in e_i\}$. Then $E(T(v)) = \{e_i \setminus \{v\} : i \in I\}$, and I induces a valid ordering of $E(T(v))$, with $\alpha(v)$ being the first edge. So $T(v)$ is a $(k-1)$-tree. Since v belongs to at most $\Delta_1(T)$ edges in T, we know that $T(v)$ has at most $\Delta_1(T)$ edges, and thus at most $\Delta_1(T) + k - 1$ vertices. □

5.2. **Layerings.** It is often convenient to root a 2-tree T at some vertex $r \in V(T)$, which gives rise to a rooted tree (T, r). Then one can define the *i-th layer L_i of (T, r)* as the set of all vertices at distance exactly i from r in T. The layers partition $V(T)$, and any vertex in layer $i+1$ is joined to some vertex in layer i.

We now introduce a generalisation of these notions to higher uniformities.

Definition 5.2. A *rooted k-tree* is a pair (T, r) where T is a k-tree and $r \in \partial(T)$.

Definition 5.3 (Layering)**.** Let (T, r) be a rooted k-tree. A *layering* for (T, r) is a tuple $\mathcal{L} = (L_1, ..., L_m)$, for some $m \in \mathbb{N}$, such that $\{L_1, ..., L_m\}$ is a partition of $V(T)$, and

- (L1) $|r \cap L_i| = 1$ for all $i \in [k-1]$, and $|L_1| = 1$,
- (L2) for each $v \in L_{i+1}$ with $1 \leqslant i < m$ there are $w \in L_i$, $e \in E(T)$ such that $\{v, w\} \subseteq e$, and
- (L3) for each $e \in E(T)$, there is $j \in [m]$ such that $|e \cap L_i| = 1$ for each $j \leqslant i < j + k$.

We call the tuple $(T, r, \mathcal{L})$ a *layered* k-tree.

Note that a layering $(L_1, ..., L_m)$ of (T, r) is the preimage of the tight path on m vertices under a homomorphism which maps all of L_i to the i-th vertex of the tight path.

Lemma 5.4. *Every rooted k-tree (T, r) has a layering.*

Proof. In some valid ordering of $V(T)$, let v be the last vertex and e the last edge. Let e' be the parent of e, and let w be the only vertex in $e' \setminus e$.

Now if $v \notin r$, then by induction, $(T - v, r)$ has a layering, which we can extend to a layering of (T, r) by either adding v to the layer L_i that hosts w, or (if all other vertices of e' lie in later layers than w) by adding v to L_{i+k}. If $v \in r$, then set $r' := (r \setminus \{v\}) \cup \{w\}$. Again, by induction,

$(T - v, r')$ has a layering $(L'_1, \dots, L'_m)$. We extend this layering to a layering of T by adding v to the layer that hosts w. Additionally, if w lies in L'_1, we move w to layer L'_k. In either case, it easy to check that (L1)–(L3) hold for our layering of T. □

Note that Definition 5.3 (L2) gives that $|L_{i+1}| \leqslant \Delta_1(T)|L_i|$ for all $i \in [m-1]$. So by (L1) we have the following easy observation.

Lemma 5.5. *Let $(T, r, \mathcal{L})$ be a layered k-tree with $\mathcal{L} = (L_1, \dots, L_m)$. Then $|L_i| \leqslant (\Delta_1(T))^{i-1}$ for all $i \in [m]$.* □

Recall that k-trees are k-partite, and each k-tree T admits a unique k-partition $\{V_1, \dots, V_k\}$ of $V(T)$. Given a layering $\mathcal{L} = (L_1, \dots, L_m)$ of (T, r), it is clear from Definition 5.3 (L2) that (after relabelling the partition classes) each V_i contains all layers $L_{i+k\mathbb{N}}$. This, together with Lemma 5.5 implies that the sizes of the partition classes of a k-tree cannot differ too much, as detailed in the following lemma.

Lemma 5.6. *Let $\Delta, k \geqslant 2$ and let T be a k-tree with k-partition $\{V_1, \dots, V_k\}$ and with $\Delta_1(T) \leqslant \Delta$. Then $|V_i| \leqslant \Delta^{k-1}(|V_j| + 1)$ for each $i, j \in [k]$.* □

5.3. **Pseudopaths in k-trees.** A basic fact about 2-trees is that every two vertices are joined by a unique path. We now introduce pseudopaths, which play a similar role in hypertrees.

Definition 5.7 (Pseudopath)**.** A k-tree P is a *pseudopath* (of uniformity k) if $E(P)$ has a valid ordering $e_1, \dots, e_t$ such that for every $i < t$, the only child of edge e_i is e_{i+1}. Given a k-graph H and distinct tuples $f, f' \in \partial H$, an *(f, f')-pseudopath* in H is a pseudopath $P \subseteq H$ with a valid ordering $e_1, \dots, e_t$ such that $f \subseteq e_i$, $f' \subseteq e_j$ if and only if $(i, j) = (1, t)$.

Observe that if an (f, f')-pseudopath P has at least two edges, then exactly two vertices of $V(P)$ have degree 1, and these vertices are contained in f and f' respectively. We use this to show existence and uniqueness of pseudopaths in hypertrees.

Lemma 5.8. *Let T be a k-tree. Then for any $f, f' \in \partial T$ there is a unique (f, f')-pseudopath in T.*

Proof. We use induction on $|E(T)|$. If $|E(T)| \leqslant 1$, the statement clearly holds. Otherwise let v and e be the last vertex and last edge in a valid ordering of $V(T)$. If $v \notin f \cup f'$, then by induction, the tree $T - v$ contains a unique (f, f')-pseudopath. This path remains unique in T, since v has degree 1 in T, and therefore, v can only appear as the first or last vertex on any pseudopath.

So assume $v \in f \cup f'$. If $v \in f \cap f'$, then $f, f' \subseteq e$, and therefore, e is a (f, f')-pseudopath, and it is unique. We can therefore suppose that $v \in f \setminus f'$. Then $f \subseteq e$. By induction, $T - v$ contains a unique $(e \setminus \{v\}, f')$-pseudopath P', which can be extended to an (f, f')-pseudopath P by adding e. Since v has degree 1 in T, any (f, f')-pseudopath in T contains e. So, as P' was unique, P is unique too. □

A set x of $k - 1$ vertices is said to *lie on* a pseudopath P, if either P is an (x, f)-path for some f, or x is contained in exactly two of the edges of P.

Definition 5.9 (Distance)**.** Given a k-tree T and $f, f' \in \partial T$, the *distance* $d_T(f, f')$ between f and f' is the number of edges in the unique (f, f')-pseudopath connecting f with f'.

Note that $d_T(f, f') \geqslant 1$ for all distinct $f, f' \in \partial T$, with equality if and only if $|f \cap f'| = k - 2$ and $f \cup f' \in E(T)$.

We finish this subsection with observations on layered pseudopaths.

Lemma 5.10. *Let P be an (f, f')-pseudopath of uniformity k with valid ordering $e_1, \dots, e_t$, and let $\mathcal{L} = (L_1, \dots, L_m)$ be a layering for (P, f). Then, setting $r(j) = \min\{i : L_i \cap e_j \neq \emptyset\}$, we have*

(i) *$r(j + 1) - r(j) \in \{0, 1\}$ for all $j \in [t - 1]$, and*
(ii) *$|L_i| \leqslant k\Delta_1(T)$ for all $i \in [m]$.*

Proof. In order to see (i), note that otherwise, there is an index $j \in [t-1]$ such that $r(j+1) - r(j) \geqslant 2$ (as $r(j+1) - r(j) \geqslant 0$ by (L2)). So by (L3), $e_j \cap L_{r(j)} \neq \emptyset \neq e_j \cap L_{r(j)+1}$ while $e_{j+1} \cap L_{r(j)} = \emptyset = e_{j+1} \cap L_{r(j)+1}$, which is impossible since $|e_j \cap e_{j+1}| = k-1$.

For (ii), set $\Delta := \Delta_1(T)$ and observe that since P is a pseudopath, for every $x \in V(P)$ there are $j \leqslant |E(P)|$, $d < \Delta$ such that $x \in e_i$ if and only if $j \leqslant i \leqslant j+d$. In particular, because of (i) and (L3), we have

$$e_i \cap L_{r_j} = \emptyset \text{ for all } j \in [t] \text{ and } i \geqslant j + \Delta. \tag{5.1}$$

Now assume for contradiction that there is an index $i \in [m]$ with $|L_i| > k\Delta$. Note that each vertex in L_i belongs to an edge that by (L3) meets the k levels $L_\ell, L_{\ell+1}, \ldots, L_{\ell+k-1}$ for some $\ell \in \{i-k+1, \ldots, i\}$. So, there is an index $\ell \in \{i-k+1, \ldots, i\}$ such that more than Δ edges meet all of the levels $L_\ell, L_{\ell+1}, \ldots, L_{\ell+k-1}$. Let $j \in [t]$ be minimum with the property that $r(j) = \ell$. Then by (5.1), only edges $e_j, e_{j+1}, \ldots, e_{j+\Delta-1}$ may meet L_ℓ. As these are only Δ edges, we arrive at a contradiction, as desired. □

5.4. **Cutting k-trees.** We will now show how to partition a k-tree into smaller k-subtrees of controlled size. Given a layered k-tree $(T, r, \mathcal{L})$, with $\mathcal{L} = (L_1, \ldots, L_m)$, and given $x \in \partial T$, we say x is *$\mathcal{L}$-layered* if $|x \cap L_i| = 1$ for each $i = j, \ldots, j+k-2$ for some $j \in [m]$, that is, x meets $k-1$ consecutive layers of $\mathcal{L}$. In that case we say that j is the *rank* of x.

Definition 5.11 (Induced k-subtree)**.** Let $(T, r, \mathcal{L})$ be a layered k-tree, with $\mathcal{L} = (L_1, \ldots, L_m)$, and let $x \in \partial T$ be $\mathcal{L}$-layered. The tree T_x *induced by* x is the k-subtree of T spanned by $\bigcup_{i \geqslant 0} E_i$ where $E_0 := \{x \cup \{v\} : \alpha(v) = x\}$ and E_{i+1} contains all children of edges in E_i. Observe that T_x might be edgeless. Write $T - T_x$ for the tree obtained from T by deleting all edges in $E(T_x)$, and all vertices in $V(T_x) \setminus x$.

Clearly, if T is rooted at r, then $T_r = T$. Observe that if (T_x, x) is an induced k-subtree of (T, r), and $y \in \partial T_x$ is $\mathcal{L}$-layered, then the induced k-subtree $((T_x)_y, y)$ of (T_x, x) is also an induced k-subtree of (T, r), and we have $((T_x)_y, y) = (T_y, y)$. Note that for each $y \in \partial T_x$, the set x lies on the unique (r, y)-pseudopath in T. Moreover, T_x inherits a valid ordering and a layering from $(T, r, \mathcal{L})$, with layers $L_j \cap V(T_x)$, which we call $\mathcal{L}^x$ the *inherited layering* of (T_x, x).

The following lemma will be useful in a moment.

Lemma 5.12. *Let $(T, r, \mathcal{L})$ be a layered k-tree with $\mathcal{L} = (L_1, \ldots, L_m)$, $\Delta_1(T) \leqslant \Delta$ and $k \geqslant 2$. Let $F \subseteq E(T)$ be the set of all edges meeting L_1 and let $S := \{e \setminus L_1 : e \in F\} \subseteq \partial T$. Then*

(i) *each $s \in S$ is $\mathcal{L}$-layered and has rank 2,*
(ii) *$|F| = |S| \leqslant \Delta$, and*
(iii) *$E(T) = F \cup \bigcup_{s \in S} E(T_s)$.*

Proof. As $\Delta_1(T) \leqslant \Delta$ and $|L_1| = 1$, we have $|F| \leqslant \Delta$. The other properties are easy to see. □

Definition 5.13 ((β, d)-decomposition)**.** Let $\Delta, k \geqslant 2$, and let $(T, r, \mathcal{L})$ be a layered k-tree. For $\beta \in (0, 1)$ and $d \geqslant 1$, a *(β, d)-decomposition of* $(T, r, \mathcal{L})$ is a tuple $(D_i, s_i)_{1 \leqslant i \leqslant m}$ of rooted k-subtrees of T such that

(i) $m \leqslant 2\Delta^d/\beta$,
(ii) $E(T) = \bigcup_{1 \leqslant i \leqslant m} E(D_i)$,
(iii) $|E(D_i)| \leqslant \beta|E(T)|$ for each $1 \leqslant i \leqslant m$,
(iv) $s_1 = r$ and each s_i is $\mathcal{L}$-layered,
(v) $(V(D_\ell) \setminus s_\ell) \cap V(D_i) = \emptyset$ for all $1 \leqslant i < \ell \leqslant m$, and
(vi) for each $2 \leqslant \ell \leqslant m$, there is an $i < \ell$ such that $s_\ell \in \partial D_i$, and the rank of s_ℓ in D_i is at least d (in the inherited layering of D_i from $\mathcal{L}$).

Proposition 5.14. *Let $\Delta, k \geqslant 2$, $d \geqslant 1$, $\beta \in (0, 1)$, and let $(T, r, \mathcal{L})$ be a layered k-tree with $t \geqslant 2\Delta^d\beta^{-1}$ edges satisfying $\Delta_1(T) \leqslant \Delta$. Then T has a (β, d)-decomposition.*

Proof. We will find the trees in $(D_i, s_i)_{1 \leqslant i \leqslant m}$ inductively. At the end of each step $j \geqslant 0$, we will have found trees $D_1, D_2, \ldots, D_j$ fulfilling properties (iii)–(vi) from Definition 5.13, with m replaced by j. Moreover, there will be a set $X_j \subseteq \partial T$ such that

(a) $E(T) = \bigcup_{1 \leqslant i \leqslant j} E(D_i) \cup \bigcup_{x \in X_j} E(T_x)$,
(b) for each $x \in X_j \setminus \{r\}$, there is a unique $i \leqslant j$ such that $x \in \partial D_i$, the rank of x in D_i is at least d, and $(V(T_x) \setminus x) \cap \bigcup_{1 \leqslant i \leqslant j} V(D_i) = \emptyset$,
(c) $\beta t \geqslant |E(D_i)| \geqslant \beta t/(2\Delta^d)$ for each $1 \leqslant i \leqslant j$, and
(d) $|E(T_x)| \geqslant \beta t/(2\Delta^d)$ for each $x \in X_j$.

Note that (c) guarantees that we stop in some step $m \leqslant 2\Delta^d/\beta$ with $X_m = \emptyset$. This, together with (a) and (b), ensures (i) and (ii) hold.

We start the procedure setting $X_0 = \{r\}$, with all properties trivially fulfilled. Now assume we are in step $j \geqslant 1$. Choose any $x \in X_{j-1}$. By (d), we have $|E(T_x)| \geqslant \beta t/(2\Delta^{d+1})$. If $|E(T_x)| \leqslant \beta t$, then set $D_j := T_x$, $s_i := x$ and $X_j := X_{j-1} \setminus \{x\}$ and terminate step j. Otherwise, apply Lemma 5.12 to (T_x, x), obtaining a set F_1 of edges, and a set S_1 of $\mathcal{L}$-layered elements of $\partial T_x \subseteq \partial T$ of rank 2 in T_x. Apply Lemma 5.12 to all trees T_s with $s \in S_1$, thus generating a set F_2 of edges and a set S_2, such that each $s \in S_2$ is $\mathcal{L}$-layered and has rank 3 in T_x. Continue in this manner until reaching a set S_{d-1} of $\mathcal{L}$-layered elements of rank d, and set $F := \bigcup_{1 \leqslant i \leqslant d-1} F_i$. Note that $|S_{d-1}| \leqslant |F| \leqslant \Delta^d \leqslant \beta t/2$ and the edges in F span a k-tree T_F rooted at x. Next, for each $s \in S_{d-1}$, in order, consider the tree T_s. If $|E(T_s)| < \beta t/(2\Delta^d)$, then add T_s to T_F and delete s from S_d, and continue to examine the next $s \in S_d$. At the end of this process, we obtain a tree $B_1 \supseteq T_F$ and a set $Z_1 \subseteq S_d$. Note that $|E(B_1)| \leqslant |F| + |S_{d-1}|(\beta t/(2\Delta^d)) \leqslant \Delta^d + \beta t/2 \leqslant \beta t$, and $|E(T_z)| \geqslant \beta t/(2\Delta^d)$ for each $z \in Z_1$.

Now, successively, for $i \geqslant 1$, choose any $z \in Z_i$ and apply Lemma 5.12 to T_z. Add the resulting edges to B_i, obtaining the set $B_i' \supseteq B_i$, and let S be the subset of ∂T from the lemma. For each $s \in S$, check whether $|E(T_s)| < \beta t/(2\Delta^d)$, and if this is the case, then add T_s to B_i' and delete s from S. After processing all $s \in S$, this results in a set S', and a tree B_{i+1}. Set $Z_{i+1} := (Z_i \cup S') \setminus \{z\}$. Then $|E(B_{i+1})| \leqslant |E(B_i)| + \Delta + |S|(\beta t/(2\Delta^d)) \leqslant |E(B_i)| + \beta t/(\Delta^{d-1})$ and $|E(T_z)| \geqslant \beta t/(2\Delta^d)$ for each $z \in Z_i$.

We continue until we reach the first index h with $|E(B_h)| \geqslant \beta t/2$ (this must happen at some point, since in each step, at least one edge from $E(T_x)$ is added to $E(B_i)$, and $|E(T_x)| > \beta t$). Then $|E(B_h)| \leqslant \beta t$. Set $D_j := B_h$, $s_j = x$, and set $X_j := (X_{j-1} \cup Z_h) \setminus \{x\}$. By construction, (a)–(d) and (iii)–(vi) from Definition 5.13 hold for X_j and $D_1, ..., D_j$. □

6. Tools

In this section we collect some tools that will be needed for the proof of Theorem 1.2.

6.1. **The weak hypergraph regularity lemma.** Let H be a k-graph and let $V_1, ..., V_k$ be pairwise disjoint subsets of $V(H)$. Let $H[V_1, ..., V_k]$ be the k-partite subhypergraph of H induced by all edges that intersect all sets V_i. The *density* of $H[V_1, ..., V_k]$ is defined as

$$d(V_1, ..., V_k) := \frac{e_H(V_1, ..., V_k)}{|V_1| \cdots |V_k|},$$

where $e_H(V_1, ..., V_k)$ denotes the number of edges in $H[V_1, ..., V_k]$. For $\varepsilon, d > 0$, we say a k-tuple $(V_1, ..., V_k)$ of pairwise disjoint non-empty subsets of $V(H)$ is (ε, d)-*regular* if

$$|d(W_1, ..., W_k) - d| \leqslant \varepsilon$$

holds for all k-tuples of subsets $W_i \subseteq V_i$ satisfying $|W_1| \cdots |W_k| \geqslant \varepsilon |V_1| \cdots |V_k|$. A k-tuple $(V_1, ..., V_k)$ will be called ε-*regular* if it is (ε, d)-regular for some $d \geqslant 0$.

The *weak regularity lemma for hypergraphs* ensures that the vertex set of every k-graph can be partitioned into a bounded number of clusters, such that almost all k-tuples of these clusters are ε-regular. We will use the lemma in the following form (see [20, Theorem 9]).

Theorem 6.1 (Weak Hypergraph Regularity Lemma). *Let $k \geqslant 2$ and let $1/n, 1/T_0 \ll 1/t_0, 1/k, \varepsilon$. For every k-graph H on n vertices there exists a partition $\{V_0, V_1, ..., V_t\}$ of $V(H)$ such that*

(i) *$t_0 \leqslant t \leqslant T_0$,*
(ii) *$|V_0| \leqslant \varepsilon n$ and $|V_1| = \cdots = |V_t|$, and*
(iii) *for all but at most $\varepsilon \binom{t}{k}$ sets $\{i_1, ..., i_k\} \subseteq [t]$, the k-tuple $(V_{i_1}, ..., V_{i_k})$ is ε-regular.*

Any partition $\mathcal{P} = \{V_0, V_1, ..., V_t\}$ of $V(H)$ satisfying (i)–(iii) will be called an *ε-regular* partition of H. Given $d > 0$, we define the *d-reduced k-graph $R_d(H)$ of H with respect to $\mathcal{P}$* as follows. Its vertex set is $[t] = \{1, ..., t\}$, and its edges are the k-sets $\{i_1, ..., i_k\}$ such that $d_H(V_{i_1}, ..., V_{i_k}) \geqslant d$ and $(V_{i_1}, ..., V_{i_k})$ is ε-regular. We will also refer to $R_d(H)$ as "the" *d-reduced k-graph of H*, and note that even if $R_d(H)$ depends on the choice of $\mathcal{P}$, it will always be clear from the context what we mean.

We will need to find almost-perfect matchings in the reduced k-graph. For $k = 2$, it is easy to find one using graph regularity, and for $k \geqslant 3$ its existence may be deduced from Claims 4.4 and 4.5 of [26].

Lemma 6.2. *Let $k \geqslant 2$, $0 < 1/n \ll 1/t \ll \varepsilon \ll 1/k, \gamma, \eta$, and let $d \ll \gamma$. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geqslant (1/2 + \gamma)n$. Let $\mathcal{P} = \{V_0, V_1, ..., V_t\}$ be an ε-regular partition of $V(H)$. Then the d-reduced k-graph $R_d(H)$ has a matching covering at least $(1 - \eta)t$ vertices.*

6.2. **Degenerate hypergraphs and extensible edges.** Given $k \geqslant 2$ and $s \in \mathbb{N}$, let $K^{(k)}(s)$ denote the complete k-partite k-graph with each class of size s. To be precise, $V(K^{(k)}(s))$ is partitioned in k clusters $V_1, ..., V_k$ of size s each, and its edges are precisely the k-sets which intersect each V_i exactly once. The following result, due to Erdős [11], is a hypergraph version of the classical Kővári–Sós–Turán theorem.

Lemma 6.3. *Let $1/n \ll 1/k, 1/s, \varepsilon$. Let H be a k-graph with n vertices and at least εn^k edges. Then H contains a copy of $K^{(k)}(s)$ as a subgraph.*

Note that for any $k \geqslant 2$, the complete k-partite k-graph $K^{(k)}(2)$ has $2k$ vertices and 2^k edges. Given a k-graph H and an edge $e \in H$, let $d_H^K(e)$ be the number of copies of $K^{(k)}(2)$ in H in which e participates. Note that $d_H^K(e) \leqslant \binom{n-k}{k}$ always.

Definition 6.4 (θ-extensible edge)**.** Given a k-graph H on n vertices and $\theta > 0$ we say an edge $e \in H$ is *θ-extensible* if $d_H^K(e) \geqslant \theta\binom{n-k}{k}$.

Extensible edges will be useful in our embedding of k-trees. We show that in an appropriately dense k-graph most edges are extensible.

Lemma 6.5. *Let $1/n, \theta \ll \varepsilon, 1/k$. In any k-graph on n vertices, all but at most $\varepsilon\binom{n}{k}$ edges are not θ-extensible.*

Proof. Lemma 6.3 implies that the Turán density of $K^{(k)}(2)$ is zero. Hence, by standard supersaturation arguments [19, Lemma 2.1] there exist n_0 and $\alpha > 0$ such that every k-graph on $n \geqslant n_0$ vertices and at least $\varepsilon\binom{n}{k}$ edges has at least $\alpha\binom{n}{2k}$ copies of $K^{(k)}(2)$. To prove the lemma we shall use $n \geqslant n_0$ and $\theta \leqslant (k!)^2 2^k \alpha/(2k!)$.

Indeed, let H be any k-graph on n vertices and let $H' \subseteq H$ be the k-graph formed by the non-θ-extensible edges of H. To find a contradiction, suppose that H' has at least $\varepsilon\binom{n}{k}$ edges. By the choice of n_0 and α, we know that H' contains at least $\alpha\binom{n}{2k}$ copies of $K^{(k)}(2)$. Note that H' has at most $\binom{n}{k}$ edges and recall that each copy of $K^{(k)}(2)$ has 2^k edges. Therefore, a double-counting argument shows that some edge e in H' participates in at least $2^k\alpha\binom{n}{2k}/\binom{n}{k} \geqslant \theta\binom{n-k}{k}$ copies of $K^{(k)}(2)$. So, $d_H^K(e) \geqslant \theta\binom{n-k}{k}$, or in other words, e is a θ-extensible edge of H. This contradicts the definition of H'. □

6.3. **Reservoirs.** Let H be a k-graph H, and let $F \subseteq \partial H$. Recall that $\deg_H(F)$ denotes the joint degree of F, as defined in (1.1). For $U \subseteq V(H)$, we let

$$\deg_H(F) = \big|\{v \in U : f \cup \{v\} \in H \text{ for each } f \in F\}\big|. \tag{6.1}$$

Similarly, recalling previously we defined $d_H^K(e)$ as the number of copies of $K^{(k)}(2)$ in the k-graph H that contain the edge $e \in E(H)$, we define $d_H^K(e, U)$ as the number of copies of $K^{(k)}(2)$ in $H[U \cup e]$ that contain e.

Definition 6.6 (Reservoir)**.** Let H be a k-graph on n vertices, and let $\gamma, \mu > 0$. We say that a set $U \subseteq V(H)$ is a *(γ, μ, h)-reservoir for H* if

(i) $|U| = (\gamma \pm \mu)n$,
(ii) for every $F \subseteq \partial(H)$ with $|F| \leqslant h$ we have $\deg_H(F, U) \geqslant (\deg_H(F)/n - \mu)|U|$, and
(iii) for every $e \in H$, we have $d_H^K(e, U) \geqslant (d_H^K(e)/\binom{n-k}{k} - \mu)\binom{|U|-k}{k}$.

Lemma 6.7 (Reservoir Lemma)**.** *Let $1/n \ll \mu \ll \gamma \leqslant 1$ and let $h \geqslant 1$. Then every k-graph H on n vertices has a (γ, μ, h)-reservoir.*

The proof of Lemma 6.7 is probabilistic, and we will use the following standard concentration inequalities for random variables.

Theorem 6.8 (Chernoff's inequality [18, Theorem 2.1])**.** *Let $0 < \alpha < 3\,\mathbf{E}[X]/2$ and $X \sim \mathrm{Bin}(n, p)$ be a binomial random variable. Then $\Pr(|X - \mathbf{E}[X]| > \alpha) < 2\exp(-\alpha^2/(3\,\mathbf{E}[X]))$.*

Theorem 6.9 (McDiarmid's inequality [24])**.** *Suppose $X_1, ..., X_m$ are independent Bernoulli random variables and $b_1, ..., b_m \in [0, B]$. Suppose X is a real-valued random variable determined by $X_1, ..., X_m$ such that changing the outcome of X_i changes X by at most b_i for all $1 \leqslant i \leqslant m$. Then, for all $\lambda > 0$, we have*

$$\Pr(|X - \mathbf{E}[X]| > \lambda) \leqslant 2\exp\left(-\frac{2\lambda^2}{B\sum_{i=1}^m b_i}\right).$$

Proof of Lemma 6.7. Choose a set $U \subseteq V(H)$ randomly by independently including each vertex of $V(H)$ with probability $p = \gamma$. With non-zero probability U will satisfy all of the properties (i)–(iii) simultaneously, which shows the desired set U exists.

Indeed, $\mathbf{E}[|U|] = pn$. Thus, using Chernoff's inequality (Theorem 6.8) with $\alpha = (n^{1/3}\gamma)^{-1}$ we get that $|U| = \gamma n \pm n^{2/3}$ fails to hold with probability at most $2\exp(-n^{1/3}/(3\gamma))$. Since n is sufficiently large, $|U| = \gamma n \pm n^{2/3}$ holds with probability at least $1 - 1/n$ and we will assume those bounds on $|U|$ from now on. Note also this implies (i) holds for U.

Now we verify (ii) holds. Let $F \subseteq \partial H$ of size at most h and note that $\mathbf{E}[\deg_H(F, U)] = p\deg_H(F)$. If $\deg_H(F, U) < \mu n$ then there is nothing to show, so we assume otherwise. In particular, $\mathbf{E}[\deg_H(F, U)] \geqslant \gamma\mu n$. If $\deg_H(F, U) < (\deg_H(F)/n - \mu)|U|$, then $\deg_H(F, U) \leqslant (\deg_H(F)/n - \mu)(\gamma n + n^{2/3}) \leqslant \mathbf{E}[\deg_H(F, U)] - \mu\gamma n/2$ since n is large. Apply Chernoff's inequality with $\alpha = \mu\gamma n/2 \leqslant 3\,\mathbf{E}[\deg_H(F, U)]/2$ to get

$$\begin{aligned}\Pr[\deg_H(F, U) < (\deg_H(F)/n - \mu)|U|] &\leqslant \Pr[|\deg_H(F, U) - \mathbf{E}[\deg_H(F, U)]| > \mu\gamma n/2] \\ &\leqslant 2\exp\left(-\frac{(\mu\gamma)^2 n}{12}\right).\end{aligned}$$

Since $|\partial H| \leqslant n^{k-1}$ and $|F| \leqslant h$, there are at most $n^{h(k-1)}$ possible choices for F. Then a union bound shows that (ii) fails to hold with probability at most $2n^{h(k-1)}\exp(-(\beta\gamma)^2 n/12) < 1/n$, where the last inequality holds since n is large.

To see (iii), fix an edge $e \in H$. If $d_H^K(e) < \mu\binom{n-k}{k}$ then there is nothing to show, so assume otherwise. Let $X = d_H^K(e, U)$ and note that $\mathbf{E}[X] = p^k d_H^K(e)$. Since $|U| = pn \pm n^{2/3}$, we have $\binom{|U|-k}{k} = (1 + o(1))\gamma^k\binom{n-k}{k}$. The presence of a vertex in U can affect X by at most n^{k-1}. Thus, we can apply McDiarmid's inequality (Theorem 6.9) with $B = b_i = n^{k-1}$ for all i (and that n is sufficiently large) to see that

$$\begin{aligned}\Pr\left[\deg_H^K(e, U) < (\deg_H^K(F)/\binom{n-k}{k} - \mu)\binom{|U|-k}{k}\right] &\leqslant \Pr\left[X < \mathbf{E}[X] - \mu\gamma^k\binom{n-k}{k}/2\right] \\ &\leqslant 2\exp\left(-\mu^2\gamma^{2k}n^{2k}/(2k!n^{2k-1})\right),\end{aligned}$$

which is less than $1/n^{k+1}$ since n is sufficiently large. Since there are at most n^k edges in H, we see (iii) fails with probability at most $1/n$, as required. □

7. Connections

For this section, the following notion will be essential. A k-graph H is *ℓ-large* if every two $f, f' \in \partial H^o$ have at least ℓ common neighbours. For instance, k-graphs H on n vertices with minimum codegree at least $(1/2+\gamma)n$ are $2\gamma n$-large, and $(\varrho, 2, \varepsilon)$-typical graphs H on n vertices are $(\varrho^2 - \varepsilon)n$-large. For $U \subseteq V(H)$, we say that H is (ℓ, U)-large if every two $f, f' \in \partial H^o$ have at least ℓ common neighbours in U. (Later, U will be a reservoir.)

The main result of the current section is Lemma 7.1, which essentially says that in any $\Omega(n)$-large k-graph H we can connect any two elements of ∂H^o by a walk (actually, many such walks) of length exactly ℓ, where ℓ is a number only depending on k. We observe that Lemma 7.1 can be seen as a strengthening of the 'Connecting lemma' of Rdl, Ruciński and Szemerédi [26, Lemma 2.4]. For our approach however it is important that we can control the precise length of the walk (instead of having an upper bound).

Recall that the interior $\mathrm{int}(W)$ of a walk W corresponds to the set of vertices $V(W) \setminus (\mathrm{sta}(W) \cup \mathrm{ter}(W))$. For our purposes we will need to find walks whose length and number of internal vertices are under control.

Lemma 7.1 (Connecting Lemma)**.** *For integers k, ℓ with $k \geqslant 2$ and $\ell \geqslant (2k+1)\lfloor k/2 \rfloor + 2k$, there exists $q \leqslant \ell$ with the following property. Let $1/n \ll \alpha \ll \gamma \ll 1/k, 1/\ell$. Let H be a k-graph on n vertices which is $(\gamma n, U)$-large for some $U \subseteq V(H)$, and let $f, f' \in \partial^o H$. Then there are at least αn^q many walks W, each of length ℓ and with q internal vertices, which go from f to f' and are internally disjoint from $(V(H) \setminus U) \cup f \cup f'$.*

To prove Lemma 7.1, it will be useful to find a walk which swaps the order in which the vertices of a given edge appear. This will be achieved in the following two short lemmas.

Lemma 7.2. *Let $k \geqslant 2$ and let $(b_1, ..., b_k)$ be an ordered edge in a k-graph H. Let $j \leqslant \lfloor k/2 \rfloor$, and let $u \in N_H(b_1, ..., b_{j-1}, b_{j+1}, ..., b_k) \cap N_H(b_1, ..., b_{k-j}, b_{k-j+2}, ..., b_k)$. Then $H[\bigcup_{1 \leqslant i \leqslant k} \{b_i\} \cup \{u\}]$ contains a walk of length $2k+1$ from $(b_1, ..., b_k)$ to*

$$(b_1, ..., b_{j-1}, b_{k-j+1}, b_{j+1}, b_{j+2}, ..., b_{k-j}, b_j, b_{k-j+2}, ..., b_k).$$

Proof. It suffices to consider the walk

$$b_1 \cdots b_k b_1 \cdots b_{j-1} u b_{j+1} \cdots b_{k-j} b_j, b_{k-j+2} \cdots b_k b_1 \cdots b_{j-1} b_{k-j+1} b_{j+1} \cdots b_{k-j} b_j b_{k-j+2} \cdots b_k,$$

which uses $3k$ vertices and thus has length $2k+1$. □

Lemma 7.3. *Let $k \geqslant 2$ and let $(a_1, ..., a_k)$ be an edge in a k-graph H. For each $j \leqslant \lfloor k/2 \rfloor$, let $u_j \in N_H(a_1, ..., a_{j-1}, a_{j+1}, ...a_k) \cap N_H(a_1, ..., a_{k-j}, a_{k-j+2}, ...a_k)$. Then $H[\bigcup_{1 \leqslant i \leqslant k} \{a_i\} \cup \bigcup_{1 \leqslant j \leqslant \lfloor k/2 \rfloor} \{u_j\}]$ contains a walk of length $2k\lfloor k/2 \rfloor + \lfloor k/2 \rfloor$ from $(a_1, ..., a_k)$ to $(a_k, ..., a_1)$.*

Proof. We use Lemma 7.2 successively, for all $j \leqslant \lfloor k/2 \rfloor$, thus swapping the vertices a_j and a_{k-j+1} in the walk, until we reach $(a_k, ..., a_1)$. This gives a walk of length $\lfloor k/2 \rfloor (2k+1)$. □

Now, we can prove Lemma 7.1.

Proof of Lemma 7.1. Set $\ell_0 := (2k+1)\lfloor k/2 \rfloor + 2k$. Let $f'_R = (x'_{k-1}, ..., x'_1)$ be the reverse of the tuple f'. Greedily construct a tight path P_1 of length $\ell - \ell_0$, starting at f'_R, ending at some $f'' = (y_1, ..., y_{k-1})$, and using only vertices in $U \setminus (f \cup f')$. This can be done, since H is $(\gamma n, U)$-large every $(k-1)$-set has at least $\delta_2(H) \geqslant \gamma n$ neighbours in U, at each step we need to avoid at most $|f \cup f'| - (\ell - \ell_0) \leqslant 2k + \ell$ vertices, so $1/n \ll 1/k, 1/\ell$ implies there are many choices at each step. Set $W := V(P_1) \setminus f$. Set $q := k + \lfloor k/2 \rfloor + \ell - \ell_0 = k + \lfloor k/2 \rfloor + |W|$.

We will first construct a single walk as promised in the lemma, afterwards we will estimate in how many ways this can be done. Let $a_1 \in (N_H(f) \cap N_H(f'') \cap U) \setminus (W \cup f \cup f')$. Having defined $a_1, ..., a_j$ for some $j \in [k-1]$, we choose an arbitrary unused vertex

$$a_{j+1} \in N_H(x_{j+1}, ..., x_{k-1}, a_1, ..., a_j) \cap N_H(y_1, ..., y_{k-1-j}, a_1, ..., a_j) \cap U.$$

Clearly $P_2 = x_1 \cdots x_{k-1} a_1 \cdots a_{k-1} a_k$ and $P_3 = a_k a_{k-1} \cdots a_1 y_{k-1} \cdots y_1$ are tight paths. Applying Lemma 7.3, we find a walk P_4 which goes from $(a_1, \cdots, a_k)$ to $(a_k, a_{k-1}, \cdots, a_1)$ only occupying unused vertices $u_1, ..., u_{\lfloor k/2 \rfloor}$ from U. (The vertices u_j exist because H is $(\gamma n, U)$-large.) Concatenating the walks P_2, P_4, P_3 and P_1 (the latter traversed in reverse order) gives a walk P from f to f'. Set $Q = W \cup \{a_i : 1 \leqslant i \leqslant k\} \cup \{u_j : 1 \leqslant j \leqslant \lfloor k/2 \rfloor\}$. Then $Q = \mathrm{int}(P)$ and

$|Q| = q$, and the length of P is $|E(P)| = k + 2k\lfloor k/2\rfloor + \lfloor k/2\rfloor + k + \ell - \ell_0 = \ell$, so P is a walk which satisfies the required properties.

Note that by construction, each vertex in the interior of P is chosen as an arbitrary unused vertex in the neighbourhood of one or two $(k-1)$-sets. Since H is $(\gamma n, U)$-large, the common neighbourhoods in U have size at least γn, and thus (here we use $1/n \ll 1/k, 1/\ell$) in every step we have at least $\gamma n - q \geqslant \gamma n/2$ possible choices in U. Thus by the previous discussion there are at least $(\gamma n/2)^q \geqslant \alpha n^q$ many different walks with the required properties, where in the last inequality we used $\alpha \ll \gamma, 1/k, 1/\ell$. □

8. Embedding large hypertrees

In this section, we will prove an embedding result (Lemma 8.1) that will allow us to embed a small rooted tree of bounded degree into a dense k-partite graph, while at the same time controlling the location of the root and the bulk of the tree quite accurately.

The following definitions will be useful. Given integers $\ell_1 \leqslant \ell_2 \leqslant m$, and a layering $\mathcal{L} = (L_1, ..., L_m)$ of a rooted k-tree (T, r), we say that $V_{[\ell_1,\ell_2]}(\mathcal{L}) := \bigcup_{\ell_1 \leqslant i \leqslant \ell_2} L_i$ is the $[\ell_1, \ell_2]$-*interval* of T. If the layering is clear from the context, we just write $V_{[\ell_1,\ell_2]}$. If, moreover, $|L_i| \leqslant M$ for each $\ell_1 \leqslant i \leqslant \ell_2$, we say $V_{[\ell_1,\ell_2]}$ is M-*bounded*. We write $f \in \partial^o T[V_{[\ell_1,\ell_2]}]$ if there is a $j \in [\ell_1, \ell_2 - k + 1]$ such that $f = (x_1, ..., x_{k-1}) \in \partial^o T$ with $x_i \in L_{j+i-1}$ for all i.

We can now state the Embedding Lemma.

Lemma 8.1 (Embedding Lemma). *Let $\Delta, k \geqslant 2$, let $\ell \geqslant \lfloor k/2\rfloor(2k+1) + 2k$ and let $1/n, \mu \ll \beta, \theta \ll 1/k, 1/\Delta, c, \gamma, d$. Let H be a γn-large k-graph on n vertices with a $(\gamma, \mu, 2)$-reservoir R and let $W_1, ..., W_k \subseteq V(H) \setminus R$, all disjoint, such that $d(W_1, ..., W_k) \geqslant d$ and $|W_i| \geqslant cn$ for each $i \in [k]$. Let $(T, r, \mathcal{L})$ be a layered k-tree on at most βn vertices with $\mathcal{L} = (L_1, ..., L_m)$ and $\Delta_1(T) \leqslant \Delta$. Then for any θ-extensible edge $e \in H$, $f \subseteq e$ of size $k-1$ and any orderings $\vec{r}, \vec{f}$ of r and f respectively, there exists an embedding $\varphi : V(T) \to f \cup R \cup W_1 \cup \cdots \cup W_k$ such that*

- (E1) $\varphi(\vec{r}) = \vec{f}$,
- (E2) $\varphi^{-1}(R \cup f) = \bigcup_{i=1}^{\ell} L_i$,
- (E3) $\varphi(V_{[\ell+1,m]}) \subseteq W_1 \cup \cdots \cup W_k$, *with* $|\varphi^{-1}(W_1)| \geqslant \cdots \geqslant |\varphi^{-1}(W_k)|$, *and*
- (E4) $\varphi(e')$ *is θ-extensible, for each* $e' \in E(T[V_{[\ell+1,m]}])$.

Roughly speaking, Lemma 8.1 states that we can embed any sufficiently small tree of bounded degree into any large k-graph using vertices only from a given extensible edge, a given reservoir, and a given dense k-partite subgraph. Property (E1) says that we can map the root of the tree into any $(k-1)$-subset of an extensible edge of the host graph, in any order. Property (E2) says that we only embed a fixed number of layers of T in the reservoir (and thus only use a constant number of its vertices). Property (E3) ensures the remaining levels of the tree are embedded into the k-partite subgraph $(W_1, ..., W_k)$ and, moreover, we can decide which of these receives most (second most, etc) of the vertices from T. Finally, Property (E4) states that all the edges from $T[V_{[\ell+1,m]}]$ are mapped to θ-extensible edges of H.

In Section 8.1 we will gather some tools for the proof of Lemma 8.1, which is postponed to Section 8.2.

8.1. **Tools for embedding.** The next standard lemma states that every dense k-partite graph contains a subgraph where every partite $(k-1)$-set has either zero or large codegree.

Lemma 8.2 (Cleaning the graph). *Let $k, m \geqslant 2$, let $d \in (0, 1)$ and let H be a k-partite k-graph, with partition classes each of size at most m. If H has at least dm^k edges, then H has a non-empty subgraph H' such that $\deg_{H'}(f) \geqslant dm/k$ for every $f \in \partial H'$.*

Proof. Starting with $H_1 := H$, proceed as follows for $i \geqslant 1$. If there is an $f \in \partial H_i$ with $\deg_{H_i}(f) < dm/k$, then obtain H_{i+1} from H by removing all edges containing f. If there is no such f, we stop and set $H' := H_i$. It only remains to show that $H' \neq \emptyset$. For this, observe that the total number of deleted edges is less than $|\partial H| dm/k \leqslant dm^k$. □

In the proof of Lemma 8.1, we will use Lemma 8.2 to clean $H[W_1, ..., W_k]$ to obtain a subgraph with many $(k-1)$-sets of large codegree. The next lemma states that in such a k-graph, one can extend any (correctly located) partial embedding of a k-tree to a larger k-tree. The proof proceeds by mapping the remaining vertices successively, using the codegree condition.

Lemma 8.3 (Extending a partial tree embedding)**.** *Let $\Delta, k, m, n \in \mathbb{N}$ with $k \geqslant 2$, and let $\delta, \beta > 0$ with $\beta \leqslant \delta/2$. Let H be a k-graph and let $W_1, ..., W_k \subseteq V(H)$ be pairwise disjoint, and of size at most n each. Let $H' \subseteq H[W_1, ..., W_k]$ such that for each $f \in \partial H'$ we have $\deg_{H'}(f) \geqslant \delta n$. Let (T, r) be a rooted k-tree with $|V(T)| \leqslant \beta n$, $\Delta_1(T) \leqslant \Delta$ and a layering $\mathcal{L} = (L_1, ..., L_m)$. Let $1 \leqslant \ell \leqslant m - k + 1$, and suppose there is an embedding φ_1 of $T_1 := T[V_{[1,\ell+k-1]}]$ in H such that*

(i) *$\varphi(L_{\ell+i}) \subseteq W_i$ for every $i \in [k-1]$, and*
(ii) *if $f \in \partial T[V_{[\ell+1,\ell+k-1]}]$ then $\varphi(f)$ has codegree at least δm in H'.*

Then there is an embedding φ of T which extends φ_1, such that for each $i \in [m-\ell]$, $\varphi(L_{\ell+i}) \subseteq W_j$ if and only if $j \equiv i \bmod k$.

Proof. In each W_i, at most $|V(T)| \leqslant \beta n \leqslant \delta n/2$ vertices are used by the partial embedding φ_1, and at most $\delta n/2$ new vertices need to be embedded in each W_i. Because of our condition on the codegrees of H', we can extend φ_1 greedily, embedding each vertex in $V(T) \setminus V(T_1)$ one by one, following any valid ordering, and choosing an unused vertex in each step. □

The next lemma, Lemma 8.4, is designed to find an embedding of a short sequence of consecutive layers of a layered k-tree, while fixing the location of the initial and final segments. Its output will be the input for Lemma 8.3, which is then used to prove Lemma 8.1.

Lemma 8.4 (Embedding the trunk of a tree)**.** *Let $1/n \ll 1/k, 1/q, 1/\ell, 1/M, \delta, \alpha$ and $2 \leqslant k \leqslant (\ell-1)/2$. Let $\mathcal{L} = (L_1, ..., L_m)$ be a layering of a rooted k-tree (T, r), and assume $V_{[t,t+\ell]}$ is M-bounded.*
If $T_I \subseteq T[V_{[t,t+\ell]}]$, H is a k-graph on n vertices, $U \subseteq V(H)$ and $F_1, F_2 \subseteq \partial^o H$ are such that

(I) *$|F_1|, |F_2| \geqslant \delta n^{k-1}$, and*
(II) *for every $f_1 \in F_1$ and $f_2 \in F_2$ there are at least αn^q many walks going from f_1 to f_2, each of length $\ell - k + 1$, each with q internal vertices all located in U, and each internally disjoint from $f_1 \cup f_2$,*

then there exists an embedding $\varphi : V(T_I) \to V(H)$ such that

(i) *$\varphi(f) \in F_1$ for each $f \in \partial^o T_I[V_{[t,t+k-2]}]$,*
(ii) *$\varphi(f) \in F_2$ for each $f \in \partial^o T_I[V_{[t+\ell-k+2,t+\ell]}]$, and*
(iii) *$\varphi(v) \in U$ for each $v \in V_{[t+k-1,t+\ell-k+1]}$.*

The proof of Lemma 8.4 can be summarised as follows. In a first step, we define an auxiliary hypergraph $\mathcal{H}'$ whose edges correspond to the interior vertices of a walk from some $f_1 \in F_1$ and $f_2 \in F_2$; our assumptions will ensure that $\mathcal{H}'$ is sufficiently dense. Secondly, we use an auxiliary colouring to discard some edges of $\mathcal{H}'$ so that the remaining hypergraph $\mathcal{H}$ has the property that the walks corresponding to its edges all have identical colour sequences. This means we can now focus on the edges, and in a final step, use supersaturation in $\mathcal{H}$ find a copy of a large complete multipartite subgraph $\mathcal{K}$ of $\mathcal{H}$. Now, as the order of the walks were encoded in the colouring, we can use $\mathcal{K}$ to embed T_I into the walks corresponding to the edges of $\mathcal{K}$.

Proof. For brevity, we define $k_1 = q + 2k - 2$ and $k_2 = \ell - k + 1$ in what follows.

Step 1: Defining an auxiliary hypergraph. We begin by defining an auxiliary k_1-graph $\mathcal{H}'$. The vertices of $\mathcal{H}'$ are the vertices of H. The vertex set of a walk W of length k_2 in H is declared an edge of $\mathcal{H}'$ if there exist $f_1, f_2 \in \partial^o H$ such that the following conditions hold:

- $\mathrm{sta}(W) \in F_1$ and $\mathrm{ter}(W) \in F_2$,
- $\mathrm{int}(W) \subseteq U$ and $|\mathrm{int}(W)| = q$, and
- $\mathrm{sta}(W)$, $\mathrm{ter}(W)$ and $\mathrm{int}(W)$ (viewed as sets) are pairwise disjoint.

We claim that

$$\mathcal{H}' \text{ has at least } \frac{\delta^2\alpha}{2\ell!} n^{k_1} \text{ edges.} \tag{8.1}$$

Indeed, by (I) there are at least $|F_1| \geqslant \delta n^{k-1}$ possible choices for f_1, which will correspond to the start of a walk W defining an edge of $\mathcal{H}'$. Each such f_1 intersects at most $(k-1)n^{k-2}$ elements of ∂H. So, as $|F_2| \geqslant \delta n^{k-1}$ and n is large, there are at least $|F_2| - (k-1)n^{k-2} \geqslant \delta n^{k-1}/2$ ways to select an end $f_2 \in F_2$ disjoint from f_1. Having chosen f_1 and f_2, by (II) there are at least αn^q many walks W going from f_1 to f_2 that could define an edge of $\mathcal{H}'$. Having fixed f_1 and f_2, since all the given walks W have length k_2, the set $\mathrm{int}(W)$ could coincide for at most $k_2! < \ell!$ many of the given walks, and thus at most $\ell!$ different walks from f_1 to f_2 yield the same edge of $\mathcal{H}'$. Thus the number of edges in $\mathcal{H}'$ is at least $\delta n^{k-1} \times (\delta n^{k-1}/2) \times \alpha n^q \times (\ell!)^{-1}$, as claimed.

Step 2: Cleaning the auxiliary hypergraph. We now define a colouring $c : V(H) \to \{1, ..., k_1\}$ by choosing a colour for each vertex independently and uniformly at random. Let $\mathcal{H}^c \subseteq \mathcal{H}'$ be spanned by all edges $X \in \mathcal{H}'$ whose corresponding walk W has the following properties:

(a) if $\mathrm{sta}(W) = (x_1, ..., x_{k-1})$, then $c(x_i) = i$ for all $i \leqslant k-1$,
(b) if $\mathrm{ter}(W) = (y_1, ..., y_{k-1})$, then $c(y_i) = k_1 - k + i$ for all $i \leqslant k-1$, and
(c) no two vertices of $\mathrm{int}(W)$ have the same colour.

For a fixed $X \in \mathcal{H}'$, the probability of belonging to $\mathcal{H}^c$ is at least $k_1^{-k_1}$. So the expected size of $\mathcal{H}^c$ is at least $|E(\mathcal{H}')|/k_1^{k_1} \geqslant \delta^2\alpha n^{k_1}/(2\ell! k_1^{k_1})$ (where we used (8.1)). We can thus fix a colouring c and $\mathcal{H}^c \subseteq \mathcal{H}'$ with properties (a)-(c) and also $|\mathcal{H}^c| \geqslant \delta^2\alpha n^{k_1}/(2\ell! k_1^{k_1})$.

Now we further restrict $\mathcal{H}^c$ to make sure that, for all edges $X \in E(\mathcal{H}^c)$ corresponding to a walk W, the interiors of the walks are all consistently coloured. All of the walks W have length k_2, and thus (seeing W as a sequence of vertices), the vertices outside of $\mathrm{sta}(W)$ and $\mathrm{ter}(W)$ correspond to $k_2 - k + 1$ vertices with possible repetitions, whose underlying set $\mathrm{int}(W)$ is coloured with different colours from $\{k, ..., k_1 - k\}$. So tracking the colours received by the vertices of the walk defines a sequence of colours, with possible repetitions, chosen among $k_1 - 2k + 1$ available colours. As there are at most $(k_1 - 2k + 1)^{k_2 - k + 1} \leqslant k_1^{k_2}$ such sequences, the pigeon-hole principle gives a subset $E(\mathcal{H}) \subseteq E(\mathcal{H}^c)$ of size at least $|E(\mathcal{H}^c)|/k_1^{k_2} \geqslant \delta^2\alpha n^{k_1}/(2\ell! k_1^{k_1+k_2})$, such that each walk W corresponding to an edge X of $\mathcal{H}$ is coloured in exactly the same way under c.

Step 3: Using supersaturation. Let $\beta = \delta^2\alpha/(2\ell! k_1^{k_1+k_2})$. By our assumptions, $1/n \ll \beta$. Since $\mathcal{H}$ is a k_1-graph with at least βn^{k_1} edges we can apply Lemma 6.3, with k_1 and $M\ell$ playing the roles of k and s, to find that $\mathcal{H}$ contains a copy $\mathcal{K}$ of $K^{(k_1)}(M\ell)$, the complete k_1-partite k_1-graph with classes of size $M\ell$. Now, take any edge in $\mathcal{K}$, and recall it gives rise to a walk $W = v_1 v_2 \cdots v_{k_2}$ in H. For all i, let V_i denote the partition class of $\mathcal{K}$ that contains v_i. Note that by construction, $V_i = V_j$ is only possible for $i, j \in \{k, ..., \ell - k + 1\}$. As all walks corresponding to edges of $\mathcal{K}$ are coloured in the same way, each of them passes through the sets V_i in the same order.

Consider any injective function $h : V(T_I) \to V(\mathcal{K})$ which maps all of L_{t+i-1} to V_i, for each $i \in \{1, ..., \ell\}$. Such a function exists, since by assumption, each L_{t+i-1} has size at most M, and since W repeats each vertex at most $k_2 \leqslant \ell$ times, whereas each V_i has $M\ell$ vertices. As $\mathcal{K}$ is complete k_1-partite, we have found the desired embedding of T_I in H. □

8.2. **Proof of Lemma 8.1.** The proof of Lemma 8.1 proceeds by separating the input T in T_1 induced by the first $\ell + k - 1$ layers, which we call the "trunk of T", and the remaining $T_2 = T \setminus T_1$, which we call the "crown of T".

The proof separates in three steps. In the first step, we will prepare the host graph H for the embedding. This will be done by removing non-extensible edges, or edges with undesirable codegree properties from $H[W_1, ..., W_k]$ to get to a 'cleaned' subgraph $H' \subseteq H$, and finding a suitable ordering of the clusters $W_1, ..., W_k$ so that the final embedding satisfies the required properties. In a second step, we will apply Lemma 8.4 (Embedding the trunk of a tree) to embed the trunk T_1. In the third and final step we extend the embedding of T_1 to an embedding of the whole tree, which is done using Lemma 8.3 (Extending a partial tree embedding). Now come the details.

Proof of Lemma 8.1. To begin, we introduce new constants $\varepsilon, \alpha > 0$ such that $\theta \ll \varepsilon, \alpha \ll d, c, 1/k$, and set $\delta = dc^k/(2k)$.

Step 1: Preparing H and T for the embedding. Obtain H'' from $H[W_1, ..., W_k]$ by removing all non-θ-extensible edges. Since $1/n, \theta \ll \varepsilon$, Lemma 6.5 implies that there are at most $\varepsilon\binom{n}{k} \leqslant \varepsilon n^k$ non-θ-extensible edges. Hence, by our choice of $\varepsilon \ll d, c, 1/k$, we have

$$e(H''[W_1, ..., W_k]) \geqslant d|W_1| \cdots |W_k| - \varepsilon n^k \geqslant dc^k n^k - \varepsilon n^k \geqslant \frac{dc^k}{2} n^k.$$

Use Lemma 8.2 to find a subgraph $H' \subset H''$ such that for every $f \in \partial H'$, $\deg_{H'}(f) \geqslant \frac{dc^k}{2k} n = \delta n$, by definition of δ.

Now turn to T. For each $i \in [k]$, let $\overline{L}_i = \bigcup_{j \geqslant 0} L_{\ell+jk+i}$. Note that $\{\overline{L}_i\}_{i\in[k]}$ partitions $V_{[\ell+1,m]}$. Let $\sigma : [k] \to [k]$ be a permutation with $|\overline{L}_{\sigma^{-1}(1)}| \geqslant \cdots \geqslant |\overline{L}_{\sigma^{-1}(k)}|$. Our plan is to embed $\overline{L}_i$ into $W_{\sigma(i)}$ for all $i \in [k]$, as this will ensure (E3).

Step 2: Embedding the trunk of the tree. Let $T_1 := T[V_{[1,\ell+k-1]}]$. We will embed T_1, using vertices in R, starting from $f \subseteq e$ and ending in $W_1 \cup ... \cup W_k$. Formally, our goal is to find an embedding φ_1 of T_1 such that

(T1) $\varphi_1(\vec{r}) = \vec{f}$,
(T2) $\varphi_1(V_{[1,\ell]} \setminus r) \subseteq R$,
(T3) for every $i \in \{1, ..., k-1\}$, $L_{\ell+i}$ is embedded in $W_{\sigma(i)}$, and
(T4) if $x \in \partial T_1[V_{\ell+1,\ell+k-1}]$ then $\deg_{H''}(\varphi_1(x)) \geqslant \delta n$.

We initially set $\varphi_1(\vec{r}) = \vec{f}$, thus ensuring (T1). In order to embed the remaining vertices in $T_1' = T_1 \setminus r$, we will use Lemma 8.4, which we will apply in a suitably defined subgraph. For this, say that a k-tuple $e' = (x_1', ..., x_k') \in R^k$ is *e-good* if $e \cup e'$ induces a copy of $K^{(k)}(2)$ with partition classes $\{x_1, x_1'\}, ..., \{x_k, x_k'\}$. Define sets $F_1, F_2 \subseteq \partial^o H$ as

$$F_1 = \{(x_2', ..., x_k') \in \partial^o H : (x_1', x_2', ..., x_k') \text{ is an } e\text{-good } k\text{-tuple for some } x_1' \in R\}, \text{and}$$
$$F_2 = \{(y_1, ..., y_{k-1}) \in W_{\sigma(1)} \times \cdots \times W_{\sigma(k-1)} : \deg_{H'}(\{y_1, ..., y_{k-1}\}) \geqslant \delta n\}.$$

Next, we show that between every $h, h' \in \partial^o H$ there are many short walks of fixed length which pass through R. To this end, note that H is γn-large and R is a $(\gamma, \mu, 2)$-reservoir, and therefore H is $((\gamma^2 - 2\mu)n, R)$-large. Moreover, by the choice of $\mu \ll \gamma$, H is actually $(\gamma^2 n/2, R)$-large. By $1/n \ll \alpha \ll \gamma \ll 1/k$, we can apply Lemma 7.1 with input $k, \gamma^2/2$, and ℓ in place of k, γ, ℓ, to see the following.

Claim 8.5. *For all $h, h' \in \partial^o H$, there are αn^q many walks W of length ℓ in $H[f \cup R \cup f']$ from f to f', each with q internal vertices in R, and internally disjoint from $f \cup f'$.*

We wish to apply Lemma 8.4 with F_1, F_2 and R in place of U. Let us check that the required hypotheses are satisfied. First we show that $|F_1|$ is large. Since e is θ-extensible and R is a $(\gamma, \mu, 2)$-reservoir, there are at least $(\theta - \mu)\binom{|U|-k}{k} \geqslant (\theta\gamma^k/(2k!))n^k$ e-good edges in R (we have used $\mu \ll \theta$). Keeping the last $(k-1)$-vertices of any such edge, we deduce that $|F_1| \geqslant (\theta\gamma^k/(2k!))n^{k-1}$. Secondly, we show that $|F_2|$ is large. For this, recall that each $(k-1)$-tuple in $\partial H'$ has codegree at least δn. So, we see that $|F_2| \geqslant \delta^{k-1} n^{k-1}$. Finally, by Claim 8.5, for each choice $f_1 \in F_1$ and $f_2 \in F_2$ there are at least αn^q walks W of length ℓ going from f_1 to f_2 in H, each satisfying $|\operatorname{int}(W)| = q$, internally disjoint from $f_1 \cup f_2$ and $\operatorname{int}(W) \subseteq R$, as required.

Note that $T_1' \subseteq T[V_{[2,\ell+k-1]}]$. Since $\mathcal{L} = \{V_1, ..., V_m\}$ is a layering of T, and $\Delta_1(T) \leqslant \Delta$, Lemma 5.5 implies that $|L_i| \leqslant \Delta^{\ell+k-1}$ for all $1 \leqslant i \leqslant \ell + k - 1$, which implies that $V_{[2,\ell+k-1]}$ is $\Delta^{\ell+k-1}$-bounded. Since $1/n \ll 1/k, 1/\Delta, \theta, \gamma, d, c, \alpha$, we are allowed to apply Lemma 8.4 (Embedding the trunk of a tree), with T_1', H, R, $\min\{(\theta\gamma^k/(2k!)), \delta^{k-1}\}$, $\Delta^{\ell+k-1}$, 2, $\ell + k - 3$ playing the roles of T_I, H, U, δ, M, t and ℓ respectively. By doing so, this gives an embedding

$\varphi_0' : V(T_1') \to V(H)$. By construction, the union of φ_0 and φ_0' gives an embedding φ_1 of T_1 satisfying (T1)–(T4).

Step 3: Embedding the crown of the tree. We need to extend the embedding φ_1 of T_1 to an embedding of all of T in $H'[W_1, ..., W_k]$. By (T4) we know that every $f \in \varphi_1(\partial T[V_{\ell+1,\ell+k-1}])$ has codegree at least δn in $H'[W_{\sigma(1)}, ..., W_{\sigma(k-1)}]$. Since $\beta \ll d, c, 1/k$ and the definition of δ, we can assume $\beta \leqslant \delta/2$. Thus we can use Lemma 8.3 (Extending a partial tree embedding) with $H', W_{\sigma(1)}, ..., W_{\sigma(k)}$ playing the role of $H, W_1, ..., W_k$ to find an embedding φ of T which extends φ_1, and for each $i \in [m - \ell]$, $L_{\ell+i}$ is embedded in W_j where $i \equiv \sigma(j) \bmod k$.

Now we verify φ satisfies the required assumptions. Since φ extends φ_1 and since $r \subseteq V(T_1)$, (E1) follows from (T1). Since all of $V(T_2) = V(T) \setminus V(T_1)$ was embedded in $W_1 \cup \cdots \cup W_k$ which is disjoint from R, we have $\varphi^{-1}(R) = \varphi_1^{-1}(R)$. From (T2)–(T3) then we deduce (E2) holds. The properties of φ imply that, for each $i \in [k]$, $\varphi(\overline{L}_i) \subseteq W_{\sigma(i)}$, and therefore $\varphi^{-1}(W_\sigma(i)) = \overline{L}_i$. By the choice of σ, (E3) holds. Finally, since $H'[W_1, ..., W_k]$ only contains θ-extensible edges, (E4) holds. □

9. Absorption

In this section, we state and prove an absorption technique for hypergraphs which will allow us to complete the embedding of an almost spanning tree. This technique is similar to the one used in [6, 7]. Let us first define useful structures both for the k-tree we want to embed, and for the host graph which is used to embed the k-tree.

Definition 9.1 (Absorbing X-tuple)**.** Let $k \geqslant 3$ and let X be a $(k-1)$-tree on $h \geqslant k-1$ vertices, with a fixed valid ordering $x_1, ..., x_h$. For a k-tree T, we say that an $(h+1)$-tuple $(v_1, ..., v_h, v^*)$ of vertices of T is an *X-tuple* if

(i) $V(T(v^*)) = \{v_1, ..., v_h\}$, and
(ii) the map $x_i \mapsto v_i$ is a hypergraph isomorphism between X and $T(v^*)$.

Let H be a k-graph on n vertices and let $(v_1, ..., v_k)$ be a k-tuple of distinct vertices of H. An *absorbing X-tuple for* $(v_1, ..., v_k)$ is an $(h+1)$-tuple $(u_1, ..., u_h, u^*)$ of vertices of H such that

(A) $\{v_1, ..., v_{k-1}, u^*\} \in H$, and
(B) there exists a copy $\tilde{X}$ of X on $\{u_1, ..., u_h\}$ such that $\tilde{X} \subseteq H(v_k) \cap H(u^*)$.

We write $\Lambda_X(v_1, ..., v_k)$ for the set of absorbing X-tuples for $(v_1, ..., v_k)$. Furthermore, we let Λ_X denote the set of all absorbing X-tuples in H, that is, Λ_X is the union of $\Lambda_X(v_1, ..., v_k)$ over all k-tuples $(v_1, ..., v_k)$ of distinct vertices of $V(H)$.
Suppose there exists an embedding $\varphi : V(T) \to V(H)$ and let $(u_1, ..., u_h, u^*)$ be an $(h+1)$-tuple of vertices of H. We say $(u_1, ..., u_h, u^*)$ is *X-covered by* φ if there exists an X-tuple $(v_1, ..., v_h, v^*)$ of vertices of T such that $\varphi(v^*) = u^*$ and $\varphi(v_i) = u_i$ for all $i \in [h]$.

The following lemma is the heart of our absorbing method.

Lemma 9.2 (Absorbing Lemma)**.** *Let $n \geqslant h \geqslant k \geqslant 3$ and let $0 < \delta < \alpha$. Let T be a k-tree on n vertices with a valid ordering of $V(T)$ given by $v_1, ..., v_n$, and let $T_0 = T[\{v_1, ..., v_{n'}\}]$ be a k-subtree of T on $n' \geqslant (1-\delta)n$ vertices. Let H be a k-graph on n vertices, and suppose there exists an embedding $\varphi_0 : V(T_0) \to V(H)$, a $(k-1)$-tree X on h vertices and a family $\mathcal{A} \subseteq \Lambda_X$ of $(h+1)$-tuples of vertices of H with the following properties:*

(i) *the tuples in $\mathcal{A}$ are pairwise vertex-disjoint,*
(ii) *every tuple in $\mathcal{A}$ is X-covered by φ_0, and*
(iii) *for every k-tuple $(v_1, ..., v_k)$ of vertices of H, $|\Lambda_X(v_1, ..., v_k) \cap \mathcal{A}| \geqslant \alpha n$.*

Then there exists an embedding of T in H.

Proof. Let $m = n - n'$ and let $\{x_1, ..., x_m\}$ be an arbitrary enumeration of $V(H) \setminus V(\varphi_0(T_0))$. For every $i \in [m]$, we set $T_i := T[\{v_1, ..., v_{n'+i}\}]$. Iteratively, for each $0 \leqslant i \leqslant m$, we will find an embedding $\varphi_i : V(T_i) \to V(H)$ and subset $\mathcal{A}_i \subseteq \mathcal{A}$ with the following properties:

(a$_i$) $\varphi_i(V(T_i)) = \varphi_0(T_0) \cup \{x_1, ..., x_i\}$,
(b$_i$) $|\mathcal{A}_i| \leqslant i$, and

(c$_i$) for every $(u_1, ..., u_h, u^*) \in \mathcal{A} \setminus \mathcal{A}_i$, $\varphi_i^{-1}(u^*) = \varphi_0^{-1}(u^*)$ and $\varphi_i^{-1}(u_j) = \varphi_0^{-1}(u_j)$ for every $j \in [h]$.

It is very easy to see that for $i = 0$ the properties hold for φ_0 and $\mathcal{A}_0 := \emptyset$. Suppose that for some $0 \leqslant i \leqslant m-1$ we have defined φ_i and $\mathcal{A}_i$ satisfying (a$_i$)–(c$_i$). We shall construct φ_{i+1} and $\mathcal{A}_{i+1}$ satisfying (a$_{i+1}$)–(c$_{i+1}$).

Since $v_1, ..., v_n$ is a valid ordering for T, there exists a unique $(k-1)$-set $\{v_{i_1}, ..., v_{i_{k-1}}\} \subseteq V(T_i)$ such that $\{v_{i_1}, ..., v_{i_{k-1}}, v_{n'+i+1}\} \in T$. Let $w_1, ..., w_{k-1} \in V(H)$ be an arbitrary labelling of $\varphi_i(\{v_{i_1}, ..., v_{i_{k-1}}\})$ and define $w_k := x_{i+1}$. Note that by assumption $|\Lambda_X(w_1, ..., w_k) \cap \mathcal{A}| \geqslant \alpha n$, $i \leqslant m \leqslant \delta n$ and $\delta < \alpha$. Thus, by (b$_i$), we have

$$|\Lambda_X(w_1, ..., w_k) \cap \mathcal{A} \setminus \mathcal{A}_i| \geqslant \alpha n - |\mathcal{A}_i| = \alpha n - i \geqslant (\alpha - \delta)n > 0.$$

Now we can select an arbitrary absorbing X-tuple $(u_1, ..., u_h, u^*) \in \mathcal{A} \setminus \mathcal{A}_i$ for $(w_1, ..., w_k)$, and define $\mathcal{A}_{i+1} := \mathcal{A}_i \cup \{(u_1, ..., u_h, u^*)\}$. Note this definition of $\mathcal{A}_{i+1}$ satisfies (b$_{i+1}$). Since $(u_1, ..., u_h, u^*)$ is an X-tuple for $(w_1, ..., w_k)$ in $\mathcal{A} \setminus \mathcal{A}_i$, then $\{u_1, ..., u_h, u^*\}$ is X-covered by φ_0 and, because of (c$_i$), it is also X-covered by φ_i. For every $x \in V(T_{i+1})$, define

$$\varphi_{i+1}(x) := \begin{cases} w_k & \text{if } x = \varphi_i^{-1}(u^*), \\ u^* & \text{if } x = v_{n'+i+1}, \\ \varphi_i(x) & \text{otherwise.} \end{cases}$$

Note that the function φ_{i+1} is injective and it satisfies (a$_{i+1}$) and (c$_{i+1}$). To finish, we check that φ_{i+1} is an embedding of $V(T_{i+1})$. Indeed, if $e \in T_{i+1}$ does not contain $v_{n'+i+1}, \varphi_i^{-1}(u^*)$, then $e \in T_i$ and $\varphi_{i+1}(e) = \varphi_i(e) \in H$ since φ_i is an embedding of T_i. If $e \in T_{i+1}$ contains $v_{n'+i+1}$, then $e = \{v_{i_1}, ..., v_{i_{k-1}}, v_{n'+i+1}\}$ and because of (A) (in the definition of absorbing X-tuples) we know that $\varphi_{i+1}(e) = \{w_1, ..., w_{k-1}, u^*\} \in H$. If $e \in T_{i+1}$ contains $\varphi_i^{-1}(u^*)$, then there exists a $(k-1)$-edge e' in $T(\varphi_i^{-1}(u^*))$ such that $e = e' \cup \{\varphi_i^{-1}(u^*)\}$ and $u^* \cup \varphi_i(e') \in H$. Note that $\varphi_{i+1}(e) = w_k \cup \varphi_i(e')$. Since $(u_1, ..., u_h, u^*)$ is an X-tuple for $(w_1, ..., w_k)$, then by (B) we know that $w_k \cup \varphi(e') \in H$ and therefore $\varphi_{i+1}(e) \in H$, as desired. Thus φ_{i+1} is an embedding of T_{i+1}.

Following this process for m steps, we find an embedding φ_m of $T_m = T$, as desired. □

In the remainder of this section we will prove a series of lemmas that allow us to build a partial embedding of a tree T in which properties (i)–(iii) of Lemma 9.2 are fulfilled.

9.1. **Finding separated tuples in k-trees.** Let $\mathcal{T}_{k,[h]}$ be the family of all non-labelled k-trees on at most h vertices, up to isomorphism. For our purposes, we need to bound $|\mathcal{T}_{k,[h]}|$ in terms of h and k. Let $\mathcal{T}_{k,h}$ be the family of all k-trees on exactly h vertices. We will bound $|\mathcal{T}_{k,h}|$ by the number of *labelled* k-trees on h vertices. For all such labelled trees T, recall that all but the first k vertices have an ancestry. Since for each vertex in T we have at most $\binom{h}{k-1}$ options for its ancestry, we thus have $|\mathcal{T}_{k,h}| \leqslant \binom{h}{k-1}^{h-k} \leqslant h^{h(k-1)}$, which in turn implies

$$|\mathcal{T}_{k,[h]}| \leqslant h^{hk}. \tag{9.1}$$

Recall that the distance between $(k-1)$-tuples of the shadow of a k-tree was given in Definition 5.9. Given a k-tree T, an $(k-1)$-tree X, and $\ell \geqslant 0$, we say that a set $\mathcal{B}$ of X-tuples of T is *ℓ-separated* they are pairwise at a distance at least ℓ, that is, for each distinct $B_i, B_j \in \mathcal{B}$, and each $f_i \in \partial T[B_i]$, $f_j \in \partial T[B_j]$, we have $d_T(f_i, f_j) \geqslant \ell$.

We now show that every bounded-degree tree contains a large ℓ-separated set of X-tuples, for some $(k-1)$-tree X.

Proposition 9.3. *Suppose $0 < \mu \ll 1/\Delta, 1/k, 1/\ell$ and $k \geqslant 2$. Let T be a k-tree on n vertices such that $\Delta_1(T) \leqslant \Delta$. Then there exists a $(k-1)$-tree $X \in \mathcal{T}_{k-1,[\Delta+k-1]}$ and an ℓ-separated set $\mathcal{B}$ of X-tuples of T with $|\mathcal{B}| \geqslant \mu n$.*

Proof. By Lemma 5.1 and the bound $\Delta_1(T) \leqslant \Delta$, for every vertex $v \in V(T)$, $T(v)$ is a $(k-1)$-tree which is in $\mathcal{T}_{k-1,[\Delta+k-1]}$. By the pigeon-hole principle, there exists a $(k-1)$-tree $X \in \mathcal{T}_{k-1,[\Delta+k-1]}$ and a subset $W' \subseteq V(T)$ of size at least $n/|\mathcal{T}_{k-1,[\Delta+k-1]}|$ such that $T(w) \cong X$ for every $w \in W'$. Note that each $w \in W'$ yields an X-tuple B_w in T which uses precisely the vertices in $T(w)$.

Let $W \subseteq W'$ be maximal so that for all distinct $w_1, w_2 \in W$, the $(k-1)$-tuples in $V(T(w_1))$ and $V(T(w_2))$ are at distance at least ℓ. Set $\mathcal{B} := \{B_w : w \in W\}$. It remains to show that $|\mathcal{B}| \geqslant \mu n$.

The assumption $\Delta_1(T) \leqslant \Delta$ implies that for every $w_1' \in W'$, there are at most $(\Delta+k)^{\ell+1}$ other vertices $w_2' \in W'$ such that $T(w_1')$ and $T(w_2')$ have distance less than ℓ. We deduce that

$$|\mathcal{B}| = |W| \geqslant \frac{|W'|}{(\Delta+k)^{\ell+1}} \geqslant \frac{n}{|\mathcal{T}_{k-1,[\Delta+k-1]}|(\Delta+k)^{\ell+1}} \geqslant \mu n,$$

where the last inequality follows from (9.1) and the assumption that $\mu \ll 1/\Delta, 1/k, 1/\ell$. □

9.2. **Finding absorbing tuples in the host graph.** Many copies of an X-tuple in a tree T will indicate parts of T that are 'flexible enough' to be interchanged. This can be used to extend a partial embedding of T into an embedding of all of T. We willl see now where such X-tuples may be embedded, assuming that the 'host graph' has large codegree.

Proposition 9.4. *Let $1/n \ll \beta \ll \gamma, 1/h, 1/k$ with $h \geqslant k \geqslant 2$. Let H be a γn-large k-graph H on n vertices, let X be a $(k-1)$-tree on h vertices and let $(v_1, ..., v_k)$ a k-tuple of distinct vertices of H. Then $|\Lambda_X(v_1, ..., v_k)| \geqslant \beta n^{h+1}$.*

Proof. We construct an absorbing X-tuple for $(v_1, ..., v_k)$ by choosing vertices one by one. First, select an arbitrary vertex $u_{h+1} \in N_H(\{v_1, ..., v_{k-1}\}) \setminus \{v_k\}$. (Since H is γn-large there are at least $\gamma n - 1 \geqslant \gamma n/2$ possible choices for u_{h+1}.) Define the $(k-1)$-graph $H' := H(v_k) \cap H(u_{h+1})$, and observe that H' is γn-large. Now select $u_1, ..., u_h$ iteratively in increasing order, as follows. First, fix a valid ordering $\{t_1, ..., t_h\}$ of the vertices of X. Select $u_1, ..., u_{k-1} \in V(H')$ arbitrarily. Then, successively for $i = k-1, ..., h+1$, select u_{i+1} in the following way: if $\{t_{j_1}, ..., t_{j_{k-2}}\}$ is the ancestry of t_{i+1}, then choose $u_{i+1} \in N_{H'}(\{u_{j_1}, ..., u_{j_{k-2}}\}) \setminus \{u_1, ..., u_i\}$ arbitrarily. By construction, $(u_1, ..., u_{\ell+1})$ is an absorbing X-tuple for $(v_1, ..., v_k)$.

Finally, as in each step there are at least $\gamma n/2$ possibilities to choose the next vertex u_i, there are at least $(\gamma n/2)^{h+1} \geqslant \beta n^{h+1}$ absorbing X-tuples for $(v_1, ..., v_k)$, as desired. □

Now we would like to find a linear-sized family $\mathcal{A} \subseteq \Lambda_X$ of vertex-disjoint X-tuples such that every k-tuple has many absorbing X-tuples in $\mathcal{A}$ (as to satisfy properties (i) and (iii) of Lemma 9.2). The following lemma can be proved by selecting independently each $(\ell+1)$-tuple in Λ_X with probability $p := \Omega(n^{-\ell})$ and showing it satisfies the required properties with positive probability, which can be done using Chernoff's inequality (Theorem 6.8) and Markov's inequality. As this strategy is standard by now and has appeared in many other absorption-based proofs (e.g. [26, Claim 3.2]), we leave the details of the proof to the reader.

Lemma 9.5. *Let $1/n \ll \alpha \ll \beta, 1/k, 1/h$ with $h \geqslant k \geqslant 2$. Let H be a k-graph on n vertices and let X be a $(k-1)$-tree on h vertices. If $|\Lambda_X(v_1, ..., v_k)| \geqslant \beta n^{h+1}$ for every k-tuple $(v_1, ..., v_k)$ of vertices of H, then there is a set $\mathcal{A} \subseteq \Lambda_X$ of at most αn disjoint $(h+1)$-tuples of vertices of H such that $|\Lambda_X(v_1, ..., v_k) \cap \mathcal{A}| \geqslant \beta\alpha n/8$ for every k-tuple $(v_1, ..., v_k)$ of vertices of H.* □

9.3. **Covering X-tuples with a partial tree embedding.** Our final step in order to use the Absorbing Lemma is to show that we can cover a large family of absorbing tuples.

Lemma 9.6 (Embedding pseudopaths). *Let $1/n \ll 1/\Delta, 1/k, 1/\ell, \gamma$ satisfying $\Delta, k \geqslant 2$, and also $\ell \geqslant (2k+1)\lfloor k/2 \rfloor + 2k$. Let H be a γn-large k-graph on n vertices. Let P be a k-uniform (f, f')-pseudopath P where we consider f, f' to be ordered tuples. Moreover, suppose $e(P) \geqslant \Delta k(\ell + 3k)$ and $\Delta_1(P) \leqslant \Delta$. Then, given any pair of disjoint $(k-1)$-tuples $x, y \in \partial^o H$, there exists an embedding $\varphi : V(P) \to V(H)$ such that $\varphi(f) = x$ and $\varphi(f') = y$.*

Proof. Suppose P has t edges $e_1, ..., e_t$, where $f \subseteq e_1$ and $f' \subseteq e_t$. Now let $\mathcal{L} = (L_1, ..., L_m)$ be a layering for (P, f), which exists by Lemma 5.4. By Lemma 5.10(ii), $|L_i| \leqslant k\Delta$ for all $i \in [m]$. Thus the number m of layers of $\mathcal{L}$ satisfies $m \geqslant |V(P)|/(k\Delta) \geqslant e(P)/(k\Delta) \geqslant \ell + 3k$.

We start our construction of the embedding by setting $\varphi(f) = x$ and $\varphi(f') = y$. As a next step, we will embed greedily the first k layers of $\mathcal{L}$ into R. Recall the definition of $r(j)$ from Lemma 5.10(i), and let j_1 be the maximum j such that $r(j) \leqslant k$. Lemma 5.10(i) implies that

$r(j_1) = k$, $r(j) \leqslant k$ for all $j \leqslant j_1$ and $r(j) > k$ for all $j > j_1$. Let P_1 be the subgraph of P spanned by the edges $e_1, ..., e_{j_1}$. Then P_1 is the 'minimum' subpath of P which contains all the edges touching the first k layers.

Let $s_1 = e_{j_1} \setminus (L_1 \cup \cdots \cup L_k)$, ordered according to the increasing layering order. Since $r(j_1) = k$, s_1 has size $k-1$. Now we embed $P_1 \setminus s_1$ making sure there are 'many' possible extensions available for s_1. First, in a greedy fashion, we count the number of embeddings of e_1 which extend $\varphi(f)$ simply by completing f to e_1 to an unused vertex outside $x \cup y$. Since H is γn-large, this can be done in at least $\gamma n - |x \cup y| \geqslant \gamma n/2$ ways. Next, we iteratively count the extensions of the embedding of e_1 to an embedding of P_1, which can be done again in a greedy fashion, again having $\gamma n/2$ choices for an unused vertex each time. Letting $n_1 = |V(P_1)| - (k-1)$, we deduce that there are at least $(\gamma n/2)^{n_1}$ embeddings of P_1 which extend $\varphi(f)$. Now, an averaging argument entails that there is an embedding of $P_1 \setminus s_1$ which extends $\varphi(f)$ and can be extended to at least $((\gamma n/2)^{n_1})/n^{n_1 - |s_1|} = (\gamma/2)^{n_1} n^{k-1}$ embeddings of P_1. Let $\varphi(P_1 \setminus s_1)$ be such an extension of $\varphi(f)$, and let $F_1 \subseteq V(H)^{k-1}$ be the set of all ordered $(k-1)$-tuples which are valid extensions of $\varphi(P_1 \setminus s_1)$ to an embedding of P_1. Then $|F_1| \geqslant (\gamma/2)^{n_1} n^{k-1}$.

Observe that if we consider the edges of P in reverse ordering (i.e. we consider e_t to be the first, e_1 to be the last edge), the resulting k-tree is a (f', f)-pseudopath. So, defining j_2 as the minimum $j \leqslant t$ such that $r(j) \geqslant m - 2k + 2$, and defining s_2 and P_2 accordingly, we can proceed as in the previous paragraph, to obtain an embedding $\varphi(P_2 \setminus s_2)$ which extends $\varphi(f')$ and can be extended to at least $(\gamma/n)^{n_2} n^{k-1}$ embeddings of P_2. Letting $F_2 \subseteq V(H)^{k-1}$ denote the set of all ordered $(k-1)$-tuples giving valid extensions of $\varphi(P_2 \setminus s_2)$ to an embedding of P_2, we have $|F_2| \geqslant (\gamma/2)^{n_1} n^{k-1}$.

We complete the embedding by using Lemma 8.4. Let $\alpha, \delta > 0$ be such that $1/n \ll \alpha, \delta \ll \gamma$. Since $m - 3k \geqslant \ell \geqslant (2k+1)\lfloor k/2 \rfloor + 2k$ and H is $(\gamma n, V(H))$-large, Lemma 7.1 outputs $q \leqslant m - 3k$ such that for each pair of disjoint tuples $f_1 \in F_1$ and $f_2 \in F_2$ there are at least αn^q walks of length $m - 3k$ connecting f_1 and f_2, each having q internal vertices, and internally disjoint from f_1, f_2. By the choice of δ we have $|F_1|, |F_2| \geqslant \delta n^{k-1}$, and finally note that the remaining vertices to be embedded correspond to $P \setminus (P_1 \cup P_2)$, whose set of vertices is completely in $V_{[k+1, m-k]}$, which is $k\Delta$-bounded. We can use Lemma 8.4 with $P, P \setminus (P_1 \cup P_2), k+1, m-2k-1, k\Delta$ playing the roles of T, T_I, t, ℓ, M to obtain an embedding φ' of the remaining vertices from $L_{k+1} \cup \cdots \cup L_{m-k}$, which extends the embedding φ to an embedding of all of P. □

Lemma 9.7 (Covering Lemma)**.** *Let $1/n \ll \alpha \ll \mu \ll \nu \ll \gamma, 1/h$ with $h, \Delta, k \geqslant 2$ and $\ell \geqslant (2k+1)\lfloor k/2 \rfloor + 2k$. Let X be a $(k-1)$-tree on h vertices. Let H be a γn-large k-graph on n vertices, and let $\mathcal{A} \subseteq \Lambda_X$ be a set of at most αn pairwise disjoint $(h+1)$-tuples of vertices of H. Let T be a k-tree on νn vertices, with $\Delta_1(T) \leqslant \Delta$ and let $\mathcal{B}$ be a $2\Delta k(\ell + 3k)$-separated set of size at least μn of X-tuples of vertices of T. Then for any $f \in \partial H$, $r \in \partial T$ there is an embedding $\varphi : V(T) \to V(H)$ such that $\varphi(r) = f$ and every tuple in $\mathcal{A}$ is X-covered by φ.*

Proof. Write $\mathcal{A} = \{A_1, ..., A_t\}$, where $1 \leqslant t \leqslant \alpha n$. We will abuse notation by treating X-tuples $B \in \mathcal{B}$ as subgraphs of T, consisting of the corresponding edges forming the X-tuple. We claim that there are $B_1, ..., B_t \in \mathcal{B}$ such that, defining $d_T(r, B_i)$ as the minimum of $d_T(r, b)$ over all $(k-1)$-sets $b \in \partial T[B_i]$, we have

$$\Delta k(\ell + 3k) \leqslant d_T(r, B_j) \leqslant d_T(r, B_i) \text{ for all } 1 \leqslant j < i \leqslant t. \tag{9.2}$$

To see this, order the elements of $\mathcal{B}$ as $B'_1, B'_2 ...$ so that $d_T(r, B'_i)$ increases. As $\mathcal{B}$ is $2\Delta k(\ell + 3k)$-separated, and using the triangle inequality, we see that

$$2\Delta k(\ell + 3k) \leqslant d_T(B'_1, B'_2) \leqslant d_T(B'_1, r) + d_T(B'_2, r) \leqslant 2 d_T(B'_2, r)$$

and thus $d_T(B'_2, r) \geqslant \Delta k(\ell + 3k)$. Since $|\mathcal{B}| \geqslant \mu n \gg \alpha n \geqslant t$, we can delete B'_1 from $\mathcal{B}$ if necessary, and delete more sets B'_i until size exactly t is reached. After relabelling we arrive at sets $B_1, ..., B_t$ satisfying (9.2).

Set $T_0 := \emptyset$. Given $i \in [t]$ and $T_{i-1} \subseteq T$, define T_i as follows. Let $t_i \in \partial T_{i-1}$ and $b_i \in \partial B_i$ such that $d_T(t_i, b_i)$ is minimised (if $i = 1$, select $t_1 = r$ instead). Let P_i be the unique (t_i, b_i)-pseudopath in T, and let $T_i = T_{i-1} \cup P_i \cup B_i$. Then $T_0, T_1, ..., T_t$ satisfy the following properties.

Claim 9.8. *For all $1 \leqslant i \leqslant t$, T_i is a subtree of T, and*

(Q1) $T_i \subseteq T_{i+1}$ *if* $i < t$,
(Q2) $(B_1 \cup \cdots \cup B_i) \subseteq T_i$, *and*
(Q3) $T_i \cap (B_{i+1} \cup \cdots \cup B_t) = \emptyset$.

Indeed, the only property which is not immediate from construction is (Q3). Suppose the property failed, and let i be minimum such that it fails. Then there exists a $j > i$ such that $B_j \cap T_i \neq \emptyset$. By minimality of i, $B_j \cap T_{i-1} = \emptyset$, and since $\mathcal{B}$ are pairwise disjoint subgraphs, $B_j \cap B_i = \emptyset$. Thus $B_j \cap (P_i \setminus B_i) \neq \emptyset$. Since P_i was the unique minimum-length pseudopath between T_{i-1} and B_i in T, this implies $d_T(r, B_j) < d_T(r, B_i)$, contradicting (9.2).

For $i \geqslant 0$, we will now construct embeddings $\varphi_i : V(T_i) \to V(H) \setminus (A_{i+1} \cup \cdots \cup A_t)$ such that for $i \geqslant 1$, φ_i extends φ_{i-1}, and such that $A_i = \varphi_i(B_i)$ is X-covered by φ_i. We start by setting $\varphi_0(r) = f$. Now assume that $i \geqslant 1$, and suppose we have embedded T_{i-1} with the embedding φ_{i-1}. By (Q3), the image of B_i has not been defined in φ_{i-1}. We begin by setting $\varphi'_i(B_i) = A_i$, in a way that A_i is X-covered by φ'_1. Recall that, by definition, T_i is the union of T_{i-1}, B_i, and a (t_i, b_i)-pseudopath P_i, for some $b_i \in \partial B_i$ and some $t_i \in \partial T_i$ (or $t_i = r$ if $r = 1$). Note that $H_i := H \setminus (A_{i+1} \cup \cdots \cup A_t)$ has at least $n' = n - (t-1)(h+1) \geqslant (1 - \alpha(h+1))n \geqslant (1 - \gamma/3)n$ vertices, as by assumption, $\alpha h \ll \gamma$. A similar calculation entails that H_i is $(\gamma n'/2)$-large. We claim that

$$d_T(t_i, b_i) \geqslant \Delta k(\ell + 3k). \tag{9.3}$$

Then, we can use Lemma 9.6 to find an embedding $\varphi_i : V(P_i) \to V(H_i)$ which extends both φ_{i-1} and φ'_i, and this completes step i.

So let us show (9.3). Assume otherwise, and let $1 \leqslant j < i - 1$ be the minimum index such that $t_i \in \partial T_j$. By minimality, we have $t_i \in \partial(P_j \cup B_j)$. If $t_i \in \partial B_j$, then P_i is a pseudopath from B_i to B_j, and since $\mathcal{B}$ is $2\Delta k(\ell+3k)$-separated we then have $d_T(r_i, t_i) \geqslant 2\Delta k(\ell+3k)$, and we are done. Assume instead that $t_i \in \partial P_j \setminus \partial B_j$. Let $b_j \in \partial T_j$ such that $d_T(b_j, r) = d_T(B_j, r)$. Note that t_i must lie on the unique pseudopath in T from b_j to r, and thus

$$d_T(B_j, t_i) + d_T(t_i, r) = d_T(B_j, r) \leqslant d_T(B_i, r) \leqslant d_T(b_i, t_i) + d_T(t_i, r),$$

where the second inequality comes from (9.2). As furthermore

$$2\Delta k(\ell + 3k) \leqslant d_T(B_j, B_i) \leqslant d_T(b_i, t_i) + d_T(B_j, t_i) < \Delta k(\ell + 3k) + d_T(B_j, t_i),$$

we obtain $d_T(b_i, t_i) \geqslant d_T(B_j, t_i) > \Delta k(\ell + 3k)$, contrary to our assumption. Thus (9.3) holds.

Having defined all φ_i, we note that each absorbing tuple in $\mathcal{A}$ is X-covered by φ_t. We extend φ_t to an embedding φ of all of $V(T)$. We can get from T_t to T by iteratively adding leaves, thus the embedding can be found in a greedy fashion since $\delta_{k-1}(H) \geqslant \gamma n$, and at each step there are at most $|V(T)| \leqslant \nu n \leqslant \gamma n/2$ used vertices. □

10. Proof of the Main Theorem

We now assemble all results from the previous sections in order to prove Theorem 1.2. The proof is divided into three main steps.

We start the first step by finding a small subtree $T' \subseteq T$ of linear size. Using Proposition 9.3 we find a $(k-1)$-tree X such that T' contains linearly many well-separated X-tuples. With the help of Proposition 9.4 and Lemma 9.5, we find a family $\mathcal{A}$ of disjoint absorbing X-tuples in the host graph H such that every k-tuple in H has linearly many absorbing X-tuples in $\mathcal{A}$. We then use the Covering Lemma (Lemma 9.7) to embed T' in H, covering every tuple in $\mathcal{A}$.

In the second step, we will find an almost spanning subtree $T'' \subseteq T - T'$ and embed it following the regularity method. The Weak Hypergraph Regularity Lemma (Theorem 6.1) gives a regular partition of the vertices of H and using Lemma 6.2, we find an almost spanning matching $\mathcal{M}$ in the corresponding reduced graph. We find a (β, d)-decomposition (Definition 5.13) of T'' and use the Embedding Lemma (Lemma 8.1) to map the small parts of this decomposition into edges of $\mathcal{M}$[1]. In the third and last step, we finish the embedding by using the Absorbing Lemma (Lemma 9.2).

We begin with a lemma which essentially covers all of the second step outlined above. First, we need a definition. Let H be a k-graph on m vertices and let $\varepsilon, d > 0$. We say H is (ε, d)-*uniformly dense* if for all pairwise disjoint sets $W_1, ..., W_k \subseteq V(H)$ we have

$$e(W_1, ..., W_k) \geqslant d|W_1| \dots |W_k| - \varepsilon m^k. \tag{10.1}$$

If H is k-partite, with parts $V_1, ..., V_k$, we say H is k*-partite* (ε, d)*-uniformly dense* if for all sets $W_1 \subseteq V_1, ..., W_k \subseteq V_k$ the bound (10.1) holds.

Definition 10.1 (Uniformly dense perfect matching)**.** For $\varepsilon, d > 0$ and $t \in \mathbb{N}$, we say $\mathcal{M} = \{(V_1^i, ..., V_k^i)\}_{i\in[t]}$ is an (ε, d)*-uniformly dense perfect matching* of a k-graph H if

- (M1) $\{V_a^i\}_{a\in[k], i\in[t]}$ partitions $V(H)$,
- (M2) $|V_a^i| = |V_b^j|$ for all $i, j \in [t]$, $a, b \in [k]$, and
- (M3) $H[V_1^i, ..., V_k^i]$ is k-partite (ε, d)-uniformly dense for each $i \in [t]$.

The following proposition will be used in the second step of the proof of Theorem 1.2, providing us with an embedding of an almost spanning tree in any graph having an (ε, d)-uniformly dense perfect matching.

Proposition 10.2 (Embedding almost spanning trees)**.** *Let $\Delta, k \geqslant 2$, let $\theta \ll 1/T_0$, and let $1/n \ll \mu \ll \theta \ll \varepsilon, \delta \ll \alpha, \gamma, d, 1/k, 1/\Delta$. Let H be a γn-large k-graph on n vertices with a $(\delta, \mu, 2)$-reservoir R. Let $\mathcal{M}$ be an (ε, d)-uniformly dense perfect matching of $H - R$ with $|\mathcal{M}| \leqslant T_0$. Let (T, r) be a k-tree with $|V(T)| \leqslant (1 - \alpha)n$ and $\Delta_1(T) \leqslant \Delta$. Then H contains T. Moreover, for any θ-extensible edge e, any $f \in \partial^o H$ with $f \subseteq e$ can be chosen as the image of r.*

Proof. Let $\ell = \lfloor k/2 \rfloor(2k + 1) + 2k + 1$. Introduce new constants c, β satisfying $\mu \ll \beta \ll \varepsilon \ll c \ll \alpha, 1/k$ and $\beta \ll 1/T_0$.

Let $\mathcal{L}$ be a layering for (T, r), which exists by Lemma 5.4. We invoke Proposition 5.14, with parameters β, Δ and ℓ, to obtain a (β, ℓ)-decomposition of $(T, r, \mathcal{L})$ into $p \leqslant 2\Delta^\ell/\beta$ parts. That is, we find a collection of rooted trees $\{(D_i, s_i)\}_{1\leqslant i\leqslant p}$ such that

- (D1) $E(T) = \bigcup_{i\in[p]} E(D_i)$,
- (D2) $|E(D_i)| \leqslant \beta n$ for each $i \in [p]$,
- (D3) $s_1 = r$ and s_i is $\mathcal{L}$-layered,
- (D4) $(V(D_j) \setminus s_j) \cap V(D_i) = \emptyset$ for all $1 \leqslant i < j \leqslant p$, and
- (D5) for each $2 \leqslant j \leqslant p$ there is a unique $i < j$ such that $s_j \in \partial D_i$ and the rank of s_j in D_i is at least ℓ (in the inherited layering of D_i from $\mathcal{L}$).

Let $e \in H$ be an arbitrary θ-extensible edge in H, and let $f \subseteq e \cap \partial^o H$. Say $\mathcal{M} = \{(V_1^h, ..., V_k^h)\}_{h\in[t]}$ are the clusters of the given uniformly dense perfect matching and set $m := |V_1^1|$.

We begin our embedding by setting $\varphi_0(r) := f$. Now we will construct successively, for all $i \in [p]$, an embedding $\varphi_i : V(D_1 \cup \dots \cup D_i) \to V(H)$ such that φ_i extends φ_{i-1}, and

- (P1) if defined, $\varphi_i(s_j)$ is contained in a θ-extensible edge for all $j > i$,
- (P2) $|\varphi_i^{-1}(R)| \leqslant i\Delta^{\ell+1}$, and
- (P3) for every $j \in [t]$, $\max_{a,b\in[k]} \left||\varphi_i^{-1}(V_a^j)| - |\varphi_i^{-1}(V_b^j)|\right| \leqslant cm$.

Having done this, then φ_p will be the desired embedding of T.

In step i, assume we have constructed φ_{i-1} satisfying (P1)–(P3). Our aim is to embed (D_i, s_i). We claim that s_i is already embedded into some $(k-1)$-tuple $\varphi_{i-1}(s_i)$ contained in some θ-extensible edge. Indeed, if $i = 1$, we have $s_1 = r$ by (D3) and we are done by the choice of φ_0. Otherwise, if $i \geqslant 2$, then (D5) implies $\varphi_{i-1}(s_i)$ is defined and we are done by (P1).

We claim that there is a $j \in [t]$ such that for each $a \in [k]$

$$|V_a^j| \geqslant |\varphi_{i-1}^{-1}(V_a^j)| + cm. \tag{10.2}$$

[1]Actually, for the second step, we will consider a slightly more general setting than the one given by a regular partition. This will allow us to treat in an unified way the proof of Theorem 1.2 and Theorem 1.4.

Indeed, otherwise, for each $j \in [t]$ there is an $a \in [k]$ such that (10.2) does not hold. Then, by (P3), also all other V_b^j are almost full, and we calculate

$$|\varphi_{i-1}^{-1}(V(H))| \geqslant \sum_{j\in[t],a\in[k]} |\varphi_{i-1}^{-1}(V_a^j)| > \sum_{j\in[t],a\in[k]} (|V_a^j| - 2cm) = n - |R| - 2cmtk \geqslant (1-\alpha)n,$$

a contradiction as $|V(T)| \leqslant (1-\alpha)n$. So, (10.2) holds for, say, index j.

Let σ be a bijection satisfying

$$|W_1| \geqslant \cdots \geqslant |W_k|, \tag{10.3}$$

where $W_a := V_{\sigma(a)}^j \setminus \varphi_{i-1}(V(T))$. Because of (10.2), $|W_i| \geqslant cm$ for each $i \in [k]$. Therefore, using that $H[V_1^j, ..., V_k^j]$ is k-partite (ε, d)-uniformly dense on km vertices and $\varepsilon \ll c, d, 1/k$, we have

$$e(H[W_1, ..., W_k]) \geqslant d|W_1|\cdots|W_k| - \varepsilon(km)^k \geqslant \frac{d}{2}|W_1|\cdots|W_k|. \tag{10.4}$$

Also note that since $m = |V(H-R)|/(kT_0)$ and R is a $(\delta, \mu, 2)$-reservoir, we have $m \geqslant n/(2kT_0)$ and thus $|W_a| \geqslant cn/(2kT_0)$ holds for each $1 \leqslant a \leqslant k$.

Set $R_{i-1} := R \setminus \varphi_{i-1}(D_1 \cup \cdots \cup D_{i-1})$. Recall that $i \leqslant p \leqslant 2\Delta^\ell\beta^{-1}$. Using (P2) we get, $|R_{i-1}| \geqslant |R| - p\Delta^{\ell+1} \geqslant |R| - 2\Delta^{2\ell+1}\beta^{-1}$. Since R is a $(\delta, \mu, 2)$-reservoir for H and $1/n \ll 1/\Delta, 1/k, \beta, \mu, \delta$, we deduce R_{i-1} is a $(\delta, 2\mu, 2)$-reservoir for H. Let $\mathcal{L}_i = (L_1, ..., L_u)$ be the inherited layering $\mathcal{L}_{s_i}$ for (D_i, s_i) from $\mathcal{L}$, this is well-defined since s_i is $\mathcal{L}$-layered by (D3). Let $\vec{s_i} = (s_{i,1}, ..., s_{i,k}) \in \partial T_i$ be an arbitrary ordering of s_i, and let $\vec{f} = \varphi_{i-1}(\vec{s_i}) \in \partial H$.

Use Lemma 8.1 (Embedding Lemma) with

object/parameter	$\ell - 1$	R_{i-1}	δ	2μ	$c/(2kT_0)$	$d/2$	D_i	s_i	$\mathcal{L}_i$	$\varphi_{i-1}(s_i)$	$\vec{s_i}$	$\vec{f}$
playing the role of	ℓ	R	γ	μ	c	d	T	r	$\mathcal{L}$	f	$\vec{s_i}$	$\vec{f}$

to find an embedding φ_i' of D_i such that

(E1) $\varphi_i'(\vec{s_i}) = \varphi_{i-1}(\vec{s_i})$,
(E2) $(\varphi_i')^{-1}(R_{i-1} \cup \varphi_{i-1}(s_i)) = \bigcup_{r=1}^{\ell-1} L_r$,
(E3) $\varphi_i'(V_{[\ell,u]}) \subseteq W_1 \cup \cdots \cup W_k$, with $|(\varphi_i')^{-1}(W_1)| \geqslant \cdots \geqslant |(\varphi_i')^{-1}(W_k)|$,
(E4) $\varphi_i'(e')$ is θ-extensible, for each e' in $E(D_i[V_{[\ell,u]}])$.

By (D4)–(D5) and (E1), we get that $\varphi_i = \varphi_{i-1} \cup \varphi_i'$ is an extension of φ_{i-1} which embeds $T[V(D_1 \cup \cdots \cup D_i)]$. It is only left to check that φ_i satisfies (P1)–(P3).

We check (P1) holds for φ_i. Since (P1) holds for $i-1$, we only need to consider those $j > i$ with $s_j \in \partial D_i$. By (D5) each $s_j \in \partial D_i$ with $j > i$ has rank at least ℓ in $\mathcal{L}_i$, namely it must be contained in $V_{[\ell,u]}$. Thus, by (E4), we know that all such $\varphi_i(s_j)$ are contained in a θ-extensible edge, as required.

To see (P2) holds, we note that since (P2) holds for $i-1$, $|\varphi_i^{-1}(R \setminus R_{i-1})| = |\varphi_{i-1}^{-1}(R \setminus R_{i-1})| \leqslant (i-1)\Delta^{\ell+1}$, so it only remains to show that $|\varphi_i^{-1}(R_{i-1})| \leqslant \Delta^{\ell+1}$. Note that by Lemma 5.5 we have $|L_r| \leqslant \Delta^{r-1}$ holds for all $1 \leqslant r \leqslant \ell-1$, and together with (E2) we get $|\varphi_i^{-1}(R_{i-1})| \leqslant \sum_{r=1}^{\ell-1}|L_r| \leqslant \sum_{r=1}^{\ell-1}\Delta^{r-1} \leqslant \Delta^{\ell+1}$, as required.

Finally, for (P3), observe that (E3) implies that

$$|V(D_i)| \geqslant |\varphi_i^{-1}(W_1) \setminus \varphi_{i-1}^{-1}(W_1)| \geqslant \cdots \geqslant |\varphi_i^{-1}(W_k) \setminus \varphi_{i-1}^{-1}(W_k)|.$$

Using (10.3) we get, for any $a, b \in [k]$ with $a < b$,

$$\Big||\varphi_i^{-1}(V_a^j)| - |\varphi_i^{-1}(V_b^j)|\Big| \leqslant \max\Big\{\Big||\varphi_{i-1}^{-1}(V_a^j)| - |\varphi_{i-1}^{-1}(V_b^j)|\Big|, |V(D_i)|\Big\}.$$

Since (P3) holds for $i-1$, the first term in the maximum is bounded from above by cn, and $|V(D_i)| \leqslant \beta n \leqslant cn$, follows from (D2) and $\beta \ll c$. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We begin by noting that $\delta_2(H) \geqslant (1/2+\gamma)n$ implies H is $(2\gamma n)$-large.

Step 1: Finding the absorbing structures. For the remainder of the proof, we choose $\ell = (2k+1)\lfloor k/2 \rfloor + 2k$ and also

$$1/n_0 \ll \nu_3 \ll \nu_2 \ll \delta, \nu_1 \ll \alpha \ll \theta_0 \ll \gamma, 1/k, 1/\Delta, 1/\ell.$$

Let T be any given tree on $n \geqslant n_0$ vertices such that $\Delta_1(T) \leqslant \Delta$. We choose an arbitrary tuple $r \in \partial T$ as the root of T and, using Lemma 5.4, we find a layering $\mathcal{L} = (L_1, ..., L_m)$ for (T, r). Let $r' \in \partial T$ be an $\mathcal{L}$-layered tuple such that $|E(T_{r'})| \geqslant \alpha n/\Delta$ with the highest possible rank. Then, using Lemma 5.12 for $(T_{r'}, r', \mathcal{L}_{r'})$, and by the maximality of r' we deduce that

$$\frac{\alpha n}{\Delta} \leqslant |E(T_{r'})| \leqslant \alpha n. \tag{10.5}$$

Since $\delta \ll 1/k, 1/\Delta, 1/\ell$, we can use Proposition 9.3 with $2\Delta k(\ell+3k)$ and δ in place of ℓ and μ, respectively. This yields a $(k-1)$-tree X with $h \leqslant \Delta+k-1$ vertices, and a $2\Delta k(\ell+3k)$-separated set $\mathcal{B}$ of X-tuples of $T_{r'}$ such that $|\mathcal{B}| \geqslant \delta n$. Note that H is $(2\gamma n)$-large, and additionally $1/n \ll \nu_1 \ll \gamma, 1/k, 1/\Delta$ and $h \leqslant \Delta + k - 1$. Thus Proposition 9.4 (with ν_1 playing the role of β) tells us that $|\Lambda_X(v_1, ..., v_k)| \geqslant \nu_1 n^{h+1}$ for every k-tuple of distinct vertices $(v_1, ..., v_k) \in V(H)^k$. The hierarchy $\nu_2 \ll \nu_1, 1/k, 1/h$ allows us to use Lemma 9.5 with parameters ν_1, ν_2 in place of β, α respectively, to deduce the existence of a family $\mathcal{A} \subseteq \Lambda_X$ of size at most $\nu_2 n$, such that

$$\text{for every } k\text{-tuple } (v_1, ..., v_k) \text{ we have } |\Lambda_X(v_1, ..., v_k) \cap \mathcal{A}| \geqslant \frac{\nu_1\nu_2}{8} n \geqslant 2\nu_3 n, \tag{10.6}$$

where the last inequality follows from $\nu_3 \ll \nu_2, \nu_1$.

Bounding very crudely, we deduce H has at least $\gamma\binom{n}{k}$ edges. Since $\theta_0 \ll \gamma, 1/k$, an application of Lemma 6.5 implies the existence of a θ_0-extensible edge e_0 which is vertex-disjoint of all the subgraphs in $\mathcal{A}$. Let $f_0 \subseteq e_0$ be an arbitrary $(k-1)$-set. Finally, since $1/n \ll \nu_2 \ll \delta \ll \alpha \ll \gamma, 1/h$, we can use Lemma 9.7 with $\nu_2, \delta, \alpha, 2\gamma, r', f_0$ playing the roles of $\alpha, \mu, \nu, \gamma, r, f$ respectively to find an embedding $\varphi' : V(T_{r'}) \to V(H) \setminus (e_0 \setminus f_0)$ which X-covers each tuple in $\mathcal{A}$, such that $\varphi'(r') = f_0 \subseteq e_0$, and $e_0 \setminus f_0$ is disjoint from $\varphi'(V(T_{r'}))$.

Step 2: Finding an almost spanning tree. We introduce new constants by letting

$$1/n \ll \mu \ll \theta \ll 1/T_0 \ll 1/t_0 \ll \varepsilon' \ll \eta \ll \varepsilon \ll \gamma' \ll d \ll \nu_3.$$

Let $H' = H - \varphi'(V(T_{r'})) + e_0$ and $n' = |V(H')|$. Then by (10.5) we have $|V(T_{r'})| \leqslant \alpha n + k - 1 \leqslant 2\alpha n$, and thus $n' \geqslant (1-2\alpha)n$. Note that $e_0 \subseteq V(H')$.

The choice of $\alpha \ll \gamma$ ensures that H' has minimum codegree at least $(1/2 + 2\gamma/3)n'$. Using $1/n \ll \mu \ll \gamma'$, Lemma 6.7 provides us with a $(\gamma', \mu, 2)$-reservoir R for H'. Now set $H'' = H' - R$ and $n'' = |V(H'')|$. Since $\gamma' \ll \alpha$, we deduce $n'' \geqslant (1-3\alpha)n$.

Now we prepare the setup to apply regularity tools. Since $1/n, 1/T_0 \ll 1/t_0, 1/k, \varepsilon'$, an application of the Weak Hypergraph Regularity Lemma (Theorem 6.1) to H'', with parameters ε' and t_0 as input, yields an ε'-regular partition $\mathcal{P} = \{V_0, V_1, ..., V_t\}$ of $V(H'')$, for some $t_0 \leqslant t \leqslant T_0$. Using Lemma 6.2, $1/t_0 \ll \varepsilon' \ll 1/k, \gamma, \eta$ and $d \ll \gamma$, we know that the (ε', d)-reduced graph $R_d(H'')$ contains a matching $\mathcal{M}$ covering at least $(1-\eta)t$ clusters. Since each edge $(V_{i_1}, ..., V_{i_k}) \in \mathcal{M}$ is (ε', d')-regular for some $d' \geqslant d$, by our choice of $\varepsilon' \ll \varepsilon$, we deduce that $H''[V_{i_1}, ..., V_{i_k}]$ is k-partite (ε, d)-uniformly dense. Let $V_{\mathcal{M}} \subseteq V(H'')$ consist of the union of clusters covered by $\mathcal{M}$ in the reduced graph. Thus, $H''[V_{\mathcal{M}}]$ has an (ε, d)-uniformly dense perfect matching with $p \leqslant T_0/k$ edges.

We wish to apply Proposition 10.2. For this we need a k-graph having a reservoir and a uniformly dense perfect matching, which we plan to be R and $V_{\mathcal{M}}$, respectively. We also need the k-graph to contain our desired root e_0, which is why we set $R' = (R \cup e_0) \setminus V_{\mathcal{M}}$. Let H^* be the induced subgraph of H' on $R' \cup V_{\mathcal{M}}$ and set $n^* = |V(H^*)|$.

We now check the requirements of Proposition 10.2 are satisfied with this choice of $H^*, R', V_{\mathcal{M}}$. Clearly $\{R', V_{\mathcal{M}}\}$ partitions $V(H^*)$ and $e_0 \subseteq V(H^*)$. As $\mathcal{P}$ is an ε'-regular partition of $H'' = H' - R$, and $\mathcal{M}$ leaves at most ηt clusters uncovered, each having size at most n''/t, we see that

$$n' - n^* \leqslant |V_0| + \sum_{V_i \notin V(\mathcal{M})} |V_i| \leqslant \varepsilon' n'' + \eta t n''/t \leqslant \varepsilon n',$$

where we used $\varepsilon', \nu \ll \varepsilon$ and $n'' \leqslant n'$ in the last inequality. This implies $n^* \geqslant (1-\varepsilon)n'$. Since $\varepsilon \ll \gamma$ and since $\delta_{k-1}(H') \geqslant (1/2 + 2\gamma/3)n'$, we deduce H^* has minimum codegree at least $(1/2 + \gamma/2)n^*$. In particular, H^* is γn^*-large. Since R is a $(\gamma', \mu, 2)$-reservoir for H', we use $n^* \geqslant (1-\varepsilon)n'$ and $1/n \ll \mu, 1/k$ to deduce R' has size $\gamma' n' \pm (\mu n' + k) = (\gamma^* \pm 2\mu)n^*$, for some $\gamma^* \leqslant 2\gamma'$. Thus R' is a $(\gamma^*, 2\mu, 2)$-reservoir for H^*. Finally, recall that e_0 is θ_0-extensible in H. As $1/n \ll \theta \ll \varepsilon \ll \alpha \ll \theta_0$, we have $n - n^* \leqslant 4\alpha n$, implying that e_0 is θ-extensible in H^*.

We set $T' = T - (T_{r'} - r')$ and root T' at r'. Let $v_1, ..., v_{|V(T')|}$ be a valid ordering of (T', r') with $r' = \{v_1, ..., v_{k-1}\}$. Let $m^* = \lceil (1-\nu_2)n^* \rceil$ and $T^* = T'[v_1, ..., v_{m^*}]$, such that $|V(T^*)| = m^*$. An application of Proposition 10.2 with

object/parameter	n^*	2μ	γ^*	$\nu_3/2$	d	H^*	R'	e_0	f_0
playing the role of	n	μ	δ	α	d	H	R	e	f

yields an embedding $\varphi^* : V(T^*) \to V(H^*)$ with $\varphi''(r') = f_0 = \varphi'(r)$.

Step 3: Finishing the embedding. The embedding $\varphi' \cup \varphi^*$ of $T_{r'} \cup T^*$ fulfils all the conditions of Lemma 9.2 (as$|V(T_{r'} \cup T^*)| \geqslant m^* + |V(T_{r'})| - k - 1 \geqslant (1-\nu_3)n$ and by (10.6)). So, we can use the Absorbing lemma (Lemma 9.2) to embed the remaining vertices $v_{m^*+1}, ..., v_{|V(T')|}$. □

11. Spanning trees in quasirandom hypergraphs

In this section, we prove Theorem 1.4. Actually, we prove a slightly stronger statement since we only need the weaker notion of being *uniformly dense* (given right before Definition 10.1).

Theorem 11.1. *Let $0 < 1/n \ll \delta \ll \eta, \gamma, 1/\Delta, 1/k$ and let H_1 and H_2 be k-graphs on the same vertex set of size n. Assume H_1 is (η, δ)-uniformly dense and H_2 is γn-large. Then $H := H_1 \cup H_2$ contains any k-tree T on n vertices with $\Delta_1(T) \leqslant \Delta$.*

Proof (sketch). Since the proof is essentially the same as the proof of Theorem 1.2, we only outline the major differences. In particular, step 1 is virtually the same, noting that the only property used there is that the host graph was that H is $2\gamma n$-large, while here H_2 is γn-large.

The difference in step 2 is that of applying regularity to obtain the uniformly dense perfect matching, we use that H_1 is uniformly dense. Indeed, suppose we have already found the equivalent of H' with a reservoir R, and we wish to find a uniformly dense matching covering most vertices in $H'' = H' - R$. Move at most $k-1$ vertices from H'' to R we can assume $|V(H'')|$ is divisible by k. Fix any partition $V_1, ..., V_k$ into equal-sized parts. Since H_1 is (η, δ)-uniformly dense, $\mathcal{M} = \{(V_1, ..., V_k)\}$ is an (η, δ)-uniformly dense perfect matching with only one edge. Having found $\mathcal{M}$, then we can proceed with the remainder of step 2 and step 3 exactly as before. □

Let us now show how Theorem 11.1 implies the same result for host hypergraphs satisfing certain *quasirandom properties*, namely *weak quasirandomness* (Corollary 11.2) and *typicality* (Theorem 1.4).

The study of quasirandomness for graphs evolved in the late 1980's, a milestone being the seminal result of Chung, Graham and Wilson [8] relating uniform edge distribution to other 'random-like' properties. For the history of quasirandomness for hypergraphs we refer to the exposition in [1]. The weakest form of quasirandomness studied in the literature is as follows. Call a k-graph H *weakly (η, δ)-quasirandom* if for every $U \subseteq V(H)$, the number $e(U)$ of edges entirely contained in U satisfies

$$\left| e(U) - \eta \binom{|U|}{k} \right| \leqslant \delta n^k. \tag{11.1}$$

An inclusion-exclusion argument shows that every weakly (η, δ)-quasirandom k-graph is $(\eta, 2^k\delta)$-uniformly dense. Thus the following is an immediate corollary of Theorem 11.1.

Corollary 11.2. *Let $0 < 1/n \ll \delta \ll \eta, \gamma, 1/\Delta, 1/k$ and let H_1 and H_2 be k-graphs on the same vertex set of size n. Assume H_1 is weakly (η, δ)-quasirandom and H_2 is γn-large. Then $H := H_1 \cup H_2$ contains any k-tree T on n vertices with $\Delta_1(T) \leqslant \Delta$.*

As mentioned in the introduction, being γn-large in Corollary 11.2 cannot be replaced with a lower bound of type $\Omega(n^{k-1})$ on the minimum degree of H, as evidenced by an example by Arajo, Piga and Schacht [4, Section 8.2].

Let us now turn to the notion of *typicality*, which was defined in the introduction. Although weakly quasirandomness does not imply typicality, one can show[2] that the converse is true: any $(\eta, 2, \delta)$-typical k-graph is weakly (η, δ')-quasirandom, for some constant δ'. This is the only ingredient we need to prove Theorem 1.4.

Proof of Theorem 1.4. Let $1/n_0 \ll \varepsilon' \ll \varepsilon$. As discussed above, H is weakly (ϱ, ε')-quasirandom. Also note that every $(\varrho, 2, \varepsilon)$-typical graph on n vertices is $(\varrho^2 - \varepsilon)n$-large. Thus the theorem follows from Corollary 11.2, by taking $H = H_1 = H_2$. □

12. Further questions

12.1. **Degree variations.** For any $1 \leqslant j < k-1$, one can define the *minimum j-degree* $\delta_j(H)$ of a graph H in analogy to the minimum codegree $\delta_{k-1}(H)$ as defined in the introduction. Also, define $\mathrm{st}_j(k)$ as the smallest $\delta > 0$ such that for every $\Delta, k \geqslant 2$ and $\mu > 0$, every large enough k-graph H with $\delta_j(H) \geqslant (\delta+\mu)\binom{n-j}{k-j}$ contains every k-tree T of the same order and with $\Delta_1(T) \leqslant \Delta$. Then by Theorem 1.2, we have $\mathrm{st}_{k-1}(k) \leqslant 1/2$ for all $k \geqslant 2$, and by Proposition 1.3 this is tight.

It would be interesting to understand $\mathrm{st}_j(k)$ for $j < k-1$. If instead of a spanning tree we are looking for a spanning cycle, then some results are known for the analogous problem. In k-graphs, a *tight Hamilton cycle* in a k-graph H on n vertices is a sequence of distinct vertices $v_1, \ldots, v_n$ such that, for each $i \in \{1, \ldots, n\}$, the edge $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$ is present in H (indices understood modulo n). Then $\mathrm{hc}_j(k)$ is defined accordingly. It is known that $\mathrm{hc}_{k-2}(k) = 5/9$ for all $k \geqslant 3$ [23, 25]. For general j, k satisfying $1 \leqslant j \leqslant k-3$ the current best lower [17] and upper [23] bounds are

$$1 - \frac{1}{\sqrt{k-j}} \leqslant \mathrm{hc}_j(k) \leqslant 1 - \frac{1}{2(k-j)}.$$

Furthermore, the result of Rödl, Ruciński and Szemerédi [26] mentioned in the introduction states that $\mathrm{hc}_{k-1}(k) = 1/2$ for all $k \geqslant 2$. Thus $\mathrm{hc}_{k-1}(k) = \mathrm{st}_{k-1}(k) = 1/2$ for all $k \geqslant 2$, and we believe the same should be true in general.

Conjecture 12.1. *For all $k > j \geqslant 1$, we have $\mathrm{hc}_j(k) = \mathrm{st}_j(k)$.*

Variations of the maximum degree condition required for the tree T would also be interesting to study.

12.2. **Trees of larger maximum degree.** Another possible generalisation Theorem 1.2 would be to relax the condition on $\Delta_1(T) = O(1)$, allowing it to grow with n. In the graph case, Komlós, Sárközy and Szemerédi [22] strengthened their earlier result, Theorem 1.1, considerably by showing that, with the same minimum degree conditions, one can find all trees of maximum degree at most $cn/\log n$, for a fixed $c > 0$ depending on the approximation γ only. They also gave an example showing this bound is tight up to a multiplicative factor.

A natural adaptation of their example shows that, for $k \geqslant 3$, the bound on $\Delta_1(T)$ in Theorem 1.2 cannot be changed to $O(n/\log n)$. Indeed, let T be the k-tree consisting of a vertex v whose link graph is a $(k-1)$-tight path P of length $c \log n$, for some sufficiently small constant $c > 0$. For each set S of $k-1$ consecutive vertices in P we add $n/(c \log n)$ new vertices adjacent to S. Then T has maximum degree $\Theta(n/\log n)$, and a straightforward calculation shows that, with high probability, the binomial random k-graph of edge density $p = 0.9$ does not contain an ordered set U of size $c \log n$ such that each vertex outside U is adjacent to some $k-1$ consecutive vertices in U.

[2] Consider the k-graph F with $V(F) = [k] \times \{0, 1\}$ and edges of the forms $\{(1,0), (2, x_2), \ldots, (k, x_k)\}$ and $\{(1,1), (2, x_2), \ldots, (k, x_k)\}$, where $x_i \in \{0, 1\}$ for $i \in \{2, \ldots, k\}$. It is clear that in any $(\eta, 2, \delta)$-typical k-graph H one can estimate the density of H and the number of F-copies, up to an error term depending on δ. Hence H satisfies the property $\mathrm{disc}_{\mathcal{Q},\eta}$ for $\mathcal{Q} = \{\{1\}, \{2, \ldots, k\}\}$, meaning that H is weakly quasirandom (see [1, 30] for details).

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