A Variant of the Erdős-Sós Conjecture

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Abstract

A well-known conjecture of Erdős and Sós states that every graph with average degree exceeding $m-1$ contains every tree with m edges as a subgraph. We propose a variant of this conjecture, which states that every graph of maximum degree exceeding m and minimum degree at least $\lfloor \frac{2m}{3} \rfloor$ contains every tree with m edges.

As evidence for our conjecture we show (i) for every m there is a $g(m)$ such that the weakening of the conjecture obtained by replacing the first m by $g(m)$ holds, and (ii) there is a $\gamma > 0$ such that the weakening of the conjecture obtained by replacing $\lfloor \frac{2m}{3} \rfloor$ by $(1-\gamma)m$ holds.

1 Introduction

A recurring topic in extremal graph theory is the use of degree conditions (such as minimum/average degree bounds) on a graph to prove that it contains certain subgraphs. For instance, every graph of minimum degree exceeding $m-1$ contains a copy of each tree with m edges. (Embed the root anywhere, and greedily continue embedding vertices whose parents are already embedded.)

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In 1963, Erdős and Sós conjectured the following strengthening of this fact: if a graph has *average* degree exceeding $m-1$ then it contains every tree with m edges as a subgraph. Their conjecture has attracted a fair amount of attention over the last decades. Partial solutions are given in [BD96, Hax01, SW97], and in the early 1990's, Ajtai, Komlós, Simonovits and Szemerédi announced a proof of this result for sufficiently large m. In order to see that the Erdős-Sós conjecture is best possible, observe that no $(m-1)$-regular graph contains the star $K_{1,m}$ as a subgraph. Alternatively, consider a graph that consists of several disjoint copies of the complete graph K_m; this graph contains no tree with m edges as a subgraph.

The related Loebl-Komlós-Sós conjecture from 1995 [EFLS95] states that if a graph has median degree at least m then it contains every tree with m edges as a subgraph. This conjecture had also received considerable attention [AKS95, Coo09, HP15, PS12, Zha11], and recently, an approximate version was shown in [HKP$^+$17a, HKP$^+$17b, HKP$^+$17c, HKP$^+$17d] (see also [HPS$^+$15]). Note that the examples above demonstrate that the Loebl-Komlós-Sós conjecture is tight as well.

In this paper we propose a new conjecture for tree embeddings under degree assumptions. We consider the minimum and maximum degrees rather than the average or median degrees.

Conjecture 1.1 *If a graph has maximum degree at least m and minimum degree at least $\lfloor\frac{2m}{3}\rfloor$ then it contains every tree with m edges as a subgraph.*

We remark that every graph of average degree exceeding $m-1$ has a subgraph of minimum degree at least $\frac{m}{2}$. Thus, replacing $\lfloor\frac{2m}{3}\rfloor$ by $\frac{m}{2}$ in our conjecture would give a strengthening of the Erdős-Sós conjecture. However, two simple examples show that the value $\lfloor\frac{2m}{3}\rfloor$ here is best possible. In both examples we consider the tree T with $3k+1$ vertices obtained from three stars on k vertices by adding a new vertex v adjacent to their centers. In the first example, G is the graph obtained from two copies of K_{2k-1} by adding a universal vertex. In the second example, G is the graph obtained by adding a universal vertex to $K_{2k-2,2k-2}$.

Nevertheless, focussing on the minimum degree of the graphs in question, could be an effective technique for approaching the Erdős-Sós conjecture. Indeed, it might be possible to prove a natural common generalization of Conjecture 1.1 and the Erdős-Sós Conjecture which makes no mention of the average degree.

Note that Conjecture 1.1 holds for paths (even with the weaker bound of $\frac{m}{2}$ on the minimum degree), because of the well-known Dirac-type result that every connected graph G of minimum degree $\delta(G)$ has a path on $\min\{2\delta(G)+1,|V(G)|\}$ vertices. It also holds for trees with many leaves (see below).

As further evidence for Conjecture 1.1, we prove the following two weakenings.

Theorem 1.2 *There is a function g such that if a graph has maximum degree at least $g(m)$ and minimum degree at least $\lfloor\frac{2m}{3}\rfloor$ then it contains every tree T with m edges as a subgraph.*

Theorem 1.3 *There is a $\gamma > 0$ such that if a graph has maximum degree at least m and minimum degree at least $(1-\gamma)m$ then it contains every tree T with m edges as a subgraph.*

After proving some useful results on trees in Section 2, we prove Theorem 1.2 in Section 3 and Theorem 1.3 in Section 4. While the proof of the first theorem is not very hard, the proof of the second theorem is much more complicated. We dedicate the remainder of the introduction to a sketch of some of the ideas used in both our proofs. For a more detailed sketch of the proof of Theorem 1.3 we refer the reader to the beginning of Section 4.

Let us start with an easy observation that involves trees having a vertex s that is adjacent to many leaves. We can embed s in a maximum-degree vertex $f(s)$ of the host graph, and then embed the rest of the tree, except for the leaves adjacent to s, in a greedy fashion. Finally, we embed the leaves at s, exploiting the large degree of $f(s)$. Note that this procedure gives a proof of both our theorems, and of Conjecture 1.1, for all trees that contain a vertex adjacent to at least $\lceil \frac{m}{3} \rceil$ leaves. It also proves Theorem 1.3 for all trees that contain a vertex adjacent to at least $\lceil \gamma m \rceil$ leaves. In particular, this proves the conjecture and Theorem 1.2 for trees having a vertex of degree at least $\lceil \frac{2m}{3} \rceil$, and Theorem 1.3 for trees having a vertex of degree at least $\lceil \frac{(1+\gamma)m}{2} \rceil$.

The proof of both of our theorems for the remaining trees splits into two cases depending on whether or not the host graph G has a small dense subgraph. To illuminate why small dense subgraphs are important let us now prove the conjecture for host graphs which do not contain any connected subgraphs with $m+1$ vertices having average degree at least 2, that is, host graphs of girth at least $m+2$. If we greedily embed a tree with m edges in such a graph by embedding the vertices in breadth-first order, treating all the children of each vertex as a consecutive block, we see that for every non-root vertex s we have embedded, the girth condition ensures that its image $f(s)$ is adjacent to the image of exactly one vertex of the tree (namely the parent of s). Since s has degree at most $\lfloor \frac{2m}{3} \rfloor$, we will be able to embed its children into the unoccupied neighbours of $f(s)$. So the greedy embedding strategy succeeds in graphs of girth at least $m+2$.

Without the girth condition imposed in the illustrating example in the previous paragraph, but still assuming that our graph is relatively sparse and has no dense subgraphs (this is the first of the two cases mentioned above), we can still show that only a few vertices have many occupied neighbours. Our approach in this case is to try and stay away from these vertices when embedding the rest of the graph. In order to do so, we exploit the well known fact (see Section 2) that every tree T with m edges has a vertex z such that at most one component of $T-z$ has more than a third of the vertices of T, and if such a component exists, it has at most two thirds of the vertices of T. The same is true replacing 'a third' with γ and 'two thirds' with $1-\gamma$. This means that we can split the components of $T-z$ conveniently into two sets, such that the one containing more vertices can be embedded greedily using the minimum degree of the host graph, while embedding z into a maximum degree vertex. Now, for embedding the remaining vertices we need to stay away from the occupied vertices. In proving Theorem 1.2 this is relatively easy to do because $f(z)$ has huge degree, and so we have a lot of flexibility when placing the neighbours of z. In proving Theorem 1.3, $f(z)$ may only have m neighbours which makes things harder. In this case we need to be more careful during the first phase of the embedding. Here, the higher minimum degree comes in handy.

Turning to graphs with small dense subgraphs (the second case mentioned above), we only discuss the proof of Theorem 1.3 here, as the approach taken in the proof of Theorem 1.2 is fairly straightforward. In that proof, we focus on the densest subgraphs of the host graph with at most $m+1$ vertices. For every such maximum-density subgraph H, if H has minimum degree d then every vertex outside of H sees at most $d+1$ vertices of H. Furthermore, because H is small and dense it turns out that we can embed trees with significantly more than d vertices in H.

So we can often embed significantly more than $d+\varepsilon m$ vertices of the tree in H and just slightly less than $(1-\varepsilon)m-d$ in $G-H$ which has minimum degree at least $(1-\varepsilon)m-d-1$. In order to do so, we split the tree T, by determining a cutvertex z, and grouping the components of $T-z$ into two sets of components, $\mathcal{C}_1$ and $\mathcal{C}_2$, whose sizes fit well with our embedding plans.

There are some further complications: we need to consider some extensions of these small dense graphs, some dense bipartite graphs, and a partition of the graph into such dense pieces. For more on these difficulties, see Section 4. We hope our description here is enough to give a flavour of the proof.

Finally, we mention that recent work [RS19a, RS19b] has partially confirmed Conjecture 1.1.

2 Some Properties of Trees

In this section we prove some useful results on trees. Our first aim is to find a relatively large stable set whose vertices have degree at most 2 in the tree. The small degree of the vertices in this set means that when embedding into a small dense subgraph H, we will be able to embed them last, after (carefully) embedding the rest of the vertices, thereby embedding many more than $\delta(H)$ vertices into H.

Lemma 2.1 *Every rooted tree T with at least two vertices contains a stable set S_T of size $\lceil |V(T)|/6 \rceil$ not containing the root such that:*

(a) every vertex in S_T is a leaf, or a vertex of degree 2 whose parent is also a non-root vertex of degree 2, and

(b) no child of a vertex in S_T is the parent of some other vertex of S_T.

Proof. Letting ℓ be the number of non-root leaves of T, we see that removing the root of T and all vertices of degree greater than 2 in T splits the non-root vertices of degree 1 and 2 in T into fewer than 2ℓ paths of with total number of vertices at least $|V(T)| - \ell$. We can put every third vertex within each of these paths into S_T, as long as we start with the second from the root. We can thereby ensure that $|S_T| \geqslant (|V(T)| - 3\ell)/3$. On the other hand, we can simply put all the non-root leaves of T into S_T, so $|S_T| \geqslant \ell$. The result follows. □

Also, it turns out that matchings in the tree we wish to embed can be useful when embedding into a dense subgraph. This is because we can embed matched vertices one right after the other, that way their embedding happens under almost identical circumstances (with respect to the 'used' or 'unused' parts of the host graph). In addition, for the first vertex of a matching edge we can choose an image with high degree into some set we wish to use for the second vertex.

Lemma 2.2 *For every tree T, and every $1 \leqslant \ell \leqslant |V(T)|/2$, either T contains at least $|V(T)| - 2\ell + 2$ leaves, or for every vertex v of T, there is a subtree of T with 2ℓ vertices which contains v and has a perfect matching.*

Proof. Consider a maximum subtree T' of T containing v which has a perfect matching. If some component of $T - T'$ has at least two vertices, then adding two adjacent vertices of this component to T', including the one joined to T' by an edge, contradicts the maximality of T'. So all vertices in $V(T - T')$ are leaves, and since $|V(T')|$ is even, the result follows. □

Finally, we prove a much used observation that allows us to split the tree into subtrees whose sizes we can control.

Observation 2.3 *Let T be a tree on t vertices.*

(a) There is $z \in V(T)$ such that every component of $T - z$ has $t/2$ or fewer vertices.

(b) For any $t' < \frac{t}{2}$, either every component of $T - z$ has fewer than t' vertices or there is a vertex $v_{t'}$ of $T - z$ such that the component of $T - v_{t'}$ containing z has at most $t - t'$ vertices and every other component has fewer than t' vertices.

Proof. For (a), we root the tree and let z be the vertex furthest from the root such that the subtree formed by z and its descendants contains at least half the vertices.

For (b), we can assume there is a component C_1 of $T - z$ having at least t' vertices. We root the tree at z and let $v_{t'}$ be the vertex furthest from z in C_1 such that the subtree formed by $v_{t'}$ and its descendants contains at least t' vertices. □

A *separator* for a tree T on t vertices is a vertex z such that each component of $T - z$ has at most $t/2$ vertices. Note that the choice of such a z is unique or there are two such choices which are endpoints of an edge e such that each component of $T - e$ contains $t/2$ vertices.

3 The Proof of Theorem 1.2

Define $g(m) := (m+1)^{2m+6} + 1$ and consider a counterexample (m, G) minimizing $|E(G)|$. Let v be a vertex of G of degree $g(m)$ and note that minimality implies that if uw is an edge of $G - v$ then one of u or w has degree $\lfloor \frac{2m}{3} \rfloor$. Let $t = m+1$ and let $t' = m - \lfloor \frac{2m}{3} \rfloor$. We assume $t \geqslant 3$ (otherwise the proof is easy). We split the proof into two cases as follows.

Case 1: G contains a $K_{t^3,t'}$.

Let A be the smaller side of this complete bipartite graph and let B be the larger side. Minimality implies that every vertex in B has degree $\lfloor \frac{2m}{3} \rfloor$.

Thus, for any vertex b of B, there are fewer than t^2 vertices of B which are adjacent to b or have a common neighbour with b which has degree at most t. So, we can choose a stable subset B' of t vertices of B, such that no vertex of degree less than t sees two vertices of B'.

We take any subtree of T with $2t'+1$ vertices and embed it in $A \cup B'$ using more vertices of B' than of A. We can now greedily complete the embedding, since by the choice of B', every (used or unused) vertex of degree less than t has degree at least $\lfloor 2m/3 \rfloor - 1 = m - (t'+1)$ into $G - B'$, while at least $t'+1$ vertices are already embedded into B'.

Case 2: G contains no $K_{t^3,t'}$.

Note that in this case, for any subset S of V that contains at least t' vertices, we have

$$\text{less than } t^3\binom{|S|}{t'} \text{ vertices of } G \text{ see } t' \text{ or more vertices of } S. \tag{1}$$

Applying Observation 2.3 with our chosen value of t', we see that there is a vertex w of T such that no component of $T - w$ has more than $\frac{2m}{3}$ vertices, and all but the largest component have fewer than t' vertices.

We embed w into v. We greedily embed the largest component of $T - w$ into G. We then embed the remaining components of $T - w$, which have size at most $t' - 1$. Whenever we come to embed such a component K, we proceed as follows.

Let A_0 be the set of vertices into which we have already embedded a vertex of T (before starting to embed K). Successively, for $i = 1, \ldots, t'$, let $A_i \subseteq V(G) - \bigcup_{j<i} A_j$ consist of all those vertices that have degree less than $t' - 1$ in $G - \bigcup_{j<i} A_j$. Note that $A_0, \ldots, A_{t'}$ are pairwise disjoint.

Now, each vertex of A_1 has degree at least $\lfloor 2m/3 \rfloor - (t'-2) \geqslant t'$ into A_0. Hence, we can use (1) to see that

$$|A_1| \leqslant t^3\binom{|A_0|}{t'}.$$

For $i \geqslant 2$, note that if $v \in A_i$, then (since $v \notin A_{i-1}$), we know that v has a neighbour in A_{i-1}. So for $i \geqslant 2$, the definition of A_{i-1} gives that $|A_i| \leqslant (t'-2)|A_{i-1}|$. Therefore,

$$|\bigcup_{i=0}^{t'} A_i| \leqslant \sum_{i=0}^{t'} (t'-2)^i t^3 \binom{|A_0|}{t'} \leqslant t^{2t+4}.$$

So, since $g(m) > t^{2t+4}$, there is a neighbour of v outside of $\bigcup_{i=0}^{t'} A_i$ in which we can embed the neighbour x of w in K. We now use the degree condition on the sets A_i to greedily embed K levelwise, allowing each level j (that is, the jth neighbourhood of x) to use vertices in $G - \bigcup_{i=0}^{t'-j} A_i$. This way we ensure that A_0 is not used for our embedding of K.

Iterating this process for each yet unembedded component of $T - w$ proves Theorem 1.2.

4 The Proof of Theorem 1.3

Let us start by giving an overview of our proof. Our proof has five parts. In the first part, in Subsection 4.1, we show that if all the subgraphs of the host graph G with at most $m+1$ vertices are really sparse, then we can find the desired embedding (this is done in Lemma 4.1). Thus we can assume that G has a subgraph of size at most $m+1$ that is reasonably dense, i.e. has average degree linear in m.

In Subsection 4.4, we show how to use such a subgraph. If we cannot find the desired embedding of T, then we find a very dense subgraph H of G. More precisely, either H has at most $m+1$ vertices, and is almost complete, in the sense that at every vertex there is only a small fraction of the possible edges missing, or H is almost complete bipartite (in the same sense), with each of its sides having size at most m.

Such a subgraph H can be very useful for embedding a part of the tree, as its extreme density allows us to accommodate more vertices of T than we would expect if we were only using the minimum degree. For technical reasons, it will be convenient to explain this approach in detail already in Subsections 4.2 and 4.3 (before actually finding H in Subsection 4.4). A series of lemmas given in these two subsections covers the range of possible situations that we might have to deal with in a later stage of the proof, when we wish to embed parts of T into such a graph H.

In the last part, in Subsection 4.5, we put everything together. We find a maximal set of disjoint very dense subgraphs H_i, knowing that at least one such subgraph is guaranteed to exist by what we said above. (Actually, our H_i will be slight expansions of the subgraphs found in Subsection 4.4.) We show that if we cannot embed T using the results of Subsections 4.2 and 4.3, there are only very few edges between the different subgraphs H_i, and between the union of the H_i and the leftover of the graph G. By the results of Subsection 4.1, this means that there is no such leftover (as it would have to contain another very dense graph). Thus one of the H_i contains a vertex of degree at least m. Again making use of results of Subsections 4.2 and 4.3, we show we can embed T. This completes the overview of our proof.

We close this subsection with some preliminaries. We often iteratively construct an embedding f of T in such a way that the embedded subtree is always connected. In this case, whenever we come to embed a vertex s of T, there is a unique embedded neighbour $p(s)$ of s and we need only ensure that s is embedded in a neighbour of $f(p(s))$ which has not yet been used in the embedding. We refer to this as a *good* iterative construction process.

Note that we can and do assume that no vertex of T is incident to more than γm leaves, as otherwise we can simply embed this vertex into a maximum degree vertex, greedily embed the tree except for the leaves incident to it and then embed these leaves. For this reason, all our lemmas are stated with this assumption.

4.1 Locally Sparse Graphs

A graph is *locally m-sparse* if it contains no subgraph with at most $m+1$ vertices and average degree exceeding $\frac{m}{25}$. The main result of this section is the following:

Lemma 4.1 *Suppose T is a tree with at most m edges each of whose vertices is adjacent to at most $\frac{m}{20}$ leaves and G is a locally m-sparse graph of minimum degree at least $\frac{19m}{20}$. Then for any vertex w of G and separator z for T, we can find an embedding f of T in G such that $f(z) = w$.*

Once we have proved Lemma 4.1, we can continue our proof only considering host graphs G that are not locally m-sparse. Before proving Lemma 4.1, we show an auxiliary result.

Lemma 4.2 *Let G be a locally m-sparse graph of minimum degree at least $\frac{19}{20}m$. Then for any $S \subseteq V(G)$ with $|S| \leqslant m-1$ there is a set $S' \supseteq S$ such that $G - S'$ has minimum degree at least $\frac{m}{2}$ and $|S'| \leqslant |S| + \frac{m}{20}$.*

Proof. Assume for a contradiction that there is no such set S'. For any set $S' \supseteq S$ such that $|S'| \leqslant |S| + \frac{m}{20}$, there is a vertex a of $G - S'$ having degree less than $\frac{m}{2}$ in $G - S'$, and so more than $\delta(G) - \frac{m}{2} = \frac{9m}{20}$ neighbours in S'. In particular $|S| \geqslant \frac{9m}{20}$, and we can find a set A with $\lceil \frac{m}{20} \rceil$ vertices such that every vertex in A has at least $\frac{9m}{20}$ neighbours in $S \cup A$. (Find A by successively adding suitable vertices). We choose any set $B \subseteq S$ of size $|A| - 1$, and note that the set $(S - B) \cup A$ has at most m vertices and induces more than $(\frac{9m}{20} - |B|)|A| \geqslant \frac{8m^2}{400} = \frac{m^2}{50}$ edges, and thus, its average degree is above $\frac{m}{25}$. This contradicts G being locally m-sparse.

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Proof of Lemma 4.1. We let F be the union of some of the components of $T-z$ which together have between $\frac{m}{4}$ and $\frac{m}{2}$ vertices. If we can embed $T-F$ into a set $f(V(T-F))$ that avoids at least $|F|-1+\frac{m}{20}$ neighbours of w, then by applying Lemma 4.2 to $S_0 = f(V(T-F))$, we obtain a set S'_0 that avoids at least $|F|-1$ neighbours of w and such that $G - S'_0$ has minimum degree $\frac{m}{2}$. Now z is adjacent to at most $\frac{m}{20}$ leaves it is adjacent to at most $|F|-1$ vertices of F, and so we can embed all neighbours of z in F into $N(w) - S'_0$. Then, since $|F| \leqslant m/2$, we can extend this embedding greedily into an embedding of all of T. Hence, fixing any set $N \subseteq N(w)$ with $|N| = \lceil \frac{19m}{20} \rceil$, it suffices to embed $T-F$ using at least $\frac{m}{10}$ vertices outside N.

Choose any set $S \subseteq N+w$ containing w of size $\lceil \frac{2m}{3} \rceil$, and consider the set $S' \supseteq S$ given by Lemma 4.2. Then $|S'-w| \leqslant \frac{3m}{4}$ and the vertices outside S' have degree at least $\frac{m}{2}$ into $G-S'$. We now embed into $N-S'$ either all or $|N-S'|$ of the neighbours of z in $T-F$, and then embed the corresponding components of $T-F-z$ greedily, trying to avoid S' as much as possible. (Since $|T-F-z| \leqslant \frac{3m}{4} \leqslant \frac{19m}{20}$, we can clearly embed all of these components in G.)

If we never used any vertex in S', then, depending on how many neighbours of z we embedded into $N-S'$, we embedded either all of $T-F-z$ in $G-S'$, or at least $|N-S'| - \frac{m}{20} \geqslant \frac{m}{10}$ vertices of $T-F-z$ in $G-N$ (since by assumption, z has at most $\frac{m}{20}$ leaf neighbours). Since z is not in N, in the first case we found the desired embedding, and in the second case we can greedily continue to find it.

So assume we used S', and let x be the first vertex we embedded there. Then, since we tried to avoid S', the parent of x is embedded in a vertex that has at least $\frac{m}{2}$ neighbours in $G-S'$ that have already been used for our embedding. At least $\frac{m}{2} - |N-S'|$ of these vertices are outside N. But $|N-S'| \leqslant |N-S| \leqslant \frac{19m}{20} - \frac{2m}{3}$. Hence at least $\frac{m}{2} - \frac{19m}{20} + \frac{2m}{3} > \frac{m}{10}$ vertices outside N have been used for embedding $T-F$, which is as desired. We greedily continue our embedding of $T-F$. $\square$

4.2 Filling Small Almost Complete Subgraphs

We now prove some auxiliary lemmas which are at the heart of the whole proof. They allow us to use almost complete subgraphs H' of the host graph G in order to embed some suitable subtree T' of T. The point of these lemmas is that T' is allowed to be substantially larger than the minimum degree of H'.

Lemma 4.3 *Let $0 < \varepsilon < \frac{1}{200}$, let H' be a graph with minimum degree at least $(1-2\varepsilon)(|V(H')|-1)$, and let T' be a tree with m' edges, rooted at z, with $(|V(H')|-1)/2 \leqslant m' \leqslant (1-\varepsilon)(|V(H')|-1)$. If each vertex of T' is incident to at most $\varepsilon m'/2$ leaves, then we can embed T' in H', choosing any vertex as the image of z.*

Proof. Using Lemma 2.1, we find a stable set S in T' of $\lceil \frac{m'+1}{6} \rceil$ non-root vertices which are leaves or vertices of degree 2 whose parents are non-root vertices of degree 2, such that no child of a vertex in S is the parent of some other vertex of S. By the definition of S, for any vertex in $T' - S$ that has more than one child in S, all its children in S are leaves. Hence

$$\text{each vertex in } T' - S \text{ has at most } \varepsilon m'/2 \text{ children in } S. \tag{2}$$

So, we can choose a set $S' \subseteq S$ with $|S|/2 \leqslant |S'| \leqslant |S|/2 + \varepsilon m'/2$ such that no vertex of $S - S'$ is closer to z than any vertex of S', and such that for any given vertex in T', either all or none of its children in S belong to S'.

Since our assumption on ε ensures that

$$|S - S'| \geqslant \frac{m'+1}{12} - \frac{\varepsilon m'}{2} \geqslant 2\varepsilon(|V(H')| - 1),$$

the minimum degree of H' is large enough to allow us to use a good iterative construction process to greedily embed the component of $T' - N(S - S')$ that contains z and all of S'. (In particular, children of vertices in S' are embedded, but vertices from $S - S'$ and their parents are not.) We immediately unembed the vertices of S'. Note that $|T' - S'| \leqslant (1 - 2\varepsilon)(|V(H')| - 1)$, so it is possible to greedily embed the remainder of $T' - S'$. However, our plan is to embed the remainder of $T' - S'$, in a way that the vertices of S' can be embedded afterwards. So we do it cautiously.

Call a vertex $u \in V(H')$ *good* for a vertex $s \in S'$, if u is adjacent to both of the images of the two neighbours of s in T'. Let *Bad* be the set of all vertices $u \in V(H')$ with the property that

$$\text{there are more than } \frac{|S'|}{2} \text{ vertices in } S' \text{ for which } u \text{ is not good.}$$

Since for each vertex $s \in S'$ there are at most $4\varepsilon(|V(H')| - 1)$ vertices $u \in V(H')$ that are not good for s, it follows that there are at most $4\varepsilon(|V(H')| - 1)|S'|$ pairs $s \in S', u \in V(H')$ such that u is not good for s. Therefore,

$$|Bad| \leqslant 8\varepsilon(|V(H')| - 1). \tag{3}$$

We now proceed our embedding of $T' - S'$ in the following manner. When we are about to embed any vertex p which has one or more children in $S - S'$, we try to embed p into a vertex with many unused neighbours in *Bad*. Note that since each unused vertex of *Bad* has at most $2\varepsilon(|V(H')| - 1)$ non-neighbours, there are at most $4\varepsilon(|V(H')| - 1)$ vertices which see less than half of the unused vertices of *Bad*. So, as the image of the parent of p has more than

$$\begin{aligned}(1 - 2\varepsilon)(|V(H')| - 1) - |V(T') - \{p\} - S'| &\geqslant -\varepsilon(|V(H')| - 1) + |S'| \\ &\geqslant \frac{m'+1}{12} - \varepsilon(|V(H')| - 1) \\ &> 4\varepsilon(|V(H')| - 1)\end{aligned}$$

unused neighbours (here we use our upper bounds on ε and m'), we can embed p into a vertex which sees more than half of the unused vertices of Bad. We immediately embed all children of p in S trying to embed as many as possible into unused vertices of Bad. By (2), we will be able to embed all these children, unless Bad has less than $\varepsilon m'$ unused vertices. Hence as long as Bad has less than $\varepsilon m'$ unused vertices, we embed the vertices of $S - S'$ in Bad. But, since $|S - S'| \geqslant \frac{m'}{12} - \frac{\varepsilon m'}{2} \geqslant |Bad| - \varepsilon m'$, eventually Bad will have less than $\varepsilon m'$ unused vertices. After that, we embed all the vertices greedily. Doing so, when we finish the embedding of $T' - S'$, we have used up all but at most $\varepsilon m'$ vertices of Bad.

It remains to embed S'. Consider the auxiliary bipartite graph between S' and the set U of the so far unused vertices in H', i.e. the graph that has an edge su for $s \in S'$, $u \in U$, if u is good for s. By Hall's theorem, if we cannot embed S' in H', then in the auxiliary graph there is a (non-empty) set $W \subseteq S'$ whose neighbourhood is smaller than $|W|$. In other words, there is a subset $U_W \subseteq U$ such that $|U_W| < |W|$, and no vertex in $U - U_W$ is good for any vertex in W.

Because of our assumption on the minimum degree of H', we know that $|U - U_W| \leqslant 4\varepsilon(|V(H')| - 1)$. On the other hand, by the other assumptions of the lemma,

$$|U| \geqslant |S'| + \varepsilon(|V(H')| - 1), \tag{4}$$

and thus

$$|S' - W| < |S'| - |U_W| \leqslant |S'| - |U| + 4\varepsilon(|V(H')| - 1) \leqslant 3\varepsilon(|V(H')| - 1).$$

So $|W| \geqslant |S'|/2$ (because $|S'| \geqslant \frac{m'}{12} \geqslant 6\varepsilon(|V(H')| - 1)$), and therefore, $U - U_W \subseteq Bad$. Since U contains at most $\varepsilon m' < \varepsilon(|V(H')| - 1)$ vertices of Bad (as we used all other vertices of Bad earlier), we deduce from (4) that $|S'| < |U_W| < |W|$, a contradiction. So we can embed all of S' as planned. $\square$

Observe that in the previous proof, we could have embedded an even larger tree T' in H', if we knew that the set Bad could be filled up completely during the middle stage of the embedding, when we try to put as many vertices of $S - S'$ as possible into Bad. In fact, the term $\varepsilon(|V(H')| - 1)$ from (4) (which comes from the assumption that $m' \leqslant (1 - \varepsilon)(|V(H')| - 1)$) is only needed to make up for the unused vertices of Bad, in the inequality of the second-to-last line of the proof.

Under certain circumstances, we *can* fill up Bad completely, or up to a very small fraction. This is the content of the next two lemmas.

Lemma 4.4 *Let $0 < \varepsilon < \frac{1}{200}$, let H' be a graph with $m' + 1$ vertices of minimum degree at least $(1 - 2\varepsilon)m'$, and let v be a vertex of H' which sees all of $V(H') - v$. If T' is a tree with at most m' edges such that each vertex of T' is incident to at most $\varepsilon m'/2$ leaves, then we can embed T' in H'.*

Proof. Clearly we can assume T' is not a single vertex, so $\frac{\varepsilon m'}{2} \geqslant 1$. We repeatedly subdivide an edge from a leaf until T' has exactly m' edges. Clearly, it is enough to prove the result for such T'.

We will proceed very much as in the proof of Lemma 4.3, with two small differences. Firstly, we avoid v in our embedding throughout the process. As before, we stop right before reaching the parents of vertices in $S - S'$, and then unembed the vertices from S'. We define the set Bad as in the proof of Lemma 4.3, and observe that $|Bad| \leqslant 8\varepsilon|V(H')|$. The next step is a little different from the proof of Lemma 4.3: When embedding the rest of $T' - S'$, every time we consider the parent of a vertex in $S - S'$ we are happy if we embed at least half of its children in vertices of Bad. Since we always embed in a vertex which sees half of Bad, if we fail, then the current parent p has more children in $S - S'$ than there are vertices in Bad. In this case, we embed p in v (this is possible as v sees all of $V(H') - v$), and use up all the vertices of Bad for embedding the children of p in $S - S'$. Observe that we are bound to find such a vertex p, because $\frac{|S-S'|}{2} > |Bad|$. We then embed the rest of $T' - S'$ greedily.

Now continue as in the proof of Lemma 4.3, and embed S'. Note that although in (4), we only get $|U| \geqslant |S'|$ instead of $|U| \geqslant |S'| + \varepsilon|V(H')|$, we compensate for this shortcoming by having filled up all of Bad. Namely, from $U - U_W \subseteq Bad$ we can deduce that $U = U_W$, and thus obtain $|S'| = |U_W| < |W|$, a contradiction which shows that we can embed all of S' as planned. □

The next lemma goes one step further than the previous lemmas, embedding the tree in- and outside the dense subgraph.

Lemma 4.5 *For sufficiently small positive γ the following holds. Let T be a tree with m edges none of whose vertices is incident to more than γm leaves. Let H' be a subgraph of G with at most $m + 1 + 3\gamma m$ vertices such that (i) both H' and $G - H'$ have minimum degree at least $m - 3\gamma m$, and (ii) there is a vertex v of H' with degree at least m in G. Then we can embed T in G.*

Proof. We can assume that v does not see m vertices of H', as otherwise we are done by applying Lemma 4.4 to $N(v) \cap H'$. We let $a = m - |N(v) \cap H'|$, and note that v has at least $a \geqslant 1$ neighbours outside of H'. We embed a separator z for T into v.

If the sum s of the sizes of the a largest components of $T - z$ is at least $3\gamma m$, or if $T - z$ has less than a components, then we can choose some subset of these components that has between $3\gamma m$ and $\frac{m}{2}$ vertices. We embed these components greedily in $G - H'$, putting neighbours of z into neighbours of v, and then embed the rest of T greedily in H' (note we can do so because of condition (i) of the lemma), and are done.

So assume from now on that $T - z$ has at least a components and that

$$s \leqslant 3\gamma m. \tag{5}$$

Letting F be the union of the a largest components, all the components of $T - F - z$ have size at most $\frac{s}{a}$. In particular,

$$\text{no vertex of } T - F \text{ other than } z \text{ has degree exceeding } \frac{s}{a}. \tag{6}$$

Also note that

$$s \geqslant 2a, \tag{7}$$

since there are at most γm singleton components of $T - z$, and by (5), these cannot be part of F.

We embed F into $G - H'$ and proceed as in the proof of Lemma 4.4, to embed $T - F$ into $H' \cap N(v)$ with one important difference, which we explain momentarily.

As before, we stop right before reaching the parents of the vertices in $S - S'$, and then unembed the vertices from S'. We define the set Bad as in the proof of Lemma 4.4, and observe that $|Bad| \leqslant 24\gamma m$. We continue embedding the rest of $T' - S'$, and as in Lemma 4.4, every time we consider the parent p of a vertex in S we are happy if we embed at least half of its children in vertices of Bad. Let us call such a parent p a *happy parent*. Since we always embed in a vertex which sees half of the unused vertices of Bad, if we cannot embed at least half of the children of p in Bad, then we can use up half the currently unused vertices of Bad by embedding children of p. Let us call such a parent p an *unhappy parent*.

Next, we determine the size of the set of unused vertices of Bad at the end of this process. Observe that at least half of the vertices of $S - S'$ with happy parents get embedded in Bad, and thus, at most $2|Bad|$ vertices of $S - S'$ can have happy parents. So, at least $\frac{m(1-3\gamma)}{12} - \gamma m - 2|Bad| \geqslant \frac{m}{15}$ vertices of $S - S'$ have unhappy parents. Thus, by (6) there are at least $\frac{am}{15s}$ unhappy parents.

So, setting $r = \frac{m}{s}$, we see that the number of unused vertices of Bad at the end of the process is at most $24\gamma r s 2^{\frac{-ra}{15}}$. Since $a \geqslant 1$, and, by (5), r is at least $\frac{1}{3\gamma}$, if γ is sufficiently small then there are at most $\frac{s}{4}$ unused vertices of Bad left.

Now, note we are only embedding $|T - z - F| = m - s$ vertices into $N(v) \cap H'$, and the size of $N(v) \cap H'$ is at least $m - a$, which by (7) is at least $m - \frac{s}{2}$. This means we have more vertices in which to embed than vertices we need to embed even if we throw the unused vertices of Bad away. So, we can continue as in the proof of Lemma 4.3, and embed S'. $\square$

4.3 Filling Small Almost Complete Bipartite Subgraphs

This section has a similar aim as the previous section. Instead of small almost complete subgraphs, we now focus on small almost complete bipartite subgraphs of the host graph G.

We chose to start this subsection with the following lemma, because of the strong similarities of its proof with the proofs from the previous subsection.

Lemma 4.6 *Let $0 < \varepsilon < \frac{1}{200}$, let T' be a tree with m' edges such that each vertex of T' has at most $\frac{\varepsilon m'}{2}$ leaf children. Let (C, D) be the unique 2-colouring of T' with $|C| \leqslant |D|$. Let $H' = ((A, B), E)$ be a bipartite graph of minimum degree at least $(1-3\varepsilon)m'$ such that both A and B have at most $\lfloor(1+\varepsilon)m'\rfloor$ vertices, B has at least $|D|$ vertices, and A contains a vertex v which sees all of B. Then we can embed T' in H'.*

Proof. We can assume that $m' \geqslant \frac{2}{\varepsilon}$ or the tree must be a singleton and we are done. Because of the minimum degree condition on H', we can greedily embed T' unless $|C| < 3\varepsilon m' + 1 \leqslant 4\varepsilon m'$, so we assume this is the case. This implies that there are at least $(1-8\varepsilon)m'$ leaves of T' in D (for this, observe that rooting T' arbitrarily, every non-leaf vertex in D has at least one child in C). We let T'' be T' with these leaves removed. Our plan is to embed C in A and D in B, starting with T''.

We use a good embedding algorithm to begin embedding T'' in $H' - v$, starting with a vertex of C. We pause the procedure the first time that the set X of vertices embedded in A has edges to more than half of the vertices of $T' - T''$. We let L be the set of neighbours of X in $T' - T''$. Note that $(\frac{1}{2} - 4\varepsilon)m' \leqslant |L| < (\frac{1}{2} + \frac{1}{2}\varepsilon)m'$, by our assumption on the number of leaf children at each vertex.

Let $f(X)$ be the image of X. We assign each vertex x of X a weight w_x which is the number of vertices of L it is incident to. For every $X' \subseteq X$, we set $w(X') = \sum_{x \in X'} w_x$. Note that $w(X) = |L|$. Call a vertex $b \in B$ *bad* if there is a set $X' \subseteq X$ with $w(X') \geqslant \frac{|L|}{2}$ such that b has no neighbour in $f(X')$. We let Bad be the set of all bad vertices of B.

We claim that Bad contains at most $8\varepsilon m'$ vertices. Indeed, otherwise every vertex from $f(X)$ sees more than half the vertices of Bad. Consider the graph we obtain from blowing up each of the vertices $f(x) \in f(X)$ to a set $f'(x)$ of size w_x (together with all adjacent edges). Then it is still true that every vertex in the set $f'(X) := \bigcup_{f(x) \in f(X)} f'(x)$ sees more than half the vertices of Bad. So by double-edge counting we see that on average, each vertex of Bad sees more than half of the vertices of $f'(X)$. Thus in the original graph, each vertex of Bad sees, on average, a set $f(Y)$ with $w(Y) \geqslant \frac{|L|}{2}$. So there is at least one vertex in Bad actually seeing such a set $f(Y)$, contrary to the definition of Bad.

We shall now attempt to embed the remaining vertices of $T' - L$ so that we will be able to apply Hall's Theorem to finish the embedding by embedding L. For this, we embed the remaining $T' - L$ using all vertices of Bad, we proceed as follows. Embed the rest of $T' - L$ in a greedy fashion, with the precaution that whenever we embed a vertex of $T' - L$, we immediately embed all of its leaf children. Also, we avoid v for the time being. As in the proof of Lemma 4.4, we see that we can choose images for the vertices of C that see at least half of the unused vertices of Bad. Then we can embed half of the leaf children of each vertex x of C into vertices of Bad until we reach a vertex $c \in C$ which has more children than there are unused vertices of Bad. Since $|L| > 2|Bad|$, there is such a c. We embed c into v and fill up the unused vertices of Bad with the leaf children of c. We continue greedily to embed all of $T' - L$. Let $f(T' - L)$ be the image of $T' - L$.

Now, by Hall's Theorem, to embed L in $B \setminus f(T' - L)$, it is sufficient to prove that $w(X') \leqslant |N(X') \setminus f(T' - L)|$ for all subset X' of X. Let X' be a subset of X. If $w(X') \leqslant |L|/2$ then, since H has minimum degree at least $(1 - 3\varepsilon)m'$, $N(X') \setminus f(T' - L)| \geqslant (1 - 3\varepsilon)m' - m' + |L| \geqslant (\frac{1}{2} - 7\varepsilon)m' \geqslant (\frac{1}{4} + \frac{1}{4}\varepsilon)m' \geqslant |L|/2$. If $w(X') \geqslant |L|/2$, then since $Bad \subseteq f(T' - L)$, by definition of Bad, we have $N(X') \setminus f(T' - L) = B \setminus f(T' - L)$ and so $|N(X') \setminus f(T' - L)| \geqslant |L|$, because $|B| \geqslant |D|$. In both cases, $w(X') \leqslant |N(X') \setminus f(T' - L)|$. This completes the proof. □

The ideas for the proofs of the remaining lemmas in this (and the subsequent subsections) are substantially different (although we still use Hall's theorem). An important tool is Lemma 4.7, which is needed for Lemma 4.8 below, and also for Lemma 4.10 of Section 4.4.

For Lemma 4.7, let us introduce good orderings of parents. For a tree T and a subset L of its leaves, consider the set of parents P of L. Order the vertices of P as $p_1, \ldots, p_m$ so that p_i has at least as many leaf children as p_{i+1}. Call any such ordering a *good ordering* of P.

Lemma 4.7 *Let G be a graph with $\delta(G) \geqslant \frac{9m}{10}$, and let T be a tree with m edges such that no vertex of T is incident to more than $\frac{m}{6}$ leaves. Let L be a subset of the leaves of T such that $|L| \geqslant \frac{9m}{10}$. Suppose there is a good ordering $p_1, \ldots, p_a$ of the parents P of L, and an embedding of $T - L$ in G such that for each $i \leqslant \lfloor a/2 \rfloor$, we have*

$$|N(f(p_{2i-1})) \cup N(f(p_{2i}))| \geqslant m. \tag{8}$$

Then we can extend the embedding of $T - L$ to an embedding T in G.

Proof. First of all, note that since no vertex has more than $\frac{m}{6}$ leaf children, for any set $S \subseteq P$ containing at most one of p_{2i-1}, p_{2i}, for each $i \leqslant \lfloor a/2 \rfloor$,

$$\text{there are at most } \frac{m}{6} \text{ more leaves under } S \text{ than under } P - S, \tag{9}$$

where we write 'leaves under X' for leaves that are children of vertices in X.

We use Hall's theorem to show we can embed the vertices of L. For this, consider the auxiliary bipartite graph H spanned between the set P' that arises from blowing up the image of each $p \in P$ to a set A_p of size equal to the number of leaf children of p, and the set of unused vertices in G. For $a \in A_p$, the edge ab is present if p is adjacent to b.

A matching saturating P' shows we can complete the embedding, so assume there is no such matching. By Hall's theorem, there is a set $S' \subseteq P'$ with $|N_H(S')| < |S'|$. Because of (8), S' can only contain vertices from one of $A_{p_{2i-1}}, A_{p_{2i}}$, for each $i \leqslant$

$\lfloor a/2 \rfloor$, and so, by (9), we know that $|S'| \leqslant |P' - S'| + \frac{m}{6}$. Thus $|S'| \leqslant \frac{2m}{3}$. But, as $\delta(G) \geqslant \frac{9m}{10}$, and since for the embedding of $T - L$ we used at most $\frac{m}{10}$ vertices, it follows that $|N(S')| \geqslant |S'|$, a contradiction. □

We continue with an analogue of Lemma 4.3 for bipartite host graphs. For its proof, we will make use of Lemma 4.7.

Lemma 4.8 *Let $0 < \varepsilon < \frac{1}{200}$, let $H' = ((A, B), E)$ be a bipartite graph of minimum degree at least $(1 - \varepsilon)m'$ such that A has at most $\lfloor (1 + \varepsilon)m' \rfloor$ vertices and B has exactly this many vertices. Let T' be a tree with m' edges such that each vertex of T' has at most $\frac{m'}{6}$ leaf children. Then we can embed T' in H'.*

Proof. We let (C, D) be the unique 2-colouring of T' with $|C| \leqslant |D|$. Because of the minimum degree condition on H', we can greedily embed T' unless $|C| < \varepsilon m' + 1$, so we assume this is the case. Note that $|C| \geqslant 2$ as T' is not a star. Thus, we obtain $|C| < 2\varepsilon m'$. We also obtain that the set L of leaves of T' in D has size at least $(1 - 2\varepsilon)m'$ (for this, observe that rooting T' at a vertex of C, every non-leaf vertex in D has at least one child in C). Set $T'' := T' - L$. We will embed C in A and D in B. Consider a good ordering $c_1, \ldots, c_a$ of the parents of leaves in L. We want to embed T'' using an embedding f such that for every $i \leqslant \lfloor \frac{a}{2} \rfloor$, we have $|N_B(f(c_{2i-1})) \cup N_B(f(c_{2i}))| \geqslant m'$. Then we are done with Lemma 4.7.

As we embed T'', when we embed a vertex $c = c_i$ of C paired with a vertex $c' = c_{i\pm 1}$ which is already embedded, we choose as $f(c)$ an unused vertex with the largest number of neighbours in $B - N(f(c'))$. Let us next estimate how large this number of neighbours will be.

Note that $|N(f(c')) \cap B| \geqslant (1 - \varepsilon)m'$ (by the minimum degree condition on H'), and so, we have $|B - N(f(c'))| \leqslant 2\varepsilon m'$ (by our assumption on the size of B). Also, each vertex in $B - N(f(c'))$ misses at most $2\varepsilon m'$ vertices of A (again by the minimum degree condition). Therefore, straightforward double-counting of non-edges between A and $B - N(f(c'))$ gives that there is a set $A' \subseteq A$ containing at least half the vertices of A such that each vertex in A' misses at most

$$\frac{4\varepsilon^2 (m')^2}{|A|/2} \leqslant 16\varepsilon^2 m' \leqslant \varepsilon m'$$

vertices of $B - N(f(c'))$. (For the first inequality, observe that $m' \leqslant 2|A|$ because of the minimum degree condition.)

So, since we only embed $|C| \ll |A'|$ vertices in A, we will be able to choose an image $f(c)$ that sees all but at most $\lfloor \varepsilon m' \rfloor$ vertices of $B - N(f(c'))$. Then, $|N_B(f(c)) \cup N_B(f(c'))| \geqslant m'$ as desired. We thus find the desired embedding of T'' and hence of T. □

The next lemma is an analogue of Lemma 4.5 for bipartite graphs.

Lemma 4.9 *Let $0 < \varepsilon < \frac{1}{200}$ and let $H' = (A, B)$ be a bipartite subgraph of a graph G. Suppose H' has minimum degree at least $(1-\varepsilon)m'$, both A and B contain at most $(1+\varepsilon)m'$ vertices, A contains a vertex v which has degree at least m' in G, and every vertex of $G - H'$ sees at least $(1-2\varepsilon)m'$ vertices of $G - H'$. Let T' be a tree with m' edges such that each vertex of T' has at most $\frac{\varepsilon m'}{2}$ leaf children. Then we can embed T' in G.*

Proof. We let (C, D) be the unique 2-colouring of T' with $|C| \leqslant |D|$. Set $B' := N(v) \cap B$ and $a := m' - |B'|$. Since we cannot embed the tree into (A, B') using Lemma 4.6, we know that $|B'| < |D|$, and thus

$$|C| = m' + 1 - |D| \leqslant m' - |B'| = a.$$

We embed a separator z for T' into v. We will embed the leaf children of z at the end of the process, which we can do because of our degree bound on v. Let $K_1, \dots, K_\ell$ be the non-singleton components of $T' - z$. Every K_i contains a vertex of C, and thus $\ell \leqslant a$.

Since z is a separator, we know that

$$\big||D \cap V(K_i)| - |C \cap V(K_i)|\big| \leqslant \frac{m'-2}{2} \tag{10}$$

for all $i \leqslant \ell$. We let w_i be the root of K_i, i.e. the vertex of K_i adjacent to z. We will embed the roots w_i into neighbours of v in G and then embed the rest of the tree greedily in H'.

First suppose that v has at least a neighbours in A. Successively embed the roots w_i, in a way that ensures we can keep the embedding as balanced as possible at each step. This means that when we are about to embed w_i, we choose an image for w_i in either A or B, so that the larger colour class of K_i will be forced to be embedded in that set among A, B that when we finish our embedding will contain less of $\bigcup_{j<i} V(K_j)$. (If both A, B will contain the same number of vertices from $\bigcup_{j<i} V(K_j)$, for instance when $i = 1$, we just arbitrarily choose either A or B for embedding w_i.)

Next, embed greedily the remainder of the components K_i. This can be done since the way we embedded the roots w_i, together with (10), ensures that

$$\big||D \cap \bigcup_{j \leqslant i} V(K_j)| - |C \cap \bigcup_{j \leqslant i} V(K_j)|\big| \leqslant \frac{m'-2}{2}$$

for each $i \leqslant \ell$. Thus, throughout the embedding process of the K_i, we use at most $\frac{3m'}{4}$ vertices on each side A, B.

Now, if v has fewer than a neighbours in A, we attempt to perform the same procedure. If we run out of neighbours of v in A during the embedding of the roots

w_i, then we start to embed roots w_i which were to be embedded into A into $N(v)-H'$ (this is possible as v has degree at least m'). We will embed the corresponding K_i in $G-H'$, using the large minimum degree of $G-H'$. If at any point the total size of the components embedded in $G-H'$ exceeds $\frac{m'}{4}$, then we stop embedding roots w_i in $G-H'$. Instead, we embed the remaining w_i in B and the remaining K_i in H' (this is possible because of the minimum degree of H'). We will be able to embed the components whose roots are embedded in $G-H'$ because they have at most $\frac{3m'}{4}$ vertices and this graph has minimum degree at least $(1-2\varepsilon)m'$. □

4.4 Graphs Without Very Dense Subgraphs

The main result of this section is Lemma 4.11. It says that if, in the situation of Theorem 1.3, we cannot embed T in G, then either G is locally m-sparse (a situation we dealt with in Subsection 4.1), or G contains at least one clique or bipartite (m,δ)-dense subgraph (see below for the definition). In the Subsections 4.2 and 4.3, we saw how to use these subgraphs. Everything will be put together in the last part of our proof, in Subsection 4.5.

Let us now define the subgraphs we are looking for. A subgraph H of G is *clique* (m,α)*-dense* if it has at most $m+1$ vertices and minimum degree at least $(1-\alpha^{1/14})m$. A connected bipartite subgraph H of G is *bipartite* (m,α)*-dense* if it has minimum degree at least $(1-\alpha^{1/14})m$ and each side of its (unique) bipartition has at most m vertices.

We first treat the case that T has many leaves. For this case, we need to make use of Lemma 4.7 from Subsection 4.3.

Lemma 4.10 *For every sufficiently small $\alpha>0$ the following holds. Suppose G is a graph of minimum degree at least $(1-\alpha)m$ with no clique (m,α)-dense subgraph and no bipartite (m,α)-dense subgraph, and let T be a tree with at most m edges. If T has at least $(1-\alpha^{1/7})m$ leaves, but no vertex of T is incident to more than $\frac{m}{6}$ leaves, then we can embed T in G.*

Proof. Let L be the set of leaves of T and fix any good ordering $p_1,\dots,p_a$ of the parents of L. We claim that we can embed all of $T-L$ in G, via a good embedding f, while maintaining that, for each $i\leqslant\lfloor a/2\rfloor$, we have

$$|N(f(p_{2i-1}))\cup N(f(p_{2i}))|\geqslant m. \tag{11}$$

Then, Lemma 4.7 guarantees our partial embedding can be extended to an embedding of all of T. So we only need to prove we can find f satisfying (11).

For this, suppose that $p=p_j$ is the first vertex of $T-L$ that cannot be embedded without violating (11). Then there is an already embedded vertex $p'=p_{j\pm1}$ such that the pair p,p' violates (11) for any embedding of p. Let q be the parent of p, and let A be a subset of size $\lceil(1-\alpha-\alpha^{1/7})m\rceil$ of the unused neighbours of $f(q)$. (Note that there are that many unused neighbours of $f(q)$ because $|V(T-L)| \leqslant \alpha^{1/7}m+1$ by assumption.) Let B be the set of (used and unused) neighbours of $f(p')$. Since (11) is violated for any embedding of p, we know that $|B| \leqslant m$ and that

$$\text{every vertex in } A \text{ has degree less than } m. \tag{12}$$

Since $\delta(G) \geqslant (1-\alpha)m$, we have $|B| \geqslant (1-\alpha)m$, and also, since (11) is violated, every vertex of A has at least $(1-2\alpha)m$ neighbours in B. So, there is a set $B' \subseteq B$ of size at least $(1-\sqrt{2\alpha})|B|$ such that each vertex in B' has degree at least $(1-\sqrt{2\alpha})|A|$ into A. Note that $|B'| \geqslant (1-\sqrt{2\alpha})|B| \geqslant (1-\sqrt{2\alpha})(1-\alpha)m \geqslant (1-2\sqrt{\alpha})m$.

Assume for a contradiction that $A - B'$ has size at most $\alpha^{1/7}m$. Then every vertex of $A \cap B'$ has degree at least

$$(1-\sqrt{2\alpha})|A| - \alpha^{1/7}m \geqslant (1-\sqrt{2\alpha})(1-\alpha-\alpha^{1/7})m - \alpha^{1/7}m \geqslant (1-\alpha^{1/14})m$$

in $G[A \cap B']$. Hence $G[A \cap B']$ is clique (m,α)-dense, a contradiction.

Hence $A - B'$ has size at least $\alpha^{1/7}m$. Then $A \cap B' = \emptyset$, because the degree (in G) of any vertex $v \in A \cap B'$ would exceed

$$\begin{aligned} |A \cup B'| - 2\alpha m - \sqrt{2\alpha}|A| &\geqslant |B'| + \alpha^{1/7}m - 2\alpha m - \sqrt{2\alpha}m \\ &\geqslant (1-2\sqrt{\alpha})m + \alpha^{1/7}m - 2\alpha m - \sqrt{2\alpha}m \\ &\geqslant m. \end{aligned}$$

contradicting (12). So, the bipartite subgraph of G with sides $A - B'$ and $B' - A$ is bipartite (m,α)-dense, a contradiction. This proves the existence of an embedding satisfying (11), completing our proof.

□

We now use Lemma 4.10 together with Lemma 4.3 from the previous section to prove the main result of this section:

Lemma 4.11 *For every sufficiently small positive constant α, and $m \geqslant \alpha^{-2}$, the following holds for each tree T with at most m edges none of whose vertices has more than αm leaf children. If G is a graph of minimum degree at least $(1-\alpha)m$ that is not locally m-sparse and contains neither a clique (m,α)-dense subgraph nor a bipartite (m,α)-dense subgraph then we can embed T in G.*

Proof. If T has less than $m-1$ edges, then the tree obtained from T by adding a path of length $m - |V(T)|$ on any vertex of T also satisfies the hypothesis of the lemma. Thus, it suffices to prove the result for trees with $m-1$ or m edges. Henceforth, we assume that T has $m-1$ or m edges.

We choose α small enough to satisfy certain inequalities in the proof.

By Lemma 4.10, we may assume that

$$T \text{ has fewer than } (1-\alpha^{1/7})m \text{ leaves.} \tag{13}$$

We let H be the densest subgraph of G with at most $m+1$ vertices. We let $\delta := \delta(H)$ be the minimum degree of H, let a be its average degree and let w be some minimum degree vertex of H. Note that $a \geqslant \frac{m}{25}$, since G is not locally m-sparse. So, as $\delta > \frac{a}{2}$ (by our choice of H),

$$\delta \geqslant \frac{m}{50}. \tag{14}$$

Also,

$$\text{no vertex } y \text{ outside of } H \text{ sees more than } \delta+1 \text{ vertices of } H, \tag{15}$$

as otherwise $H - w + y$ contradicts our choice of H. Furthermore, we can assume that

$$\delta < (1-\alpha^{1/14})m \tag{16}$$

as otherwise H is a clique (m, α)-dense subgraph.

We apply Observation 2.3 to obtain a vertex z such that the largest component of $T-z$ has fewer than $m(1-\alpha^{1/3})$ vertices and every other component has fewer than $\alpha^{1/3}m+1$ vertices. We let F be a forest consisting of the union of some components of $T-z$ with between $\alpha^{1/3}m$ and $2\alpha^{1/3}m$ vertices. Note that since z has at most αm leaf children (by the assumptions of the lemma), and since $\frac{|F|-\alpha m}{2} \geqslant \alpha m$

$$z \text{ has at least } \alpha m \text{ non-neighbours in } F. \tag{17}$$

We embed z into w and the neighbours of z in F into $G-H$; this is possible because by (16) w has at least $\delta(G) - \delta \geqslant (\alpha^{1/14} - \alpha)m \geqslant 2\alpha^{1/3}m$ neighbours in $G-H$. We leave the remaining at least αm vertices of F to embed at the end of the process.

By (13), we know $T-F$ has fewer than $(1-\alpha^{1/7})m < (1-9\alpha^{1/3})m$ leaves. Hence, by Lemma 2.2, we can choose a subtree T' of $T-F$ containing z which has $2\lceil\alpha^{1/3}m\rceil + 2$ vertices and a perfect matching.

As we are about to explain, we claim that either

(i) there are $u, u' \in V(H)$ such that $d_H(u) \leqslant \delta + 3\alpha m$, and $N_H(u')$ contains a set A of $\lceil\delta - 4\alpha^{1/3}m\rceil$ vertices each of which sees at most $\delta + 7\alpha^{1/3}m$ vertices of H at least $\delta - 4\alpha^{1/3}m$ of which are in $N_H(u)$, or

(ii) we can construct an embedding of T' so that for every $x \in V(H)$ with $d_H(x) < \delta + 3\alpha m$, we have used at least $3\alpha m$ vertices outside the closed neighbourhood of x.

We will show that if (i) does not hold in H, then we can find an embedding as in (ii). To do so, we root T' at z and consider a good iterative construction process for T' into H in which (a) we embed the two vertices of each matching edge in consecutive iterations, and (b) we embed each vertex q in a randomly chosen unused element of $N(f(p(q)))$. Using our lower bound of α^{-2} on m, we shall prove that with positive probability for every $x \in V(H)$ with $d_H(x) < \delta + 3\alpha m$, we have used at least $3\alpha m$ vertices outside the closed neighbourhood of x.

So consider a vertex x such that $|N_H(x)| < \delta + 3\alpha m$. Let us first estimate the probability that for a fixed matching edge $e = \{v_1, v_2\}$ (of the perfect matching of T') which does not contain z, we embed the second endpoint v_2 of e outside $N_H(x)$. For this, we let A be the set of all vertices that are neighbours of the image of $p(v_1)$ and see at most $\delta + 7\alpha^{1/3}m$ vertices of H at least $\delta - 4\alpha^{1/3}m$ of which are in $N_H(x)$. Since we assume (i) does not hold (for $u = x$ and $u' = f(p(v_1))$), we know that $|A| < \lceil \delta - 4\alpha^{1/3}m \rceil$, while there are at least $\delta - |V(T') - v_1 - v_2| \geqslant \delta - 2\lceil \alpha^{1/3}m \rceil$ available possible images for the first endpoint v_1. Thus, irrespective of the embedding to this point, the probability that v_1 is embedded in a vertex outside A is at least $2\alpha^{1/3}$. Therefore, again irrespective of the embedding to this point, the probability we embed v_2 outside $N_H(x)$ is at least $4\alpha^{2/3}$. (For this, observe that every vertex outside A has at least $4\alpha^{1/3}m$ neighbours in $H - N_H(x)$ and that at least $2\alpha^{1/3}m$ of them are unused.) We have shown[1] that the number of non-neighbours of x used in the embedding is a random variable whose value dominates $\mathrm{Bin}(\lceil \alpha^{1/3}m \rceil, 4\alpha^{2/3})$, where the *binomial random variable* $\mathrm{Bin}(n, p)$ is the sum of n independent 0–1 random variables, each equal to 1 with probability p.

Thus the probability that there are less than $3\alpha m$ such non-neighbours is bounded from above by the probability that $\mathrm{Bin}(\lceil \alpha^{1/3}m \rceil, 4\alpha^{2/3})$ is less than $3\alpha m$. Chernoff's Bound (see [AS08, McD89]) states that for every $t \in [0, np]$,

$$\mathbf{Pr}\left(|\,\mathrm{Bin}(n,p) - np| > t\right) < 2\exp\left(-\frac{t^2}{3np}\right).$$

Hence the probability that the number of non-neighbours of x used in the embedding is less than $3\alpha m$ is less than $2\exp(-\alpha m/12)$.

Since the number of such vertices x (vertices with less than $\delta + 3\alpha m$ neighbours in H) is at most $m+1$, the probability that there is a vertex x with $|N_H(x)| < \delta + 3\alpha m$ such that less than $3\alpha m$ non-neighbours of x are used in the embedding is at most $(m+1) \times 2\exp(-\frac{1}{12}\alpha m) \leqslant 2(m+1)\exp(-\frac{1}{12}m^{1/2})$ because $m \geqslant \alpha^{-2}$. Since we assumed m to be sufficiently large (since it is at least α^{-2}), this is less than 1, and so there is an embedding as in (ii).

If we find an embedding as in (ii), then we can continue our good iterative construction process on the rest of $T - F$, always embedding in a vertex of H if possible. Clearly, we embed at least $\delta + 3\alpha m + 1$ vertices in H. At this point, making use of (15), we can greedily embed F in the unused vertices of $G - H$.

So we will from now on assume that (i) holds. Then, we can find a subset B of $\lceil \delta - 1.5\alpha^{1/6}m \rceil$ vertices of $N_H(u)$ each of which sees at least $\delta - 7\alpha^{1/6}m$ vertices

[1] We can decide for each matching edge e when we come to it, whether or not its second endpoint is in $N_H(x)$, and then choose the embedding of its two endpoints conditional on our decision. We can make this decision by considering a random variable z_e which is 1 with probability $4\alpha^{2/3}$. If $z_e = 1$ we do not put the second endpoint of e in $N_H(x)$, otherwise we may or may not put this second endpoint in $N_H(X)$. The z_e are independent.

of A. Indeed, otherwise there are at least

$$1.5\alpha^{1/6}m \cdot (|A| - (\delta - 7\alpha^{1/6}m)) \geqslant 1.5\alpha^{1/6}m \cdot 3\alpha^{1/6}m \geqslant 4.5\alpha^{1/3}m^2$$

non-edges between A and $N_H(u)$, but the way A was chosen allows for at most

$$|A| \cdot (|N_H(u)| - (\delta - 4\alpha^{1/3}m)) \leqslant \lceil \delta - 4\alpha^{1/3}m \rceil \cdot (4\alpha^{1/3}m + 3\alpha m) < 4.5\alpha^{1/3}m^2$$

such non-edges (here, we use that $\delta < m$ by (16)). Clearly every vertex of A sees at least $\delta - 7\alpha^{1/6}m$ vertices of B.

Let us recapitulate the situation as follows. We found sets $A, B \subseteq V(H)$ such that

$$|A| = \lceil \delta - 4\alpha^{1/3}m \rceil, \ |B| = \lceil \delta - 1.5\alpha^{1/6}m \rceil \tag{18}$$

and

$$\text{the minimum degree from } A \text{ to } B \text{ and from } B \text{ to } A \text{ is at least } \delta - 7\alpha^{1/6}m. \tag{19}$$

Case 1: $A - B$ and $B - A$ both have size at least $25\alpha^{1/6}m$.

Let $[A - B, B - A]$ denote the bipartite subgraph of G spanned by the edges between $A - B$ and $B - A$. Then

$$[A - B, B - A] \text{ has minimum degree at least } 17\alpha^{1/6}m. \tag{20}$$

Furthermore, each vertex of $A \cap B$ sees at least $|B| - 7\alpha^{1/6}m + |A - B| - 7\alpha^{1/6}m \geqslant |B| + 11\alpha^{1/6}m$ vertices of $A \cup B$, and thus,

$$\text{each vertex of } A \cap B \text{ sees at least } \delta + 9\alpha^{1/6}m \text{ vertices of } A \cup B. \tag{21}$$

By (13), $T - F$ has fewer than $(1 - \alpha^{1/7})m$ leaves, and by definition $|T - F|2\alpha^{1/3}$. Hence, $T - F$ has fewer than $|T - F| - 33\alpha^{1/6}m$ leaves. So, by Lemma 2.2, we can find a subtree T^* of $T - F$ with $2\lceil 16\alpha^{1/6}m \rceil$ vertices which contains z and has a perfect matching and hence a 2-colouring with colour classes of equal size. Using (20), we embed T^* into $[A - B, B - A]$, with z in $A - B$.

We claim that at this point, for every vertex x of $A \cup B$ with less than $\delta + \alpha^{1/6}m$ neighbours in $A \cup B$,

$$\text{we have embedded at least } 8\alpha^{1/6}m \text{ vertices in non-neighbours of } x. \tag{22}$$

For this, it suffices to observe that $x \notin A \cap B$ by (21), and if $x \in A$, say, then we embedded at least $16\alpha^{1/6}m$ vertices in $A - B$, but x only sees at most $\delta + \alpha^{1/6}m - d_B(x) \leqslant 8\alpha^{1/6}m$ of these. (Here we used (19) for the bound on $d_B(x)$.)

We continue embedding $T - F$ into $H[A \cup B]$ until we have embedded at least $\delta + \alpha^{1/6}m + 1$ vertices into it, which we can do because of (19) and (22). By definition of F, z has at most $2\alpha^{1/3}m$ neighbours in F. We can embed these into $G - V(H)$, since $f(z)$ has at least

$$\delta(G) - \delta > (1-\alpha)m - (1-\alpha^{1/14})m \geqslant 2\alpha^{1/3}m$$

neighbours outside H (we used (16) for the first inequality). We can then complete greedily the embedding of $T - F$ as at least αm vertices of F have not yet been embedded by (17). Finally, complete the embedding of F in $G - V(H)$; this is possible because in $G - V(H)$ every vertex has degree at least $(1-\alpha)m - \delta - 1$ by (15), and at most $m - \delta - \alpha^{1/6}m$ vertices of T are embedded in $G - V(H)$.

Case 2: One of $A - B$ or $B - A$ has size at most $25\alpha^{1/6}m$.

Since by (19), each vertex of A misses at most $7\alpha^{1/6}m$ vertices of B, and vice versa, $G[A \cap B]$ has minimum degree at least $|A \cap B| - 7\alpha^{1/6}m$. We consider a largest induced subgraph H' of G with at most $m+1$ vertices and at most $7\alpha^{1/6}m|V(H')|$ non-adjacent pairs of vertices, chosen so as to maximize the number of edges in H'. So, if H' has minimum degree δ' then

$$\text{every vertex outside } H' \text{ has degree at most } \delta' + 1 \text{ in } H'. \tag{23}$$

Note that since $G[A \cap B]$ is one possible choice for H',

$$|V(H')| \geqslant |A \cap B| \geqslant \min\{|A|, |B|\} - 25\alpha^{1/6}m \geqslant \delta - 27\alpha^{1/6}m > \frac{m}{100}, \tag{24}$$

where we used (18) in the second-to-last inequality and (14) in the last one. We obtain a subgraph H^* of H' by iteratively deleting vertices which are non-adjacent to more than $\frac{\alpha^{1/13}m}{3}$ vertices in the current subgraph. Then the minimum degree m^* of H^* is bounded by

$$\delta^* \geqslant |V(H^*)| - \frac{\alpha^{1/13}m}{3}. \tag{25}$$

Clearly we delete at most $\frac{\alpha^{1/13}m}{10}$ vertices, that is,

$$|V(H')| - |V(H^*)| \leqslant \frac{\alpha^{1/13}m}{10}. \tag{26}$$

If $|V(H^*)|$ exceeds $(1-\alpha^{1/13})m$ then as H^* has minimum degree at least $\delta^* \geqslant |V(H^*)| - \frac{\alpha^{1/13}m}{3} \geqslant (1-\alpha^{1/14})m$, we obtain that H^* is an (m,α)-dense clique, contradicting our assumption that no such exist. So we can assume that

$$|V(H^*)| \leqslant (1-\alpha^{1/13})m. \tag{27}$$

Observation 2.3 implies we can choose a vertex z^* of T such that the largest component of $T - z^*$ contains at most $(1-\frac{\alpha^{1/13}}{2})m$ vertices and every other component of $T - z^*$ contains fewer than $\frac{\alpha^{1/13}m}{2}$ vertices. We choose a smallest possible forest F^* consisting of the union of components of $T - z^*$ whose total size is between $\frac{\alpha^{1/13}m}{2}$ and $\alpha^{1/13}m$. We note that since z^* is incident to at most αm leaves,

$$F^* \text{ contains at least } \frac{\alpha^{1/13}m}{6} \text{ non-neighbours of } z^*. \tag{28}$$

First suppose δ' (the minimum degree of H') is at most $|V(H')| - 1 - \frac{2\alpha^{1/13}m}{3}$. We use a good iterative construction process to embed $T - F^*$ into G with z^* in a vertex of H^* and using vertices of H^* when possible. By (25), we use at least $|V(H^*)| - \frac{\alpha^{1/13}m}{3} + 1$ vertices of H^* before embedding any of $T - F^*$ outside H^*. When we are about to first embed a vertex outside of H^*, we proceed as follows.

We start by embedding the neighbours of z^* in F^* into $G - V(H')$. Observe that this can be done, since because of (28), we know that z^* has at most $\frac{5}{6}\alpha^{1/13}m$ neighbours in F^*, while $f(z^*)$ has at least

$$(1-\alpha)m - |V(H^*)| - |V(H' - H^*)| \geqslant (\frac{9}{10}\alpha^{1/13} - \alpha)m$$

neighbours in $G - H'$ (here, we used (26) and (27)). Then we finish our embedding of $T - F^*$, just using the minimum degree of G. Finally, we embed the rest of F^* in $G - V(H')$, using (23), our assumption on δ', and the fact that we used at least $|V(H^*)| - \frac{\alpha^{1/13}m}{3} + 1$ vertices of H^*.

So we can assume that $\delta' \geqslant |V(H')| - 1 - \frac{2\alpha^{1/13}m}{3}$. Since H' is not an (m, α)-dense clique, it follows that $|V(H')| \leqslant m(1 - \frac{\alpha^{1/14}}{2})$. We choose (the unique value of) ε such that $\delta' = (1 - 2\varepsilon)(|V(H')| - 1)$. Choose a subtree T' of $T - F^*$ with $m' = (1 - \varepsilon)(|V(H')| - 1)$ edges that contains z^* and subject to this has as few leaves as possible. We note that this implies if a vertex of T' has two leaf children then all its leaf children are also leaves of $T - F^*$.

If no vertex of T' is incident to more than $\frac{\varepsilon m'}{2}$ leaves then Lemma 4.3, with $\varepsilon := (\frac{\alpha^{1/13}}{3})(\frac{m}{|V(H')|-1})$, ensures that we can embed T' in H' with z^* embedded in a vertex of minimum degree in H'. Note that for the application of Lemma 4.3, we use that because of (24) and the fact that we can make m as large as we want by making α small, we know that $\frac{m}{|V(H')|-1}$ is at most 101, ensuring that ε is sufficiently small. When we stop there are at most $\varepsilon m'$ unused vertices of H'.

If some vertex of T' is incident to more than $\frac{\varepsilon m'}{2}$ leaf children then all but one of these leaves are also leaves of T. So, by hypothesis, $\frac{\varepsilon m'}{2} < \alpha m + 2$. In this case,we just use a good iterative construction process to embed as much of T' into H' as possible where to begin we embed z^* in a minimum degree vertex of H'. When we stop there are at most $2\varepsilon m' < 4\alpha m + 8$ unused vertices of H'.

In either case, as above, we then embed all of the neighbours of z^* in F^* into $G - H'$ which we can do because of our upper bound on the size of $|V(H)|$. We then finish our embedding of $T - F^*$, just using the minimum degree of G. For the embedding of the rest of F, it is enough to observe that by our chocie of H', every vertex of $V(G) - V(H')$ misses at least $max(7\alpha^{1/6}m, 2\varepsilon m' - 1)$ vertices of H' and the number of unused vertices of H' is at most $max(\varepsilon m', \alpha m + 8)$. □

4.5 Finishing Things Off

In this section we prove Theorem 1.3. We choose $\alpha < 1/200^{15}$ sufficiently small so that Lemma 4.11 holds, and that other inequalities implicitly given in this section hold. We choose $\gamma = \alpha^2$. Note that we can assume $m \geqslant \frac{1}{\gamma} = \frac{1}{\alpha^2}$ as otherwise the graph has minimum degree greater than $m - 1$ so at least m, and we can just greedily embed T. We can also assume that no vertex has γm or more leaf children as otherwise we can embed this vertex in a maximum degree vertex, greedily embed the tree except for its leaf children and then greedily embed these children.

By Lemmas 4.1 and 4.11, we may assume G contains a clique or bipartite (m, α)-dense subgraph. For a clique (m, α)-dense subgraph D of G, by an *expansion of D* we mean a graph H obtained by iterately adding vertices (one at a time) of $G - V(D)$ which see at least $(1 - \alpha^{1/15})m$ vertices of the current expansion. For a bipartite (m, α)-dense subgraph $D = (A, B)$ of G, by an *expansion of D* we mean a graph $H = (A', B')$ obtained by iterately adding one at a time vertices v of $G - V(D)$ which see at least $(1 - \alpha^{1/15})m$ vertices of one of the sides of the current expansion; we then add v to the other side, and forget about all edges from v to this side. A *maximal expansion of D* is an expansion H as defined above of maximal size.

G contains an expansion H of a clique (m, α)-dense subgraph with $|V(H)| = 1 + \lceil (1 - \alpha^{1/15})^{-1} m \rceil$, then we can embed T within it, by Lemma 4.3, with $\varepsilon := \alpha^{1/15} < \frac{1}{200}$. (For this, observe that the minimum degree of H is at least $\lceil (1 - \alpha^{1/15})m \rceil \geqslant (1 - 2\varepsilon)(|V(H)| - 1)$, while the number of edges of the tree T is $m \leqslant (|V(H)| - 1)(1 - \varepsilon)$.) So we can assume for all expansions H of clique (m, α)-dense subgraphs of G we have

$$|V(H)| < 1 + (1 - \alpha^{1/15})^{-1} m. \tag{29}$$

Similarly, if G contains an expansion $H = ((A', B'), E')$ of a bipartite (m, α)-dense subgraph $D = ((A, B), E)$ with $\max\{|A'|, |B'|\} = \lfloor (1 + \alpha^{1/15})m \rfloor$ then we can embed T within it, by Lemma 4.8. So we can assume for each expansion $H = ((A', B'), E')$ of every bipartite (m, α)-dense subgraph of G we have

$$\max\{|A'|, |B'|\} \leqslant (1 + \alpha^{1/15})m. \tag{30}$$

We will show below that if we cannot embed T, then for each maximal expansion H of a clique or bipartite (m, α)-dense subgraph of G, it holds that

(A) no vertex of $G - H$ sees more than $2\gamma m$ vertices of H, and

(B) no vertex of H sees more than $2\gamma m$ vertices of $G - H$.

Now, assuming (A) and (B) hold, we consider a maximal sequence $D_1, \dots, D_\ell$ of clique and bipartite (m,α)-dense subgraphs of G, together with corresponding maximal expansions $H_1, \dots, H_\ell$. More precisely, we choose D_i as a clique or bipartite (m,α)-dense subgraph of $G - \bigcup_{j<i} H_j$, and let H_i be its maximal expansion in $G - \bigcup_{j<i} H_j$. Note that D_i is clique or bipartite (m,α)-dense in G and by (B), applied to the graphs H_j with $j < i$, we know that H_i is also a maximal expansion of D_i in G.

We will show below that moreover, if we cannot embed T, then

(C) no vertex of $V(G) - \bigcup_{i=1}^{\ell} H_i$ sees more than $10\gamma m$ vertices of $\bigcup_{i=1}^{\ell} H_i$.

Thus, if $G - \bigcup_{i=1}^{\ell} H_i$ is non-empty then by Lemmas 4.1 and 4.11, we can embed T within it. So, choosing a vertex $v \in V(G)$ of maximal degree, we can assume that v is contained in one of the H_i.

Now if v is in the expansion of a bipartite (m,α)-dense subgraph, then Lemma 4.9, together with (30) and (A), tells us that we can embed T. So we can assume v is in the expansion H of a clique (m,α)-dense subgraph. If $|V(H)| \leqslant 1 + (1+3\gamma)m$, then Lemma 4.5, together with (A) and (B), gives an embedding of T in G. So $|V(H)| \geqslant 1 + (1+3\gamma)m$. Setting $\varepsilon := \frac{|V(H)|-1-m}{|V(H)|-1}$ we see that (B) guarantees that the minimum degree of H is at least $(1-\gamma)m - 2\gamma m = (1-3\gamma)m \geqslant (1-2\varepsilon)(|V(H)|-1)$. Furthermore our upper bound (29) on the size of expansions ensures $\varepsilon \leqslant \alpha^{1/15} < \frac{1}{200}$. Finally our lower bound on $V(H)$, ensures that for sufficiently small γ, $\varepsilon = 1 - \frac{m}{|V(H)|-1} \geqslant 1 - \frac{1}{1+3\gamma} = \frac{3\gamma}{1+3\gamma} \geqslant 2\gamma$. Hence we can embed T using Lemma 4.3.

This completes the proof of the theorem. It only remains to show (A), (B) and (C).

To prove (A) we consider the expansion H of some clique or bipartite (m,α)-dense subgraph D of G. Note that by the definition of an expansion, $G - H$ has minimum degree at least $(\alpha^{1/15} - \gamma)m$. Let w be a vertex outside of H with maximum degree into H. Let d be the number of its neighbours in H and assume for a contradiction that $d > 2\gamma m$.

Our plan is to find an embedding of T in G, here is an outline of the proof. We distinguish between two cases: First, we treat the case that d is relatively large (almost $m/2$ or larger). In this case we embed a suitable vertex z of T in w, a few small components of $T - z$ outside of H, and the main part of T in H. The other case is that d is rather small (between $2\gamma m$ and almost $m/2$). In that case, we embed a suitable vertex z of T in w, and embed into H a set $\mathcal{C}$ of components of $T - z$ whose union contains a little bit more than d, namely $d + \gamma m$, vertices. This is possible since $T - z$ has at most γm singleton components, and so the number of neighbours of z in $\bigcup \mathcal{C}$ is at most d. We then embed the rest of T outside H. Let us now turn to the details of this plan.

Case 1: $d > (\frac{1}{2} - \frac{\alpha^{1/15}}{6})m$.

In this case we choose a vertex z of T such that the largest component of $T - z$ has at most $(1 - \frac{\alpha^{1/15}}{3})m + 1$ vertices and every other component has fewer than $\frac{\alpha^{1/15}m}{3}$ vertices (this is possible by Observation 2.3). We embed z into w. We choose some components including all the (at most γm) singleton components, so that the union of these components has between $\frac{\alpha^{1/15}m}{3}$ and $\frac{2\alpha^{1/15}m}{3}$ vertices. We embed these components greedily into $G - H$. Since the remaining components of $T - z$ each have at least two vertices, there are at most $\frac{1}{2}(1 - \frac{\alpha^{1/15}}{3})m < d$ of them. We embed the roots (neighbours of z) of these components into neighbours of w in H, preferring vertices of D.

We then proceed to embed greedily into H all those components of $T - z$ whose root was embedded in $H - D$. If such components exist, then, since there are at most $4\alpha^{1/15}m$ vertices in $H - D$ (at most $((1 - \alpha^{1/15})^{-1} + \alpha^{1/14})m \leqslant 3\alpha^{1/15}m$ if D is clique dense by (29), and at most $4\alpha^{1/15}m$ if D is bipartite dense by (30)), and since we preferred vertices of D for putting down the roots, we must have embedded at least $d - 4\alpha^{1/15}m \geqslant \frac{m}{3}$ roots of other components into D. So, as we already got rid of singleton components, there are at least $\frac{m}{3}$ vertices in components whose root is in D which we are not yet embedded. Thus, the minimum degree of H, which is $(1 - \alpha^{1/15})m$, is sufficient for embedding all components with roots in $H - D$. Finally, we embed all those components whose root was embedded in D. For this, observe that being a (m, α)-dense subgraph, D has minimum degree $(1 - \alpha^{1/14})m$, which is sufficient for embedding the rest of T (since at least $\frac{\alpha^{1/15}m}{3}$ vertices of T were already embedded outside H).

Case 2: $2\gamma m < d \leqslant (\frac{1}{2} - \frac{\alpha^{1/15}}{6})m$.

Then $G - H$ has minimum degree

$$\delta(G - H) \geqslant m - d - \gamma m. \tag{31}$$

We choose a vertex z of T such that the largest component C_{max} of $T - z$ has fewer than $1 + m - d - \gamma m$ vertices and every other component has at most $d + \gamma m$ vertices (possible by Observation 2.3), and embed z into w. Let $\mathcal{C}_{T-z}$ be the set of components of $T - z$. Take a smallest set $\mathcal{C} \subseteq \mathcal{C}_{T-z} - \{C_{max}\}$ with

$$d + \gamma m \leqslant |\bigcup_{C \in \mathcal{C}} V(C)|. \tag{32}$$

Clearly,

$$|\bigcup_{C \in \mathcal{C}} V(C)| \leqslant 2d + 2\gamma m. \tag{33}$$

We claim that moreover,

$$\text{if } \mathcal{C} \text{ has singleton components, then } |\bigcup_{C \in \mathcal{C}} V(C)| \leqslant 2d - \lfloor \gamma m \rfloor. \tag{34}$$

In order to see (34) suppose that $|\bigcup_{C\in\mathcal{C}} V(C)| \geqslant 2d - \lfloor \gamma m \rfloor + 1$. We need to show that $\mathcal{C}$ has no singleton components. For this, it suffices to observe that by the minimality of $\mathcal{C}$, for each component $C^* \in \mathcal{C}$ we have that $|\bigcup_{C\in\mathcal{C}, C\neq C^*} V(C)| \leqslant \lceil d + \gamma m \rceil - 1$. So

$$|V(C^*)| \geqslant 2d - \lfloor \gamma m \rfloor + 1 - (\lceil d + \gamma m \rceil - 1) = d - \lfloor \gamma m \rfloor - \lceil \gamma m \rceil + 2 > 1,$$

where for the last inequality we apply our hypothesis that $d > 2\gamma m$.

Next, we wish to show that

$$|\mathcal{C}| \leqslant d. \tag{35}$$

If $\mathcal{C}$ has singleton components, then there are at most $\lfloor \gamma m \rfloor$ such components, and (35) follows from the fact that by (34),

$$|\mathcal{C}| \leqslant \lfloor \gamma m \rfloor + \frac{|\bigcup_{C\in\mathcal{C}} V(C)| - \lfloor \gamma m \rfloor}{2} \leqslant \lfloor \gamma m \rfloor + \frac{2d - 2\lfloor \gamma m \rfloor}{2} = d.$$

If $\mathcal{C}$ has no singleton components, and additionally, $|\bigcup_{C\in\mathcal{C}} V(C)| \leqslant 2d$, then $|\mathcal{C}| \leqslant \frac{|\bigcup_{C\in\mathcal{C}} V(C)|}{2} \leqslant d$, as desired, so let us now assume that $|\bigcup_{C\in\mathcal{C}} V(C)| > 2d$. Then $|\mathcal{C}| \leqslant 3$, as otherwise the set $\mathcal{C}'$ obtained from $\mathcal{C}$ by deleting the smallest component satifies $|\bigcup_{C\in\mathcal{C}'} V(C)| > \frac{3}{4} \cdot 2d > d + \gamma m$ (since $d > 2\gamma m$), contradicting the minimality of $\mathcal{C}$. Moreover, since $d > 2\gamma m \geqslant 2$, we know that $d \geqslant 3$. Thus again, $|\mathcal{C}| \leqslant d$. This completes the proof of (35).

We now embed $T - z$. By (31) and by (32), the minimum degree of $G - H$ is large enough to greedily embed into $G - H$ all the components of $T - z$ that are not in $\mathcal{C}$. Next, we embed the (by (35) at most d) roots of the components from $\mathcal{C}$ into H, as above preferring vertices in D over vertices in $H - D$. We then embed all components whose root was put into $H - D$, and finally embed the components with root embedded in D. In order to see that we succeed in embedding all of T, we argue similarly as in the previous case: For the components with root in $H - D$, note that again, we must have embedded at least $d - 4\alpha^{1/15}m$ roots of non-singleton components into D, so, unless $d - 4\alpha^{1/15}m < \alpha^{1/15}m$, we can argue as above that the minimum degree of H is sufficient. On the other hand, if $d - 4\alpha^{1/15}m < \alpha^{1/15}m$, that is, if $d < 5\alpha^{1/15}m$, then by (33),

$$|\bigcup_{C\in\mathcal{C}} V(C)| \leqslant 2d + 2\gamma m < 10\alpha^{1/15}m + 2\gamma m,$$

so again, the minimum degree of H is sufficient. For the components with root in D, note that as above, the minimum degree of D is sufficient for embedding them because by (33) at least

$$m - |\bigcup_{C\in\mathcal{C}} V(C)| \geqslant m - 2d - 2\gamma m \geqslant \frac{\alpha^{1/15}m}{3} - 2\gamma m \geqslant \alpha^{1/14}m$$

vertices of T were already embedded outside H. This completes the proof of Case 2, and thus of (A).

To prove (B) we consider the expansion H of some clique or bipartite (m, α)-dense subgraph D of G. We let w be a vertex of H which has maximum degree $d_{G-H}(w)$ outside of H and set

$$d := \min\{d_{G-H}(w), \alpha^{1/15}m\}.$$

Then, H has minimum degree at least $m - d - \gamma m$ (this is clear if $d = d_{G-H}(w)$, and follows from the fact that H is an expansion in the case that $d = \alpha^{1/15}m$). We embed a separator z for T into w. We choose a minimal set $\mathcal{C}$ of components of $T - z$ containing at least $d + \gamma m$ vertices. Since at most γm of these components are singletons, we need at most $\frac{d}{2} + \gamma m < d$ components. Furthermore, their total size is at most $\frac{m}{2}$ (as the size of the components of $T - z$ is bounded by this number, since z is a separator). The minimum degree of H is clearly enough to greedily embed into H all those components of $T - z$ that are not in $\mathcal{C}$. We then embed the components from $\mathcal{C}$ into $G - H$. We embed the (at most d) neighbours of z first. After that, the minimum degree of $G - H$ (which, by (A), is at least $(1 - \gamma - 2\gamma)m \geqslant \frac{m}{2} \geqslant \bigcup_{C\in\mathcal{C}} |V(C)|$) ensures we can embed the remainder of the components from $\mathcal{C}$. This completes the proof of (B).

It remains to prove (C). We shall do so by inductively proving that if we cannot embed T, then for every j between 1 and ℓ,

(C') no vertex of $V(G) - \bigcup_{i=1}^{j} H_i$ sees more than $10\gamma m$ vertices of $\bigcup_{i=1}^{j} H_i$.

For $j = 1$, (C') holds by (A). Assuming (C') holds for $j - 1$, let us show that (C') also holds for j. By (A), no vertex of $V(G) - \bigcup_{i=1}^{j} H_i$ sees more than $10\gamma m + 2\gamma m = 12\gamma m$ vertices of $\bigcup_{i=1}^{j} H_i$. Thus, $G - \bigcup_{i=1}^{j} H_i$ has minimum degree at least $m - 13\gamma m$.

Suppose there is a vertex $w \in V(G) - \bigcup_{i=1}^{j} H_i$ which sees at least $10\gamma m$ vertices in $\bigcup_{i=1}^{j} H_i$. Our aim to show that we can then embed T. We embed a separator z for T into w. We choose a minimal set $\mathcal{C}$ of components of $T - z$ containing at least $13\gamma m$ vertices. Since at most γm of the components in $\mathcal{C}$ are singletons, we know that $|\mathcal{C}| \leqslant 7\gamma m$. Furthermore, $\bigcup_{C\in\mathcal{C}} |V(C)| \leqslant \frac{m}{2}$. We then embed the components of $T - z$ that are not in $\mathcal{C}$ greedily into $G - \bigcup_{i=1}^{j} H_i$, using the minimum degree of $G - \bigcup_{i=1}^{j} H_i$. Finally, we embed the components from $\mathcal{C}$ into $\bigcup_{i=1}^{j} H_i$, embedding the (at most $7\gamma m$) neighbours of z first, and using the minimum degree of the H_i for the rest of these components (this works since we embedded at least $\frac{m}{2}$ vertices outside of $\bigcup_{i=1}^{j} H_i$). This shows (C'), and thus completes the proof of (C).

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