Sofic (and Effective) Subshifts on f.g. Groups

Lecture 1: Symbolic Dynamics on f.g. groups: a computational approach.

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Introduction

Mini-course divided into 4 lectures

- Lecture 1: SD on f.g. groups: a computational approach.
- Lecture 2: Domino Problem, Part I: Wang tiles.
- Lecture 3: Domino Problem, Part II: f.g. groups.
- Lecture 4: Effective subshifts.

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- **Lecture 1: SD on f.g. groups: a computational approach.**
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Lecture 1: Symbolic Dynamics on f.g. groups: a computational approach.

Why subshifts on groups ?

From a computer scientist point of view:

- $\mathbb{Z}^2$-subshifts as a computational model.
- Decidability gap between $\mathbb{Z}$-subshifts and $\mathbb{Z}^2$-subshifts
- Understand where is the limit: study subshifts on other structures.
- Preserve the duality dynamical/combinatorial approach.

Why finitely generated groups ?

Two restrictions: **finitely generated** (f.g.) and **recursively presented** (r.p.) groups.

- Understand computational properties of SFTs/sofic subshifts.
- We need a finite encoding/description of the group.
- How to encode computation inside SFTs ?

Configurations and Subshits (I)

- Let A be a finite alphabet, G be a finitely generated group.
- Colorings $x : G \to A$ are called **configurations**.
- Endowed with the prodiscrete topology A^G is a **compact** and **metrizable** set.
- **Cylinders** form a clopen basis

$$[a]_g = \left\{ x \in A^G \mid x_g = a \right\}.$$

- A **pattern** is a finite intersection of cylinders, or equivalently a finite configuration $p : S \to A$
- A **metric** for the cylinder topology is

$$d(x, y) = 2^{-\inf\{|g| \mid g \in G:\, x_{\mathbf{g}} \neq y_{\mathbf{g}}\}},$$

where $|g|$ is the length of the shortest path from 1_G to g in $\Gamma(G, S)$.

Configurations and Subshits (II)

The **shift** action $\sigma : G \times A^G \to A^G$ is given by

$$(\sigma_g(x))_h = x_{g^{-1}h}.$$

The dynamical system (A^G, σ) is called the **G-fullshift over A**.

Definition

A **G-subshift** is a closed and σ-invariant subset $X \subset A^G$.

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A **G-subshift** is a closed and σ-invariant subset $X \subset A^G$.

A pattern $p \in A^S$ **appears** in a configuration $x \in A^G$ if $(\sigma_g(x))_S = p$ for some $g \in G$.

Proposition

X is a G-subshift iff there exists a set $\mathcal{F}$ of forbidden patterns s.t.

$$X = X_{\mathcal{F}} := \left\{x \in A^G \mid \text{ no pattern of } \mathcal{F} \text{ appears in } x\right\}.$$

G-SFT, block maps and sofic G-subshifts

A **block map** $\phi : A^G \to B^G$ is a continuous and σ-commuting map.

- A G-subshift X is **of finite type** (G-SFT) if there exists a finite set of forbidden patterns $\mathcal{F}$ that defines it: $X = X_{\mathcal{F}}$.
- A G-subshift X is **sofic** if there exists a G-SFT Y and a block map ϕ s.t. $X = \phi(Y)$.

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Proposition

If a G-subshift X is sofic, then there exists a nearest neighbor SFT Y and a letter-to-letter block map ϕ s.t. $X = \phi(Y)$.

Remark: These notions of G-SFT and sofic G-subshifts do not depend on the presentation of the group G.

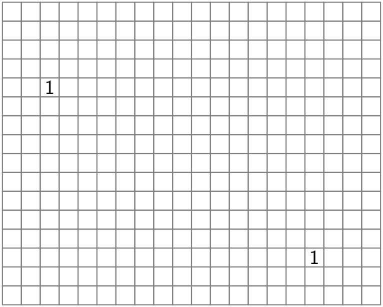
Example 1: the one-or-less subshift

$$X_{\leq 1} = \left\{x \in \{0,1\}^G \mid |\{g \in G : x_g = 1\}| \leq 1\right\}$$

Question

On which f.g. groups is the one-or-less subshift sofic ?

Sofic on multidimensional grids $\mathbb{Z}^d$



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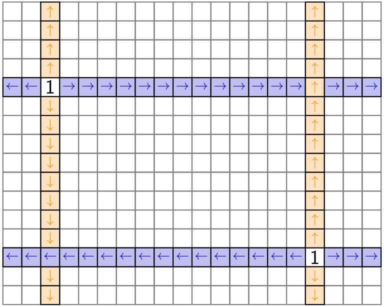

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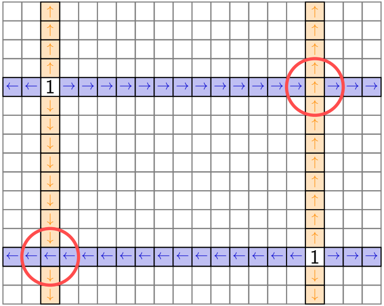

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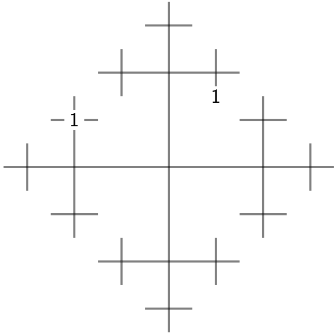

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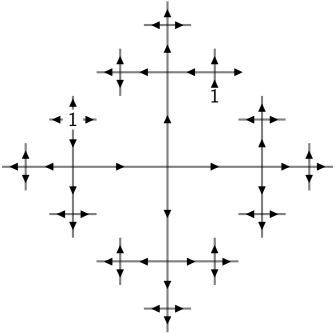

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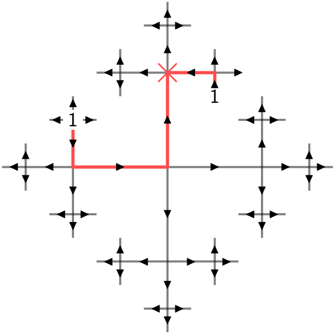

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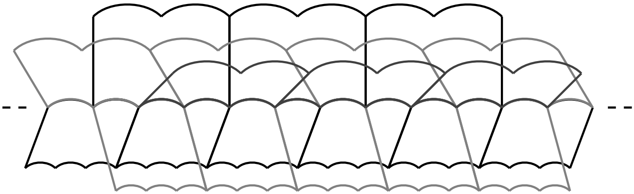
Sofic on BS(m,n)



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Proposition (Dahmani & Yaman, 2002)

- If $X_{\leq 1}$ is sofic for G_1 and G_2, then it is also sofic for $G_1 \otimes G_2$.
- Let $H \leqslant G$ be a subgroup with $[G : H] < \infty$, then $X_{\leq 1}$ is sofic for G if and only if it is sofic for H.
- If G is an hyperbolic group, then $X_{\leq 1}$ is sofic for G.
- ...

Question

Does there exists a f.g. group on which $X_{\leq 1}$ is not sofic ?

Example 2: the even shift

$$X_{\text{even}} = \left\{ x \in \{0,1\}^G \middle| \text{ finite CC of 1's have even size } \right\}.$$

Proposition

The even shift X_{even} is sofic for every f.g. group G.

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Proof: Consider the G-SFT X_k, where $k = |B_1|$, with alphabet

$A_3 =$  + rotations

$A_4 =$  + rotations

$A_5 =$  + rotations

$A_6 =$  + rotations and reflections

etc...

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Proof: Take for instance $k = 4$ (for $\mathbb{Z}^2$ or $BS(m,n)$)

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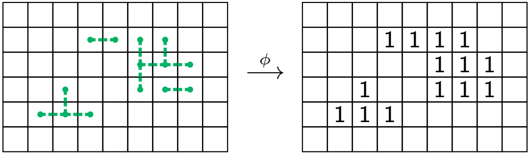
Green components have even size (handshaking lemma)$\Rightarrow \phi(X_k) \subseteq X_{\text{even}}$



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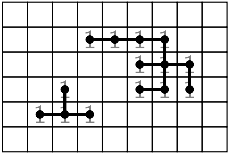

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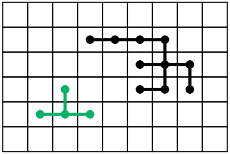

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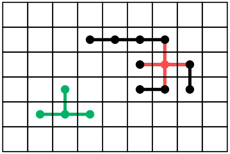

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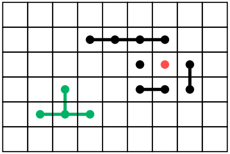

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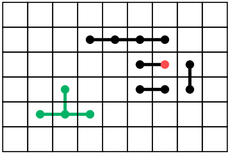

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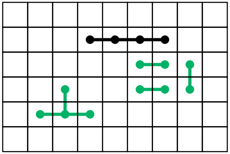

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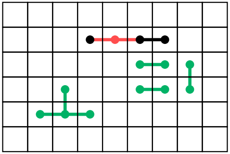

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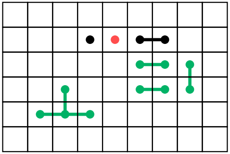

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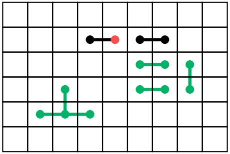

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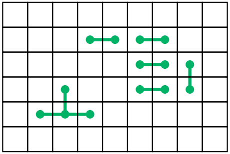

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Soficness on f.g. groups

Two previous examples:

- Exhibit the SFT cover to prove soficness. . .
- . . . and actually it is almost the only technique known !
- One-or-less subshift: illustrates how information can flow inside the group by local rules.

Periodic configurations and aperiodic subshifts (I)

The **stabilizer** of a configuration $x \in A^G$ is the set of translations that leave it unchanged

$$\mathrm{Stab}(x) = \{g \in G \mid \sigma_g(x) = x\} \leqslant G.$$

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- A configuration $x \in A^G$ is **weakly periodic** if its stabilizer is infinite.
  A configuration $x \in A^G$ is **strongly aperiodic** if x is not weakly periodic.
- A configuration $x \in A^G$ is **strongly periodic** if its stabilizer is of finite index in G
  $$[G : \mathrm{Stab}(x)] < \infty.$$
  A configuration $x \in A^G$ is **weakly aperiodic** if x is not strongly periodic.

Remark: x strongly (a)periodic $\Rightarrow$ x weakly (a)periodic

Periodic configurations and aperiodic subshifts (II)

A non-empty subshift is

- **weakly aperiodic** if it contains no strongly periodic configuration.
- **strongly aperiodic** if it contains no weakly periodic configuration.

Remark 1: X strongly aperiodic $\Rightarrow X$ weakly aperiodic.
Remark 2: On $\mathbb{Z}$ and $\mathbb{Z}^2$ the notions are equivalent **(see Lecture 2)**.

Examples:

- On $\mathbb{Z}$ there exists no (weakly/strongly) aperiodic SFT.
- On $\mathbb{Z}^2$ there exists (weakly/strongly) aperiodic SFT.

Aperiodic SFT

Questions

- Which f.g. groups admit weakly aperiodic SFT ?
- Which f.g. groups admit weakly aperiodic SFT but no strongly aperiodic SFT ?
- Which f.g. groups admit strongly aperiodic SFT ?

More about this on ***Wednesday***:

Ayse Sahin (12:10) and David Cohen (14:30)

Emptyness Problem (I)

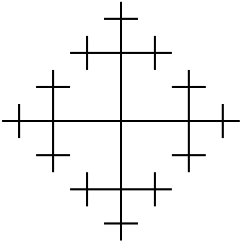
- Let $k \in \mathbb{N}^*$ and A a finite alphabet

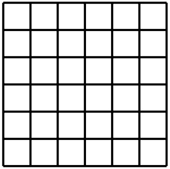
$$A_1 = \{ \quad , \quad \}.$$

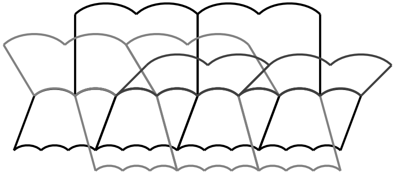
- Let $\mathcal{F}$ be a set of nearest neighbors rules.

$$\overline{\mathcal{F}_1} = \{ \quad \}$$

- Let G be a group generated by k generators.







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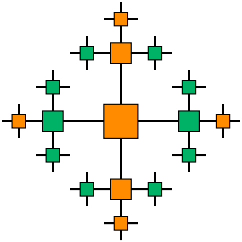

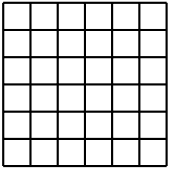

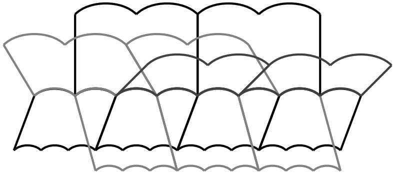

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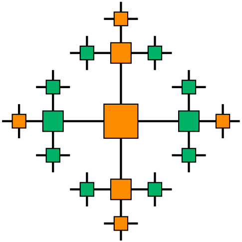
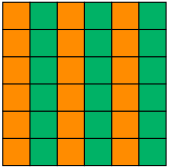
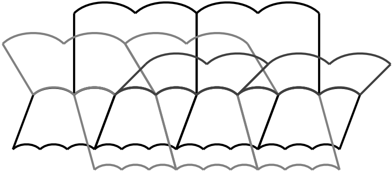
- Let G be a group generated by k generators.







- Does the G-SFT $X_{\mathcal{F}}$ contain a configuration ?

Emptyness Problem (I)

- Let $k \in \mathbb{N}^*$ and A a finite alphabet

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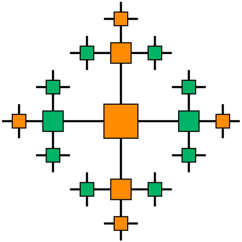

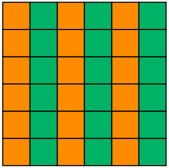

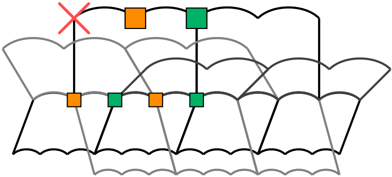

- Does the G-SFT $X_{\mathcal{F}}$ contain a configuration ?

Emptyness Problem (II)

Fix G a f.g. group and S a generating set for G.

Emptyness Problem for G-SFTs

Input: F a finite set of forbidden patterns on S.
Output: **Yes** if there exists a configuration in X_F, **No** otherwise.

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Emptyness Problem for G-SFTs

Input: F a finite set of forbidden patterns on S.
Output: **Yes** if there exists a configuration in X_F, **No** otherwise.

Question

Which f.g. groups have decidable Emptyness Problem ?

More about this on ***Tuesday*** ($\mathbb{Z}^2$) and ***Thursday***:

Lecture 2 (11:00) and Lecture 3 (09:30)

Lecture 1: Symbolic Dynamics on f.g. groups: a computational approach.

Word Problem for f.g. groups (I)

Does there exist an algorithm that decides whether two words w_1 and w_2 on the generators and their inverses represent the same element in G $(w_1 =_G w_2)$?

$$WP(G) = \left\{ w \in \left(S \cup S^{-1}\right)^* \mid w =_G 1_G \right\}.$$

Definition

A f.g. group G has **decidable WP** if there exists an algorithm that take two words w_1 and w_2 as input and outputs **Yes** if $w_1 =_G w_2$ and **No** if $w_1 \neq_G w_2$.

Remark: Decidability of WP does not depend on the choice of S.

Word Problem for f.g. groups (II)

Theorem

The word problem is decidable for the following classes

- f.g. groups defined by a single relator (Magnus, 1932)
- f.p. simple groups (Simmons, 1973)
- f.p. residually finite groups
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The word problem for a f.g. group G is **recognizable** iff G is recursively presented.

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Proposition

The word problem for a f.g. group G is **recognizable** iff G is recursively presented.

Theorem (Novikov, 1955 and Boone, 1958)

There exist f.p. groups with undecidable word problem.

Why ? $\approx$ Encode Turing machine inside the presentation of the group.

Word Problem and soficness of $X_{\leq 1}$

Proposition

If G has undecidable Word Problem, then $X_{\leq 1}$ cannot be sofic.

Proof: Wait for **Lecture 4**

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Questions

- Does there exists a f.g. group with decidable WP on which $X_{\leq 1}$ is not sofic ?
- $X_{\leq 1}$ is sofic on G iff G has decidable WP ?

Lecture 1: Symbolic Dynamics on f.g. groups: a computational approach.

Free groups and virtually free groups

Free groups $F_S = \langle S | \emptyset \rangle$
A f.g. group G is **virtually free** if it has a free subgroup of finite index.
Examples:

- The *twisted* free group $\langle a, b, c | bc = ca, ac = b^{-1}c \rangle$.
- Every semi-direct product $F \rtimes N$ with F free and N finite.
- $\mathbb{F}_2$ is virtually $\mathbb{F}_n$ for every $n \geq 2$.

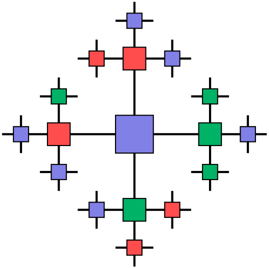
Weak periodicity

Consider the free group $\mathbb{F}_2 = \langle a, b | \emptyset \rangle$.

Theorem (Piantadossi, 2006)

Every non empty $\mathbb{F}_2$-SFT X contains a weakly periodic configuration.

Proof: Take a configuration $x \in X$.



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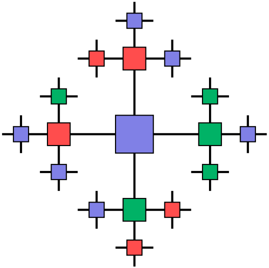

$y = \dots$  $\dots$

Consider the $\mathbb{Z}$-subshift $\pi_{\langle a \rangle}(X)$: it is a non-empty $\mathbb{Z}$-SFT, so it contains a periodic configuration y with period $p \in \mathbb{N}^*$.

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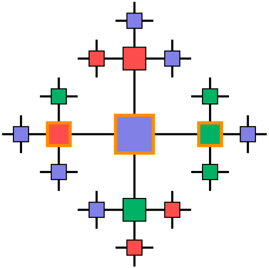

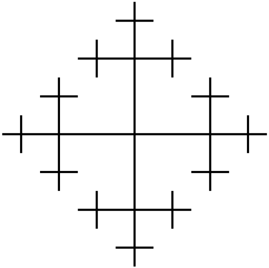

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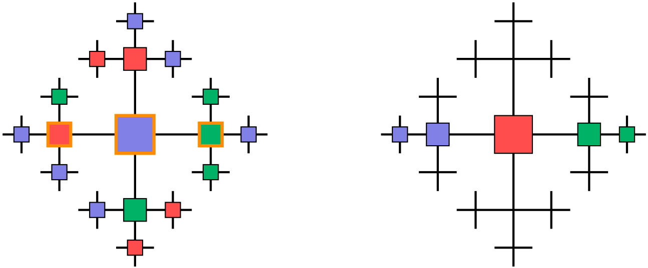

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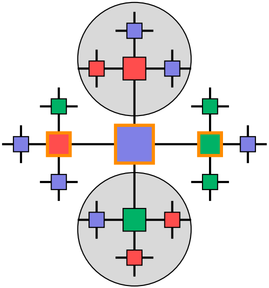

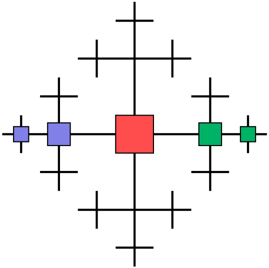

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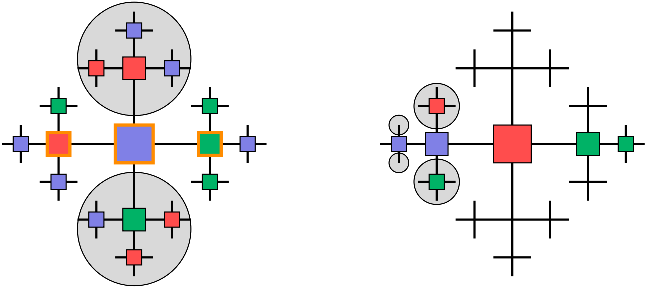

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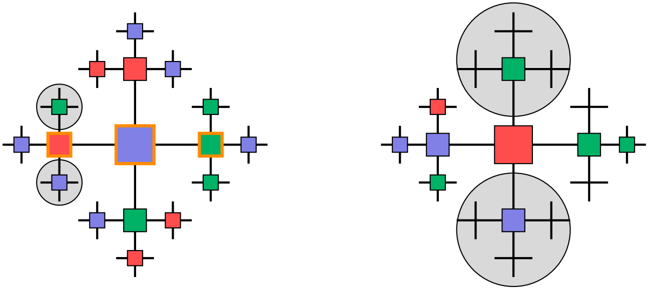

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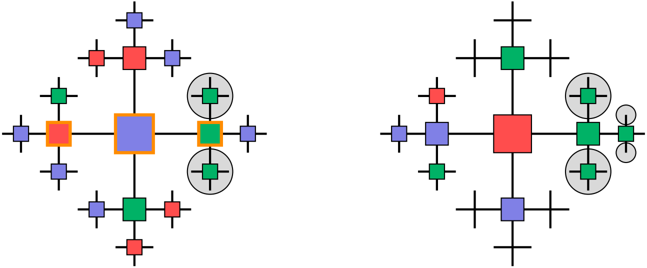

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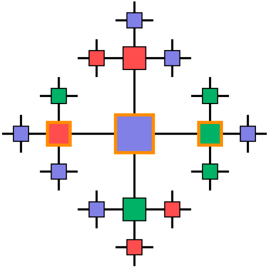

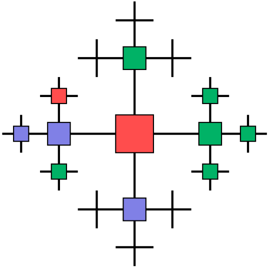

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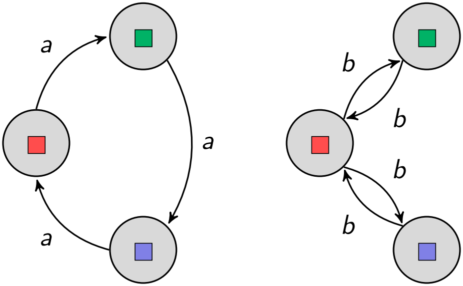
Weak aperiodicity

Consider the free group $\mathbb{F}_2 = \langle a, b | \emptyset \rangle$.

Theorem (Piantadossi, 2006)

There exists weakly aperiodic $\mathbb{F}_2$-SFTs.

Proof: Consider the following $\mathbb{F}_2$-SFT X.



There can be a period p for $x \in X$ only if $p = a^{3n}$ or $p = b^{2m}$ (but not both !).

Emptyness Problem on $\mathbb{F}_2$

Theorem

The Emptyness Problem is decidable on $\mathbb{F}_2$.

Proof: Take a n.n. SFT X on $\mathbb{F}_2$ with alphabet A.

- Erase from A all symbols that cannot be extend to a locally admissible pattern of size 1.
- Iterate until you cannot erase symbol.
- Then $A \neq \emptyset$ iff $X \neq \emptyset$.

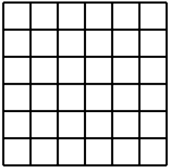
Lecture 1: Symbolic Dynamics on f.g. groups: a computational approach.

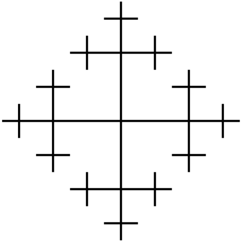
Definition

The **number of ends** of a f.g. group G is the limit

$$\lim_{n\to\infty} |CC(\Gamma_G \setminus B_n)|$$

Remark: The number of ends does not depend on the choice of Γ_G.





Proposition

A f.g. group has 0,1,2 or infinitely many ends.

Number of ends

Stallings theorem and consequences

Let G be a f.g. group. Then

- $e(G) = 0$ iff G is finite,
- if G is virtually free then $e(G) \geq 2$,
- $e(G) = 2$ iff G is virtually cyclic,
- if $e(G) = \infty$ then G contains a non-abelian free subgroup.

Number of ends and soficness

Groups with more than two ends can be **disconnected by a finite set**.

- In sofic subshifts, only a *finite amount of information* can go through this disconnecting set.
  $\Rightarrow$ use **Communication Complexity** to formalize this notion ? **(see Emmanuel Jeandel's talk)**
- Can be used to prove some subshifts with highly non-local conditions are not sofic on groups G with $e(G) \geq 2$. **(see Sebastián Barbieri's poster)**

Conclusion

- Sofic subshifts: information flow through the group.
- Computational resctriction: groups with decidable Word Problem.
- Free groups: *easy* case.

Tomorrow: more about Domino Problem on $\mathbb{Z}^2$.

Thank you for your attention !!