Random Hyperbolic Graphs: Geometry, Structural Properties, and Probabilistic Analysis

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- Slides: `www.dim.uchile.cl/∼mkiwi/latin2026.pdf`

- Outline
  - Part I: Introduction, motivation and general concepts.
  - Part II: Random Hyperbolic Graphs – Model definition.
  - Part III: Random Hyperbolic Graphs – Tools & facts.
  - Part IV: Ultra-small average distance, navigability & greedy routing.

Part I: Introduction, motivation and general concepts.

Goal

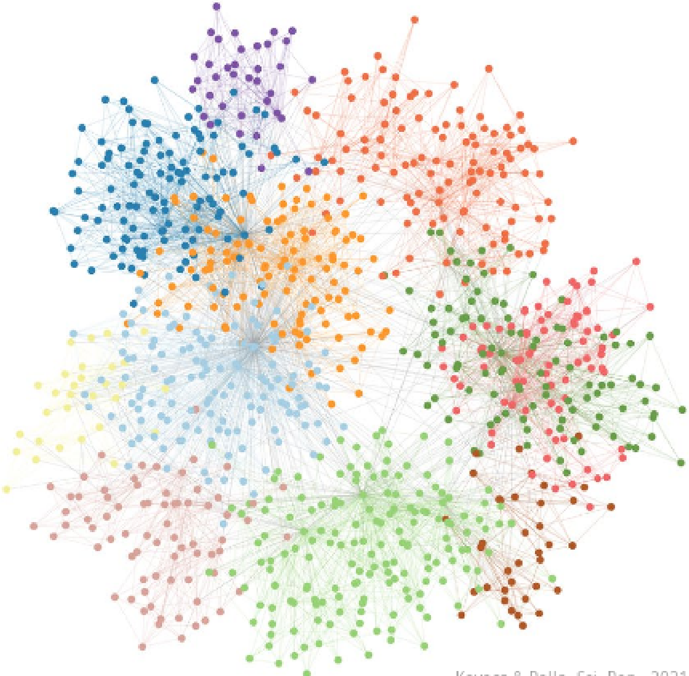
Identify models that exhibit characteristic properties of "real world networks/complex networks"

Example of networks:
- *Power grid*
- *Internet*
- *Social networks*
- *Biological interaction networks*
- *Scientific collaboration networks*
- *...*

Typical properties:
- *Sparse*
- *Heterogeneous*
- *Locally dense (exhibit clustering phenomena)*
- *Small world*
- *Navigable*
- *Scale free (with exponent between 2 and 3)*
- *...*

Also, we want models that are susceptible to mathematical analysis!

Why resort to random graph models?



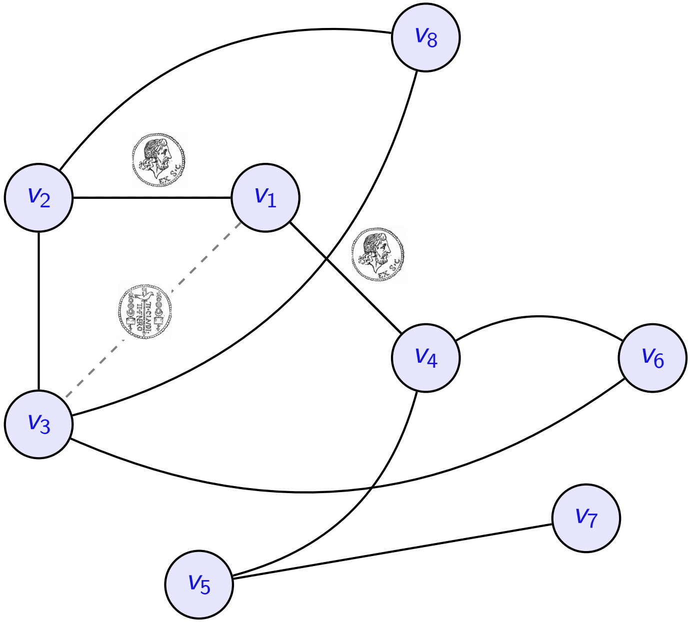
Kovacs & Palla, Sci. Rep., 2021.

Among other reasons:

- To avoid rigid and non-robust models.
- Real network statistical signature's are best captured by random models.
- Real networks grow organically with some level of unpredictability, better captured as the outcome of a stochastic process.

Random Graphs

The most famous random graph model is the Binomial Random Graph model (often referred to as Erdös–Renyi model), usually denoted $\mathcal{G}(n, p)$ where $n \in \mathbb{N}$ and $p \in [0, 1]$.



If coin associated to edge lands head, include edge.

Simple to describe yet mathematical very interesting!

Some basic characteristics of $G = \mathcal{G}(n, p)$:

- Expected size of $E(G)$ is $\binom{n}{2} p$.
- Thus, expected average degree is $(n-1)p$ – independent of vertex.

  Note: In fact, $\mathrm{dgr}_G(v)$ follows a $\mathrm{Binom}(n-1, p)$.
- Probability a vertex is isolated is $(1-p)^{n-1}$ which is $\sim e^{-c}$ if $p = c/n$ and $n \gg 1$.

Real world networks (a.k.a. social networks) are sparse. Relevant range for $\mathcal{G}(n, p)$ is

$$p = O(1/n),$$

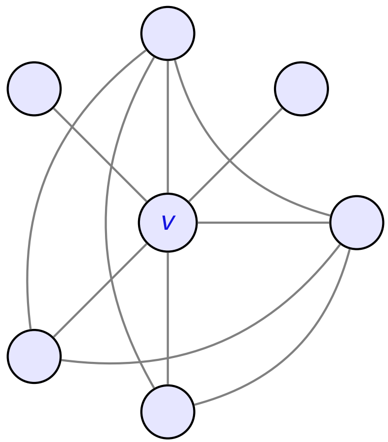
where it is disconnected, yet for $p = c/n$ with $c > 1$ a.a.s.[1] it has a unique connected component of size $L_1(G) = \Omega(n)$.

[1] That is, with probability going to 1 as $n \to \infty$.

My friends are typically friends among them.

Let $G := (V, E)$ be a graph and $v \in V$. Define the *local clustering coefficient* of $v \in V(G)$ by

$$\overline{c}(v) := \begin{cases} 0, & \text{if } \mathrm{dgr}_G(v) < 2, \\ \frac{1}{\binom{\mathrm{dgr}_G(v)}{2}} \left| \{uw \in E \mid u, w \in \mathrm{Neigh}_G(v)\} \right|, & \text{otherwise.} \end{cases}$$



$\overline{c}(v) = 5/\binom{6}{2} = 5/12$

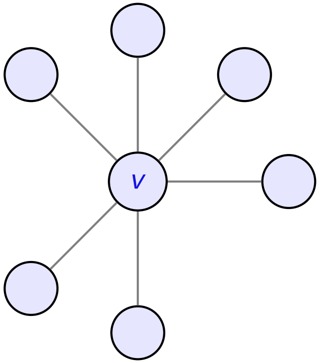
The *average local clustering coefficient* of G is “defined” as $\overline{c}(G) := \frac{1}{|V(G)|} \sum_{v \in V} \overline{c}(v)$.

Expectation of average local clustering coefficient of the Binomial model

Lemma: If $G_n := \mathcal{G}(n,p)$, then $\mathbb{E}(\overline{c}(G_n)) \leq p$.

Proof: Note that:

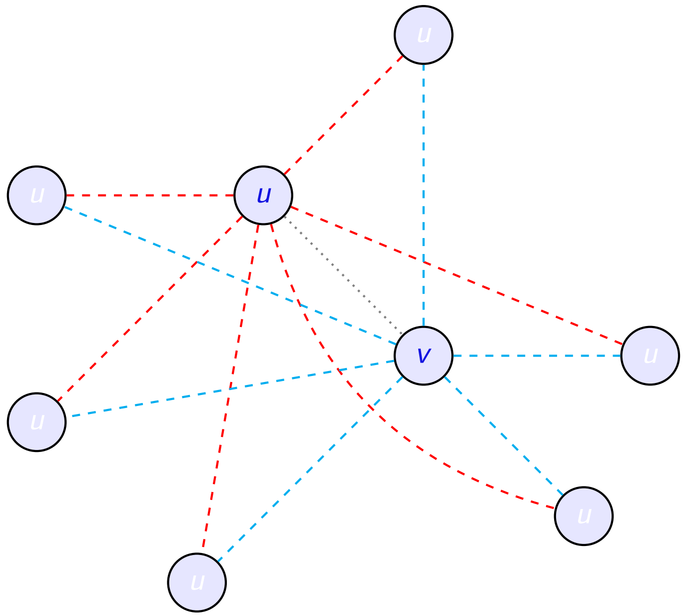
- $\mathbb{E}(\overline{c}(v)) = \sum_{d=2}^{n-1} \mathbb{E}(\overline{c}(v) \mid \mathrm{dgr}_{G_n}(v) = d)\mathbb{P}(\mathrm{dgr}_{G_n}(v) = d)$.
- If $d \geq 2$, then $\mathbb{E}(\overline{c}(v) \mid \mathrm{dgr}_{G_n}(v) = d) = \frac{1}{\binom{d}{2}}\binom{d}{2}p = p$.



- Hence, $\mathbb{E}(\overline{c}(v)) = p\sum_{d=2}^{n-1} \mathbb{P}(\mathrm{dgr}_{G_n}(v) = d) \leq p$, so $\mathbb{E}(\overline{c}(G_n)) = \frac{\sum_{v\in V(G_n)} \overline{c}(v)}{|V(G_n)|} \leq p$. □

In the sparse regime, $\mathbb{E}(\overline{c}(G_n)) \to 0$ when $n \to \infty$.

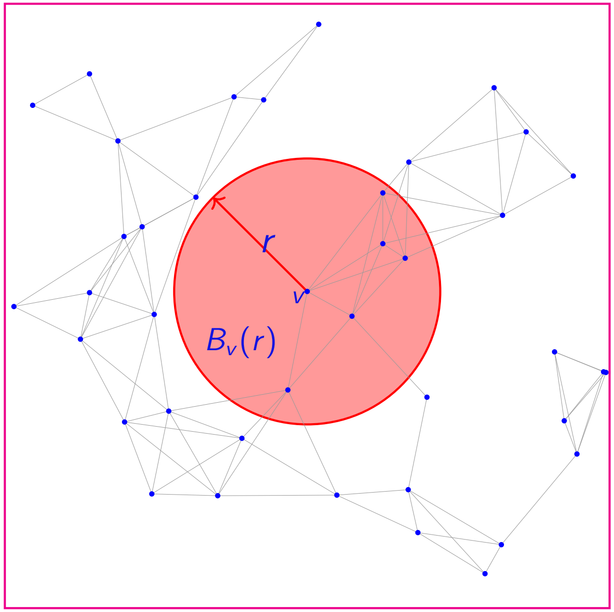
Key property of the Binomial model



Events on disjoint sets of potential edges are independent.

Geometric Graphs (GG) in $[0,1]^2$

Say we are given n points in $[0,1]^2$ and a parameter $r > 0$.



Ball $B_v(r) := \{p \in \mathbb{R}^2 \mid \|p - v\| < r\}$ has perimeter $2\pi r$ and area πr^2.

Definition of (uniform) Random GG model: $\mathcal{U}_r^E(\Omega, n)$

Model parameters: $\Omega \subseteq \mathbb{R}^2$ connected and measurable, radius $r \in \mathbb{R}_+$, graph size $n \in \mathbb{N}$.[2]

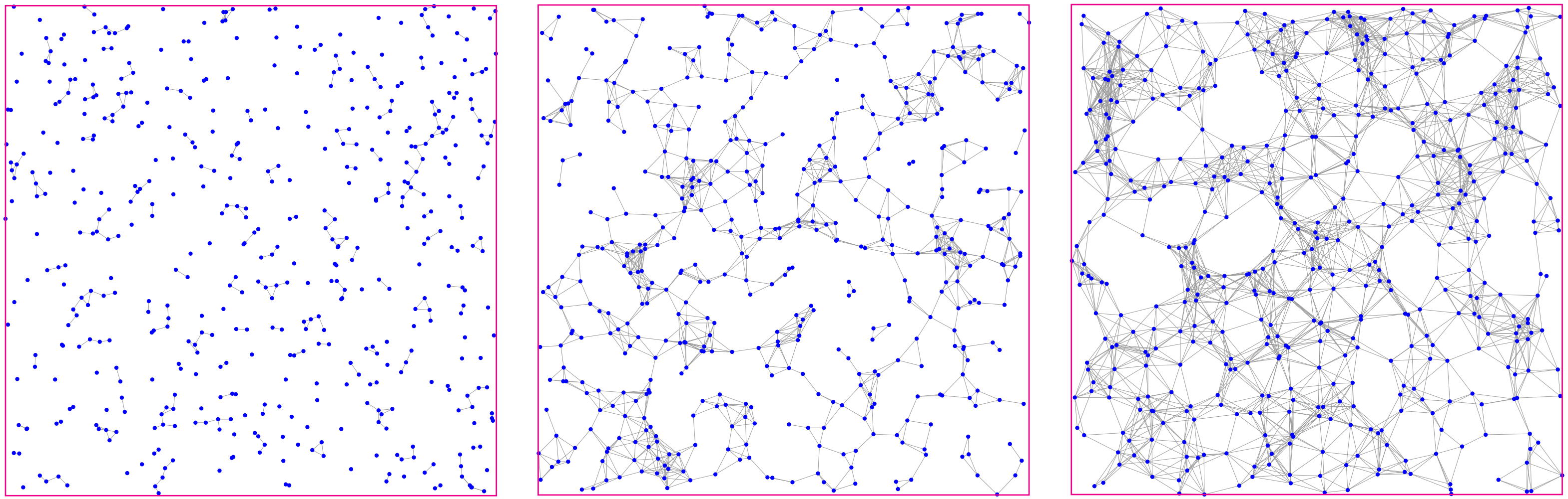
Sample a graph $G = (V, E)$ with $|V| = n$ as follows:

- Choose location of each $v \in V$ uniformly and independently in $\Omega \subseteq \mathbb{R}^2$.
- Let $uv \in E$ iff $u \in B_v(r)$ (that is, euclidean distance between u and v is at most r).

Note: The radius r might be a function of n. For example, $r = r(n) := c/n$ where $c > 0$ is a constant.

[2] In this course Ω is always a bounded set.

Examples of $\mathcal{U}_r^E([0,1]^2, n)$



$r = 0.03$ $r = 0.06$ $r = 0.09$

$n = 500$ points

What are random GGs useful for?

Good for modeling:

- radio stations / communications.
- trees / spread of diseases
- sensor networks / monitoring
- More dubious: cell phone antennas / communication

Warm up: Degrees of vertices in Random GG (RGG)

If $G = \mathcal{U}_r^E([0,1]^2, n)$, then $\mathrm{dgr}_G(v) \sim \mathrm{Binom}(n-1, p_v)$ where $p_v = \mathrm{Area}(B_v(r) \cap [0,1]^2)$, so

$$\mathbb{E}[\mathrm{dgr}_G(v)] = (n-1)p_v \qquad \text{for } \tfrac{1}{4}\pi r^2 \leq p_v \leq \pi r^2.$$

Thus, sparse regime requires $r = r(n) = O(1/\sqrt{n})$.

Note 1: Prob. v is isolated is $(1-p_v)^{n-1}$ which is $\sim e^{-p_v(n-1)} = \Omega(1)$ in the sparse regime.

Note 2: There is a critical threshold λ_c such that, if $\lambda > \lambda_c$ and $nr^2 \to \lambda$ when $n \to \infty$ (called critical regime), then a.a.s. there is a unique largest connected component $L_1(G) = \Omega(n)$.

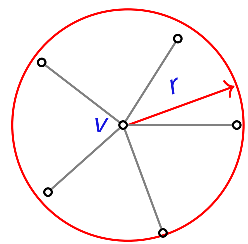
We restrict our discussion on the range above the critical regime and focus on the giant component of the RGG.

Expectation of average local clustering coefficient of RGGs

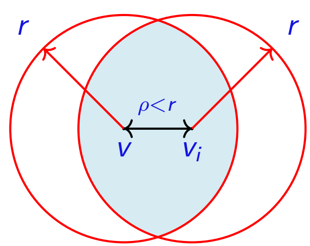
Lemma: Above the critical regime, $\mathbb{E}(\overline{c}(G_n)) = \Omega(1)$ where $G_n := \mathcal{U}^E_{r_n}([0,1]^2, n)$.[3]

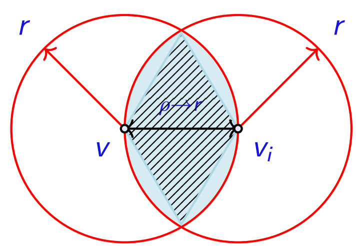
Proof (rough sketch):

- Because a.a.s. $L_1(G_n) = \Omega(n)$ and since among 6 neighbors of a vertex, a.a.s. at least 2 must be adjacent, there are $\Omega(n)$ vertices of G_n of degree at least 2.



- Fix $v \in V(G_n)$, $d = \mathrm{dgr}_{G_n}(v) \geq 2$, and $\mathrm{Neigh}_{G_n}(v) = \{v_1, \ldots, v_d\}$.



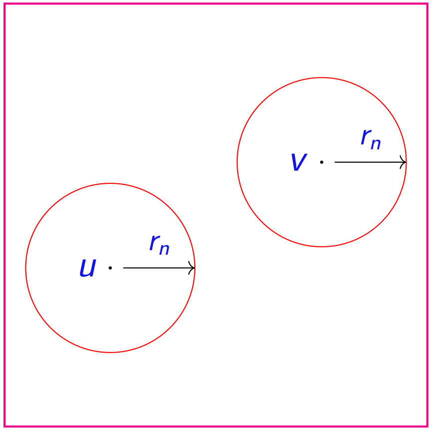


Hence, $\overline{c}(v) \geq \frac{\sqrt{3}r^2}{2\pi r^2} = \Omega(1)$. □

[3]In other words, in the sparse regime, RGGs have non-negligible clustering coefficient.

Caveats of working with $\mathcal{U}_r^E(\Omega, n)$.

Say u, v vertices of $G := \mathcal{U}_{r_n}^E([0,1]^2, n)$ are such that $B_u(r_n) \cap B_v(r_n) = \varnothing$:



Question: Are $\mathrm{dgr}_G(u)$ and $\mathrm{dgr}_G(v)$ independent?

In general, $|V(G) \cap S|$ and $|V(G) \cap S'|$ are not independent, even when $S \cap S' = \varnothing$.

Poissonized version of $\mathcal{U}_r^E(\Omega, n)$ (denoted $\mathcal{P}_r^E(\Omega, n)$)

It is more natural to consider a Poissonized version of $\mathcal{U}_r^E(\Omega, n)$.

Observe that $G := \mathcal{U}_r^E(\Omega, n)$ is obtained by sampling $v_1, \ldots, v_n$ uniformly in Ω and setting:

- $V(G) := \{v_1, \ldots, v_n\}$.
- $E(G) := \{v_i v_j \mid i \neq j, \|v_i - v_j\|_2 \leq r\}$.

If instead we

- choose $v_1, v_2, \ldots$ uniformly in Ω,
- let N be a $\mathrm{Poiss}(n)$ (independent of $\{v_1, v_2, \ldots\}$), and
- take $V(G) := \{v_1, \ldots, v_N\}$,

we obtain the so called (homogeneous) Poissonized Random GG model, denoted $\mathcal{P}_r^E(\Omega, n)$.

Note 1: The set $\mathcal{V}_n := \{v_1, \ldots, v_N\}$ is called *(homogeneous) Poisson Point Process (PPP)* of intensity n.

Note 2: Again, r might be a function of n.

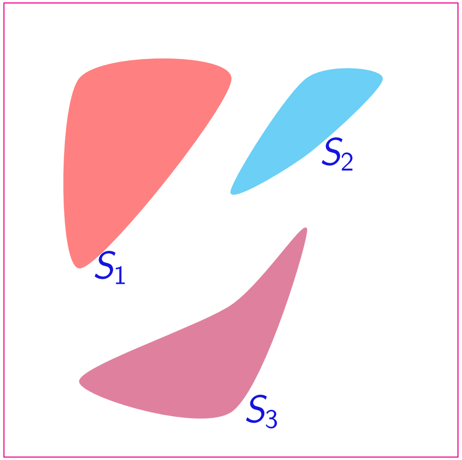
Key properties of homogeneous PPPs

For simplicity's sake, henceforth $\Omega \subseteq \mathbb{R}^2$ is such that $\mathrm{Area}(\Omega) = 1$.

If $S \subseteq \Omega$, then the following hold:

(i) If $S \subseteq \Omega$, then $|\mathcal{V}_n \cap S|$ is a $\mathrm{Poiss}(n \cdot \mathrm{Area}(S))$.

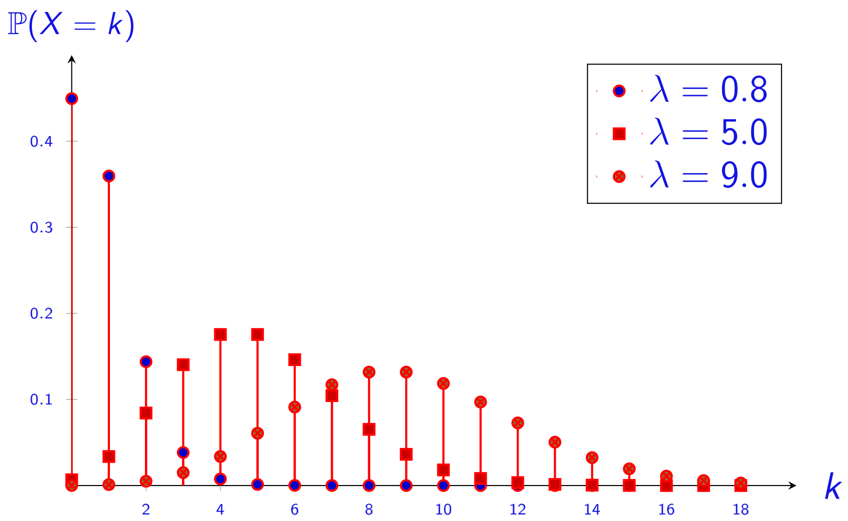
(ii) If $S_1, ..., S_m \subseteq \Omega$ are disjoint, then $|\mathcal{V}_n \cap S_1|, ..., |\mathcal{V}_n \cap S_m|$ are independent.



Poisson Random Variable

If $X \sim \mathrm{Poiss}(\lambda)$, then

- Expectation and variance of X are λ.
- Probability that $X = 0$ is $e^{-\mathbb{E}(X)} = e^{-\lambda}$.



Concentration bounds for Poisson r.v.'s

Similar in spirit to Chernoff bounds for sums of Bernoulli r.v.'s

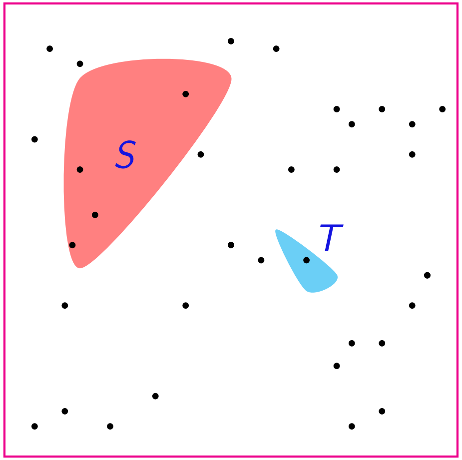
Thm: If $P \sim \mathrm{Poiss}(\mu)$ and $\epsilon > 0$, then

$$\mathbb{P}\big(P \le \mu(1-\epsilon)\big) \le e^{-\frac{1}{2}\epsilon^2\mu},$$

and

$$\mathbb{P}\big(P \ge \mu(1+\epsilon)\big) \le \big[e^{\epsilon}(1+\epsilon)^{-(1+\epsilon)}\big]^{\mu}.$$

Key consequences:



- If $\mathrm{Area}(S) = \omega((\log n)/n)$, then $|\mathcal{V}_n \cap S|$ is concentrated around $\mathbb{E}[|\mathcal{V}_n \cap S|] = n \cdot \mathrm{Area}(S)$.
- If $\mathrm{Area}(S) = \Omega((\log n)/n)$, then $\mathcal{V}_n \cap S = \varnothing$ is $1/n^{\Omega(1)}$.
- If $\mathrm{Area}(T) = o((\log n)/n)$, then the probability that $\{|\mathcal{V}_n \cap T| \geq \log n\}$ is $o(1)$.

In particular, if $\mathbb{P}, \mathbb{E}$ are taken over the random choices of $G := \mathcal{P}_r^E(\Omega, n)$, then

- $\mathbb{P}(V(G) = \varnothing) = e^{-n}$
- $|V(G)|$ is concentrated around $\mathbb{E}[|V(G)|] = n$.

Depoissonization

Key takeaway: Poissonized model is a "blurred/average" version of the uniform model.
If an event $\mathcal{E}$ has probability $P_n(\mathcal{E})$ in the Uniform model and $P_{\mathrm{Poiss}(n)}(\mathcal{E})$ in the Poisson model:

$$P_{\mathrm{Poiss}(n)}(\mathcal{E}) = \sum_{k \in \mathbb{N}} P_k(\mathcal{E}) \cdot \mathbb{P}(\mathrm{Poiss}(n) = k).$$

Quantitatively:

- "Poisson-to-Uniform" model bound:

$$P_n(\mathcal{E}) \leq e\sqrt{n} \cdot P_{\mathrm{Poiss}(n)}(\mathcal{E}).$$

  i.e., "extremely unlikely" in the Poisson world $\implies$ still "unlikely" in the uniform model.
- "Uniform-to-Poisson" model bound:

$$P_{\mathrm{Poiss}(n)}(\mathcal{E}) \leq \max_{k \in n \pm \delta\sqrt{n}} P_k(\mathcal{E}) - 2e^{-c\delta^2}$$

Inhomogeneous PPPs

Similar to homogeneous PPPs, but now there is a density function

$$\lambda : \Omega \to \mathbb{R}_+, \qquad \text{where} \qquad \mu(B) := \int_B \lambda(v)\mathrm{d}v < \infty \qquad \text{for every } B \subseteq \Omega$$

and points $v_1, v_2, \ldots$ are chosen according to $\lambda(\cdot)$.

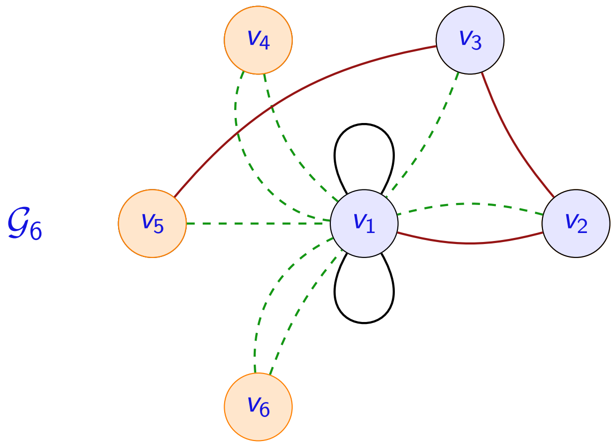
All else is as before except now:

- $\mathcal{V}_n := \{v_1, \ldots, v_N\}$ is called inhomogeneous PPP with intensity $n \cdot \lambda$, and
- the rôle of $\mathrm{Area}(S)$ is replaced by $\mu(S)$.

Note: A homog. PPP is a special case of a inhomog. PPP where $\lambda(\cdot)$ equals $1/\mathrm{Area}(\Omega)$ everywhere.

Hybrid Preferential Attachment (PA) model ($\mathcal{PA}(m, \lambda, t)$)

"The rich get richer"



$b_{t,j} \sim \mathrm{Bern}(\lambda)$ for $j = 1, ..., m$.

Uniform ($b_{t,j} = 0$)

Preferential ($b_{t,j} = 1$)

- $t = 1, 2, 3, \ldots$
- $m = 2$**:** Each new node brings two edges into the system.
- **Multiple Edges:** In this model, multi-edges between nodes are possible (e.g., $v_4 \rightarrow v_1$).

Power law degree distribution

Say the random graph sequence $(G_n)_{n\in\mathbb{N}}$ has *power law degree distribut. with exponent* $\beta > 1$ if

$$\mathbb{E}(|E_d|) = \Theta(n/d^{\beta-1})$$

where $E_d := |\{v \in V(G_n) \mid \mathrm{dgr}_{G_n}(v) \geq d\}|$.

Note: Power law degree distribution graphs with exponent $\beta > 1$ are sparse.

Properties of $\mathcal{PA}(\lambda, m, t)$

- $|V(G_t)| = t$ and $|E(G_t)| = m \cdot t$.
- (Power law degree) $\frac{1}{t}\mathbb{E}(\underbrace{|\{v \in V(G_t) \mid \mathrm{dgr}_{G_t}(v) \geq d\}|}_{E_d}) \approx \left(\frac{d}{2m}\frac{\lambda}{1-\lambda}\right)^{-2/\lambda}$.
- (Negligible clustering coefficient) $\mathbb{E}\big(\overline{c}(G_t)\big) \to 0$ when $t \to \infty$.

Summary so far

- Let $(G_n)_{n\in\mathbb{N}}$ where $G_n := \mathcal{G}(n,p)$
  Vertices are homogeneous (so, sequence is not power law).
  The sequence has constant average degree when $p = \Theta(1/n)$ *in which case it has negligible clustering coefficient.*
- Let $(G_n)_{n\in\mathbb{N}}$ where $G_n := \mathcal{U}_r^E(\Omega, n)$.
  Vertices are homogeneous (so, sequence is not power law).
  The sequence has constant average degree when $r = \Theta(1/\sqrt{n})$ *in which case it has non-negligible clustering coefficient.*
- Let $(G_t)_{t\in\mathbb{N}}$ where $G_t := \mathcal{PA}(\lambda, m, t)$.
  Vertices are non-homogeneous.
  The sequence has constant average degree when $0 < \lambda < 2$ *in which case it is power law. However, it has vanishing clustering coefficient.*

Are there good models that exhibit all desired properties?

Part II: Random Hyperbolic Graphs – Model definition

Random hyperbolic graphs (RHGs): Introduction

- Introduced by Krioukov, Papadopoulos, Kitsak, Vahdat, Boguña [Phys. Rev. '10]
- Appeal: Replicate characteristic properties observed in "real world networks" or "complex networks"

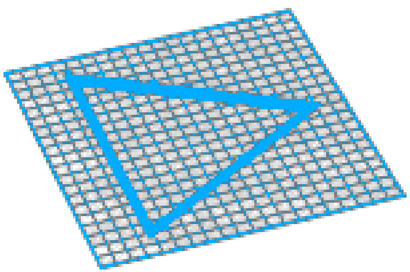
Susceptible to mathematical analysis!

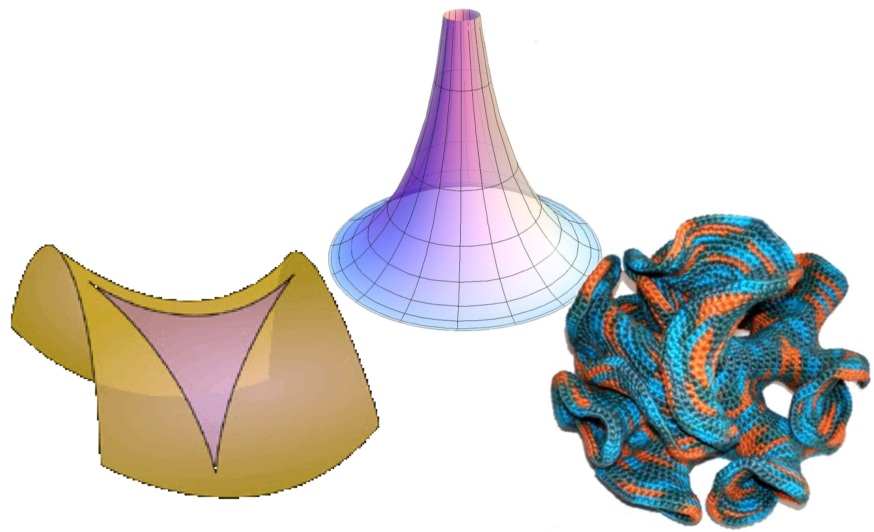
Informal definition of RHGs model

Like random geometric graphs but where the underlying space instead of being Euclidean is Hyperbolic.

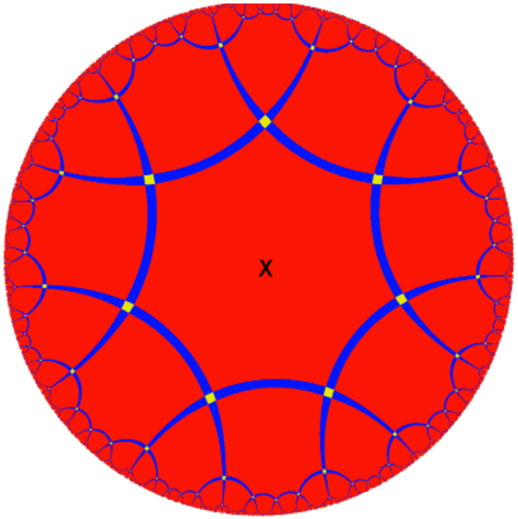
Hyperbolic plane $\mathbb{H}^2$

Euclidean plane $\mathbb{R}^2$





Poincaré disk model of $\mathbb{H}^2$



[Rendered with KaleidoTile by J. Weeks]

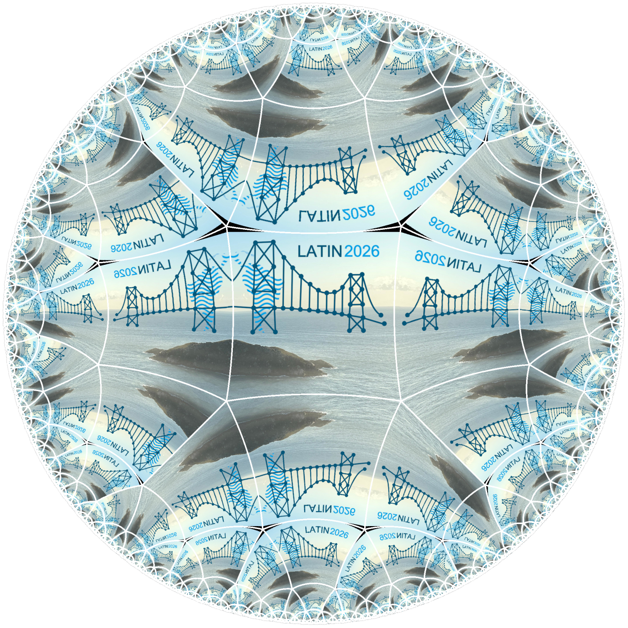
- $\mathbb{H}^2$ is represented as a unit open disk D.
- Blue curves are geodesics (arcs of circles perpendicularly incident to D).
- Each heptagon has the same area.
- Points in ∂D are at infinite distance from X.
- Points at (Euclidean) distance y from X are at hyperbolic distance r from X where

$$r = \log \frac{1+y}{1-y}.$$

Space expands at exponential rate!

Continuous analogue of regular trees.

Good for making cool pictures!



[Rendered with M. Christersson hyperbolic tiling applet]

Hyperbolic space: Summary

- There is lots of space. We all can live in a huge house with huge patios.
- Yet, everyone can be close to family, work, friends. No cars needed!
  Conferences, seminars and collaborators can be just steps away!
- You can draw trees (I mean graphs) as you have always dreamed!

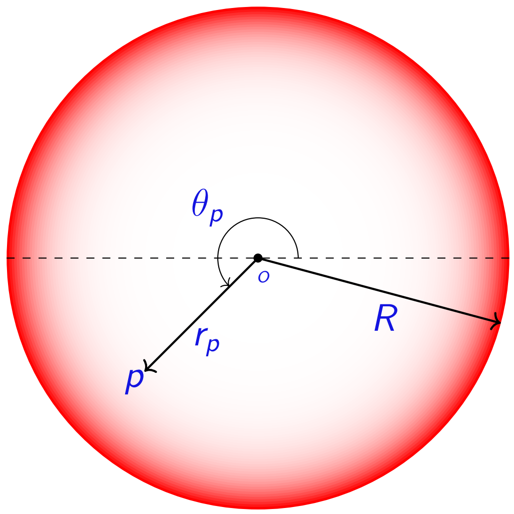
Hyperbolicland is awesome!

Full disclosure ...

Hyperbolicland can be dangerous!

“Just because you keep getting lost on the way to work is no proof that the Universe is hyperbolic!”

Gans model of $\mathbb{H}^2$



- $\mathbb{H}^2$ is represented as $\mathbb{R}^2$.
- A point p is represented in polar coordinates as (r_p, θ_p).
- r_p is the hyperbolic distance between p and the origin o.

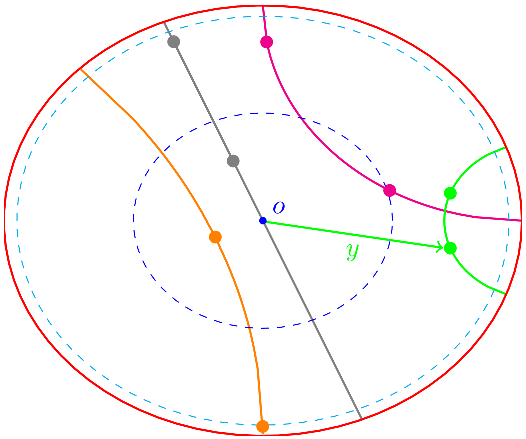
In $\mathbb{H}^2$, ball $B_o(R)$ has perimeter $2\pi \sinh(R)$ and area $2\pi\big(\cosh(R) - 1\big)$.[4] Note that: [5]

$$\pi e^R(1-e^{-2R}) \leq 2\pi \sinh(R) \leq \pi e^R \leq 2\pi \cosh(R) \leq \pi e^R(1+e^{-2R}).$$

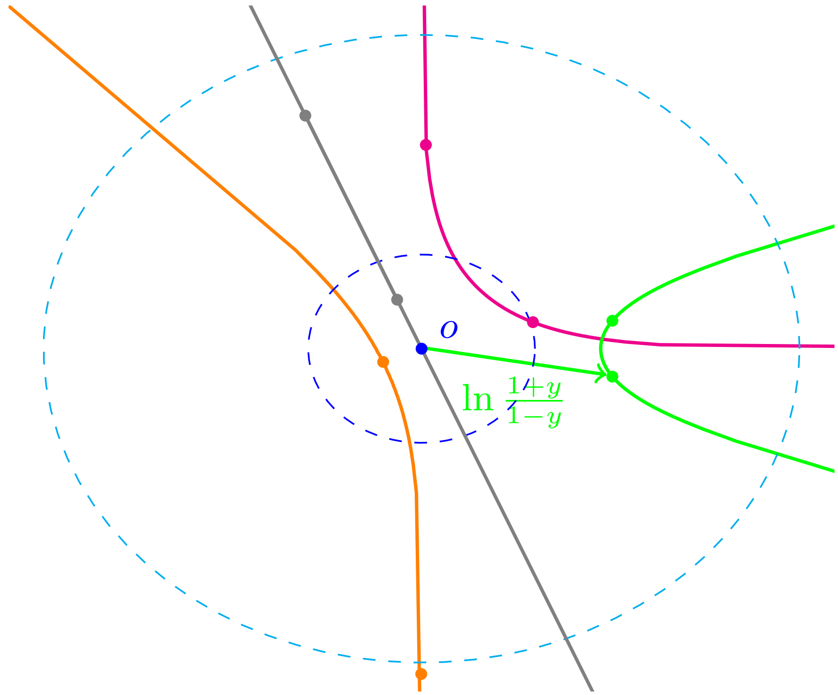
[4] Actually, true for hyperbolic space of so called curvature -1.

[5] Recall that $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.

Poincaré vs Native representation of $\mathbb{H}^2$

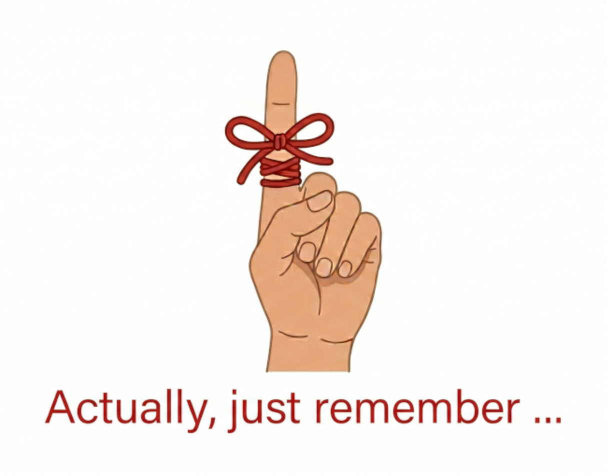


Poincaré model



Native representation.

Henceforth ...



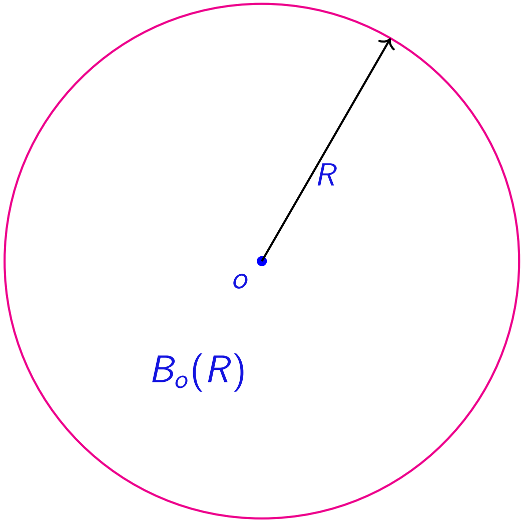
n and R are related.

Formal definition of RHG *uniform* monodel: $\mathcal{U}^H_{\alpha,\nu}(n)$

(Gugelmann, Panagiotou, Peter [ICALP'12])

Model parameters:

$\alpha, \nu \in \mathbb{R}_+, n \in \mathbb{N}_+.$



Choose an n-node graph $G = (V, E)$ as follows:

- Choose location of each $v \in V$ uniformly and independently in $B_o(R)$. Choose location (r_v, θ_v) of $v \in V$ so $\theta_v \sim \mathrm{Unif}([0, 2\pi))$ and independent of r_v with density:

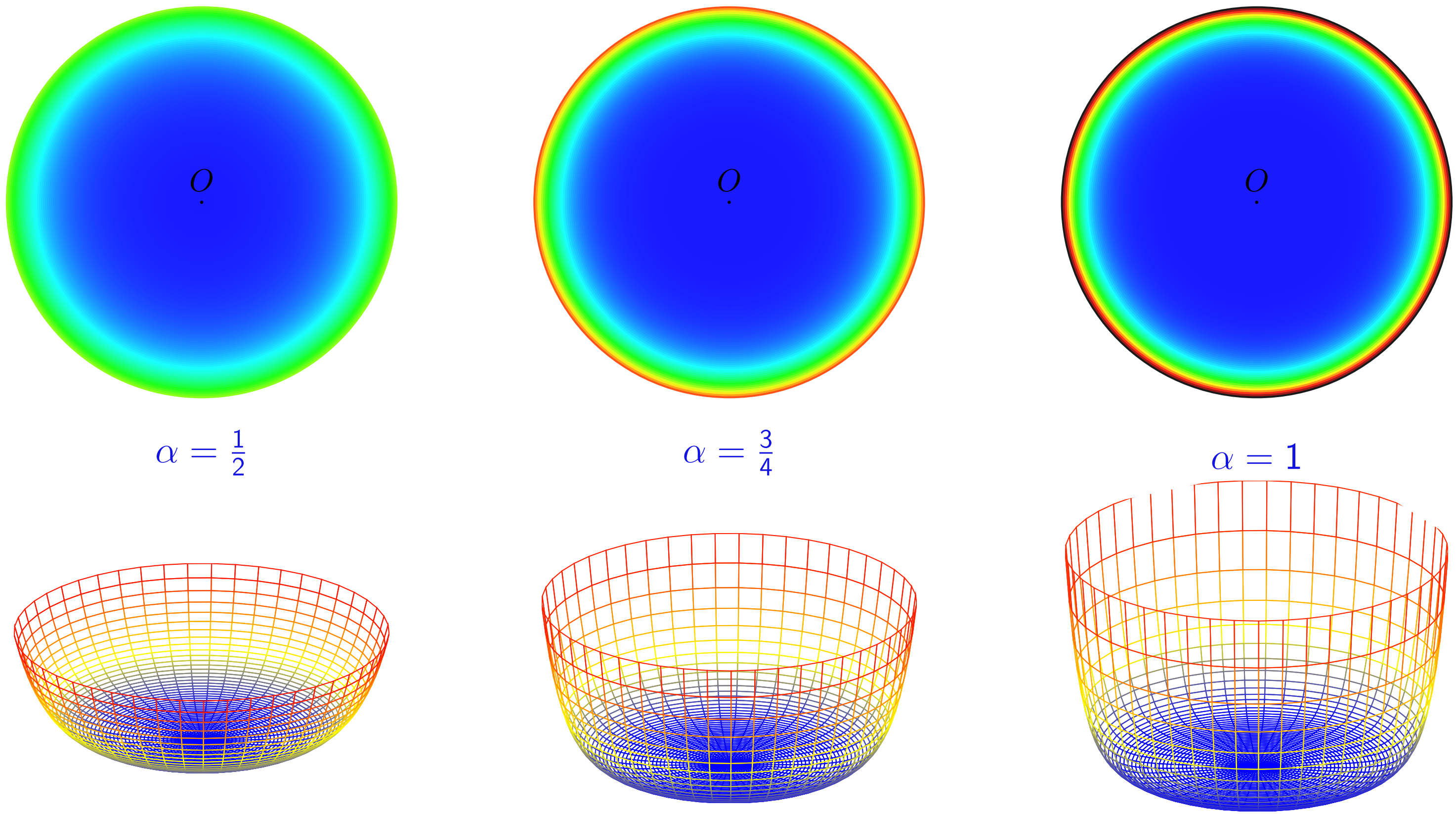
$$f(r) := \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1}, \qquad \text{if } 0 \leq r < R \text{ and } 0 \text{ otherwise.}$$

  (Note: denominator is just a normalizing constant.)
- Let $uv \in E$ iff $\mathrm{dist}_{\mathbb{H}^2}(u, v) < R$.[6]

[6]That is, hyperbolic distance between u and v is at most R.

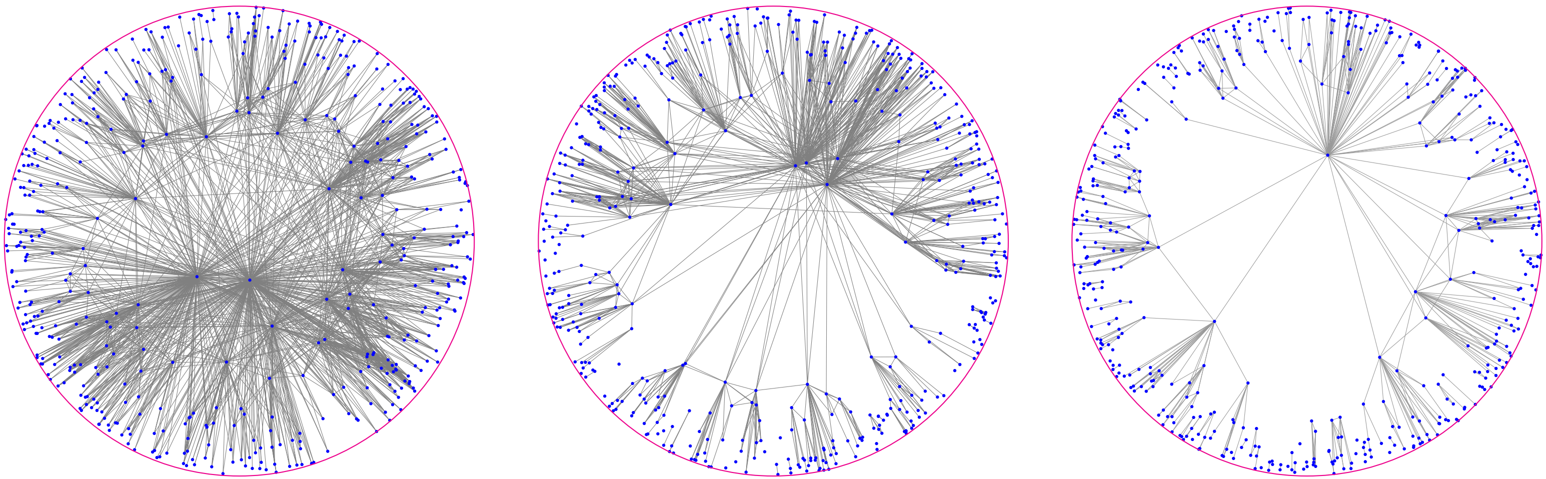
Pdf of (r_v, θ_v) and its heat plot

(Colder colors correspond to smaller density)



Examples of RHGs

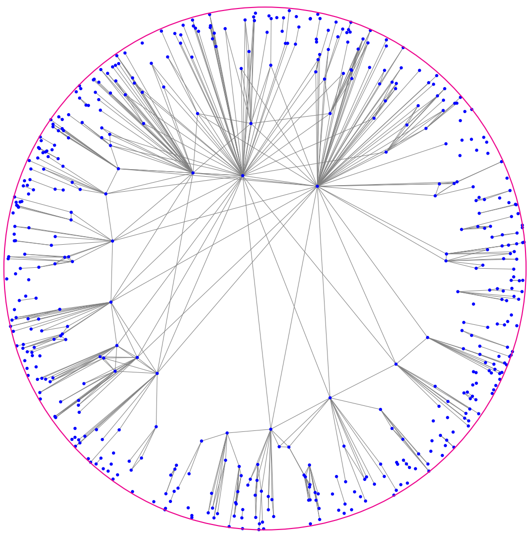
($\nu = 1$ fixed, $n = 500$)



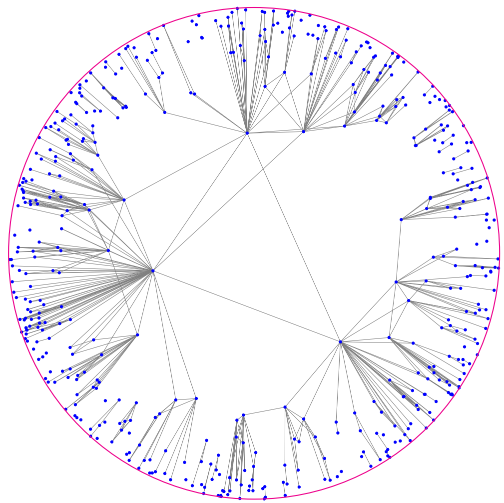
$\alpha = 0.60$ $\alpha = 0.75$ $\alpha = 0.90$

Examples of RHGs

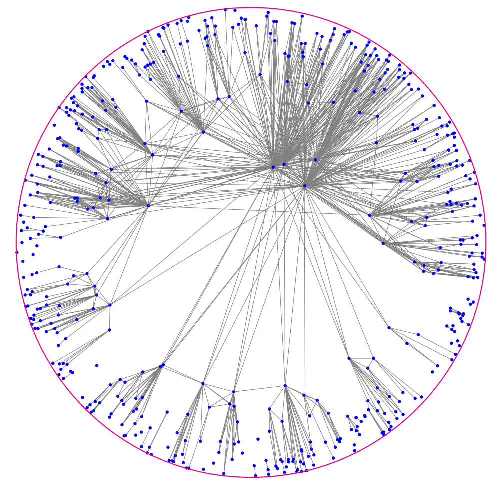
($\alpha = \frac{3}{4}$ fixed, $n = 500$)



$\nu = 0.50$



$\nu = 0.75$



$\nu = 1.00$

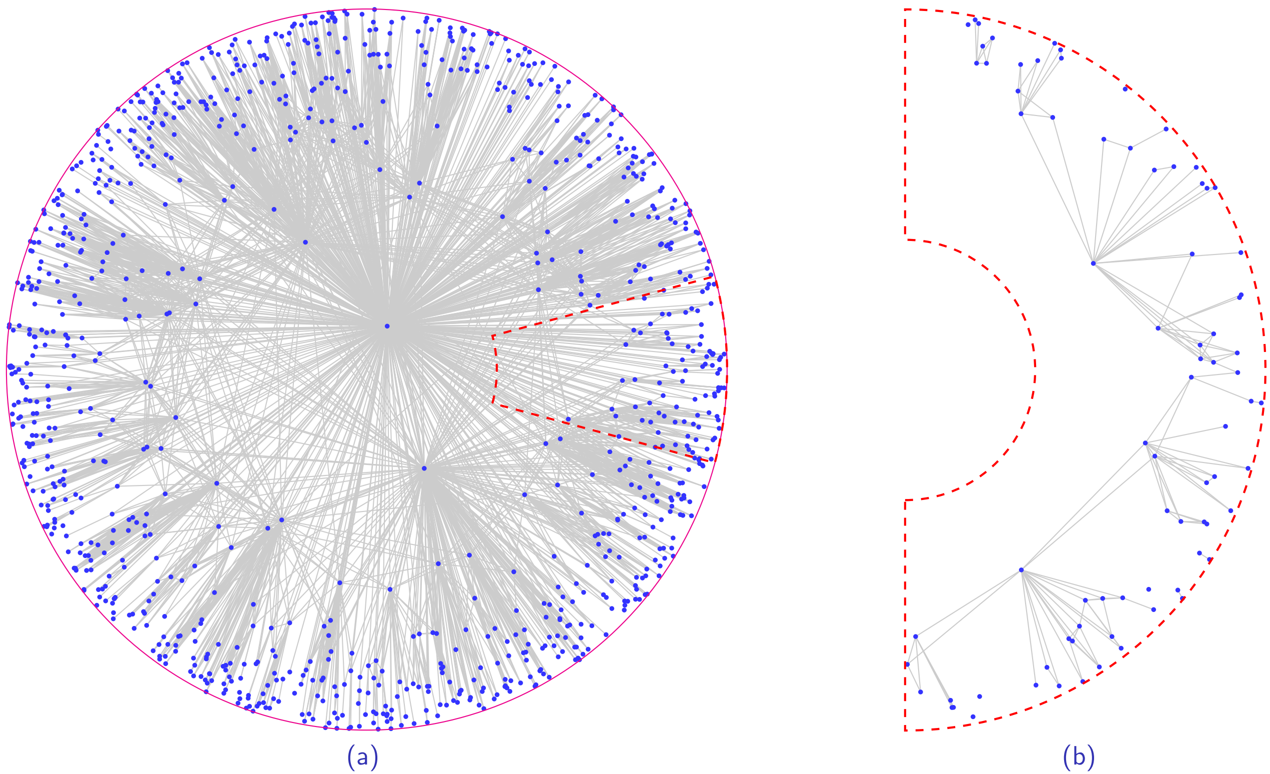
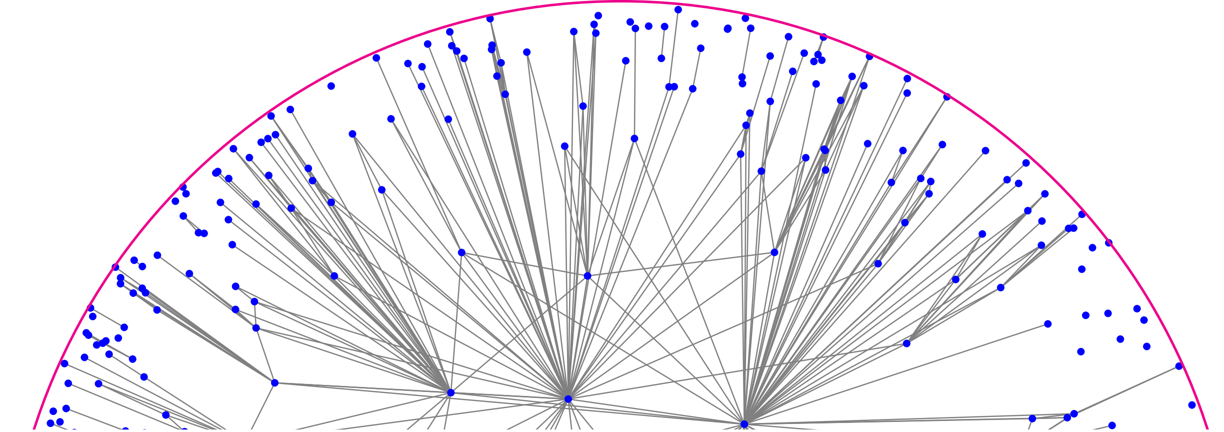
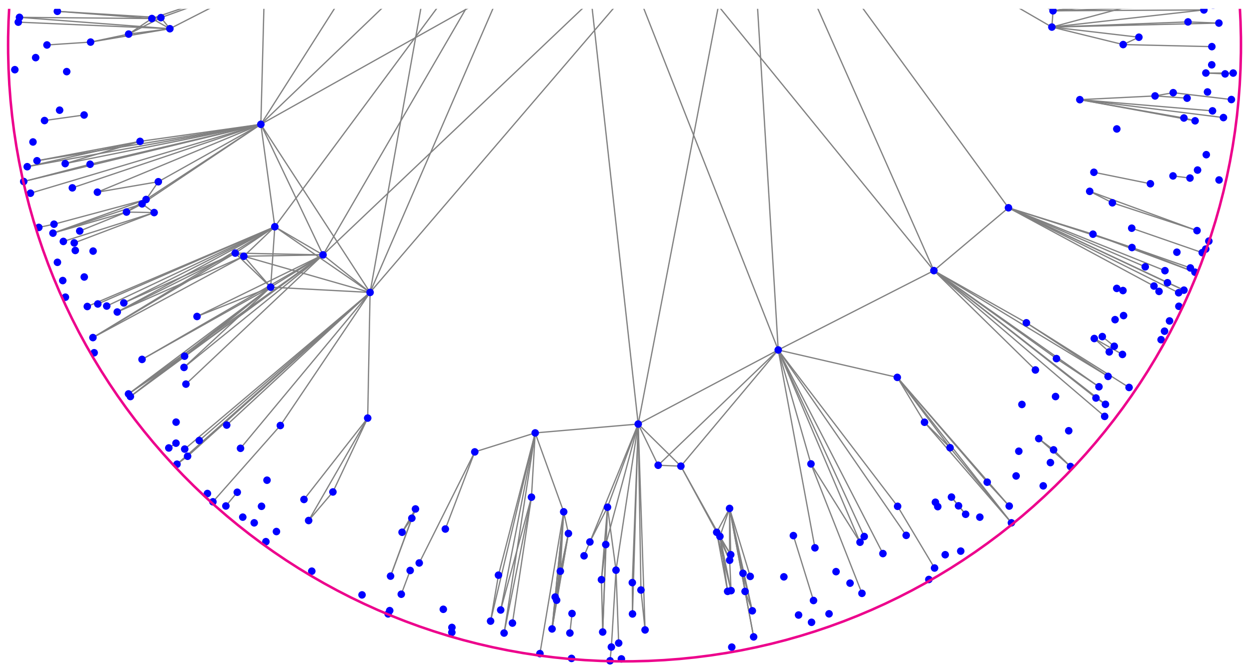


Figure: (a) An instance $\mathcal{G}_{1000}(\alpha, \nu)$ with $\alpha = 0.7$, and $\nu = 1.1$. (b) Subgraph induced by the vertices inside the dashed region in (a), where angular coordinates have been scaled by a factor of 6.



All vertices of $\mathcal{U}_{\alpha,\nu}(n)$ are within (hyperbolic) distance $R = 2\log(n/\nu)$ of each other!!

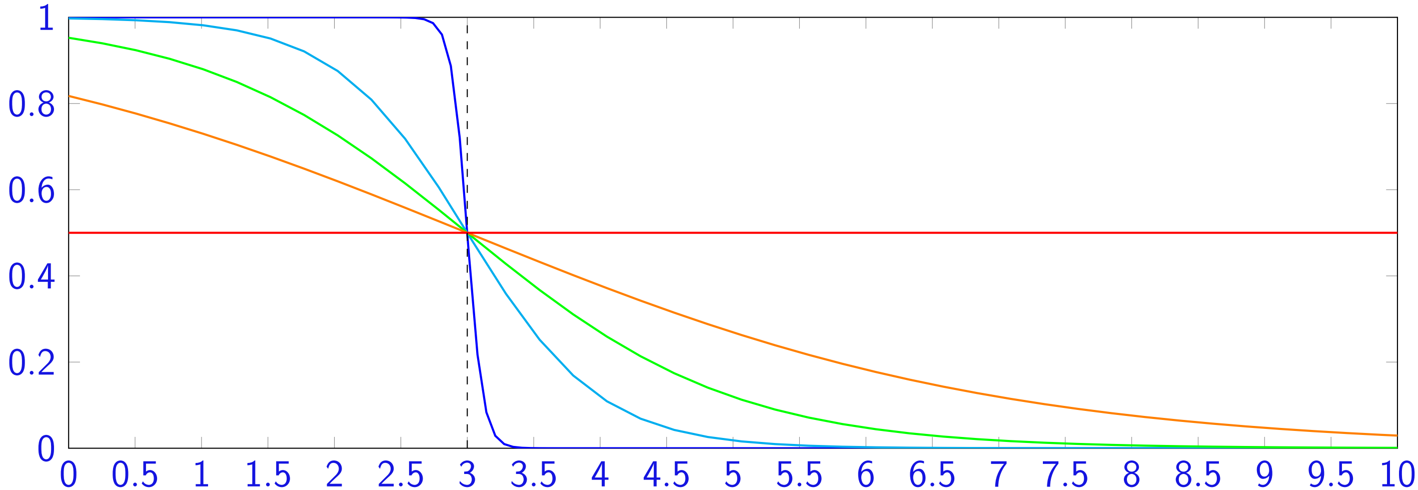


Soft version of $\mathcal{U}^H_{\alpha,\nu}(n)$

Incorporates a temperature T and a *probability of connecting* u and v:

$$p(d) := \frac{1}{1 + e^{\frac{1}{2T}(d-R)}} \qquad \text{(Logistic function)}$$

where $d := \mathrm{dist}_{\mathbb{H}^2}(u, v)$ is the (hyperbolic) distance between $u, v \in \mathbb{H}^2$.



$+\infty \approx T > T > T > T > T \approx 0$

$R = 3.0.$

Nice, but *who cares?*

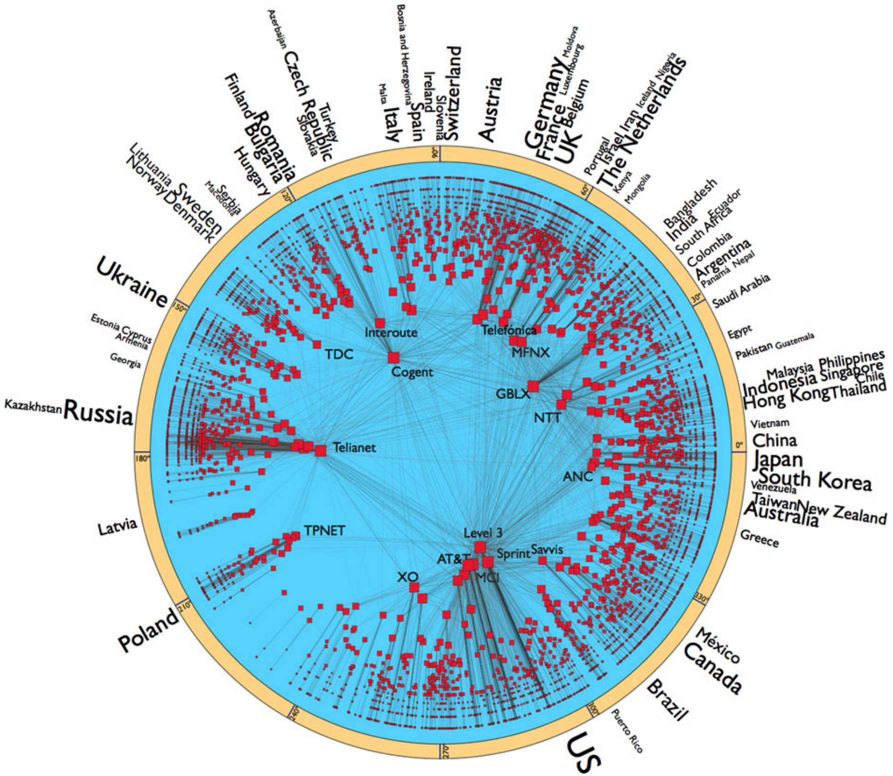
First model that "naturally" exhibits:

- Power law degree distribution, AND
- Non-negligible clustering AND
- Small-world phenomena.

But, what really drew attention ...

Mapping of Internet's Autonomous Systems (ASs)

(2009 data collected by infrastructure developed by CAIDA)



[Boguña, Papadopoulus & Krioukov. Nat. Comm. 2010)]

Data set:

- $23,752$ ASs
- $58,416$ links
- Average degree 4.92

"Maximum Likelihood" fit:

- $\alpha = 0.55$
- $R = 27$
- Temperature $T = 0.69$

Part III: Random Hyperbolic Graphs – Tools & facts

Poissonized version of $\mathcal{U}^{H}_{\alpha,\nu}(n)$ (denoted $\mathcal{P}^{H}_{\alpha,\nu}(n)$)

Let $G := \mathcal{P}^{H}_{\alpha,\nu}(n)$ be such that $V(G)$ is a inhomogeneous PPP with intensity $n \cdot f(r,\theta)$, where

$$f(r,\theta) := \frac{1}{2\pi} \cdot \frac{\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1}, \qquad \text{if } 0 \leq \theta < 2\pi \text{ and } 0 \leq r < R.$$

is the joint pdf of (r_v, θ_v) for vertex v of $\mathcal{U}^{H}_{\alpha,\nu}(n)$.

Henceforth,

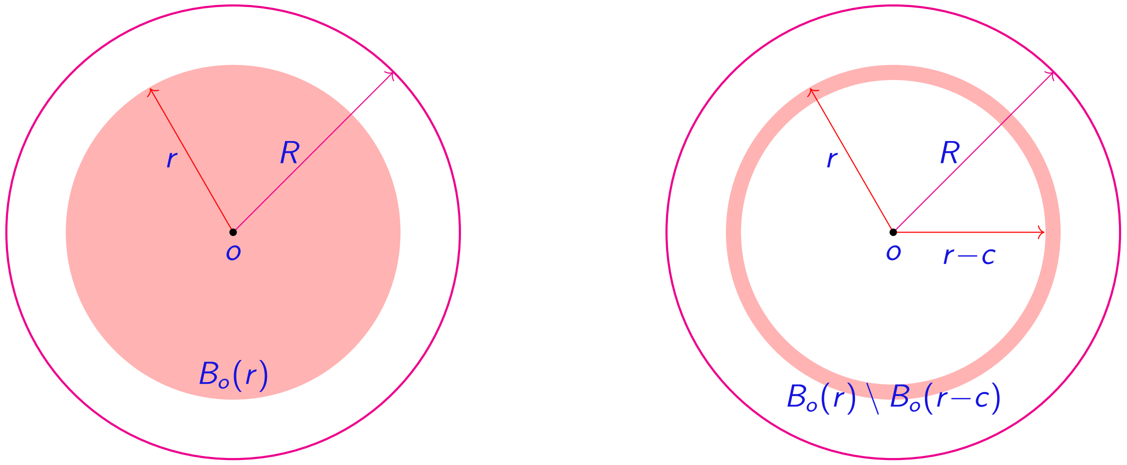
$$\mu(S) = \iint_S f(r,\theta) d\theta dr.$$

In particular,[7]

$$\mu(B_o(r)) = e^{-\alpha(R-r)}(1+o(1)).$$

So, for $c > 0$ a fixed constant,

$$\mu(B_o(r) \setminus B_o(r-c)) = \Theta(\mu(B_o(r))).$$



Consequences:

- If $G := \mathcal{P}^H_{\alpha,\nu}(n)$, then $|V(G)|$ is a $\mathrm{Poiss}(n)$ sharply concentrated around $\mathbb{E}(|V(G)|) = n$.
- For any constant $c > 0$, a.a.s., a constant fraction $\rho := \rho(c)$ of vertices of G are within distance c of $B_o(R)$'s boundary!

[7] Where $B_o(\rho) = \{p \in \mathbb{H}^2 \mid \mathrm{dist}_{\mathbb{H}^2}(o,p) < \rho\}$.

Henceforth ...

... formula for $\mu(B_o(\rho))$ is simple.

Central Clique

Let $u, v \in V(G)$. If $r_u + r_v \leq R$, then by Δ-inequality

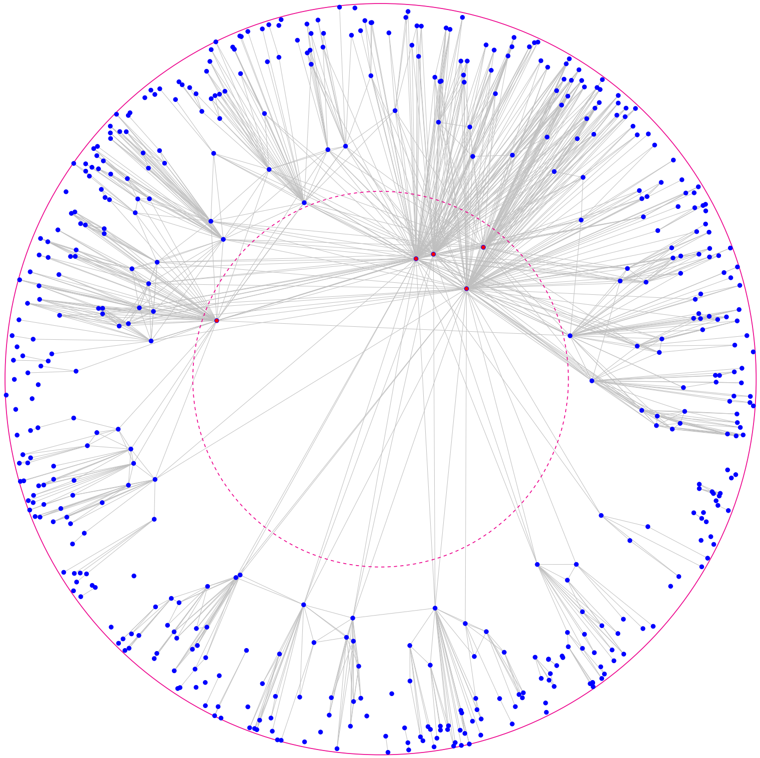
$$\mathrm{dist}_{\mathbb{H}^2}(u, v) \leq \mathrm{dist}_{\mathbb{H}^2}(u, o) + \mathrm{dist}_{\mathbb{H}^2}(o, v) = r_u + r_v \leq R,$$

and thus $uv \in E(G)$.

Consequence: $S := V(G) \cap B_o(\frac{R}{2})$ induces a clique in $G := \mathcal{P}^H_{\alpha,\nu}(n)$ of expected size

$$n \cdot \mu(B_o(\tfrac{R}{2})) = n \cdot e^{-\alpha \frac{R}{2}} = n \cdot \left(\frac{n}{\nu}\right)^{-\alpha} = \Theta(n^{1-\alpha}),$$

(so, $|S|$ is sharply concentrated around its mean).



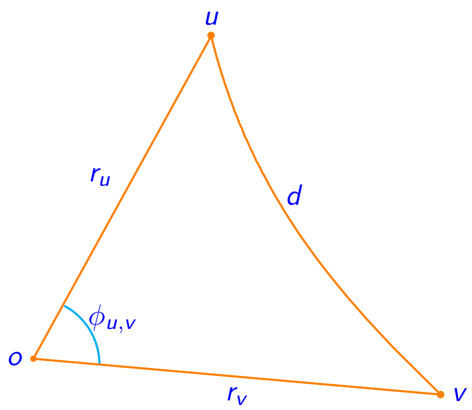
Central Clique of RHG.

Calculating distances

Hyperbolic distance from v to origin o, ... easy! Just r_v.

In general, use hyperbolic law of cosines:

$$\mathrm{dist}_{\mathbb{H}^2}(u,v) = d \iff \cosh(d) = \cosh(r_u)\cosh(r_v) - \sinh(r_u)\sinh(r_v)\cos(\phi_{u,v}).$$



Key technical observation

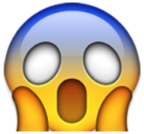
Define $\theta_R(\rho, \rho') := \sup\left\{\phi_{p,p'} \mid p, p' \in \mathbb{H}^2, \mathrm{dist}_{\mathbb{H}^2}(p, p') < R, r_p = \rho, r_{p'} = \rho'\right\}$.

Thus,

$$\phi_{u,v} \leq \theta_R(r_u, r_v) \iff \mathrm{dist}_{\mathbb{H}^2}(u, v) < R. \qquad \text{(Very useful/important!!)}$$

By the hyperbolic law of cosines, we get

$$\theta_R(r_u, r_v) = \mathrm{arcos}\left(\frac{\cosh(R) - \cosh(r_u)\cosh(r_v)}{\sinh(r_u)\sinh(r_v)}\right).$$



Also important: $\theta_R(\rho, \rho')$ is non-increasing in both ρ and ρ'.

Key technical observation (cont.)

Gugelmann et al. showed (2012) that if $r_u + r_v > R$, then

$$\theta_R(r_u, r_v) := 2e^{\frac{1}{2}(R-r_u-r_v)}(1 + \Theta(e^{R-r_u-r_v})) = \Theta(e^{\frac{1}{2}(R-r_u-r_v)}).$$

By triangle inequality, $\theta_R(r_u, r_v) = \pi$ if $r_u + r_v \leq R$.

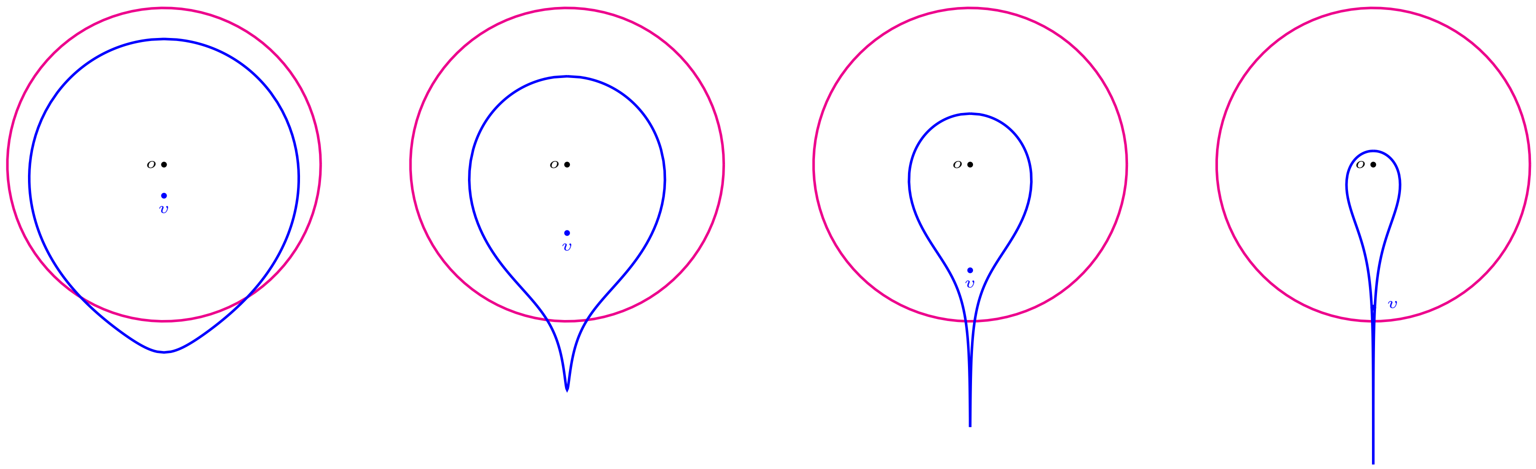
Henceforth ...

there is an easy formula for $\theta_R(\cdot, \cdot)$.

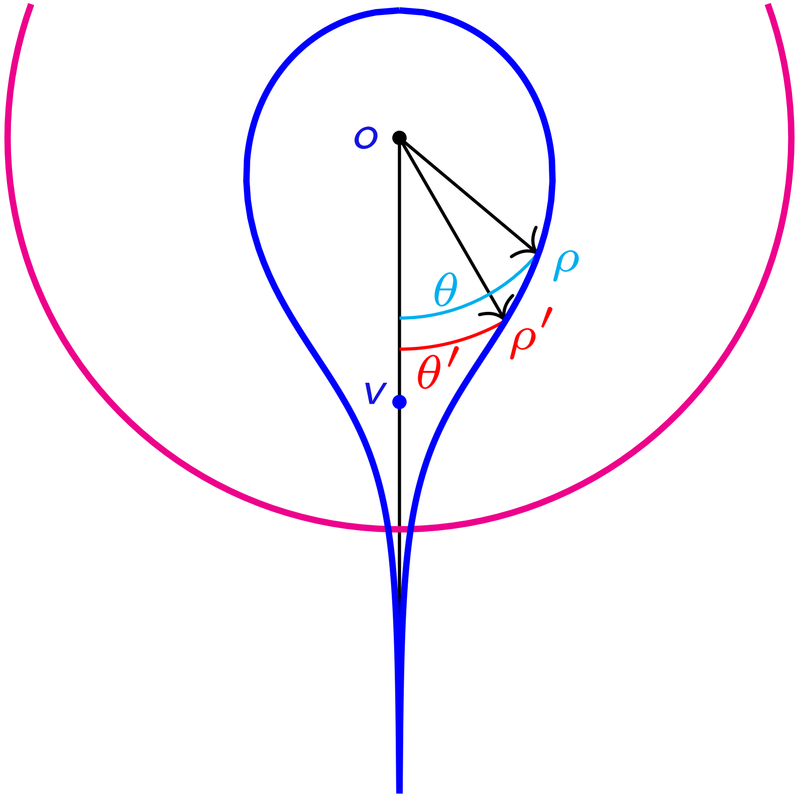
Illustration of balls centered at $v \neq o$

Observe that $B_v(R) = \{p \in \mathbb{H}^2 \mid \mathrm{dist}_{\mathbb{H}^2}(p, v) < R\} = \{p \in \mathbb{H}^2 \mid \phi_{p,v} \leq \theta_R(r_p, r_v)\}$.

Illustration of $B_v(R)$ for different $v \in B_o(R)$.



Monotonicity of $\theta_R(\cdot, \cdot)$ illustrated

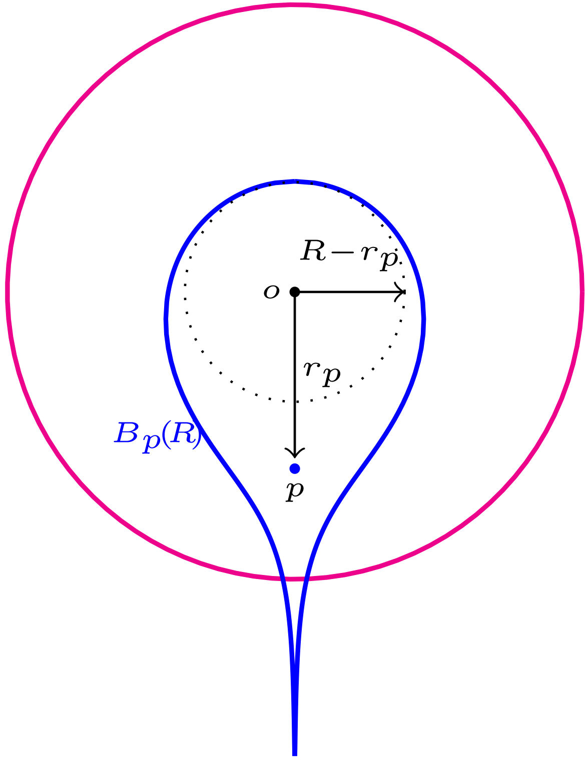


if $\rho' > \rho$, then $\theta = \theta_R(r_v, \rho) \geq \theta_R(r_v, \rho') = \theta'$

Vertex degrees

Question: Say $p \in \mathbb{H}^2$ and $G := \mathcal{P}^H_{\alpha,\nu}(n)$. What is $\mathbb{E}\big(|V(G) \cap B_p(R)|\big)$?

Answer: We have, $|V(G) \cap B_p(R)| \sim \mathrm{Poiss}(n \cdot \mu(B_p(R)))$.



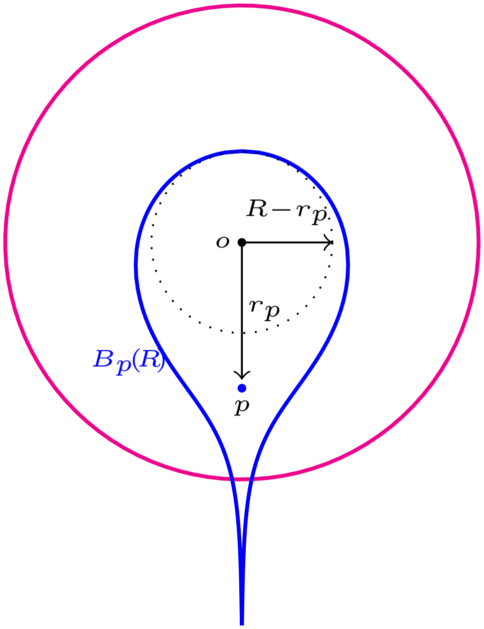
Where,

$$\mu(B_p(R)) = \int_{B_p(R)} f(r,\theta) d\theta dr$$
$$= \underbrace{\mu(B_o(R - r_p))}_{=(1+o(1))e^{-\alpha r_p}} + \int_{R-r_p}^{R} \int_{-??}^{+??} f(r,\theta) d\theta dr.$$

Vertex degree (cont.)

So $|V(G) \cap B_p(R)| \sim \mathrm{Poiss}(n \cdot \mu(B_p(R)))$ where

$$\mu(B_p(R)) = e^{-\alpha r_p}(1 + o(1)) + \int_0^R \int_{-\theta_R(r,r_p)}^{+\theta_R(r,r_p)} f(r,\theta) d\theta dr.$$



But, in fact, computing integrals is not necessary!

First, because it has already been done:

Lemma: [Gugelmann et al. (2012)] If $\alpha > \frac{1}{2}$, then

$$\mu(B_p(R)) = \frac{2}{\pi\alpha(\alpha - \frac{1}{2})} e^{-\frac{1}{2}r_p} (1 + O(e^{-(\alpha - \frac{1}{2})r_p})) = \Theta(e^{-\frac{1}{2}r_p}).$$

Second, because its not very insightful!

Corollary: Conditioned on $r_v = r$ we have $\mathbb{E}(\mathrm{dgr}_G(v)) = \Theta(n \cdot \mu(B_v(R))) = \Theta(e^{\frac{1}{2}(R-r_v)})$.

Warning: The conditioning above is made on an event of probability 0. This can be formalized via *Palm distribution*.

Campbell's theorem (see also Mecke's formula)

Calculating $\mathbb{E}(\mathrm{dgr}_G(v))$ is a special case of computing

$$\mathbb{E}\Big(\sum_{u\in\mathcal{V}_n\setminus\{v\}}\varphi(u)\Big) \qquad \text{with } \varphi(u)=\mathbf{1}_{\{u\in B_v(r)\}}.$$

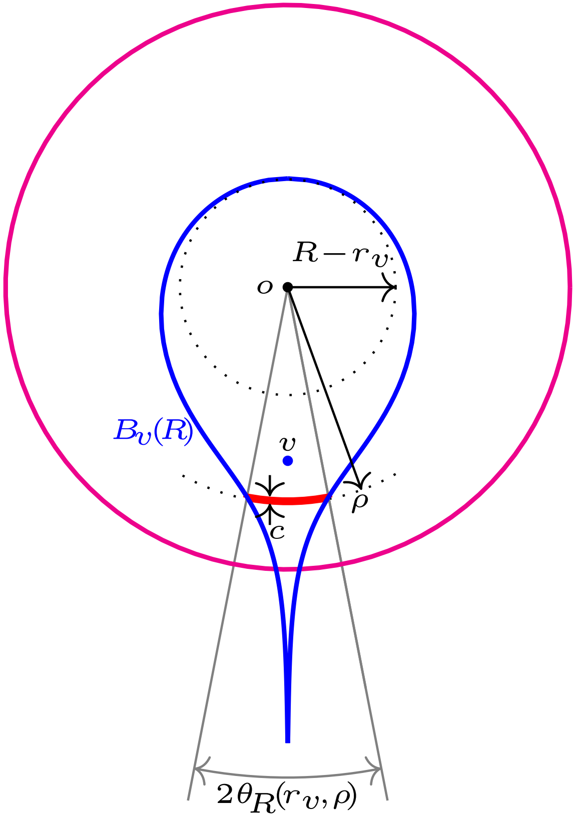
A useful result concerning PPPs of intensity λ, is that for any measurable $\varphi : B_o(R)\to\mathbb{R}$.

$$\mathbb{E}\Big(\sum_x \varphi(x)\Big)=\int_{B_o(R)}\varphi(x)\lambda(dx)$$

where the sum is over the points x of the PPP.

Additional comments

For constant $c > 0$, lets compute: $\mu\big(\underbrace{B_v(R) \cap (B_o(\rho) \setminus B_o(\rho - c))}_{A_v(\rho)}\big)$.



Observe:

$$\begin{aligned}\mu(A_v(\rho)) &\geq \mu\big(B_o(\rho) \setminus B_o(\rho - c)\big) \cdot \frac{1}{2\pi} \cdot 2\theta_R(r_v, \rho) \\ &= \Theta(\mu\big(B_o(\rho)\big) \cdot \theta_R(r_v, \rho)\big) \\ &= \Theta\big(e^{-(\alpha - \frac{1}{2})(R-\rho)} \cdot e^{-\frac{1}{2}r_v}\big).\end{aligned}$$

Similar derivation for upper bound gives:

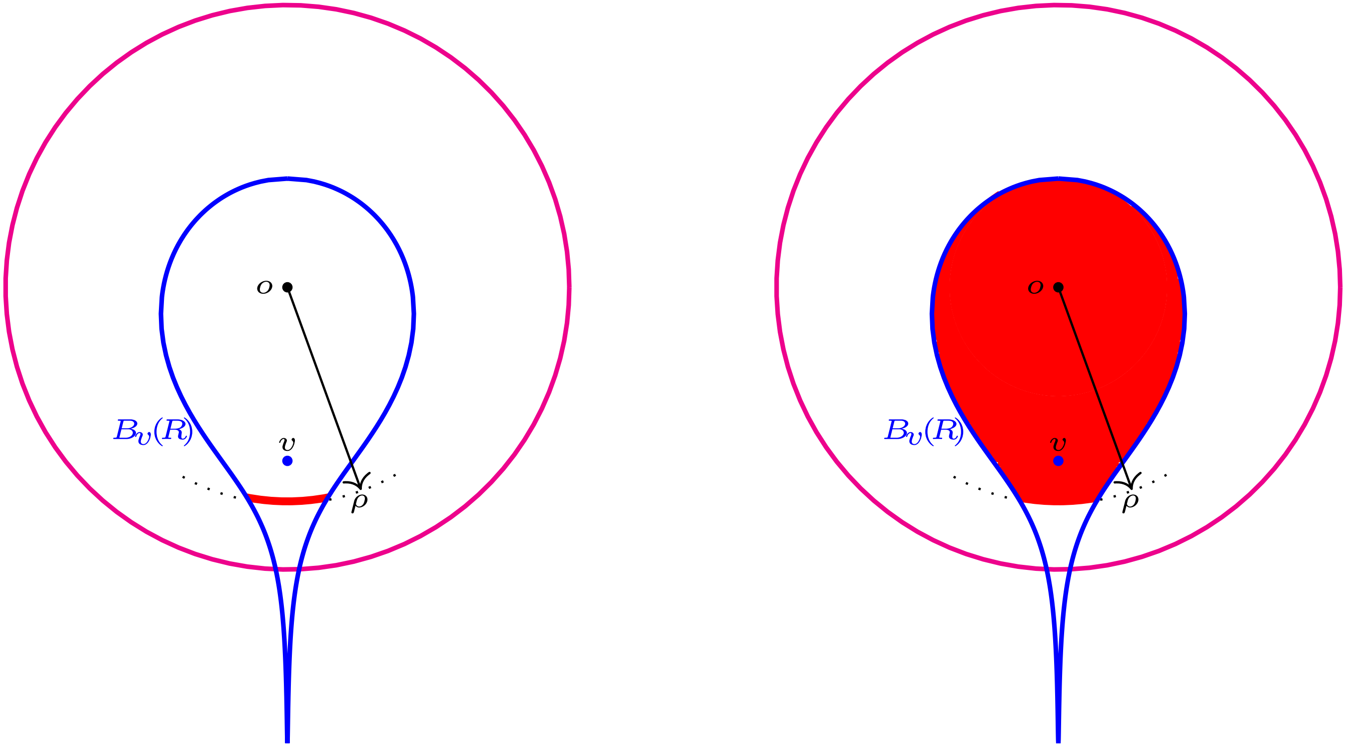
$$\mu(A_v(\rho)) = \Theta\big(e^{-(\alpha - \frac{1}{2})(R-\rho)} \cdot e^{-\frac{1}{2}r_v}\big).$$

Note: $\mu(A_v(\rho))$ is increasing in ρ when $\alpha > \frac{1}{2}$.

Moreover:

$$\mu\big(B_v(R) \cap B_o(\rho)\big) = \Theta\big(\mu\big(A_v(\rho)\big)\big).$$

Up to constants, the red regions below have the same measure (provided $\alpha > 1/2$).



Summary and consequences (assuming $\alpha > 1/2$):

- $\mathbb{E}(\mathrm{dgr}_G(v)) = \Theta(e^{\frac{1}{2}(R-r_v)})$ decreases (exponentially fast) with r_v.
- $\mathbb{E}(|\mathrm{Neigh}_G(v) \setminus B_o(R-c)|) = \Theta(\mathbb{E}(\mathrm{dgr}_G(v)))$.
- $\mathbb{E}(|\mathrm{Neigh}_G(v) \cap (B_o(\rho) \setminus B_o(\rho-c))|)$ increases (exponentially fast) with ρ.

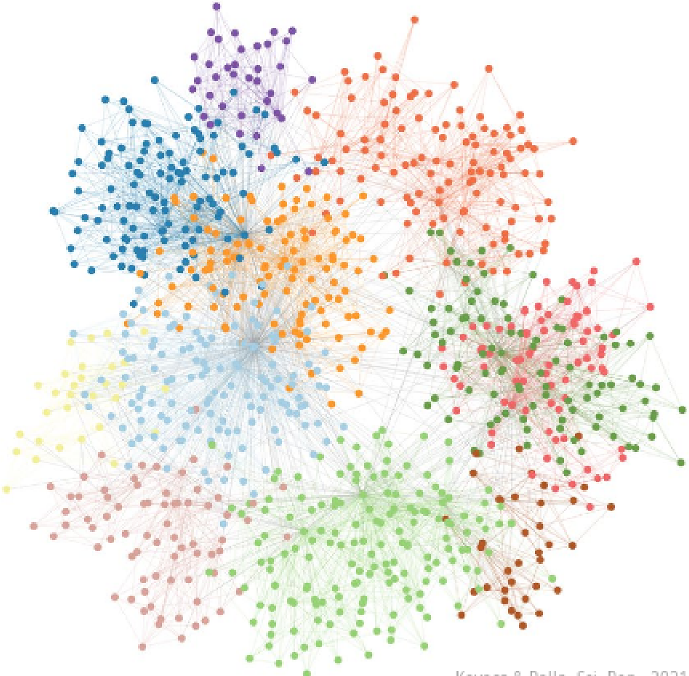
Since $n \cdot \mu(B_o(\rho)) = \Theta\big(n \cdot e^{-\alpha(R-\rho)}\big)$, for "almost all" the relevant range of ρ, the expected number of vertices with degree at least $d = \Theta(e^{\frac{1}{2}(R-\rho)})$ is

$$\Theta(n/d^{2\alpha}).$$

Thus,

- G has power law degree distribution with exponent $2\alpha + 1$.
- G is sparse (with expected average degree $\frac{2\alpha^2\nu}{\pi(\alpha-1/2)}$).

G is not sparse if $\alpha < 1/2$.



Kovacs & Palla, Sci. Rep., 2021.

Empirically observed: social networks have power law degree distribution between 2 and 3.

Henceforth ...

Do Not Forget!

$$\alpha > \frac{1}{2}.$$

RHGs are a.a.s. not connected (for $\alpha > 1/2$)

Just because:

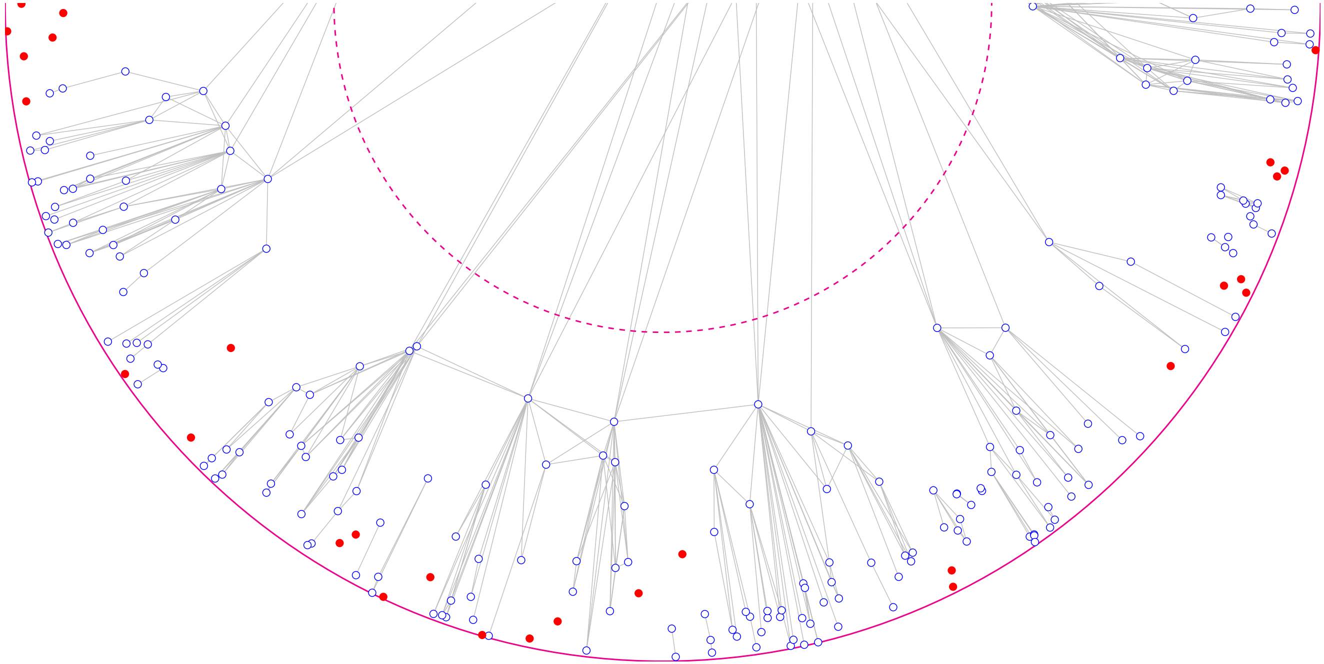
- $p \in B_o(R) \setminus B_o(R-c)$ implies that $\mu(B_p(R)) = \Theta(1/n)$.
- Hence, v is isolated with probability

$$\mathbb{P}\big(V(G) \cap B_v(R) = \varnothing\big) = e^{-n \cdot \mu(B_v(R))} = e^{-\Theta(1)}.$$

- Moreover, $|V(G) \cap (B_o(R) \setminus B_o(R-c))|$ is concentrated around

$$n \cdot \mu(B_o(R) \setminus B_o(R-c)) = \Theta\big(n \cdot \mu(B_o(R))\big) = \Theta(n).$$

So, a.a.s., a constant fraction of vertices of G are isolated.



Isolated vertices shown in red.

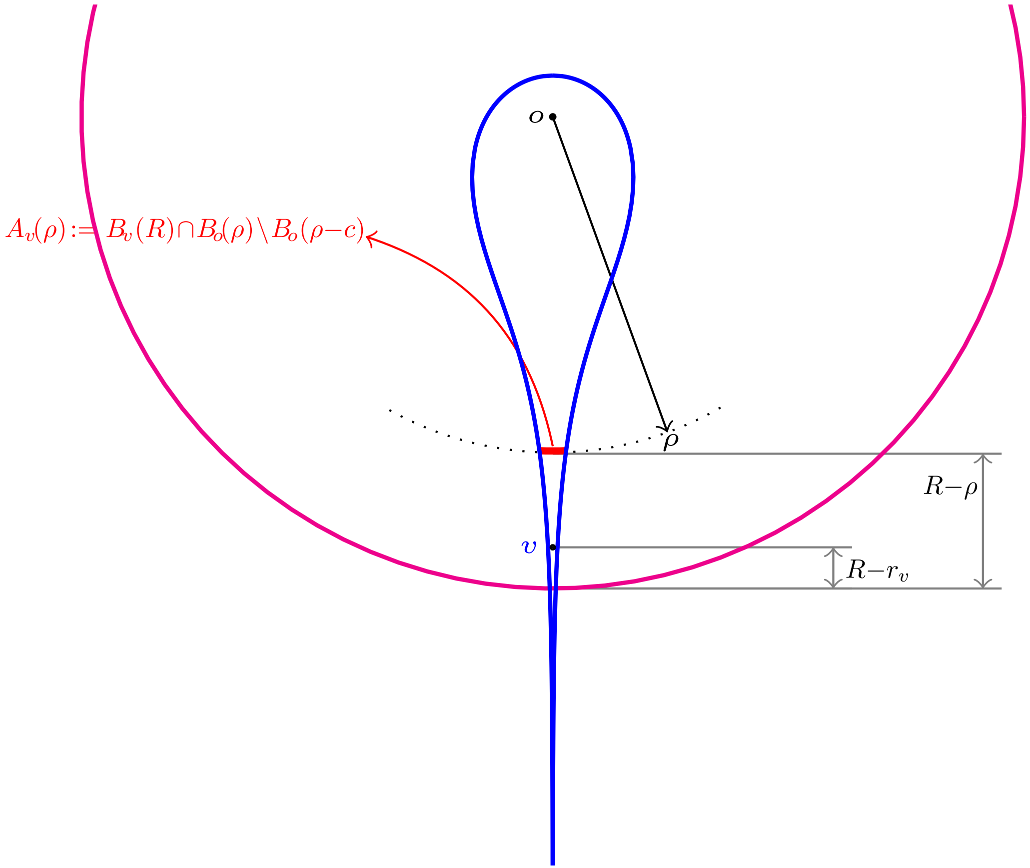
Average local clustering coefficient of RHGs

SPOILER ALERT

Well, ... not really!

Average local clustering coefficient of RHG is non-negligible, i.e., $\Omega(1)$ as for RGGs.

Neighbor of v closest to the origin



The expected number of vertices in $V(G) \cap A_v(\rho)$ is 1 iff $n \cdot \mu(A_v(\rho)) = 1$.

Neighbor of v closest to the origin

Note that:

$$n \cdot \mu(A_v(\rho)) = 1 \iff R - \rho = \frac{1}{2\alpha - 1}(R - r_v) + \Theta(1).$$

So, if we fix ρ slightly larger, say $R - \rho = \frac{1-\epsilon}{2\alpha-1}(R - r_v) + \Theta(1)$ with $\epsilon > 0$, then

$$\mathbb{P}\big(\exists v_1 \in V(G) \cap A_v(\rho)\big) = 1 - e^{-n\cdot\mu(A_v(\rho))} = 1 - \exp\big(-\Omega\big([\mathbb{E}(\mathrm{dgr}_G(v))]^{\epsilon}\big)\big),$$

and

$$R - r_{v_1} = \underbrace{\frac{1-\epsilon}{2\alpha - 1}}_{f := f(\epsilon, c)}(R - r_v) + \Theta(1) \qquad \text{and} \qquad \mathbb{E}\big(\mathrm{dgr}_G(v_1)\big) = \Theta\Big(\big[\mathbb{E}\big(\mathrm{dgr}_G(v)\big)\big]^{f}\Big)$$

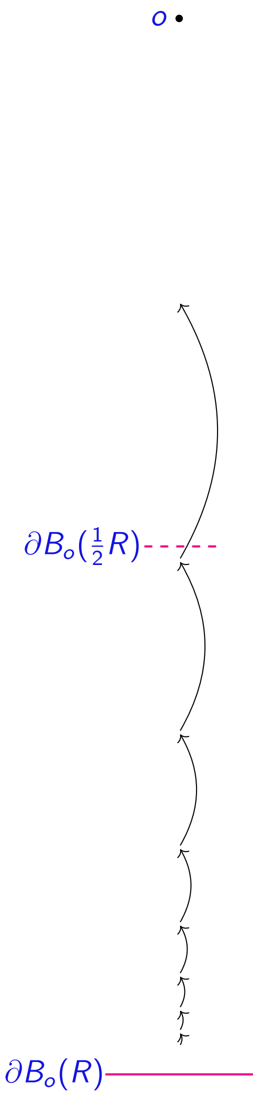
Important: We need $\alpha < 1$ for v_1 to be closer to the origin than v!

Henceforth ...

Do Not Forget!

$$\alpha < 1$$

Exploding path



Say that $v_0 = v, v_1, \ldots, v_k \in V(G)$ is a (φ, ϵ)-*exploding path* (*exploding path* for short) between v and the Central Clique if

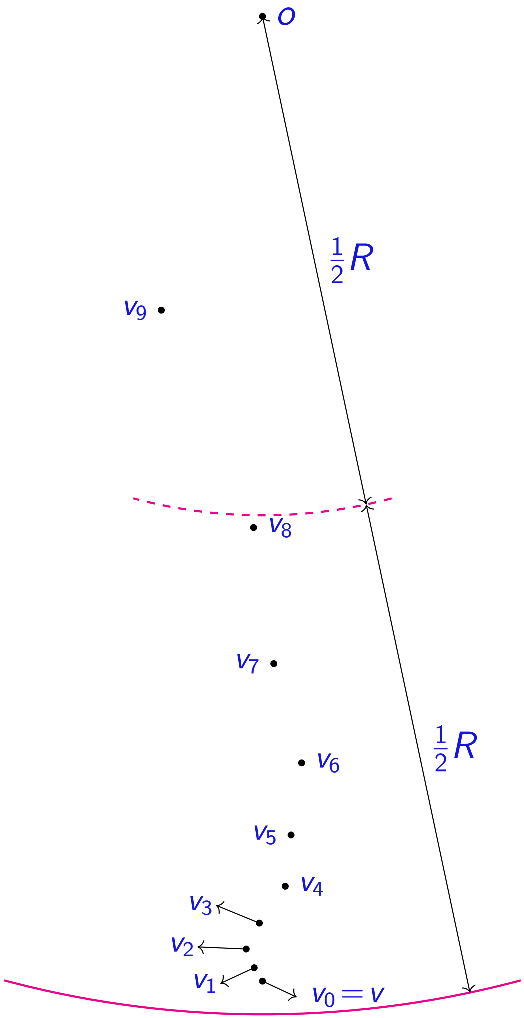
$$\forall i \in \{1, \ldots, k\}, \quad 0 \le \left(R - r_{v_i}\right) - f \cdot \left(R - r_{v_{i-1}}\right) \le \varphi,$$

and

$$r_{v_k} \le \frac{1}{2}R.$$

Unless said otherwise, assume $\varphi > 0$ is constant.

Exploding paths (cont. & illustration)



- The probability that $v = v_0$ connects via an *exploding path* with a vertex of the Central Clique is

$$\prod_{i=1}^{k} \Big(1 - \exp\Big(-\Omega\big([\mathbb{E}(\mathrm{dgr}_G(v_i))]^{\epsilon}\big)\Big)\Big)$$

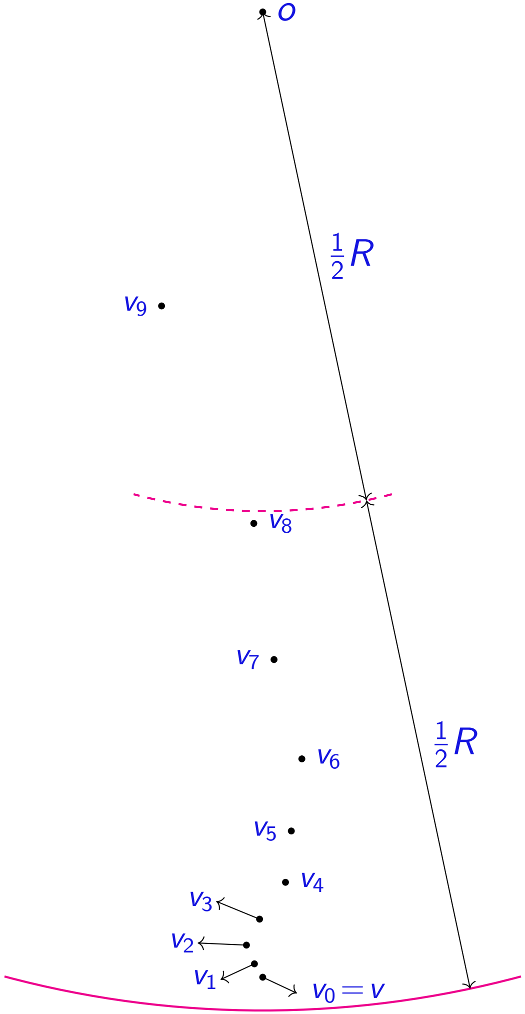
where k is the smallest integer such that

$$f^k(R - r_v) > \tfrac{1}{2}R = \log(n/\nu).$$

Hence, $k = \Theta(\log\log n)$ if $R - r_v = \Omega(1)$.

- Failure probability is dominated by failure of initial stages.

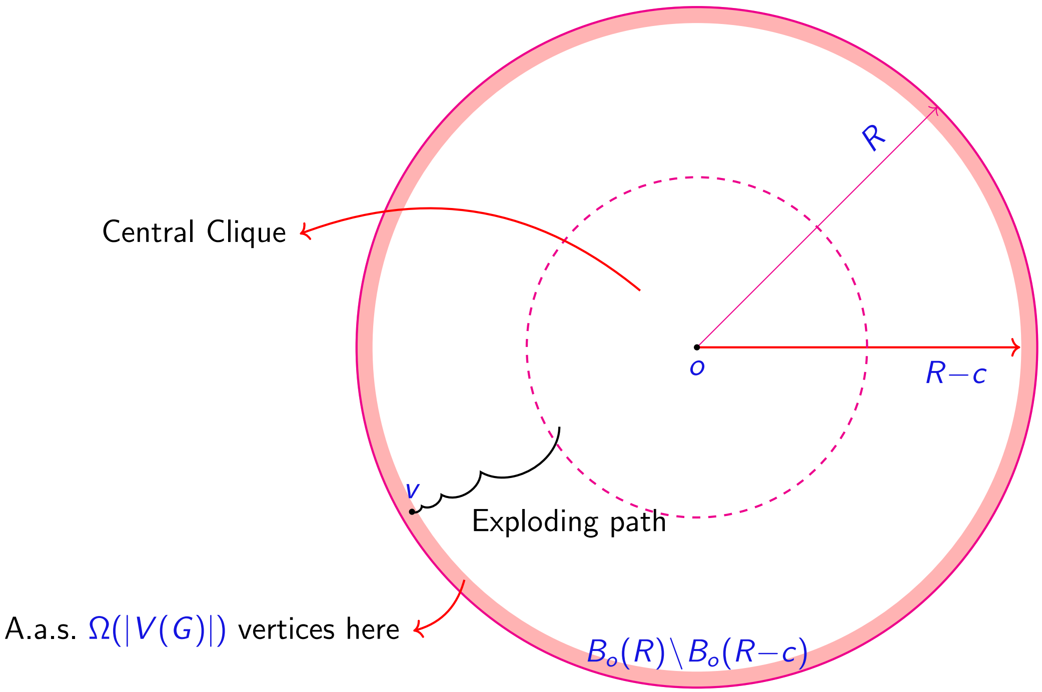
Exploding paths (cont. & summary)



There is an exploding path from $v_0 = v$ to the Central Clique

- with probability at least a strictly positive constant, if $r_v = R - \Omega(1)$.
- with probability $1 - o(1)$, if $r_v = R - \log\log R$ (recall $R = \Theta(\log n)$).

Giant component

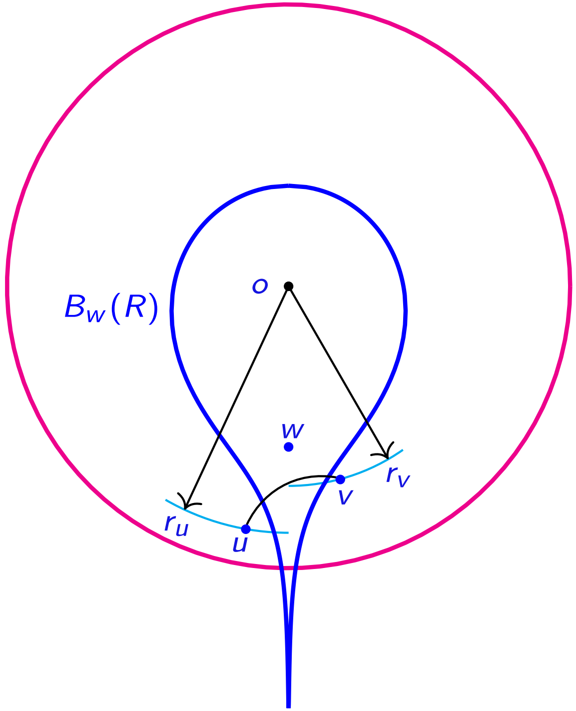


For $\alpha < 1$, existence (with probability $\Omega(1)$) of exploding path from v to central Clique

$$\Longrightarrow \mathbb{E}\big(L_1(\mathcal{P}^H_{\alpha,\nu}(n))\big) = \Omega(n).$$

For $\alpha > 1$ the largest component's expected size is $o(n)$.

Key geometric observation (case 1)



Case 1: $r_w \le \min\{r_u, r_v\}$.

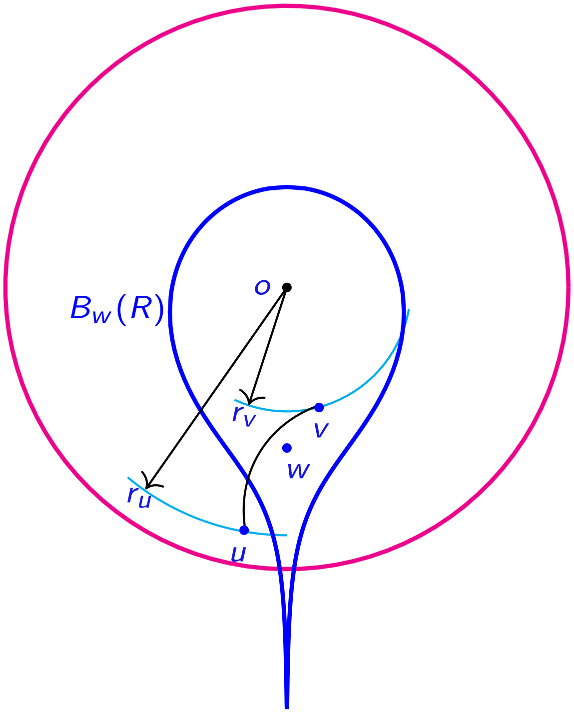
If $uv \in E(G)$ (i.e., $\mathrm{dist}_{\mathbb{H}^2}(u, v) < R$), then

$$\phi_{u,w} \le \phi_{u,v} < \theta_R(r_u, r_v) \le \theta_R(r_u, r_w),$$

so $uw \in E(G)$.

Similarly, if $uv \in E(G)$, then $vw \in E(G)$.

Key geometric observation (case 2)



Case 2: $r_v \le r_w \le r_u$.

If $uv \in E(G)$ (i.e., $\mathrm{dist}_{\mathbb{H}^2}(u, v) < R$), then

$$\phi_{w,v} \le \phi_{u,v} < \theta_R(r_u, r_v) \le \theta_R(r_w, r_v),$$

so $vw \in E(G)$.

Part IV: Ultra-small average distance, navigability & greedy routing

Ultra-small average distance (claim)

Thm: If s, t are uniformly chosen vertices in the giant of $G := \mathcal{P}^{H}_{\alpha,\nu}(n)$, then a.a.s., [ABF'17]

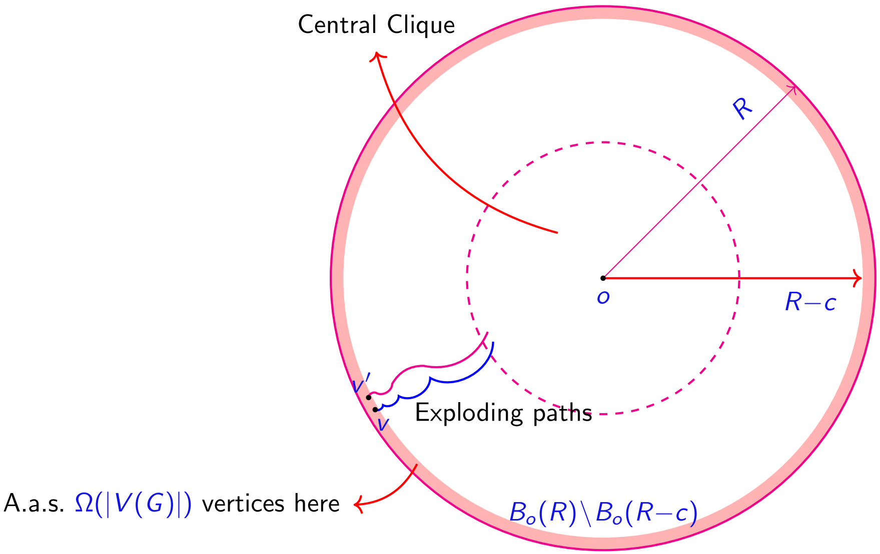
$$\mathrm{dist}_G(s, t) = O(\log\log n).$$

We showed that, with probability $1 - o(1)$, there is an exploding path from v to the Central Clique if $r_v \leq R - \log\log R$. Hence, claim holds when $s, t \in B_o(R - \log\log R)$.

It suffices to show that, with probability $1 - o(1)$, for a uniformly chosen vertex $v \notin B_o(R - \log\log R)$ not in the giant, there is a length $O(\log\log n)$ path connecting v to and exploding path to the Central Clique. Then, apply the result with $v = s$ and $v = t$.

Ultra-small average distance (intuition)

Note/Recall: The expected number v's in $B_o(R) \setminus B_o(R-c)$ connecting via an exploding path to a vertex in $B_o(R - \log\log R)$ is $\Omega(n)$.

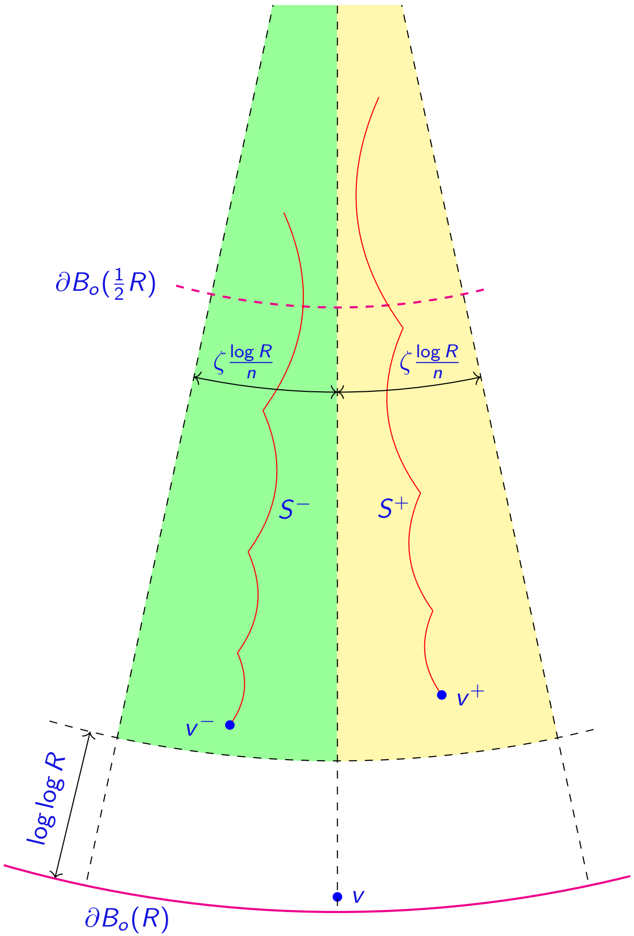


Intuitively, vertices of the giant component not in $B_o(R - \log\log R)$ "sandwiched between" exploding paths should either

- connect directly to a vertex in $B_o(R - \log\log R)$, or
- connect via a path to one of the "boundary" exploding paths – because of geometry!. One expects these paths to be short!

So, we expect that a $1 - o(1)$ fraction of vertices in the giant but not in $B_o(R - \log\log R)$ connect via "short" paths to the Central Clique.

Ultra-mall average distance (formal)



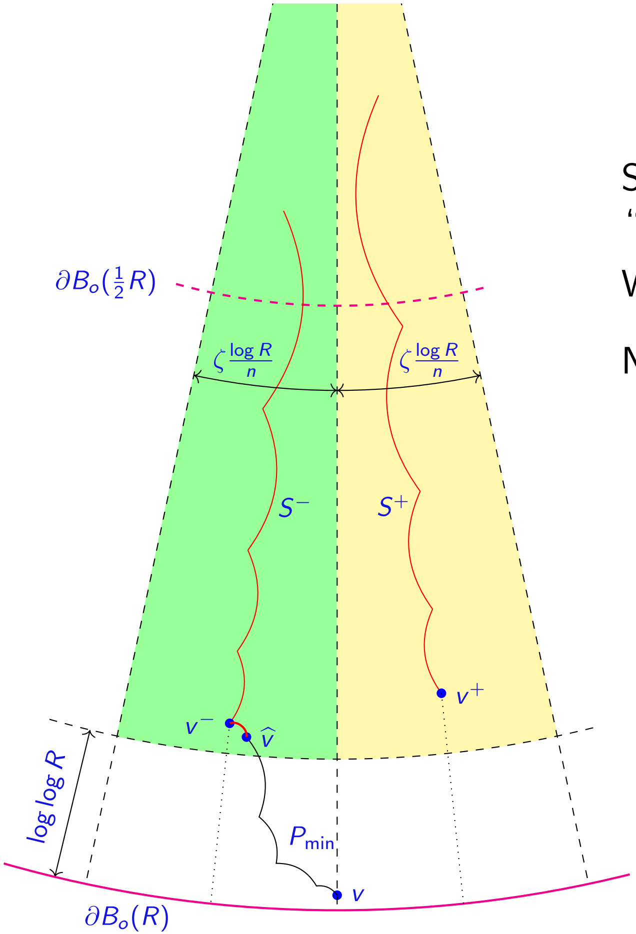
- Say v is in the giant of $G = (V, E)$, $G := \mathcal{P}^H_{\alpha,\nu}(n)$.
- Observe that

$$\begin{aligned}\mathbb{E}(|V \cap S^{\pm}|) &= n\frac{\zeta \log R}{2\pi n}\mu(B_o(R - \log\log R)) \\ &= \Omega(\log^{1-\alpha} R) \\ &= \omega(1).\end{aligned}$$

- Thus, $\mathbb{P}(S^- \cap V = \varnothing \text{ or } S^+ \cap V = \varnothing) = o(1)$.
- Hence, $\exists v^{\pm} \in V \cap S^{\pm}$ with probability $1 - o(1)$. W.l.o.g., of largest radius in $S^{\pm}$.
- With probability $1 - o(1)$ there are exploding paths between $v^{\pm}$ and the Central Clique, each of length

$$O(\log R) = O(\log\log n).$$

Ultra-mall average distance (formal - case 1)



Case 1

Shortest path P_{min} from v to $V \cap B_o(R - \log\log R)$ is "between" v^- and v^+.

W.l.o.g., assume other end-vertex $\widehat{v}$ of P_{min} is in S^-.

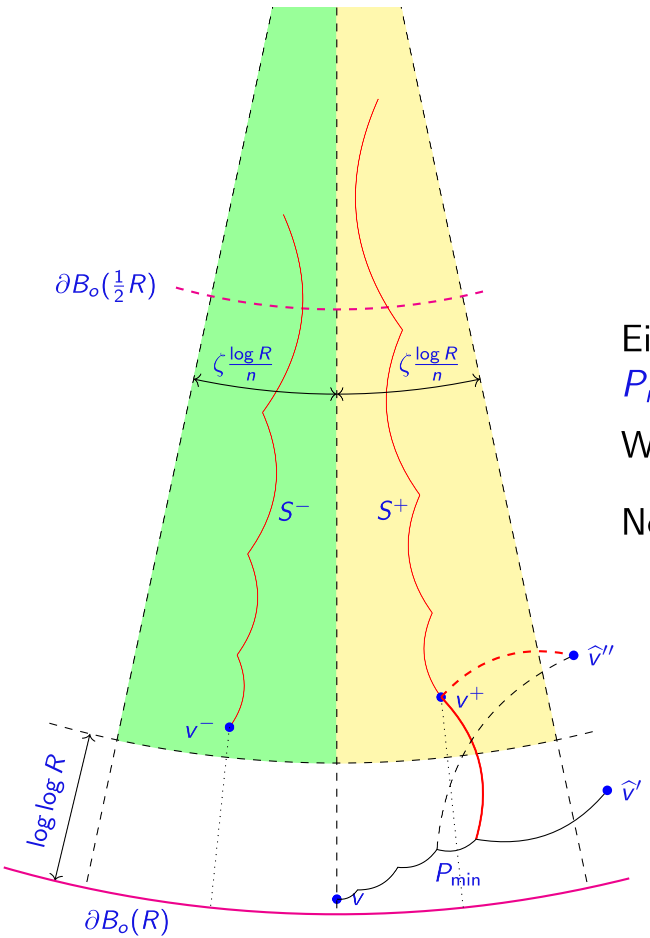
Note:

- $\phi_{v^-,\widehat{v}} \leq \frac{\zeta \log R}{n}$
- $\theta_R(r_{v^-}, r_{\widehat{v}}) \geq \underbrace{\theta_R(R - \log\log R, R - \log\log R)}_{=\Omega\left(\frac{\log R}{n}\right)}$
- For ζ sufficiently small, $v^-\widehat{v} \in E$, because

$$\phi_{v^-,\widehat{v}} \leq \theta_R(R - \log\log R, R - \log\log R)$$

Done provided length of P_{min} is $O(\log R) = O(\log\log n)$.

Ultra-mall average distance (formal - case 2)



Case 2

Either v^- or v^+ lies "between" end-vertices of edge of P_{min}

W.l.o.g., assume v^+ does.

Note:

- Geometric constraints imply there is an edge between v^+ and a vertex of P_{min}.

Done provided length of P_{min} is $O(\log R) = O(\log \log n)$.

Ultra-mall average distance (formal - conclusion)

- Say $v_0 = v, \ldots, v_\ell$ are all but "last" vertex of $P_{\min}$.
- Note that $\phi_{v_0,v_\ell} = O\left(\frac{\log R}{n}\right)$.
- By minimality, there is no edge between v_i and v_j if $|i-j|>1$, so

$$|i-j|>1 \implies \phi_{v_i,v_j} > \theta_R(r_{v_j}, r_{v_{j+2}}) \geq \theta_R(R,R) = \Omega\left(1/n\right).$$

- We get that $\ell = |P_{min}| = O(\log R) = O(\log\log n)$.

Navigability

The ability to efficiently find short paths between any two nodes using only local information, without a global map of the network.

[Baek, Porter & Parkinson. Soc. Cogn. Affect. Neurosci. 2020)]

Navigability

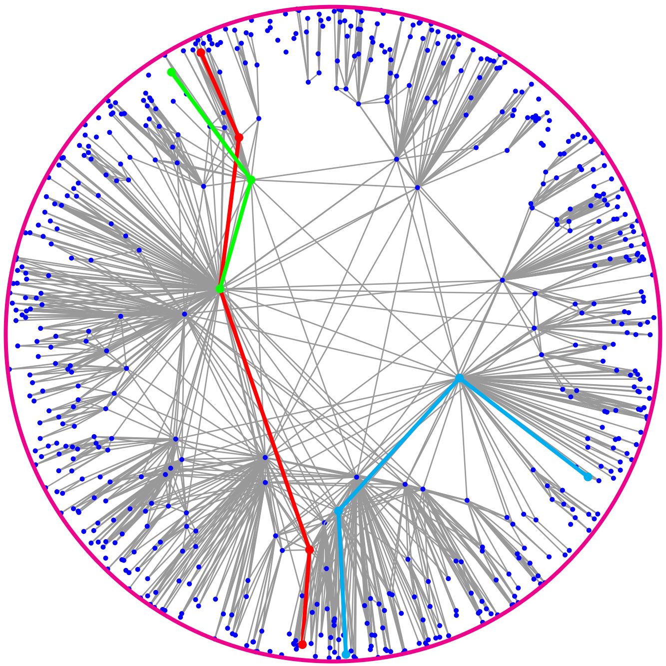
Greedy Routing Algorithm

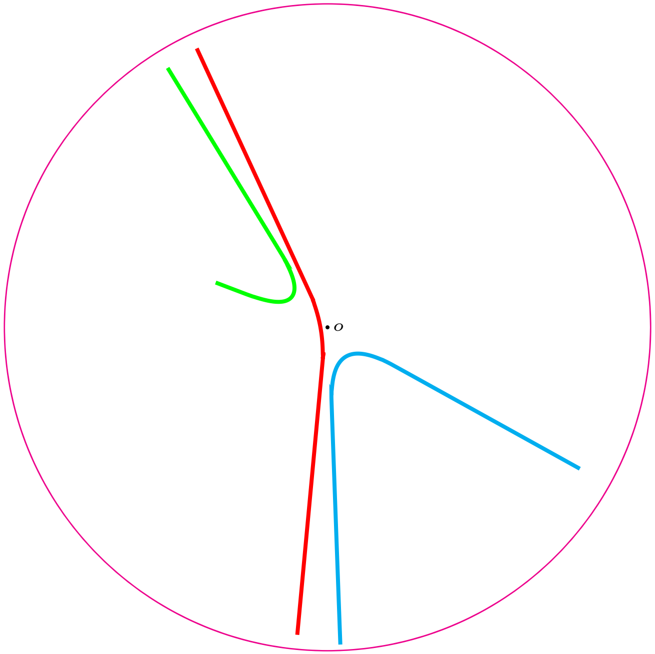
function $\textsc{Greedy}(s, t)$
 ▷ s *(source) and* t *(target) vertices of* $G := \mathcal{P}^{H}_{\alpha,\nu}(n)$ ◁

 if $s == t$ **then**
  Deliver message.
 else
  $w \leftarrow \text{argmin}\left\{\mathrm{dist}_{\mathbb{H}^2}(v, t) \mid v \in \mathrm{Neigh}_G(s)\right\}$
  if $\mathrm{dist}_{\mathbb{H}^2}(w, t) < \mathrm{dist}_{\mathbb{H}^2}(s, t)$ **then**
   $\textsc{Greedy}(w, t)$
  else
   return *failure*

Navigability

Greedy Routing Algorithm – illustration





Papadopoulos et al. [INFOCOM 2010], in a experimental study (but without "real" data) report excellent stretch (average ~ 1, max ~ 1.4) and success ratio (0.99920 for $\alpha \sim \frac{1}{2}$ to 0.92 for $\alpha \sim 1$, with α, ν of best Internet fit).

Say $P_{s,t}$ is a greedy path between s and t in $G := \mathcal{P}^{H}_{\alpha,\nu}(n)$.

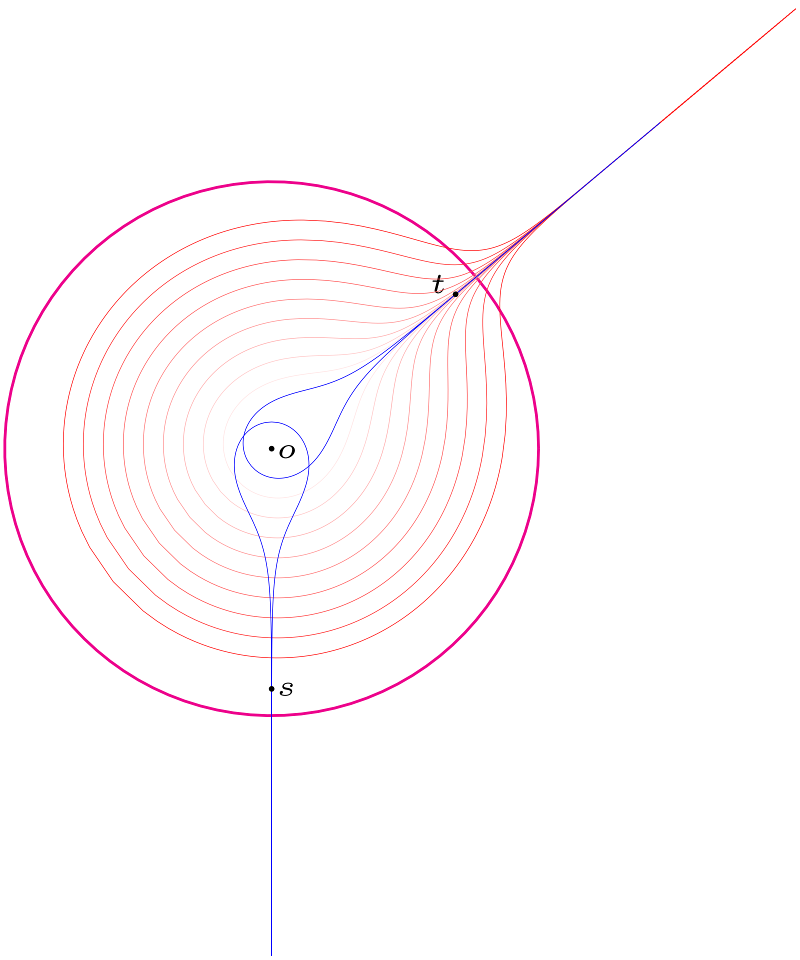
We expect that:

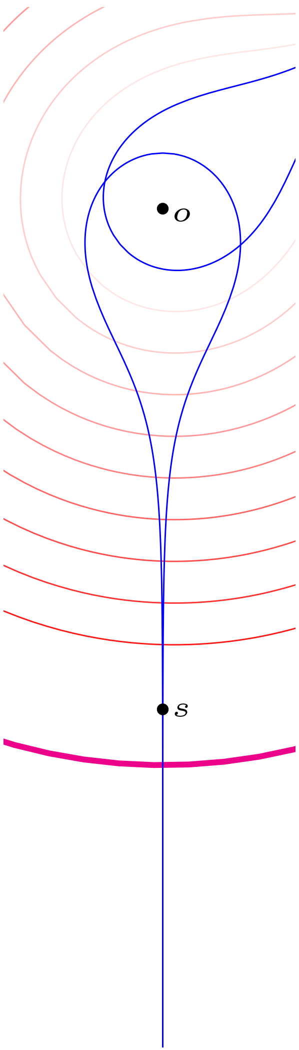
- $P_{s,t}$ "mimics" the geodesic path between s and t in $B_o(R)$.
- To span a large angle, $P_{s,t}$ must traverse a large degree (relative to $\mathrm{dgr}_G(s)$ and $\mathrm{dgr}_G(t)$) vertex.

There are two distinct phases.

Navigability - Phase I

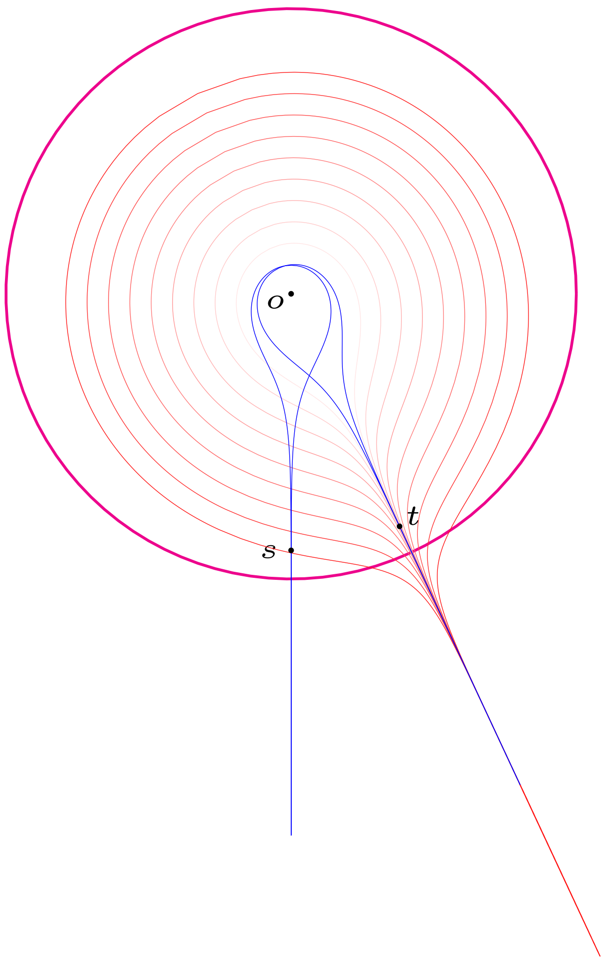
Pictures - large $\phi_{s,t}$ case

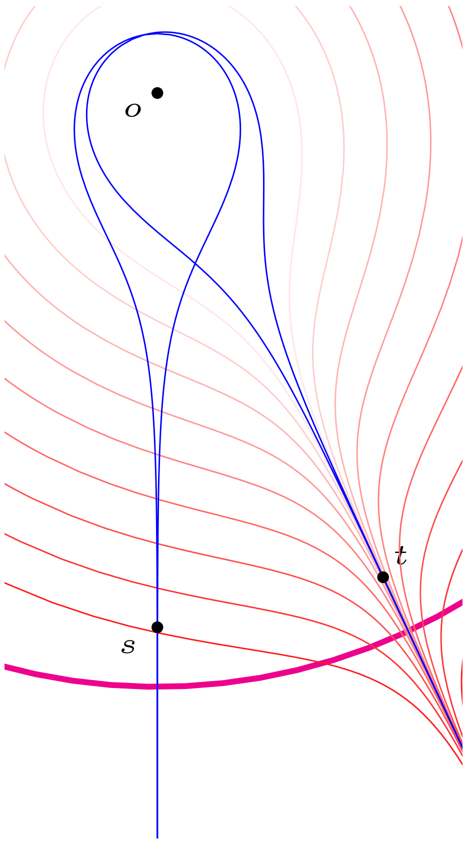




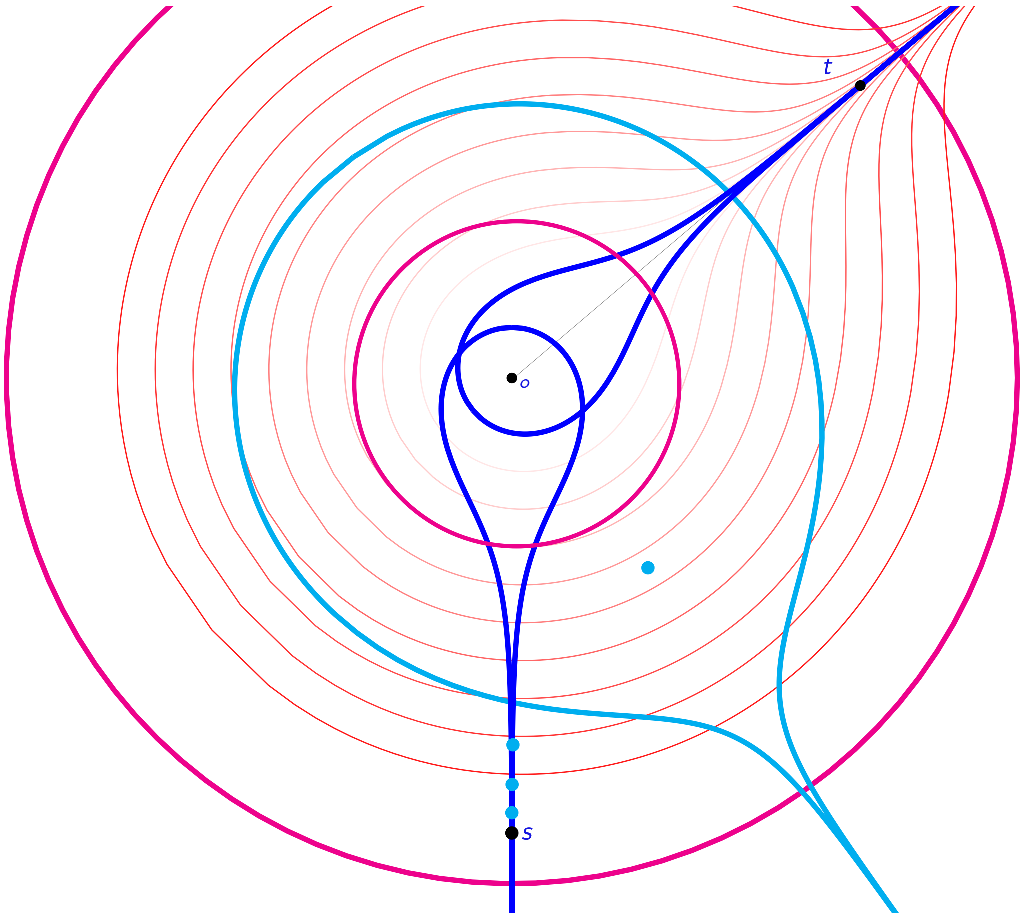
Navigability - Phase I

Pictures - small $\phi_{s,t}$ case

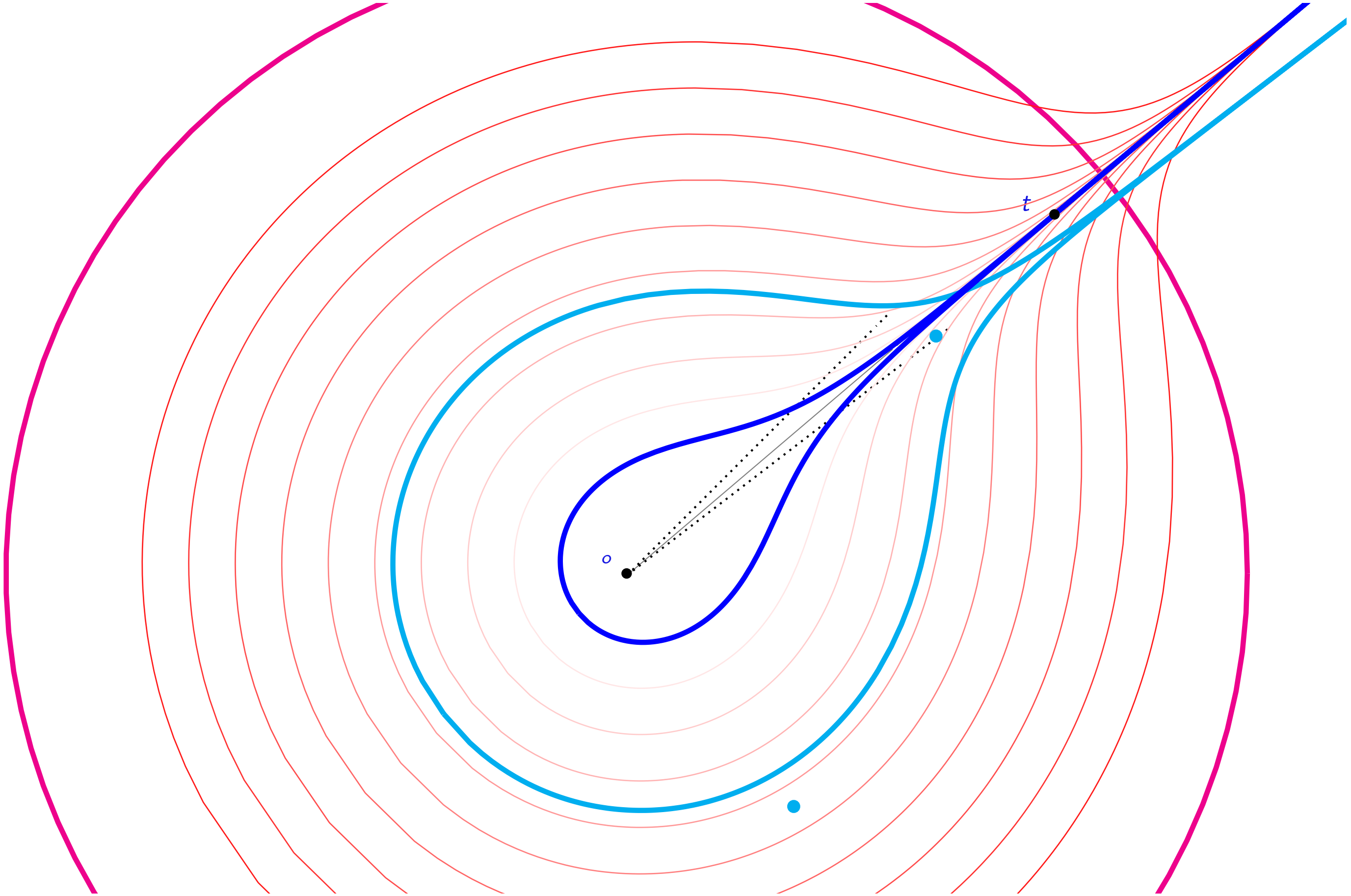




Navigability - Phase I



Navigability - Phase II



Navigability

Greedy forwarding - insights (formalized)

Roughly speaking, for uniformly chosen initial source s and destination t vertices of the giant:

- Consecutive vertices w and w' of greedy path between s and t satisfy:

$$\Phi(w') = \Theta\left((\Phi(w))^{\frac{1}{2\alpha-1}}\right)$$

for the potential $\Phi(x) := \dfrac{\mathrm{dgr}_G(x)}{n \cdot \phi_{x,t}}$ " $=$ " $\Theta\left(\dfrac{n}{\sqrt{\cosh(\mathrm{dist}_{\mathbb{H}^2}(v,t))}}\right)$.

- During Phase I, $\phi_{x,t}$ changes little and $\mathrm{dgr}_G(x)$ increases by powers of $\frac{1}{2\alpha-1}$.
- Analysis of Phase II is more delicate ...

Greedy paths succeed with probability $\Omega(1)$ and have length $\Theta(\log\log n)$. [BKL+ - PODC'17]

Several additional refinements also in [BKL+ - PODC'17].

Open problems in navigability and routing in RHG

- Congestion and the "hidden backbone"
- Dynamic navigability and node mobility
- Routing in higher dimensions: How does the probability of success change?

Literature (Math/CS)

Early results

Focused mostly on structural aspects of RHGs, e.g.:

- Average degree, degree distribution, max-degree, clustering coeff. GPP - ICALP'12
- Average distance ABF - IntMath'17
- Diameter KM - ANALCO'15 ; FK - SIDMA'18 ; MS - AdvAP'19
- Component sizes (giant and 2nd-largest component). BFM - EJC'15 ; KM - SIDMA'19 ; DM - ECP'21 ; BFKRZ - arXiv23
- Separators, treewidth and related parameters BFK - ESA'16
- Spectral-gap, conductance, min and max bisection KM - AAP'17.
- Connectivity threshold BFM - RS&A'16
- Etc.

Literature (cont.)

More recent

Several specialized algorithms on RHG's, e.g.:

- Routing, Bidirectional BFS, Exact cover, Approximate cover, Distributed coloring

BKL+ - PODC'17 ; BFF+ - ICALP'18 ; BFK - ESA'21; MR - SODA'26

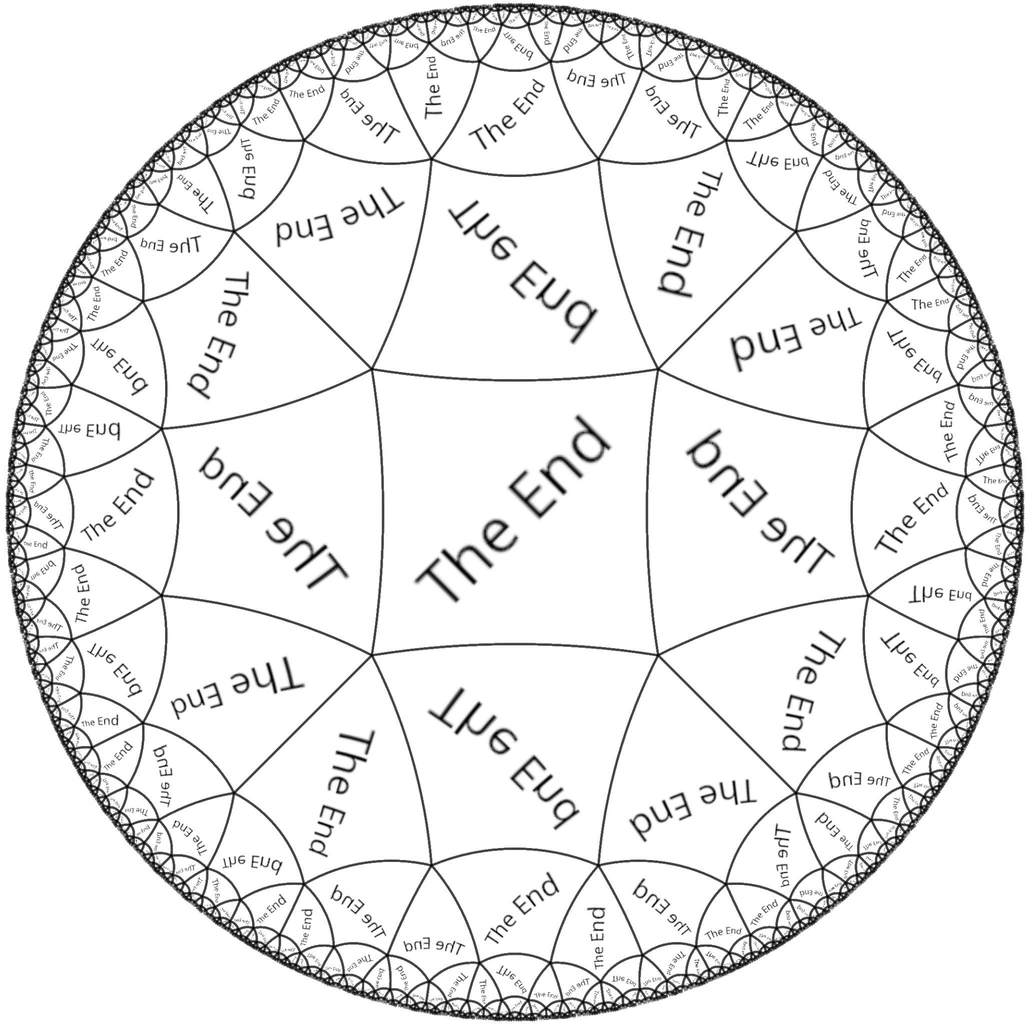
Stochastic processes on RHGs:

- Bootstrap percolation in RHGs and GIRGs CF - SP&A'16 ; KL - ICALP'16 ; KKFM - AAP'18
- Contact process LMSV - AP'21
- Scale free percolation KL - SP&A'20
- Cover and hitting time, Kirchhoff index, target time (Kemeney's constant) KSS - RS&A'24
- Ordering of large degrees Gas - AdvAP'25
- Rumor spreading KLL+ - SODA'26

Disclaimer: (Large/significant) Physics literature not mentioned!!

Spatial models of random graphs

- Geometric random graphs.
- Kleinberg's small-world model
- Hyperbolic random graphs.
- Geometric inhomogeneous random graphs (GIRGs).
- Spatial preferential attachments.
- Barthélemy's spatial network models.
- Latent space models.
- Etc.


The End