MINIMIZATION OF THE EXPECTED COMPLIANCE AS AN ALTERNATIVE APPROACH TO MULTILOAD TRUSS OPTIMIZATION.

Felipe Alvarez Daziano

Miguel Carrasco Briones

Universidad de Chile

Facultad de Ciencias Físicas y Matemáticas

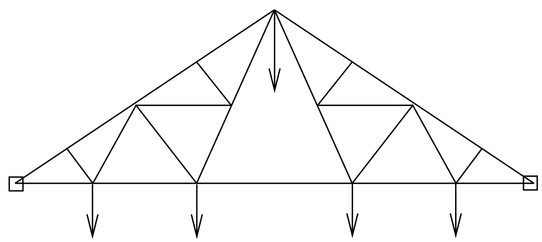
Departamento de Ingeniería Matemática

- Optimization of trusses
  - Standard minimum compliance truss design.
  - Instabilities and the standard multiload model.
  - Random Perturbations: minimizing the expected compliance.
  - Numerical examples.

Optimization of trusses

Introduction

- Definition.
- Problem: find the best structure capable to carry a external nodal force.
- Constraints: mechanical equilibrium , total volume and others . . .
- Minimize the compliance.



Minimum compliance truss design

$$(\mathcal{D}) \qquad \begin{aligned} \min_{y,\lambda,u}\ & \frac{1}{2} f^T u \\ s.t.\ & K(y,\lambda)u = f \\ & \lambda \in \Delta_m \\ & y \in V_1 \times \ldots \times V_N. \end{aligned}$$

Design variables

- λ:vector of bar volumes (topology).
- y: nodal positions (geometry).

$\frac{1}{2} f^T u$ is called *Compliance*.

Matematical formulation

- $\lambda \in \mathbb{R}^m$; m number of bars.
- $y \in \mathbb{R}^N$; N number of nodes.
- n grades of fredom.
- $f \in \mathbb{R}^n$ nodal load vector.
- $u \in \mathbb{R}^n$ nodal displacements.

$$\boxed{\begin{aligned} \min_{y,\lambda,u}\ & \frac{1}{2} f^T u \\ s.t.\ & K(y,\lambda)u = f \\ & \lambda \in \Delta_m \\ & y \in V_1 \times \ldots \times V_N. \end{aligned}}$$

- $\Delta_m = \{\lambda \in \mathbb{R}^m \mid \lambda \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1\}$
- $K(y,\lambda) \in \mathbb{R}^{n \times n}$, *stiffness matrix*.

Stiffness Matrix

$$K(y,\lambda) = \sum_{i=1}^{m} \lambda_i K_i(y)$$

where

$$K_i(y) = \frac{E_i}{l_i(y)^2}\gamma_i(y)\gamma_i(y)^T \in \mathbb{R}^{n\times n}$$

Remark K_i is dyadic ($K_i = b_i b_i^T$).

- E_i: Young modulus.
- $l_i(y)$: length of bar i.
- $\gamma_i(y)$: *cosines/sines* vector.

ground structure approach

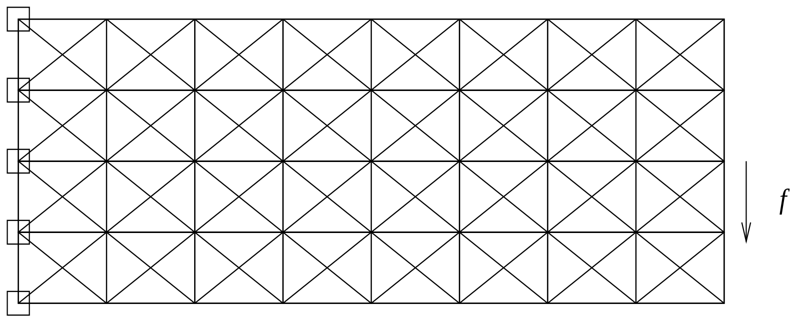
- geometric variable $\Rightarrow (\mathcal{D})$ is very difficult

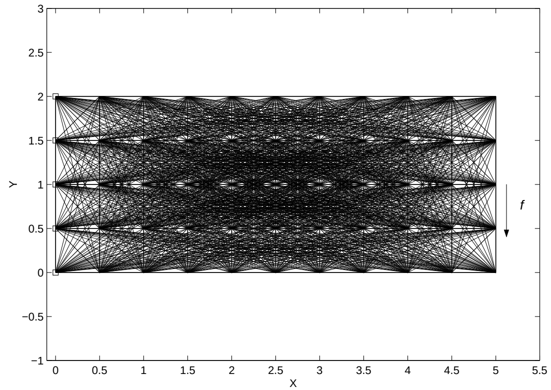
ground structure approach

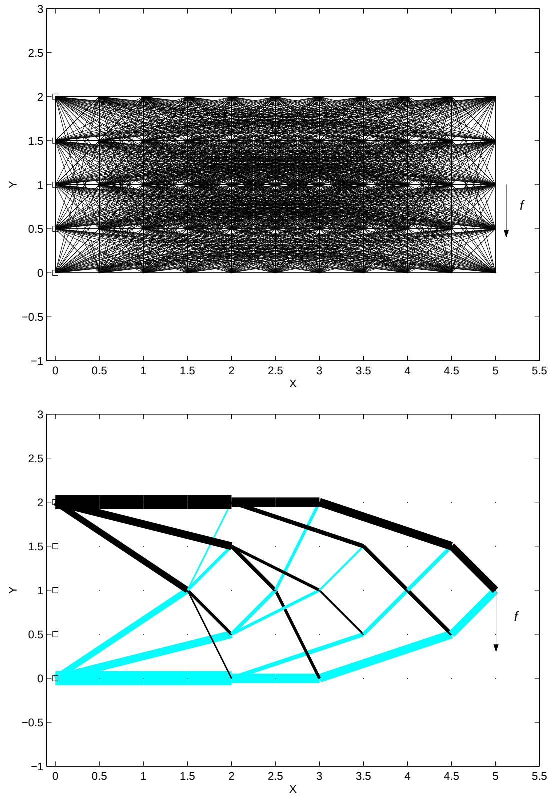
- geometric variable $\Rightarrow (\mathcal{D})$ is very difficult
- solution: *ground structure*
  - Nodal positions are fixed in the reference configuration (y is fixed).
  - We use a mesh full of nodes and bars.

ground structure approach

- geometric variable $\Rightarrow (\mathcal{D})$ is very difficult
- solution: *ground structure*
  - Nodal positions are fixed in the reference configuration (y is fixed).
  - We use a mesh full of nodes and bars.




Y
X
f



Typically a large number of bar vanish at the optimum.

$$(\mathcal{D}) \qquad \begin{aligned} \min_{\lambda\in\mathbb{R}^m,u\in\mathbb{R}^n} \ & \frac{1}{2}f^T u \\ s.t. \ & K(\lambda)u = f \\ & \lambda \in \Delta_m \end{aligned}$$

$\frac{1}{2}f^T u$ is independent of the choice u satisfying $K(\lambda)u = f$.

Minimax Formulation

$(\mathcal{P})$
$$\min_{x\in\mathbb{R}^n}\max_{1\le i\le m}\{\frac{1}{2}x^TK_ix - f^Tx\}.$$

Dual of $(\mathcal{P})$ is the design problem $(\mathcal{D})$.

advantages

- reduction of the dimention.
- The proof of equivalence between $(\mathcal{P})$ and $(\mathcal{D})$ don't use the dyadic structure of matrix K_i.

disadvantages

- Lost of differentiability of the objective function.

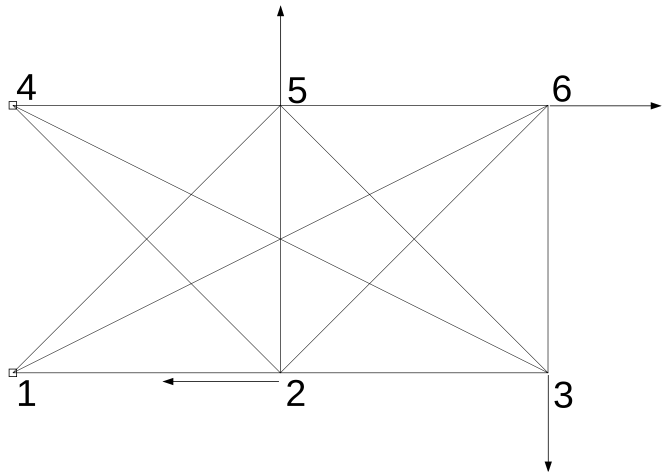
Instabilities and the standard multiload model

Example

The design problem $(\mathcal{D})$ may produce insatisfactory results in respect to mechanical stability

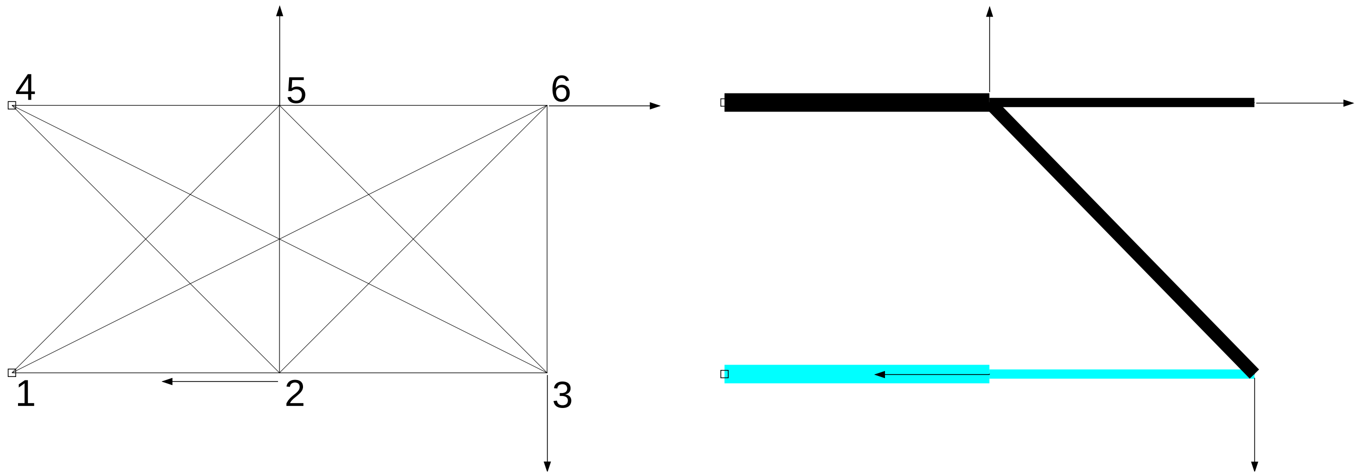
Example

The design problem $(\mathcal{D})$ may produce insatisfactory results in respect to mechanical stability



Example

The design problem $(\mathcal{D})$ may produce insatisfactory results in respect to mechanical stability



Multiload model

Standard multiload model

$$\min_{\lambda \in \mathbb{R}^m} \frac{1}{2} \sum_{j=1}^{k} \alpha_j f^{jT} u^j$$

$$s.t.\ K(\lambda) u^j = f^j, \quad j = 1, \dots, k$$

$$\lambda \in \Delta_m.$$

We minimize a weighted average of the compliances.

Random Perturbations: minimizing the expected compliance.

Random load model

Let $\xi \in \mathbb{R}^n$ be a perturbation on the load vector f.

$$\Psi(\xi,\lambda) = \begin{cases} \frac{1}{2}(f+\xi)^T x & \text{if } \lambda \in \Delta_m \text{ and } u \in \mathbb{R}^n \text{ such that} \\ & K(\lambda)u = f+\xi, \\ +\infty & \text{otherwise} \end{cases}$$

$$\Psi\colon (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \to (\mathbb{R} \cup \{+\infty\}, \bar{\mathcal{B}}(\mathbb{R}))$$

Results to be proper, lower semicontinuous and convex.

Random load model

For heach $\lambda \in \Delta_m$

$$\begin{array}{rccc} \Psi(\cdot,\lambda): & (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) & \to & (\mathbb{R} \cup \{+\infty\}, \bar{\mathcal{B}}(\mathbb{R})) \\ & \xi & \mapsto & \Psi(\xi,\lambda) \end{array}$$

is measurable.
We asume that ξ is a random variable corresponding to an uncertain nodal load perturbation

$$\begin{array}{rccc} \xi: & (\Omega, \mathcal{A}) & \to & (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \\ & \omega & \mapsto & \xi(\omega) \end{array}$$

Random load model

Minimum expected compliance design problem

$(\mathcal{D};\mathbb{P})$ $$\min_{\lambda\in\Delta^m} \mathbb{E}_\xi[\Psi(\xi,\lambda)]$$

Remark: $V_\lambda = \operatorname{Im} K(\lambda) = \{K(\lambda)u \mid u \in \mathbb{R}^n\}$

$$\mathbb{E}_\xi[\Psi(\xi,\lambda)] = \int_{V_\lambda} \Psi(\xi,\lambda) d\mathbb{P}(w) + \infty \mathbb{P}\{V_\lambda^c\}$$

if $V_\lambda \neq \mathbb{R}^n$ it may ocurr that $\mathbb{E}_\xi[\Psi(\xi,\lambda)] = +\infty$ whenever $\mathbb{P}\{f+\xi \in V_\lambda\} > 0 \Rightarrow$ λ is not a feasible member.

Example: discrete perturbation

1. $\mathbb{P}_1\{\xi \in B\} = 1$ if $0 \in B$ and 0 otherwise.
   We obtein the clasical single load model.
2. $\mathbb{P}_2$ with finite support $\mathrm{Sop}(\mathbb{P}_2) = \{\xi^1, \ldots, \xi^k\}$.

$$(\mathcal{D}; \mathbb{P}_2) \qquad \min_{\lambda \in \mathbb{R}^m} \frac{1}{2} \sum_{j=1}^{k} \alpha_j f^{jT} u^j$$

$$s.t.\ K(\lambda) u^j = f^j, \quad i = 1, \ldots, k$$

$$\lambda \in \Delta_m.$$

$(\mathcal{D}; \mathbb{P}_2)$ is the *standard multiload model* where $\alpha_j = \mathbb{P}(\xi = \xi^j)$ and $f^j = f + \xi^j$, $j = 1, \ldots, k$.

Continuos case

Theorem 0.1. *Let* $\xi : \Omega \to \mathbb{R}^n$ *be a continuous random variable with mean vector* $\mathbb{E}(\xi) = 0$ *and Variance–covariance matrix* $\mathrm{Var}(\xi) = PP^T$, *with* $P \in \mathbb{R}^{n\times k}$.
Then the corresponding minimum expected compliance design problem $(\mathcal{D};\mathbb{P})$ *is given by*

$$(\mathcal{D};\mathbb{P}) \qquad \min_{\lambda\in\mathbb{R}^m} \underbrace{\frac{1}{2}f^T u}_{\text{mean load}} + \underbrace{\frac{1}{2}\mathrm{Trace}(P^T U)}_{\text{variance}}$$

$$s.t.\; K(\lambda)u = f$$

$$K(\lambda)U = P$$

$$\lambda \in \Delta_m.$$

Continuous case

Remark $f^T u$ and $\frac{1}{2}\,\mathrm{Trace}(P^T U)$ are independent of the choice u and U satisfaying $K(\lambda)u = f$, and $K(\lambda)U = P$ respectively.
Remark $p^j,\ j = 1, \ldots, k$, columns of P
Defining

$$\hat{f} = (f^T, p^{1T}, \ldots, p^{kT})^T \in \mathbb{R}^{n(k+1)},$$

$$\hat{K}_i = \mathrm{diag}(K_i, K_i, \ldots, K_i) \in \mathbb{R}^{n(k+1)\times n(k+1)},$$

$$\hat{K}(\lambda) = \sum_{i=1}^{m} \lambda_i \hat{K}_i,$$

Then $(\mathcal{D}; \mathbb{P})$ may be written as a multiload model.

Random one dimentional preturbation

$\xi = \varepsilon d$, where

$d \in \mathbb{R}^n,\ \mathbb{E}(\varepsilon) = 0, \mathrm{Var}(\varepsilon) = \sigma^2.$

Then $\mathbb{E}(\xi) = 0$ and $\mathrm{Var}(\xi) = \sigma d d^T$

$$(\mathcal{D}; \mathbb{P}_3) \qquad \begin{aligned} \min_{\lambda \in \mathbb{R}^m} \ & \frac{1}{2} f^T u + \frac{1}{2} \sigma d^T v \\ s.t.\ & K(\lambda) u = f \\ & K(\lambda) v = \sigma d \\ & \lambda \in \Delta_m. \end{aligned}$$

Random multidimentional independent perturbation

$\xi = \sum_{j=1}^{k} \varepsilon_j d^j$, where
$d^j \in \mathbb{R}^n$, $\mathbb{E}(\varepsilon_j) = 0$ and $\mathrm{Var}(\varepsilon_j) = \sigma_j^2$.

$$
(\mathcal{D}; \mathbb{P}_4) \qquad \begin{aligned} \min_{\lambda \in \mathbb{R}^m} \; & \frac{1}{2} f^T u + \frac{1}{2} \sum_{j=1}^{r} \sigma_j d^{jT} u^j \\ s.t. \; & K(\lambda) u = f \\ & K(\lambda) u^j = \sigma_j d^j \quad j = 1, \ldots, k \\ & \lambda \in \Delta_m . \end{aligned}
$$

Stochastic model including variance

We consider

$$\min_{\lambda \in \mathbb{R}^m} \alpha \mathbb{E}_\xi[\Psi(\xi,\lambda)] + \beta \,\mathrm{Var}[\Psi(\xi,\lambda)]$$

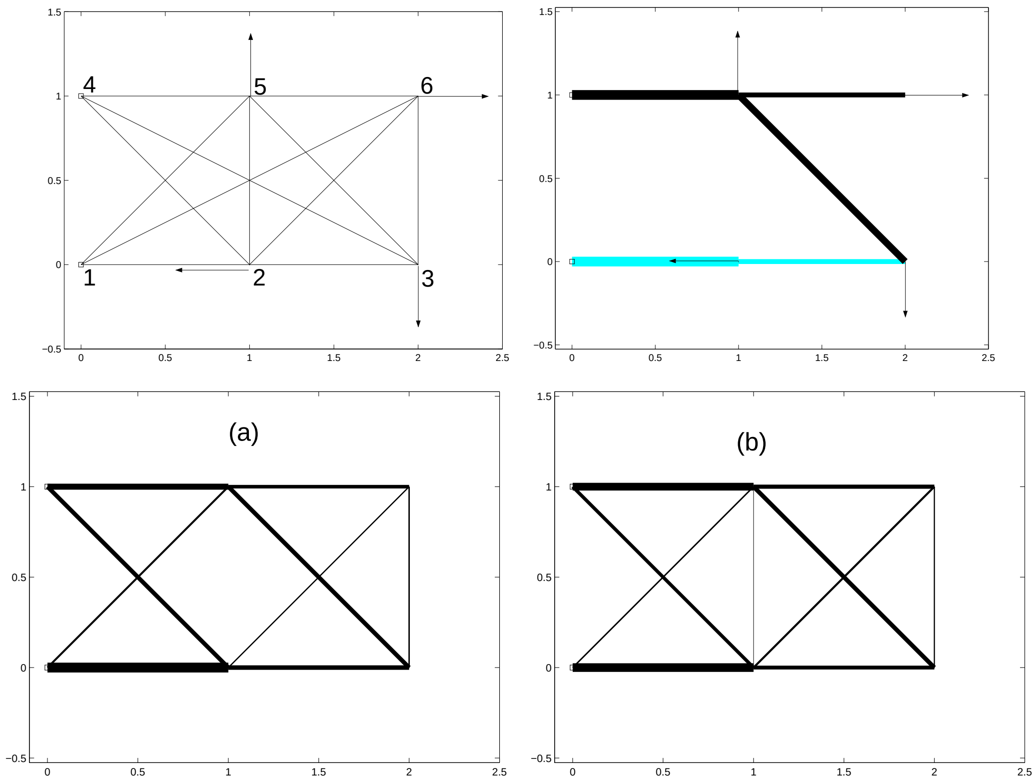
Example: $\xi \sim \mathcal{N}(0, PP^T)$.
Taking for simplicity $\alpha = 0$ and $\beta = 1$

$$(\mathcal{D}; \mathbb{P}_{\text{var}}) \quad \min_{\lambda \in \mathbb{R}^m} \frac{1}{2} \left(\mathrm{Trace}((P^T U)^2) + 2 f^T U U^T f \right)$$
$$s.t.\ K(\lambda) u = f$$
$$K(\lambda) U = P$$
$$\lambda \in \Delta_m.$$

Apparent harder to solve.

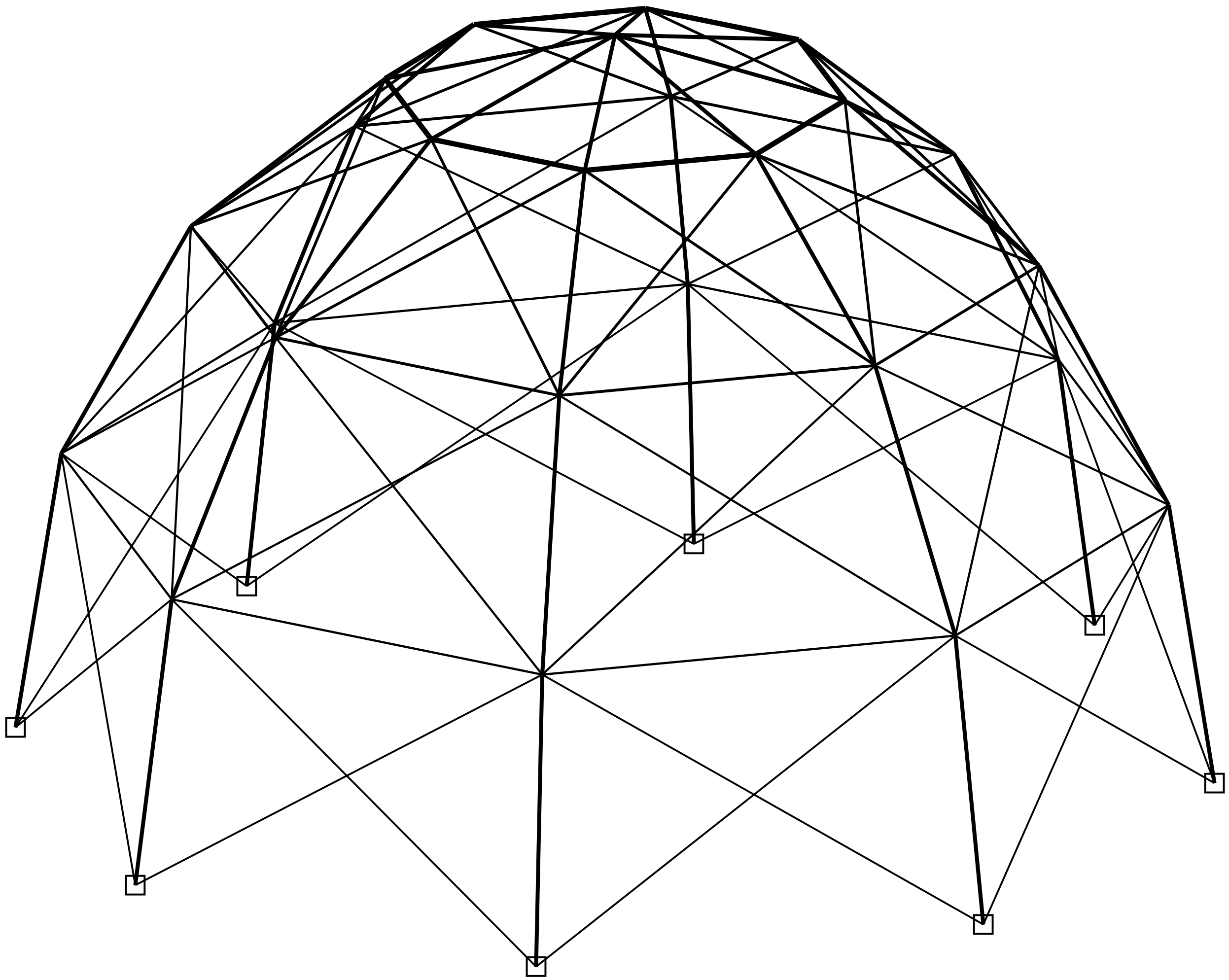
Numerical results

Toy example



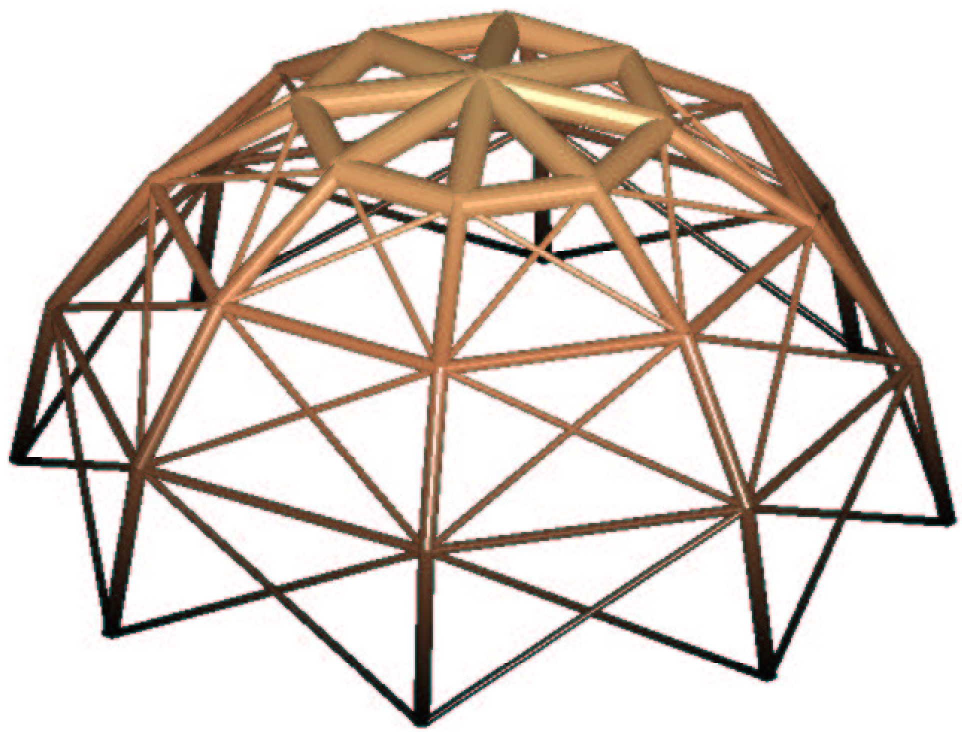
Problem	compl. mean	max compl.	min compl.
Standard single load	$\nexists$	$+\infty$	$+\infty$
(a)	0.025	0.053	0.013
(b)	0.023	0.051	0.012

Dome

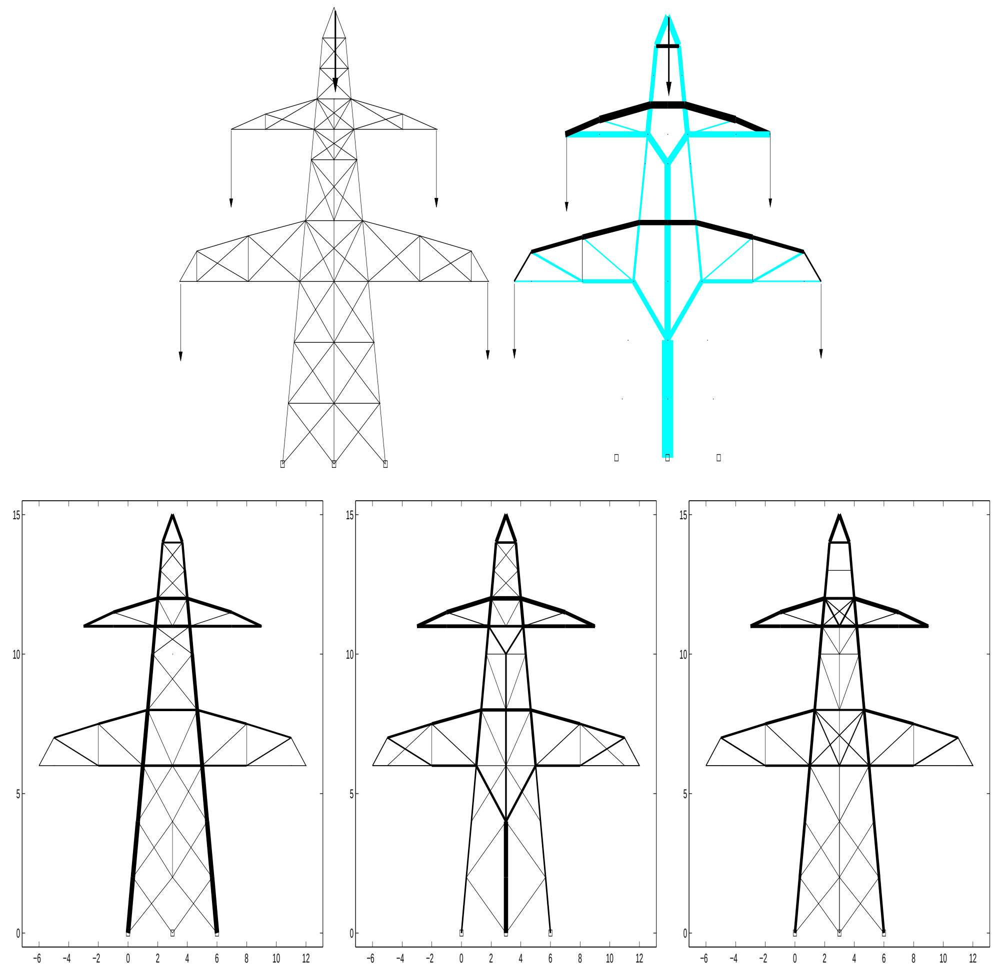


Dome

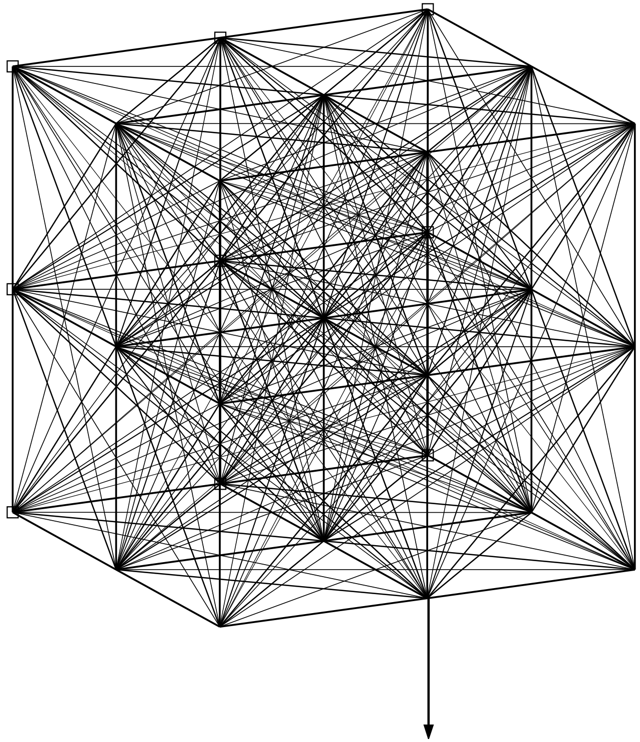
Dome

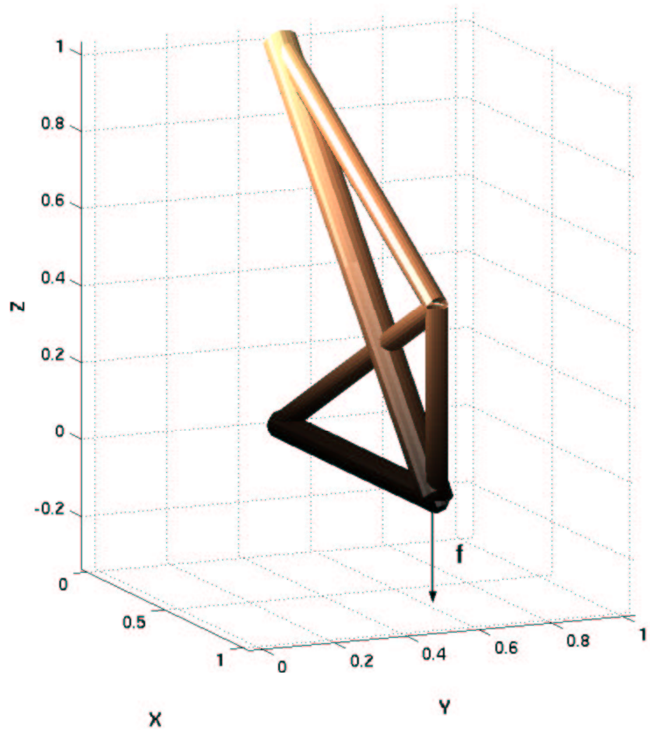


Electricity mast

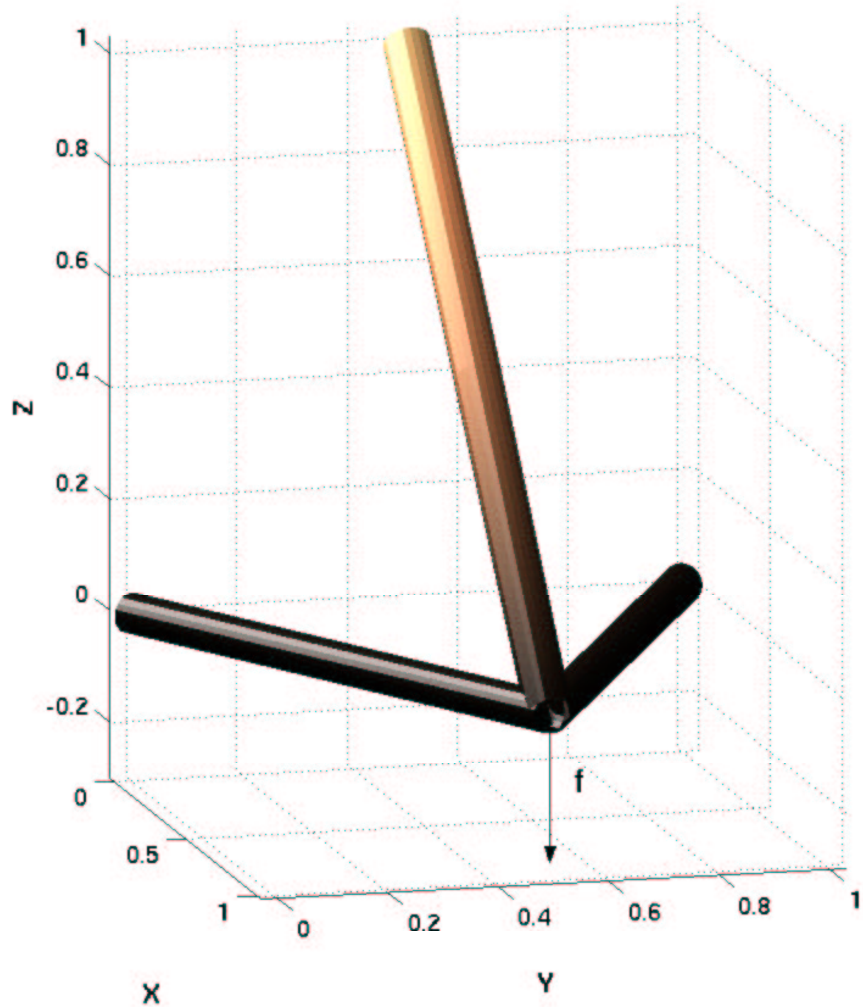


Michel 3D

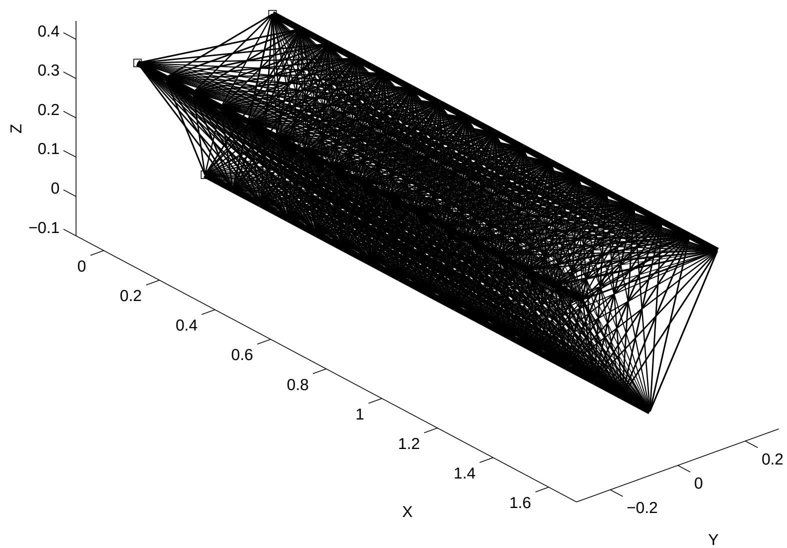




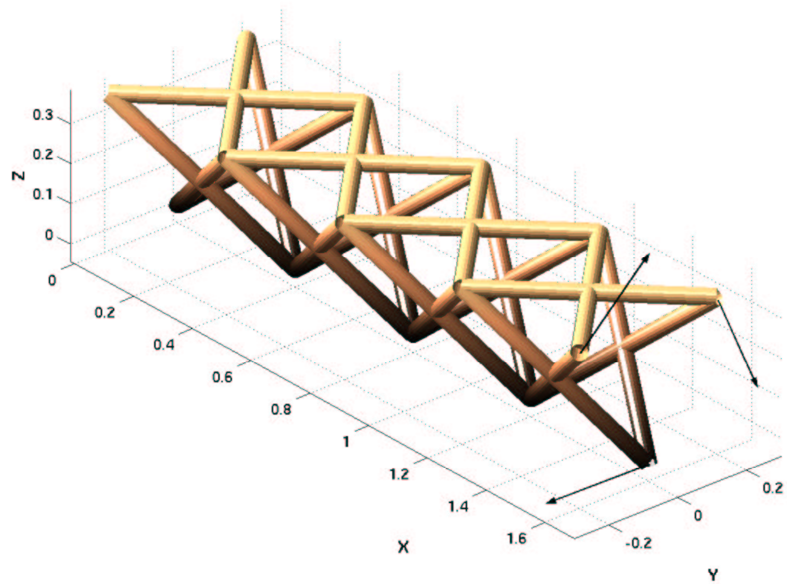
Michel 3D



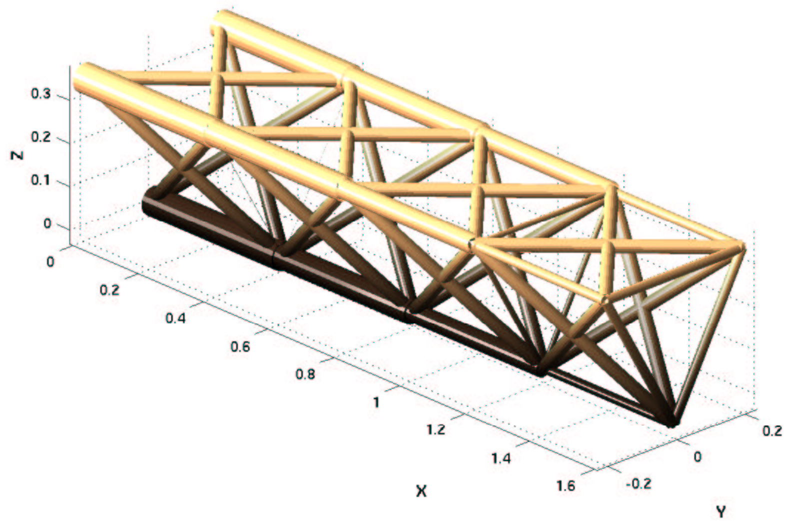
Cantilever

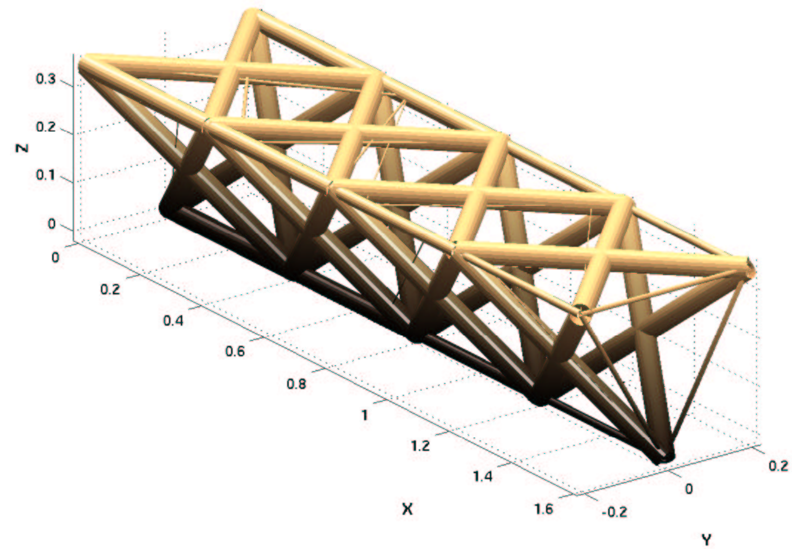


0.4
0.3
0.2
0.1
0
−0.1
Z
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
X
−0.2
0
0.2
Y



0.3
0.2
0.1
0
Z
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
X
-0.2
0
0.2
Y


0.3
0.2
Z
0.1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
X
-0.2
0
0.2
Y


0.3
0.2
Z
0.1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
X
-0.2
0
0.2
Y

Conclusions

- Classical model (one load) doesn’t get good results .
- Random model helps to avoid this problem.
- It’s equivalent to multiload one, but another interpretations in weight factors holds.
- We have to know the influence of each perturbation.
- Increase the dimention of the problem.

MINIMIZATION OF THE EXPECTED COMPLIANCE AS AN ALTERNATIVE APPROACH TO MULTILOAD TRUSS OPTIMIZATION.

Felipe Alvarez Daziano

Miguel Carrasco Briones

Universidad de Chile

Facultad de Ciencias Físicas y Matemáticas

Departamento de Ingeniería Matemática

MINIMIZATION OF THE EXPECTED COMPLIANCE AS AN ALTERNATIVE APPROACH TO MULTILOAD TRUSS OPTIMIZATION.

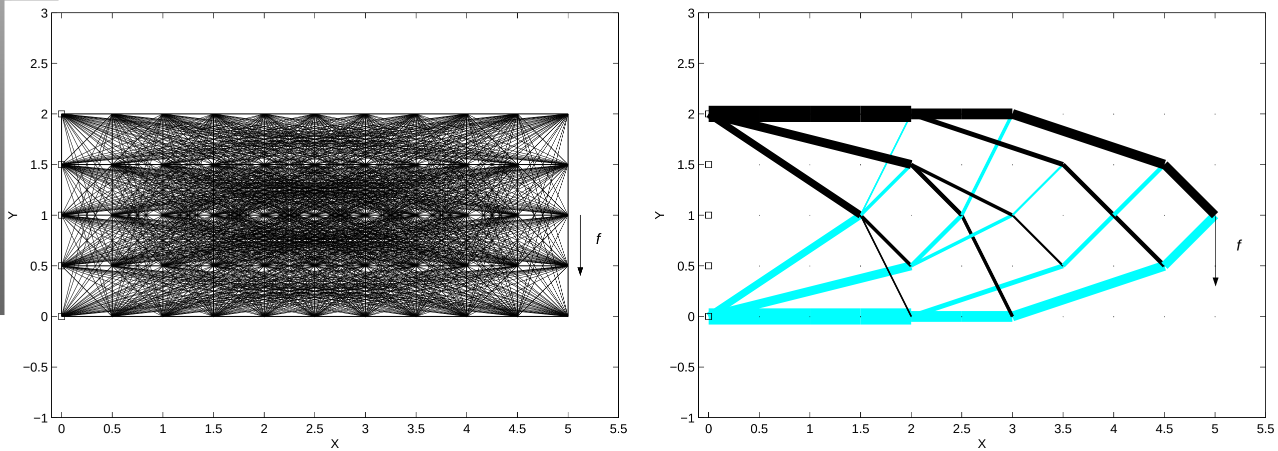
Felipe Alvarez Daziano

Miguel Carrasco Briones

Universidad de Chile

Facultad de Ciencias Físicas y Matemáticas

Departamento de Ingeniería Matemática


Y
X
f
Y
X
f

Lemma 0.1. *Let $\lambda \in \Delta_m$ and C be such that $C \geq \frac{1}{2}\max_j f_j^T u_j$ with $K(\lambda)u_j = f_j$, $j = 1,\ldots,k$. For each $f \in \mathrm{conv}\{f_1,\ldots,f_k,-f_1,\ldots,-f_k\}$, there exists $u \in \mathbb{R}^n$ satisfying $K(\lambda)u = f$ and moreover $C \geq \frac{1}{2}f^T u$.*

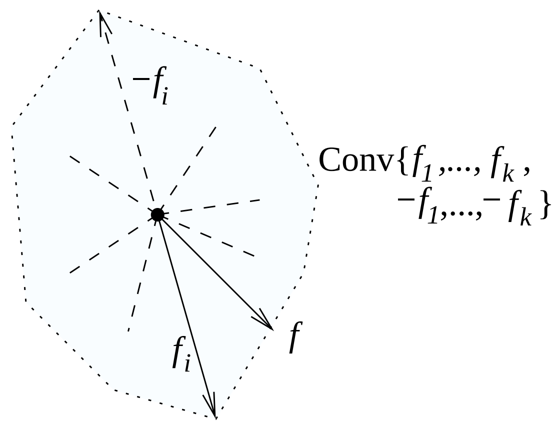
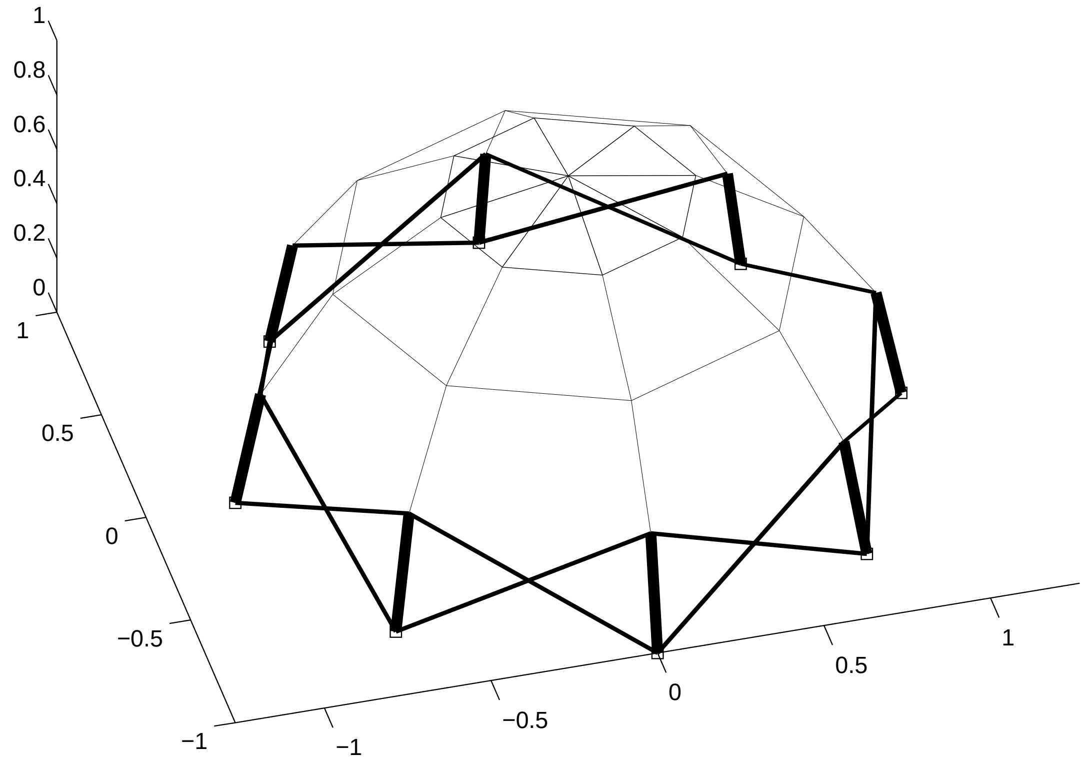
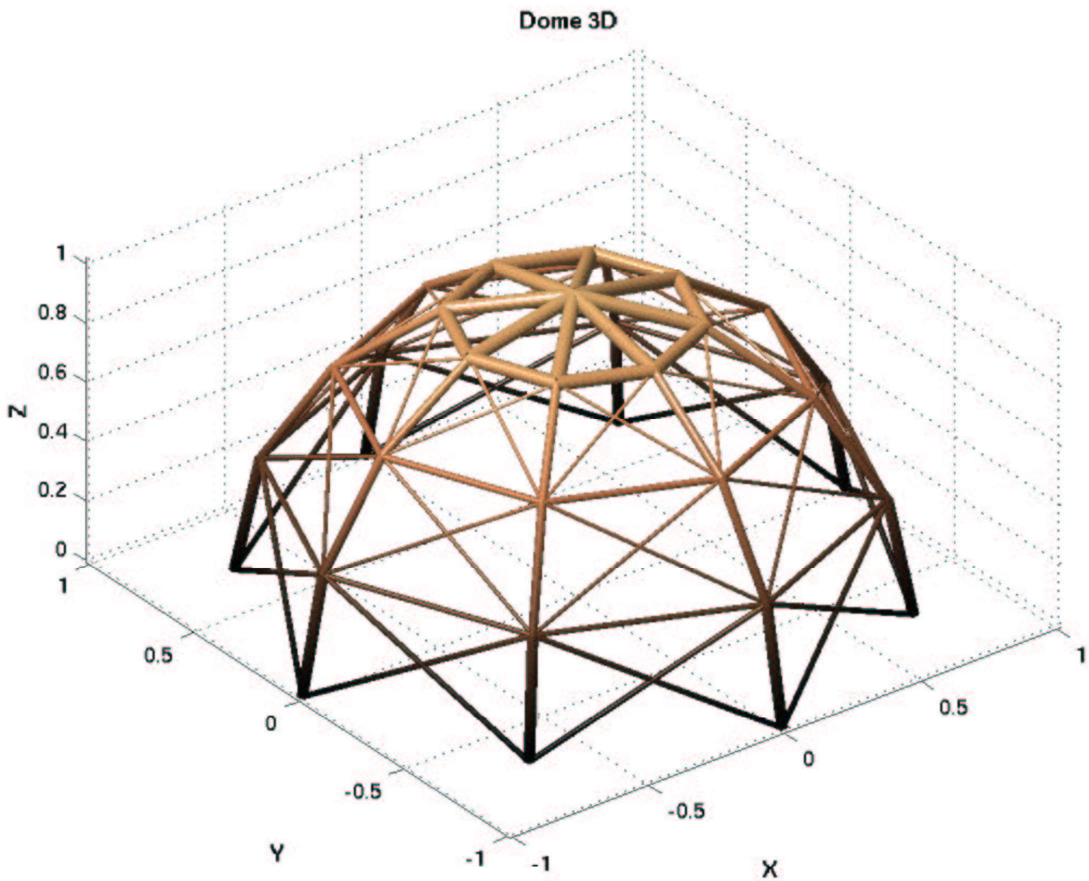


Figure 1: Convex Hull of $f_1,\ldots,f_k,-f_1,\ldots,-f_k$

self weight



self weight



Se cumple el siguiente resultado:

Theorem 0.2. *Sea (x^*, μ^*) solución del problema (QP), entonces existen $\sigma^+, \sigma^- \in \mathbb{R}^m_+$ tales que definiendo*

$$\lambda^* = \frac{1}{\mu^*}(\sigma^+ + \sigma^-)$$

se cumple que el par (x^*, λ^*) es solución de $(\mathcal{D})$, (el problema de diseño).

(QP) es equivalente a (LP)

$$
\begin{array}{lll}
(LP) & \displaystyle\min_{x\in\mathbb{R}^n} & -f^T x \\
 & s.a. & b_i^T x \leq 1 \quad i = 1,\ldots,m \\
 & & -b_i^T x \leq 1 \quad i = 1,\ldots,m
\end{array}
$$

Se cumple el siguiente resultado

Theorem 0.3. *Sea $\bar{x}$ solución del problema (LP) con vectores multiplicadores $\rho^+, \rho^- \in \mathbb{R}^m_+$, entonces*

$$x^* := \mu^* \bar{x}$$

$$\lambda_i^* := \frac{1}{\mu^*}(\rho_i^+ + \rho_i^-) \; i = 1, \ldots, m$$

con $\mu^* := \sum_{i=1}^m (\rho_i^+ + \rho_i^-)$ resuelven $(\mathcal{D})$.