



MINIMIZATION OF THE EXPECTED COMPLIANCE AS AN ALTERNATIVE APPROACH TO MULTILOAD TRUSS OPTIMIZATION.

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- Optimization of trusses
 - Standard minimum compliance truss design.
 - Instabilities and the standard multiload model.
 - Random Perturbations: Minimizing the expected compliance.
 - Numerical examples.

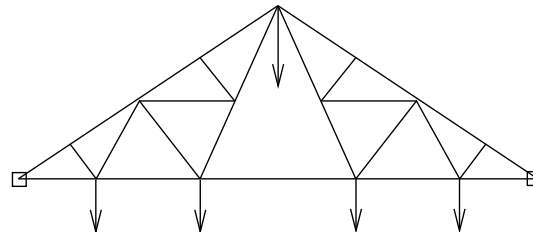


Optimization of trusses



Introduction

- Definition.
- Problem: find the best structure capable to carry a external nodal force.
- Constraints: mechanical equilibrium , total volume and others . . .
- Minimize the compliance.





Minimum compliance truss design

($\mathcal{D}$)

$$\min_{y, \lambda, u} \frac{1}{2} f^T u$$

$$s.t. \ K(y, \lambda)u = f$$

$$\lambda \in \Delta_m$$

$$y \in V_1 \times \dots \times V_N.$$

Design variables

- λ : vector of bar volumes (topology).
- y : nodal positions (geometry).

$\frac{1}{2} f^T u$ is called *Compliance*.



Mathematical formulation

- $\lambda \in \mathbb{R}^m$; m number of bars.
- $y \in \mathbb{R}^N$; N number of nodes.
- n grades of freedom.
- $f \in \mathbb{R}^n$ nodal load vector.
- $u \in \mathbb{R}^n$ nodal displacements.

$$\begin{aligned} \min_{y, \lambda, u} \quad & \frac{1}{2} f^T u \\ \text{s.t.} \quad & K(y, \lambda) u = f \\ & \lambda \in \Delta_m \\ & y \in V_1 \times \dots \times V_N. \end{aligned}$$

- $\Delta_m = \{\lambda \in \mathbb{R}^m \mid \lambda \geq 0, \sum_{i=1}^m \lambda_i = 1\}$
- $K(y, \lambda) \in \mathbb{R}^{n \times n}$, *stiffness matrix*.



$$K(y, \lambda) = \sum_{i=1}^m \lambda_i K_i(y)$$

where

$$K_i(y) = \frac{E_i}{l_i(y)^2} \gamma_i(y) \gamma_i(y)^T \in \mathbb{R}^{n \times n}$$

remmark K_i is dyadic ($K_i = b_i b_i^T$).

- E_i : Young modulus.
- $l_i(y)$: length of bar i .
- $\gamma_i(y)$: *cosines/sines* vector.



ground structure approach

- geometric variable $\Rightarrow (\mathcal{D})$ is very difficult



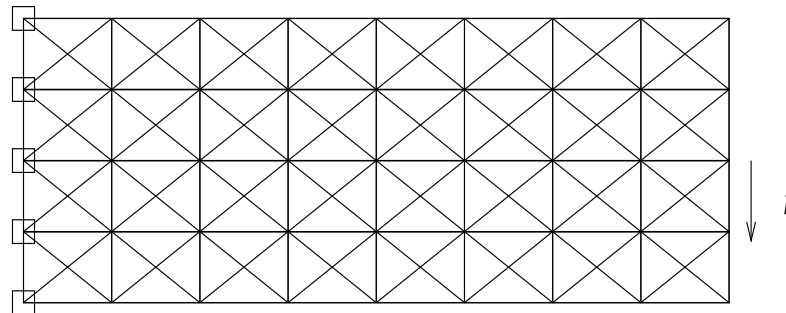
ground structure approach

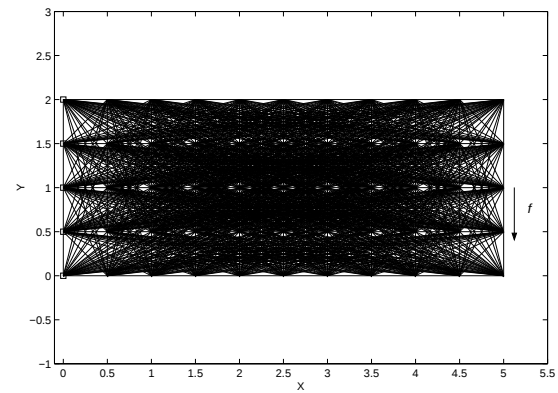
- geometric variable $\Rightarrow (\mathcal{D})$ is very difficult
- solution: *ground structure*
 - Nodal positions are fixed in the reference configuration (y is fixed).
 - We use a mesh full of nodes and bars.

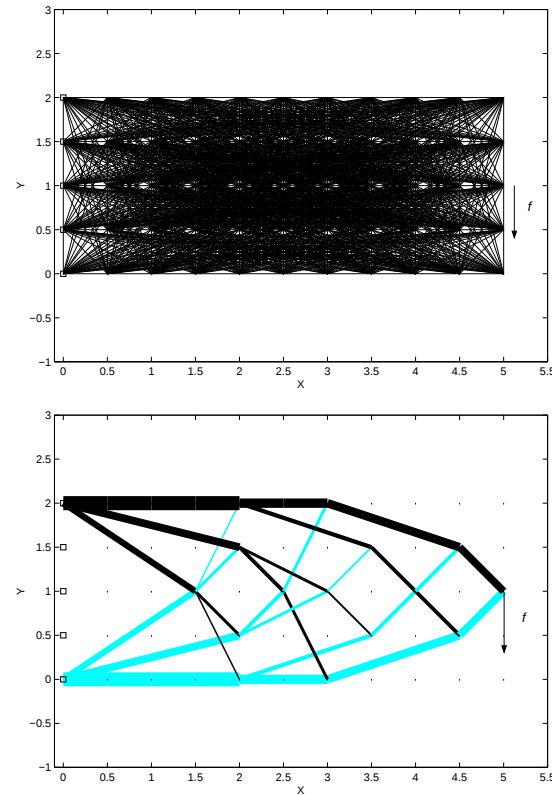


ground structure approach

- geometric variable $\Rightarrow (\mathcal{D})$ is very difficult
- solution: *ground structure*
 - Nodal positions are fixed in the reference configuration (y is fixed).
 - We use a mesh full of nodes and bars.







Typically a large number of bar vanish at the optimum.



(D)

$$\begin{aligned} \min_{\lambda \in \mathbb{R}^m, u \in \mathbb{R}^n} \quad & \frac{1}{2} f^T u \\ \text{s.t.} \quad & K(\lambda)u = f \\ & \lambda \in \Delta_m \end{aligned}$$

$\frac{1}{2} f^T u$ is independent of the choice u satisfying $K(\lambda)u = f$.



Minimax Formulation

$$(\mathcal{P}) \quad \min_{x \in \mathbb{R}^n} \max_{1 \leq i \leq m} \left\{ \frac{1}{2} x^T K_i x - f^T x \right\}.$$

Dual of $(\mathcal{P})$ is the design problem $(\mathcal{D})$.

advantages

- reduction of the dimension.
- The proof of equivalence between $(\mathcal{P})$ and $(\mathcal{D})$ don't use the dyadic structure of matrix A_i .

disadvantages

- Lost of differentiability of the objective function.



Instabilities and the standard multiload model



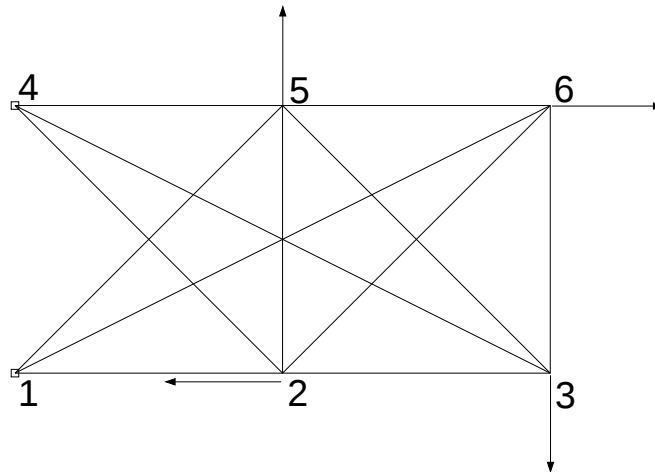
Example

The design problem ($\mathcal{D}$) may produce unsatisfactory results in respect to mechanical stability



Example

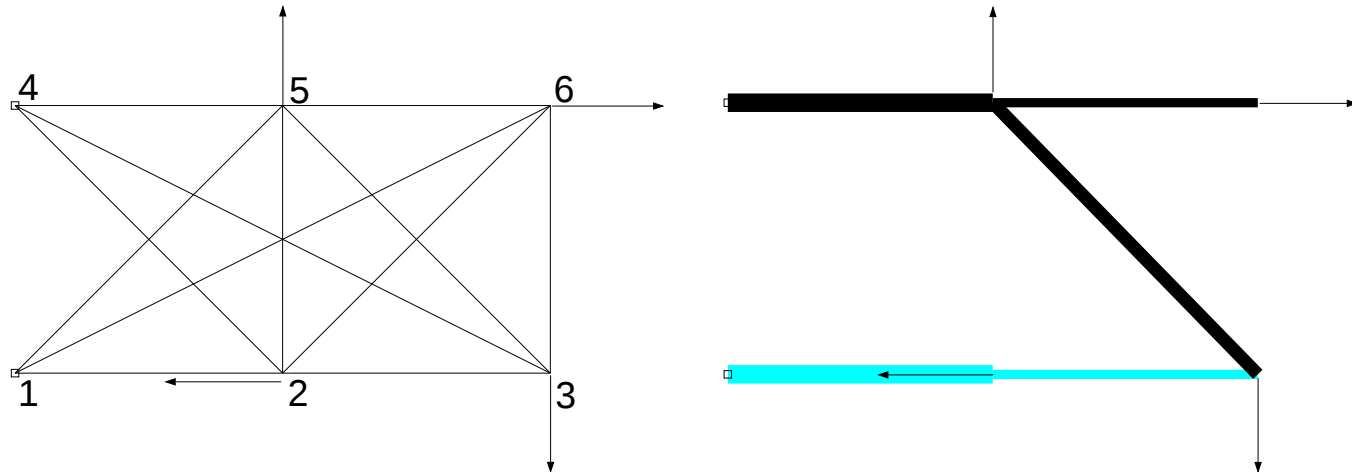
The design problem ($\mathcal{D}$) may produce unsatisfactory results in respect to mechanical stability





Example

The design problem ($\mathcal{D}$) may produce unsatisfactory results in respect to mechanical stability





Standard multiload model

$$\min_{\lambda \in \mathbb{R}^m} \frac{1}{2} \sum_{j=1}^k \alpha_j f^{jT} u^j$$

$$s.t. \quad K(\lambda) u^j = f^j, \quad j = 1, \dots, k$$

$$\lambda \in \Delta_m.$$

We minimize a weighted average of the compliances.



Random Perturbations: Minimizing the expected compliance.



Random load model

Let $\xi \in \mathbb{R}^n$ be a perturbation on the load vector f .

$$\Psi(\xi, \lambda) = \begin{cases} \frac{1}{2}(f + \xi)^T x & \text{if } \lambda \in \Delta_m \text{ and } u \in \mathbb{R}^n \text{ such that} \\ & K(\lambda)u = f + \xi, \\ +\infty & \text{otherwise} \end{cases}$$

$$\Psi: (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \rightarrow (\mathbb{R} \cup \{+\infty\}, \bar{\mathcal{B}}(\mathbb{R}))$$

Results to be proper, lower semicontinuous and convex.



Random load model

For each $\lambda \in \Delta_m$

$$\begin{array}{ccc} \Psi(\cdot, \lambda): & (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) & \rightarrow (\mathbb{R} \cup \{+\infty\}, \bar{\mathcal{B}}(\mathbb{R})) \\ & \xi & \mapsto \Psi(\xi, \lambda) \end{array}$$

is measurable.

We assume that ξ is a random variable corresponding to an uncertain nodal load perturbation

$$\begin{array}{ccc} \xi: & (\Omega, \mathcal{A}) & \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \\ & \omega & \mapsto \xi(\omega) \end{array}$$



Minimum expected compliance design problem

$$(\mathcal{D}; \mathbb{P}) \quad \min_{\lambda \in \Delta^m} \mathbb{E}_{\xi}[\Psi(\xi, \lambda)]$$

Remmark: $V_{\lambda} = \text{Im } K(\lambda) = \{K(\lambda)u \mid u \in \mathbb{R}^n\}$

$$\mathbb{E}_{\xi}[\Psi(\xi, \lambda)] = \int_{V_{\lambda}} \Psi(\xi, \lambda) d\mathbb{P}(w) + \infty \mathbb{P}\{V_{\lambda}^c\}$$

if $V_{\lambda} \neq \mathbb{R}^n$ it may occur that $\mathbb{E}_{\xi}[\Psi(\xi, \lambda)] = +\infty$ whenever $\mathbb{P}\{f + \xi \in V_{\lambda}\} > 0 \Rightarrow \lambda$ is not a feasible member.



Example: discrete perturbation

1. $\mathbb{P}_1\{\xi \in B\} = 1$ if $0 \in B$ and 0 otherwise.
We obtain the classical single load model.
2. $\mathbb{P}_2$ with finite support $\text{Sop}(\mathbb{P}_2) = \{\xi^1, \dots, \xi^k\}$.

$$\begin{aligned} (\mathcal{D}; \mathbb{P}_2) \quad & \min_{\lambda \in \mathbb{R}^m} \frac{1}{2} \sum_{j=1}^k \alpha_j f^{jT} u^j \\ & s.t. \quad K(\lambda) u^j = f^j, \quad i = 1, \dots, k \\ & \quad \lambda \in \Delta_m. \end{aligned}$$

$(\mathcal{D}; \mathbb{P}_2)$ is the *standard multiload model* where $\alpha_j = \mathbb{P}(\xi = \xi^j)$ and $f^j = f + \xi^j$, $j = 1, \dots, k$.



Continuos case



Continuous case

Theorem 0.1. *Let $\xi: \Omega \rightarrow \mathbb{R}^n$ be a continuous random variable with mean vector $\mathbb{E}(\xi) = 0$ and Variance–covariance matrix $\text{Var}(\xi) = PP^T$, with $P \in \mathbb{R}^{n \times k}$. Then the corresponding minimum expected compliance design problem $(\mathcal{D}; \mathbb{P})$ is given by*

$$\begin{aligned} (\mathcal{D}; \mathbb{P}) \quad & \min_{\lambda \in \mathbb{R}^m} \underbrace{\frac{1}{2} f^T u}_{\text{mean load}} + \underbrace{\frac{1}{2} \text{Trace}(P^T U)}_{\text{variance}} \\ & s.t. \quad K(\lambda)u = f \\ & \quad \quad K(\lambda)U = P \\ & \quad \quad \lambda \in \Delta_m. \end{aligned}$$



Continuous case

Remmark $f^T u$ and $\frac{1}{2} \text{Trace}(P^T U)$ are independant of the choice u and U satisfying $K(\lambda)u = f$, and $K(\lambda)U = P$ respectively.

Remmark p^j , $j = 1, \dots, k$, columns of P
Defining

$$\hat{f} = (f^T, p^{1T}, \dots, p^{kT})^T \in \mathbb{R}^{n(k+1)},$$

$$\hat{K}_i = \text{diag}(K_i, K_i, \dots, K_i) \in \mathbb{R}^{n(k+1) \times n(k+1)},$$

$$\hat{K}(\lambda) = \sum_{i=1}^m \lambda_i \hat{K}_i,$$

Then $(\mathcal{D}; \mathbb{P})$ may be written as a multiload model.



Random one dimensional preturbation

$\xi = \varepsilon d$, where

$d \in \mathbb{R}^n$, $\mathbb{E}(\varepsilon) = 0$, $\text{Var}(\varepsilon) = \sigma^2$.

Then $\mathbb{E}(\xi) = 0$ and $\text{Var}(\xi) = \sigma d d^T$

$$\begin{aligned} (\mathcal{D}; \mathbb{P}_3) \quad & \min_{\lambda \in \mathbb{R}^m} \frac{1}{2} f^T u + \frac{1}{2} \sigma d^T v \\ & s.t. \quad K(\lambda) u = f \\ & \quad \quad K(\lambda) v = \sigma d \\ & \quad \quad \lambda \in \Delta_m. \end{aligned}$$



Random multidimensional independent perturbation

$\xi = \sum_{j=1}^k \varepsilon_j d^j$, where
 $d^j \in \mathbb{R}^n$, $\mathbb{E}(\varepsilon_j) = 0$ and $\text{Var}(\varepsilon_j) = \sigma_j^2$.

$$(\mathcal{D}; \mathbb{P}_4) \quad \min_{\lambda \in \mathbb{R}^m} \frac{1}{2} f^T u + \frac{1}{2} \sum_{j=1}^r \sigma_j d^j T u^j$$

$$s.t. \quad K(\lambda)u = f$$

$$K(\lambda)u^j = \sigma_j d^j \quad j = 1, \dots, k$$

$$\lambda \in \Delta_m.$$



Stochastic model including variance

We consider

$$\min_{\lambda \in \mathbb{R}^m} \alpha \mathbb{E}_{\xi}[\Psi(\xi, \lambda)] + \beta \text{Var}[\Psi(\xi, \lambda)]$$

Example: $\xi \sim \mathcal{N}(0, PP^T)$.

Taking for simplicity $\alpha = 0$ and $\beta = 1$

$$\begin{aligned} (\mathcal{D}; \mathbb{P}_{\text{var}}) \quad & \min_{\lambda \in \mathbb{R}^m} \frac{1}{2} \left(\text{Trace}((P^T U)^2) + 2f^T U U^T f \right) \\ & s.t. \quad K(\lambda)u = f \\ & \quad \quad K(\lambda)U = P \\ & \quad \quad \lambda \in \Delta_m. \end{aligned}$$

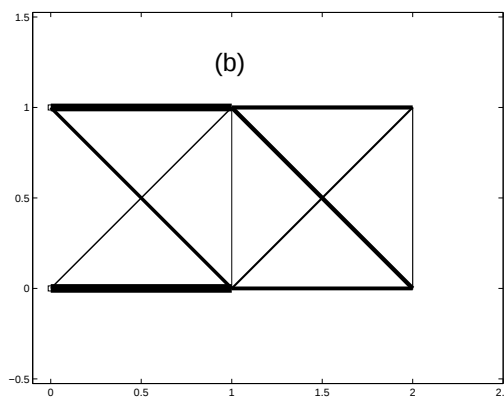
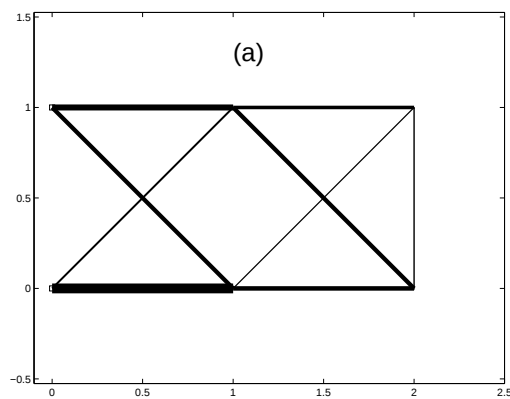
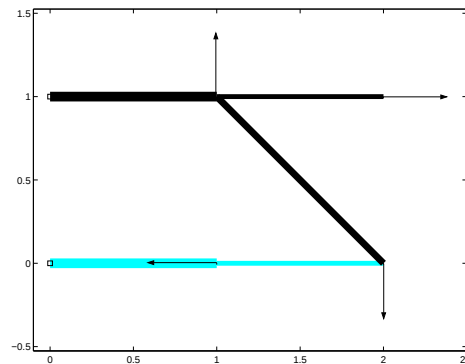
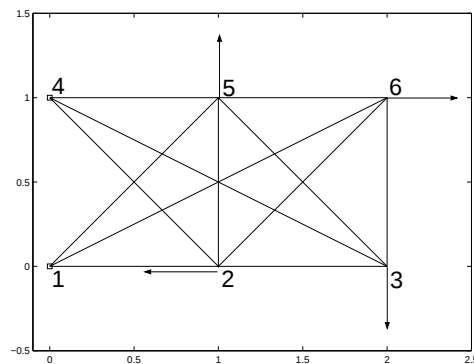
Apparent harder to solve.



Numerical results

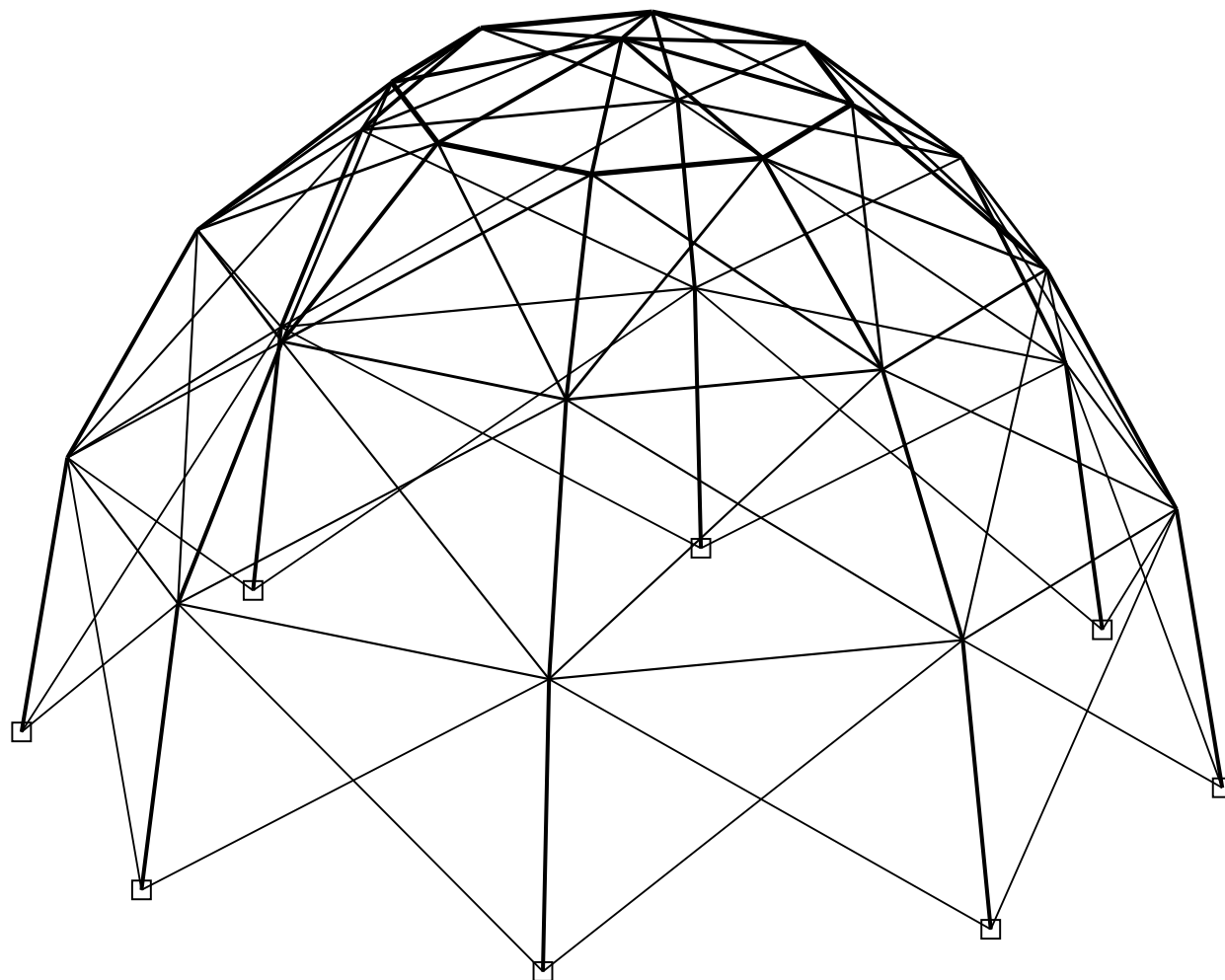


Toy example





Problem	compl. mean	max compl.	min compl.
Standard single load	$\neq$	$+\infty$	$+\infty$
(a)	0.025	0.053	0.013
(b)	0.023	0.051	0.012





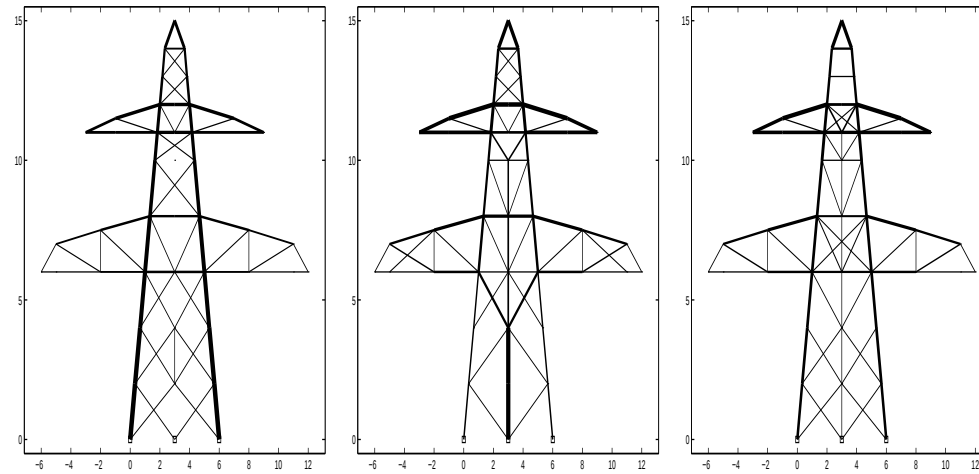
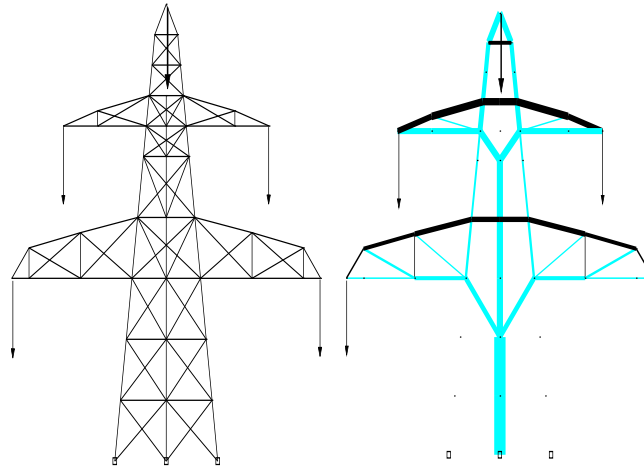
Dome

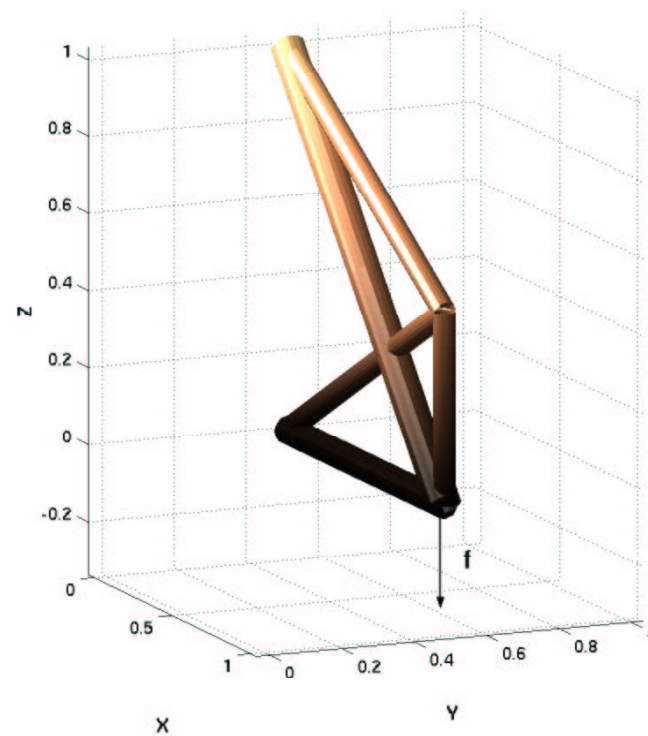
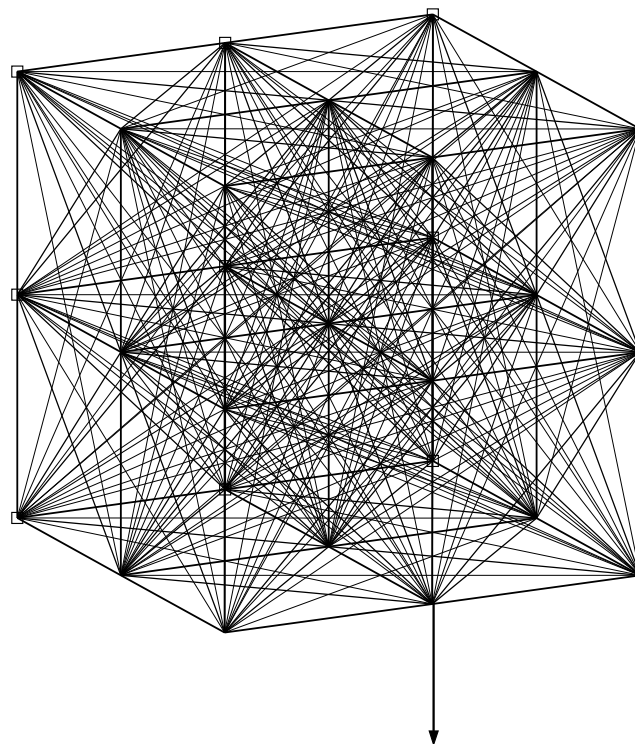


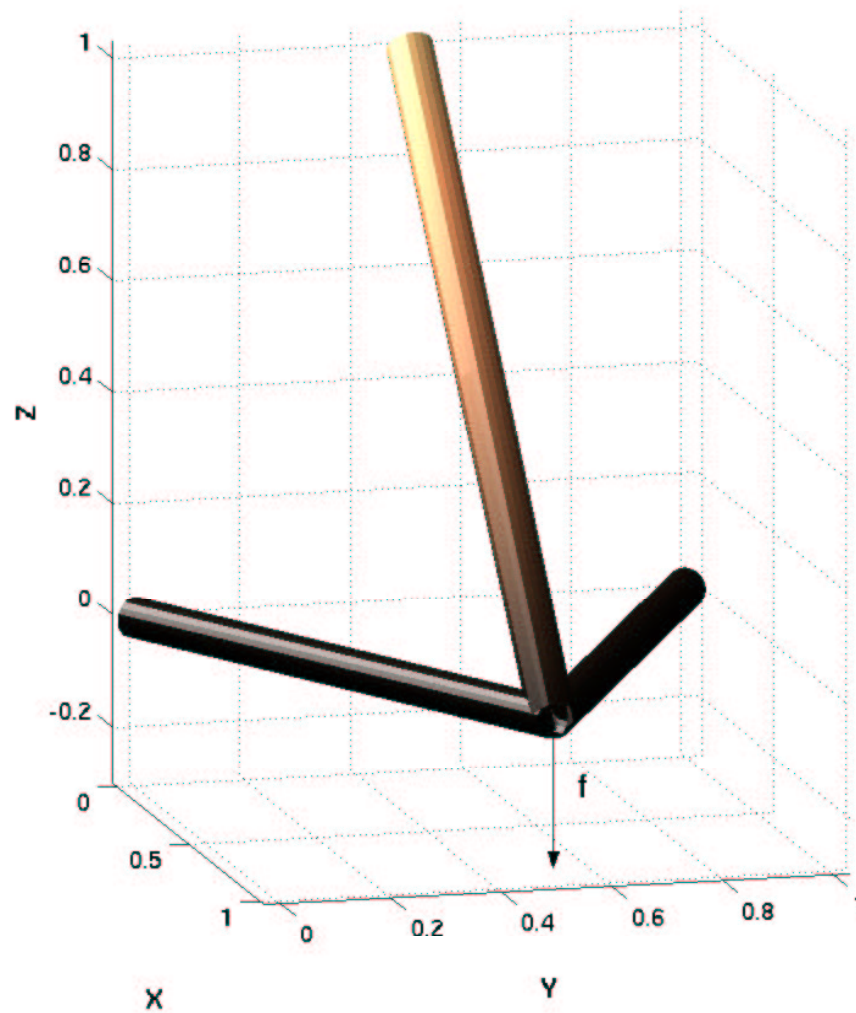




Electricity mast

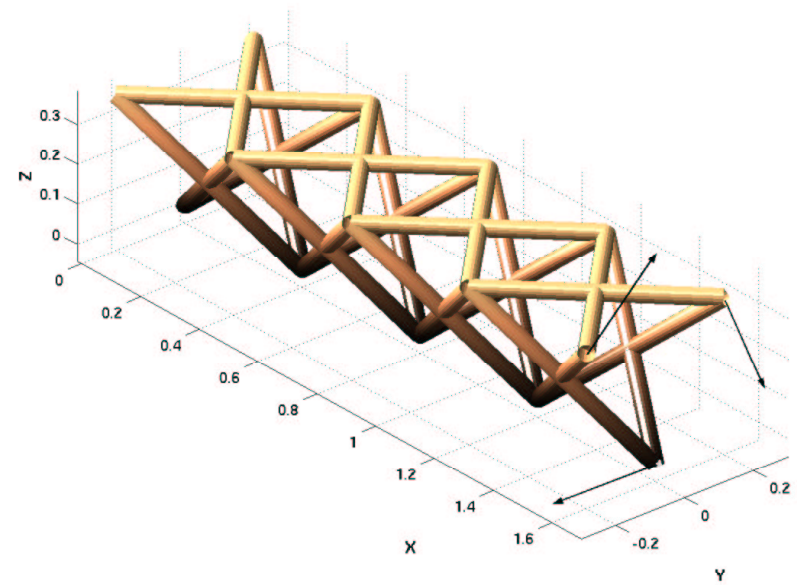
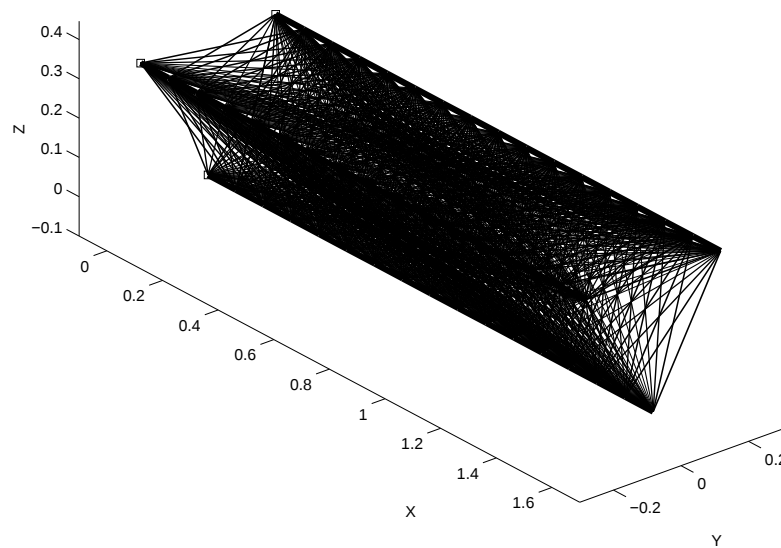


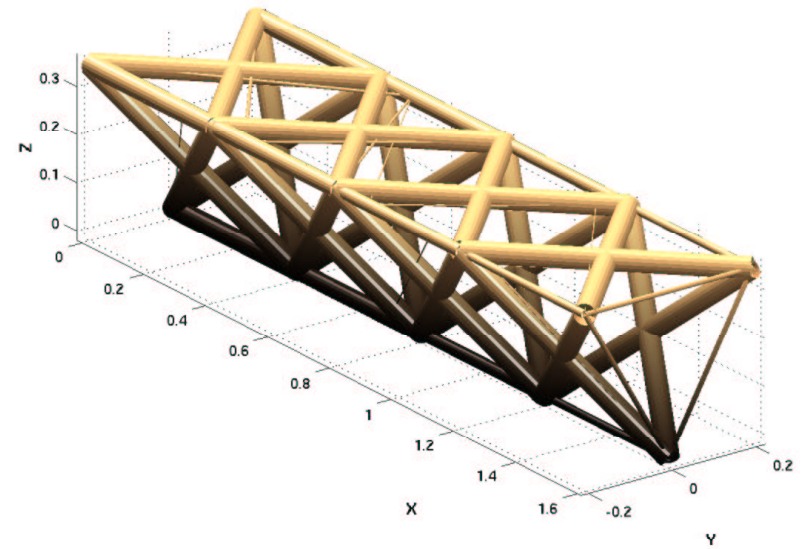
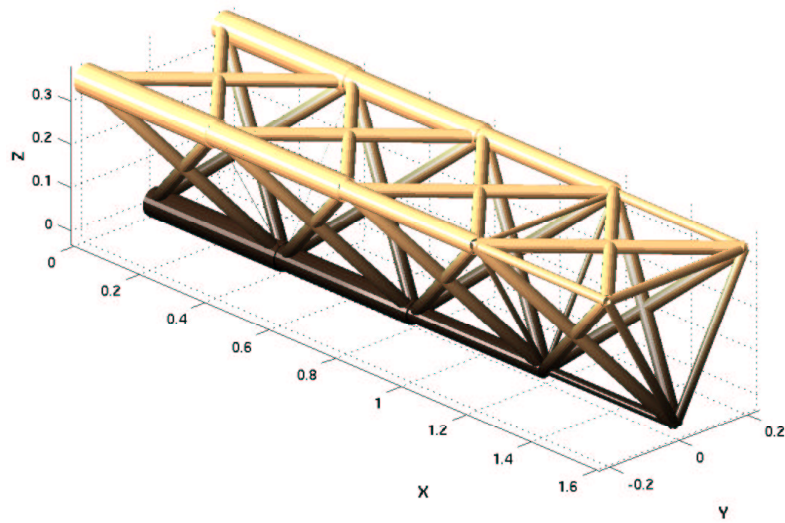






Cantilever







Conclutions

- Classical model (one load) don't get good results .
- Random model helps to avoid this problem.
- It's equivalent to multiload one, but another interpretations in weight factors holds.
- We have to know the influence of each perturbation.
- Increase the dimention of the problem.



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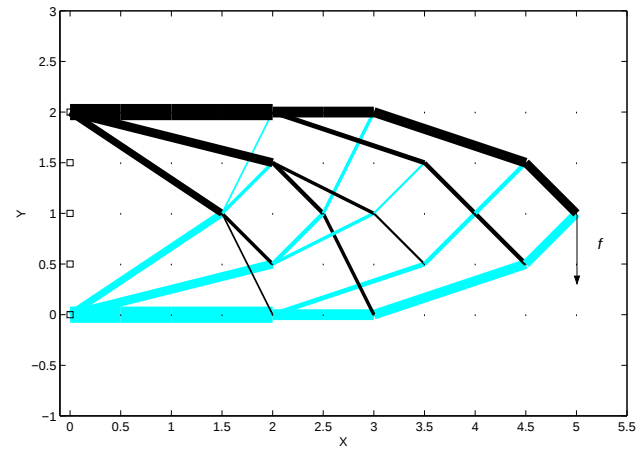
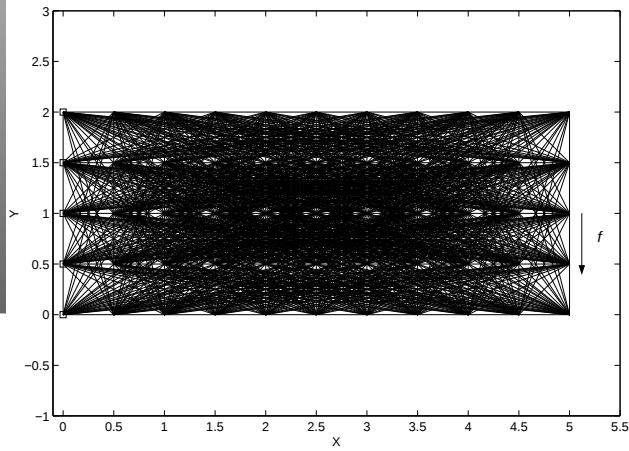
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stability result

Lemma 0.1. Let $\lambda \in \Delta_m$ and C be such that $C \geq \frac{1}{2} \max_j f_j^T u_j$ with $K(\lambda)u_j = f_j$, $j = 1, \dots, k$. For each $f \in \text{conv}\{f_1, \dots, f_k, -f_1, \dots, -f_k\}$, there exists $u \in \mathbb{R}^n$ satisfying $K(\lambda)u = f$ and moreover $C \geq \frac{1}{2} f^T u$.

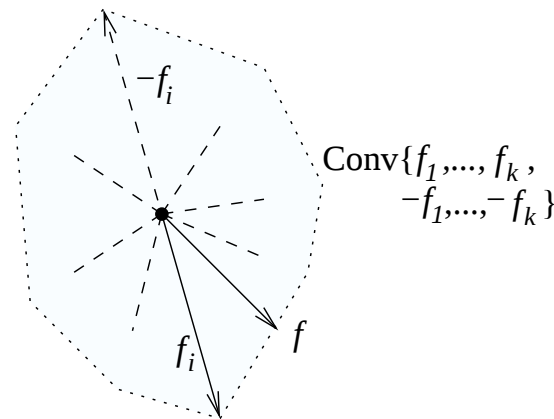
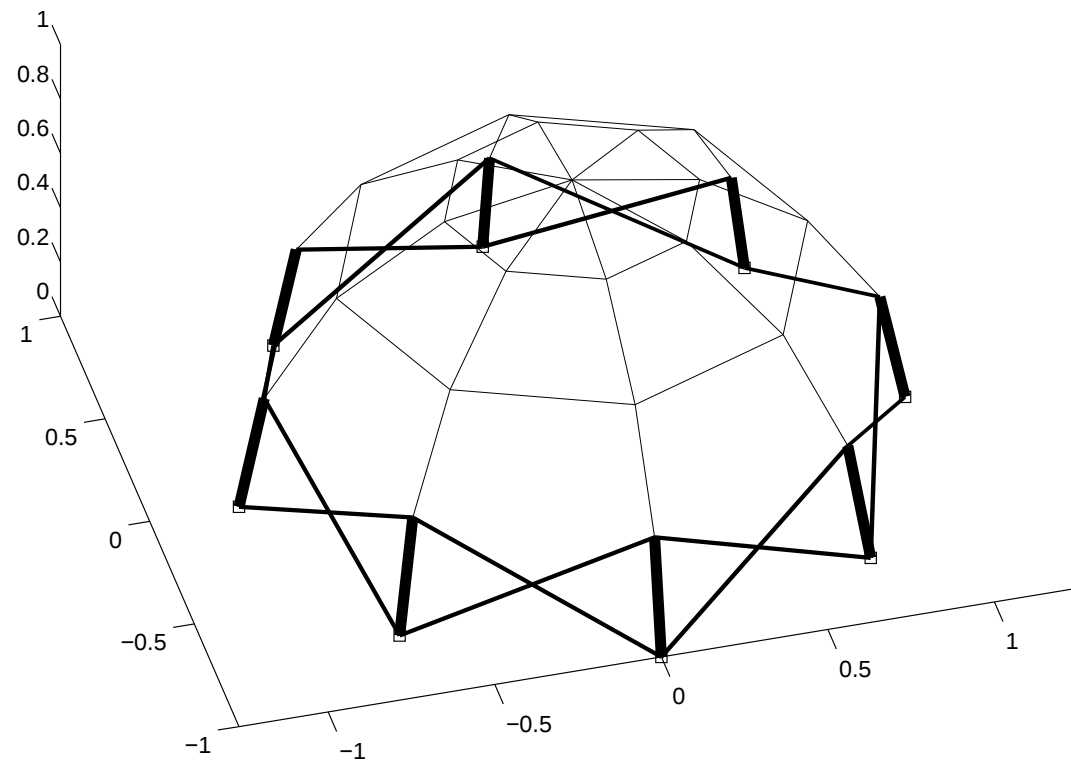


Figure 1: Convex Hull of $f_1, \dots, f_k, -f_1, \dots, -f_k$

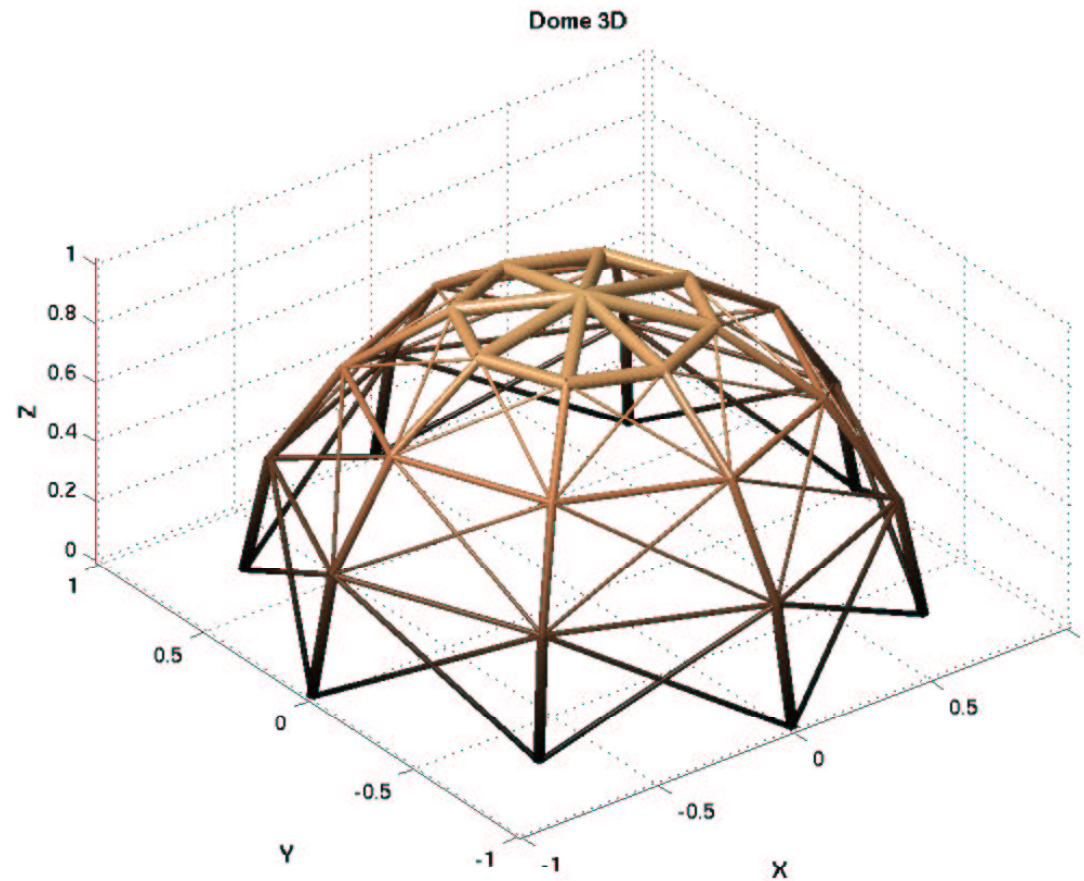


self weight





self weight





Se cumple el siguiente resultado:

Theorem 0.2. *Sea (x^*, μ^*) solución del problema (QP) , entonces existen $\sigma^+, \sigma^- \in \mathbb{R}_+^m$ tales que definiendo*

$$\lambda^* = \frac{1}{\mu^*}(\sigma^+ + \sigma^-)$$

se cumple que el par (x^, λ^*) es solución de $(\mathcal{D})$, (el problema de diseño).*



(QP) es equivalente a (LP)

$$\begin{aligned} (LP) \quad & \min_{x \in \mathbb{R}^n} \quad -f^T x \\ & s.a. \quad b_i^T x \leq 1 \quad i = 1, \dots, m \\ & \quad \quad -b_i^T x \leq 1 \quad i = 1, \dots, m \end{aligned}$$



Se cumple el siguiente resultado

Theorem 0.3. *Sea $\bar{x}$ solución del problema (LP) con vectores multiplicadores $\rho^+, \rho^- \in \mathbb{R}_+^m$, entonces*

$$x^* := \mu^* \bar{x}$$

$$\lambda_i^* := \frac{1}{\mu^*} (\rho_i^+ + \rho_i^-) \quad i = 1, \dots, m$$

con $\mu^* := \sum_{i=1}^m (\rho_i^+ + \rho_i^-)$ resuelven $(\mathcal{D})$.