p-Box: A “new” graph model

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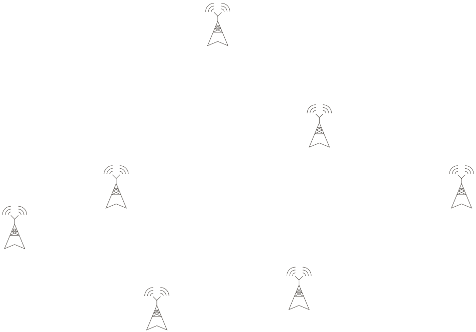
LIAFA, Paris

Wireless Sensor Networks



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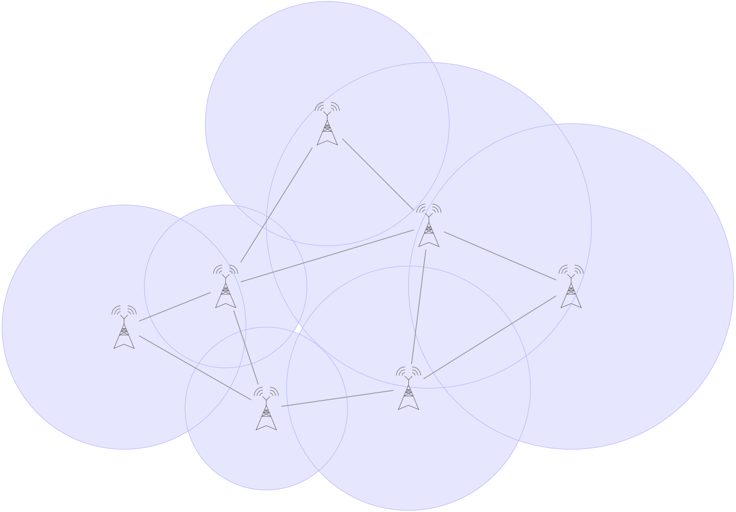

Related Graphs Classes

- Intersection Graphs: $G = (V, E)$

$$\begin{array}{rcl} V & \rightarrow & \mathcal{F} \text{ (Interval, Disk, Boxes)} \\ u & \mapsto & F_u \end{array}$$

$$uv \in E \Leftrightarrow F_u \cap F_v \neq \emptyset$$

- Tolerance Graphs:

$$u \mapsto (I_u, t_u)$$

$$uv \in E \Leftrightarrow |I_u \cap I_v| >= \min\{t_u, t_v\}.$$

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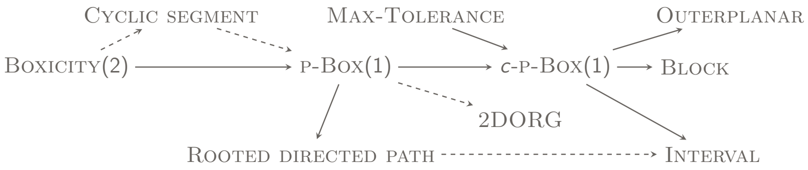
The set $\{(I_v, p_v)\}_{v \in V}$ is called (c-)p-Box (1) *realization* of G.

Today


Cyclic segment
Max-Tolerance
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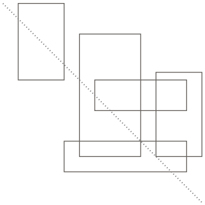

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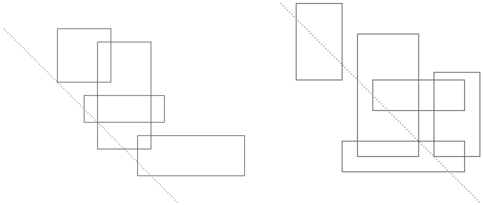

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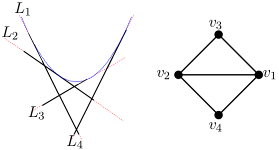

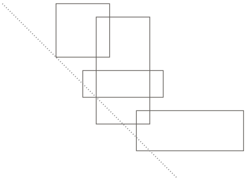

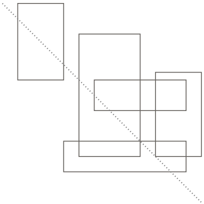

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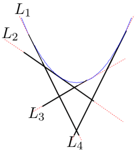

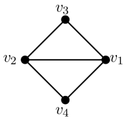

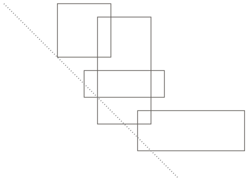
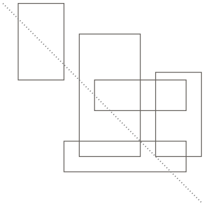

$$\text{Point-Tolerance} = \mathrm{p} - Box(1) \subsetneq DCS \subsetneq SRG$$

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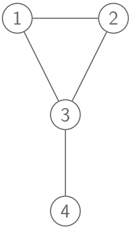
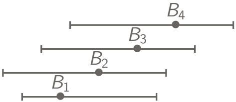
- Correa et al['13]: MHS/MIS $\in [3/2, 2]$. (Wegner conjecture)

Intersection Model





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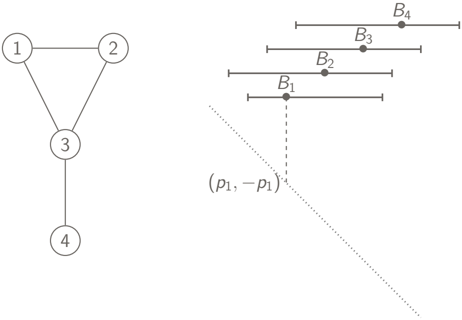

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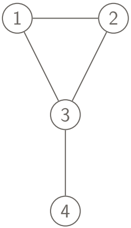

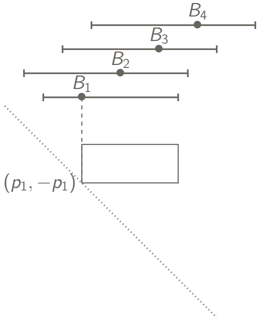

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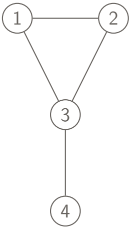

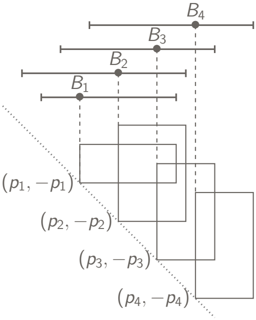

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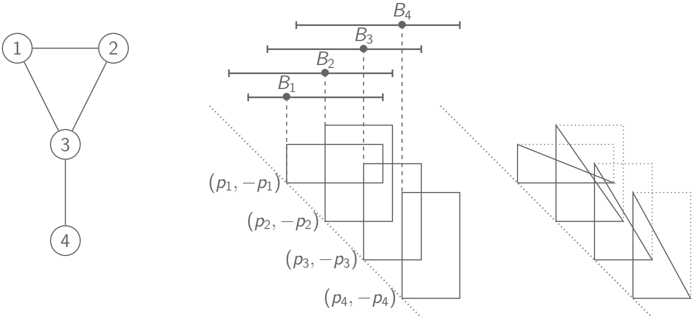

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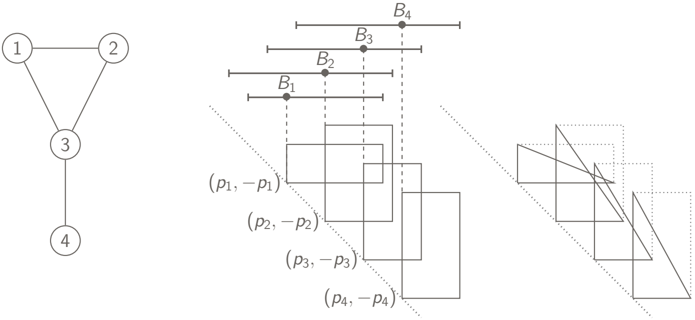
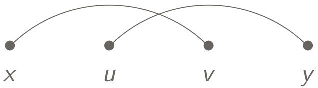


- Kaufmann et al.[SODA'06]: MAX-TOLERANCE = Intersection of isosceles triangles

$$\textsc{Max-tolerance} \supset c\text{-p-}\textsc{Box}(1).$$

Combinatorial Characterization

Positions of representative points p_v induce an order of V.



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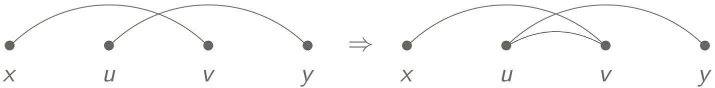

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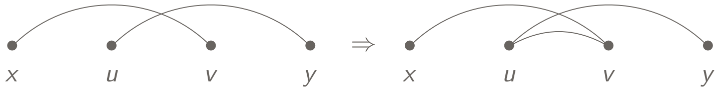
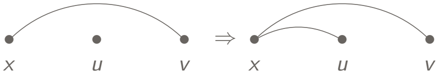


- Interval graphs : [S. Olariu '91]



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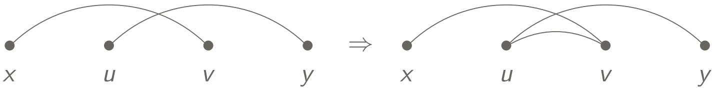

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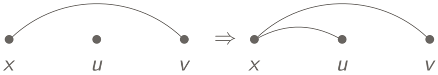

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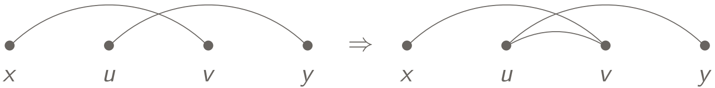

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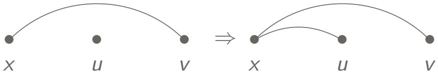

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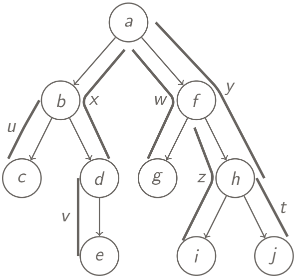
We will prove: {Interval, Outerplanar}$\in$ c-p-Box (1)

Combinatorial Characterization cont.

Rooted directed path graph : intersection graphs of of directed paths in a rooted directed tree



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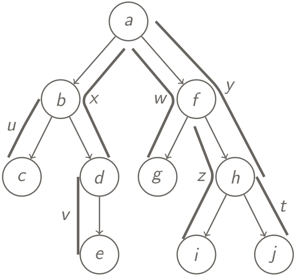

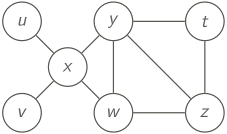

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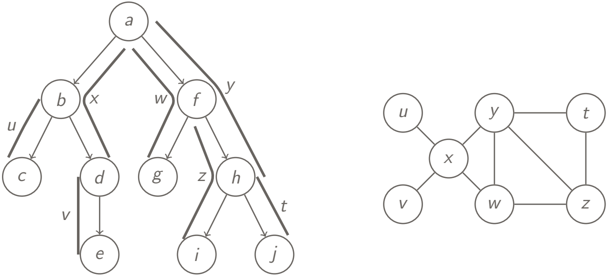

An inverse DFS on the tree is ($j\ i\ h\ g\ f\ e\ d\ c\ b\ a$) inducing the order of the vertices of the graph: ($t\ z\ y\ w\ v\ x\ u$)

The c-p-Box (1) class

Some Definitions

- Given an order π of the vertex, we note by:
  - $\ell_\pi(v)$ the most left neighbor of v.
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- Given a realization of G:
  - $L(v)$ denotes the left extreme of I_v
  - $R(v)$ denotes the right extreme of I_v
  - v is **safe** if its position p_v belongs only to its neighbors intervals.

Interval Graphs and c-p-Box (1)

Theorem

Interval $\subset$ *c-p-*Box *(1).*

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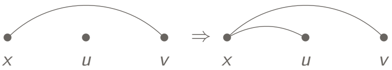

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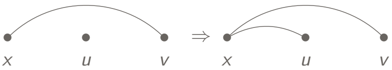

[S. Olariu 1991]

We greedily construct a realization according to order s.t. at step i:

1. $p_{k-1} < p_k$
2. $\rho(j) <_\pi \rho(k) \Rightarrow R(j) < R(k)$
3. $L(j) < p_{\ell(j)}$
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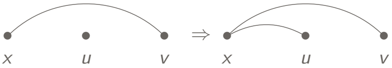

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Second, set the interval I_i s.t. it contains all its previous neighbors.

Finally, we modify, if necessary, the interval of previous vertices in order to satisfy conditions 2.

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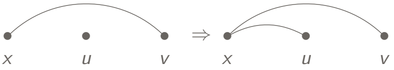

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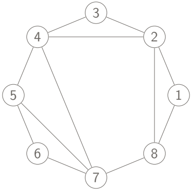
All vertices are safe with respect to previous neighbors.

Outerplanar Graphs and c-p-Box (1)

Theorem

OUTERPLANAR ⊂ *c-p*-BOX *(1)*.

- Non trivial biconnected components are dissections of polygons.



- Cycles are in *c*-p-Box (1)
- How to "glue" two cycles by an edge
- How to "glue" two biconnected components by a vertex

Outerplanar and c-p-$\textsc{Box}$ (1)

$C_n \in c$-p-$\textsc{Box}(1)$. Moreover, if π denotes the permutation induced by a realization Then, there exists a clockwise (or anticlockwise) labeling $l : V \rightarrow \{1, 2, \ldots, n\}$ such that:

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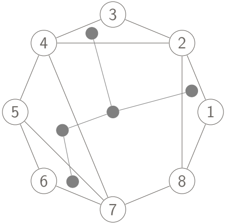
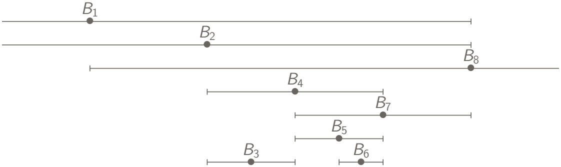
Extremes vertices are safe!

Outerplanar and c-p-Box (1)

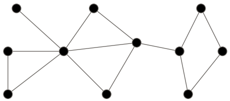
We can "glue" cycles by an edge in A DFS of weak dual

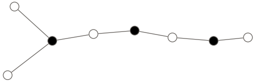


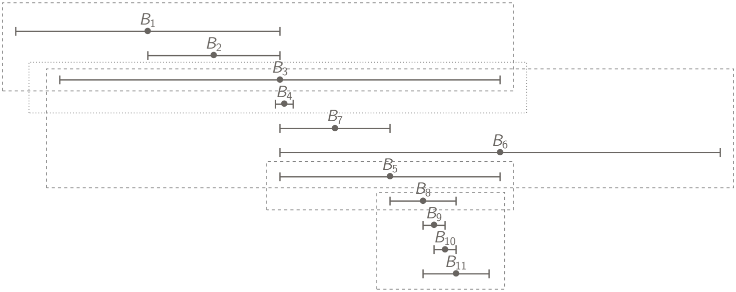


Outerplanar and c-p-Box (1)

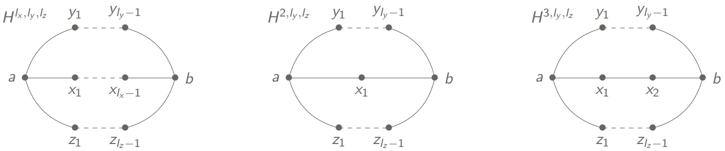
We construct a realization according to a BFS on the Block-tree of G scaling biconnected component



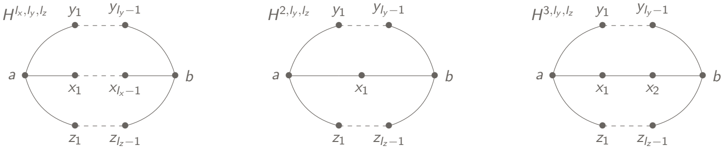


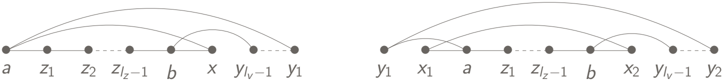


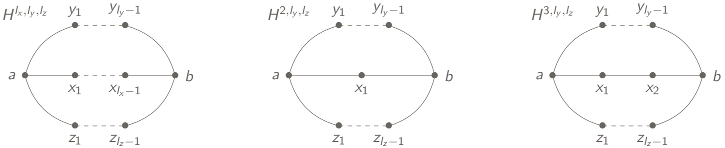
p-Box (1) $\searrow$ c-p-Box (1)

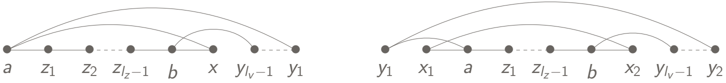


H^{l_x,l_y,l_z}
y_1
y_{l_y-1}
a
b
x_1
x_{l_x-1}
z_1
z_{l_z-1}
H^{2,l_y,l_z}
y_1
y_{l_y-1}
a
b
x_1
z_1
z_{l_z-1}
H^{3,l_y,l_z}
y_1
y_{l_y-1}
a
b
x_1
x_2
z_1
z_{l_z-1}

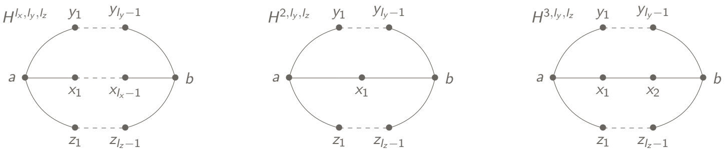

H^{l_x,l_y,l_z}
y_1
y_{l_y-1}
a
b
x_1
x_{l_x-1}
z_1
z_{l_z-1}
H^{2,l_y,l_z}
y_1
y_{l_y-1}
a
b
x_1
z_1
z_{l_z-1}
H^{3,l_y,l_z}
y_1
y_{l_y-1}
a
b
x_1
x_2
z_1
z_{l_z-1}

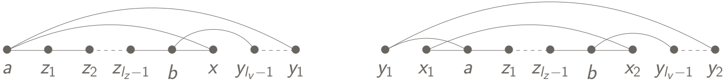

a z_1 z_2 z_{l_z-1} b x y_{l_v-1} y_1
y_1 x_1 a z_1 z_{l_z-1} b x_2 y_{l_v-1} y_2





- Any H^{l_x,l_y,l_z} graph such that $l_z \geq l_y \geq l_x \geq 2$ does not belong to c-p-Box (1).





- Any H^{l_x,l_y,l_z} graph such that $l_z \geq l_y \geq l_x \geq 2$ does not belong to c-p-Box (1).
- Any H^{l_x,l_y,l_z} graph such that $l_z \geq l_y \geq l_x > 3$ does not belong to p-Box (1).

Future Work and Open Questions

- Combinatorial characterization for c-p-$\textsc{Box}$ (1).

Future Work and Open Questions

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  - Given an order?

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Future Work and Open Questions

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  - Given an order?
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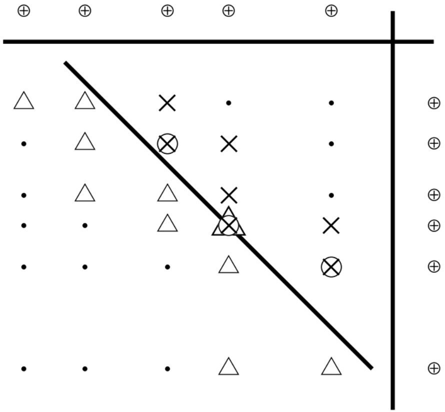
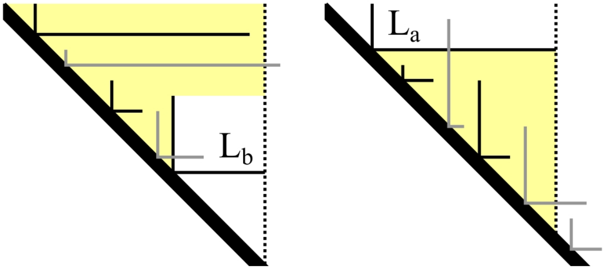
  - Unit-p-Box
  - p-Box with arcs

Duality gap p-Box (1), Correa et al. ['13]



- $MIS(G) \geq max\{|I_x|, |I_y|\} = max\{|H_x|, |H_y|\}$
- $MHS(G) \leq H \leq |H_x| + |H_y| \leq 2MIS(G)$

WMIS in p-Box (1), Cantazaro et al.['13]



- $opt[u, v] = max_{B_i \in [a,b]}\{opt[a, B_i] + w(B_i) + opt[i, b]\}$
- $O(n^2)$ pairs computed in $O(n)$

c-p-Box with given order

$$
\begin{aligned}
\min \quad & \sum_{i \in V} r_i & & (1) \\
\text{s.t.} \quad & x_i - x_{\ell_\pi(i)} \le r_i - \varepsilon_1 & 1 \le i \le n \quad & (2) \\
& x_{\rho_\pi(i)} - x_i \le r_i - \varepsilon_1 & 1 \le i \le n \quad & (3)
\end{aligned}
$$

c-p-Box with given order

$$
\begin{aligned}
\min \quad & \sum_{i \in V} r_i && && (1)\\
\text{s.t.} \quad & x_i - x_{\ell_\pi(i)} \le r_i - \varepsilon_1 && 1 \le i \le n && (2)\\
& x_{\rho_\pi(i)} - x_i \le r_i - \varepsilon_1 && 1 \le i \le n && (3)\\
& x_j - x_i \ge r_j && 1 \le i <_\pi j \le n,\ ij \notin E,\ i \curvearrowright j && (4)\\
& x_j - x_i \ge r_i && 1 \le i <_\pi j \le n,\ ij \notin E,\ i \curvearrowleft j && (5)
\end{aligned}
$$

c-p-Box with given order

$$\min \sum_{i \in V} r_i \tag{1}$$

$$\text{s.t.} \quad x_i - x_{\ell_\pi(i)} \leq r_i - \varepsilon_1 \qquad 1 \leq i \leq n \tag{2}$$

$$x_{\rho_\pi(i)} - x_i \leq r_i - \varepsilon_1 \qquad 1 \leq i \leq n \tag{3}$$

$$x_j - x_i \geq r_j \qquad 1 \leq i <_\pi j \leq n,\ ij \notin E,\ i \curvearrowright j \tag{4}$$

$$x_j - x_i \geq r_i \qquad 1 \leq i <_\pi j \leq n,\ ij \notin E,\ i \curvearrowleft j \tag{5}$$

$$x_j - x_i \leq r_i \qquad 1 \leq i <_\pi j \leq n,\ ij \notin E,\ i \between j \tag{6}$$

$$x_j - x_i \leq r_j \qquad 1 \leq i <_\pi j \leq n,\ ij \notin E,\ i \between j \tag{7}$$

c-p-Box with given order

$$
\begin{array}{llll}
\min & \sum_{i \in V} r_i & & (1) \\
\text{s.t.} & x_i - x_{\ell_\pi(i)} \leq r_i - \varepsilon_1 & 1 \leq i \leq n & (2) \\
& x_{\rho_\pi(i)} - x_i \leq r_i - \varepsilon_1 & 1 \leq i \leq n & (3) \\
& x_j - x_i \geq r_j & 1 \leq i <_\pi j \leq n,\ ij \notin E,\ i \curvearrowright j & (4) \\
& x_j - x_i \geq r_i & 1 \leq i <_\pi j \leq n,\ ij \notin E,\ i \curvearrowleft j & (5) \\
& x_j - x_i \leq r_i & 1 \leq i <_\pi j \leq n,\ ij \notin E,\ i \between j & (6) \\
& x_j - x_i \leq r_j & 1 \leq i <_\pi j \leq n,\ ij \notin E,\ i \between j & (7) \\
& x_{i+1} - x_i \geq \varepsilon_2 & 1 \leq i \leq n-1 & (8) \\
& x_1 = 0,\ x_n = L & &
\end{array}
$$

Containment Relations