Minimization of Symmetric Submodular Functions under Hereditary Constraints

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Outline

Background

Minimal Minimizers and Pendant Pairs

Algorithms

Queyranne's algorithm to find Pendant Pairs

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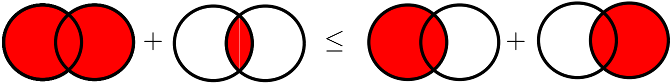
A set function $f : 2^V \to \mathbb{R}$ with ground set V is ...

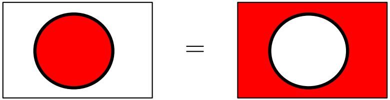
Submodular if:

$$f(A \cup B) \quad + \quad f(A \cap B) \quad \leq \quad f(A) \quad + \quad f(B)$$



Symmetric if:

$$f(A) \quad = \quad f(V \setminus A)$$

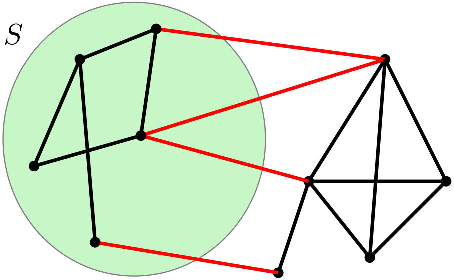


We have access to a value oracle for f.

Typical Example of a Symmetric Submodular Function (SSF)

Cut function of a weighted undirected graph:

$$f(S) = w(\delta(S)) = \sum_{e:|e \cap S|=1} w(e)$$



Hereditary families

Definition

A family $\mathcal{I} \subseteq 2^V$ is hereditary if it is closed under inclusion.
$\mathcal{I}^* = \mathcal{I} \setminus \{\emptyset\}$.

Examples

- $V = V(G)$: Graph properties closed under induced subgraphs ($\mathcal{I}^*$: stable sets, clique, k-colorable, etc.)
- $V = E(G)$: Graph properties closed under subgraphs ($\mathcal{I}^*$: matching, forest, etc.)
- Upper cardinality constraints, knapsack constraints, matroid constraints, etc.

We have access to a membership oracle for $\mathcal{I}$.

Problem:

Constrained SSF minimization

Find a nonempty set in $\mathcal{I}$ minimizing f.

We exclude the empty set since:

$$2f(A) = f(A) + f(V \setminus A) \geq f(V) + f(\emptyset) = 2f(\emptyset).$$

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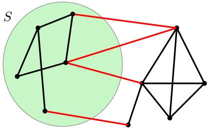
Example: Special mincuts.

Find a minimum cut $S \subseteq V$ such that $|S| \leq k$ (or S is a clique, stable, etc.)



Our results

[GS]

$O(n^3)$-algorithm for minimizing SSF on hereditary families, where $n = |V|$. (In fact, we find all the Minimal Minimizers in $O(n^3)$-time).

Compare to:

[Queyranne 98]

$O(n^3)$-algorithm for minimizing SSF.

[Svitkina-Fleischer 08]

Minimizing a general submodular function under upper cardinality constraints is NP-hard to approximate within $o(\sqrt{n/\log n})$.

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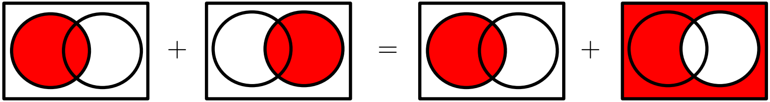
Tool: SSF are posimodular

$$f(A \setminus B) + f(B \setminus A) \leq f(A) + f(B)$$

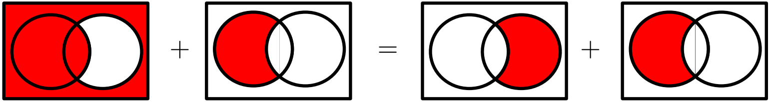
Proof.

$$f(A) \quad + \quad f(B) \quad = \quad f(A) \quad + \quad f(V \setminus B)$$



$$\geq \quad f(A \cup (V \setminus B)) + f(A \cap (V \setminus B)) = \quad f(B \setminus A) \quad + \quad f(A \setminus B)$$

$\geq$



□

Minimal Minimizers are disjoint (I)

Minimal Minimizers (MM)

S is a MM if: (i) $S \in \mathcal{I}^*$, (ii) $f(S) = \min_{X \in \mathcal{I}^*} f(X) = \mathrm{OPT}$,
and (iii) $\forall \emptyset \subset Y \subset S,\ f(S) < f(Y)$.

Lemma

The MM of $(f, \mathcal{I})$ are disjoint.

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Lemma

The MM of $(f, \mathcal{I})$ are disjoint.

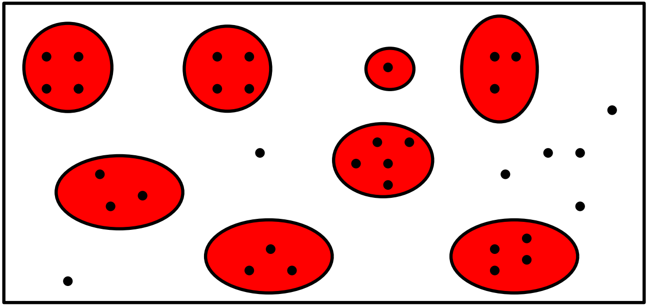
Proof.

If A and B are intersecting MM, then $A \setminus B, B \setminus A \in \mathcal{I}^*$.
By posimodularity

$$f(A \setminus B) + f(B \setminus A) \;\leq\; f(A) + f(B) = 2\mathrm{OPT},$$

then $f(A \setminus B) = f(B \setminus A) = \mathrm{OPT}$. □

Minimal Minimizers are disjoint (II)

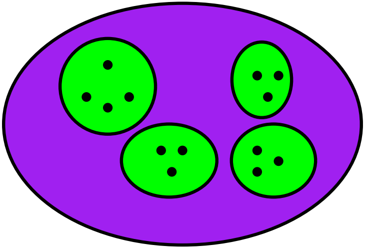


- Family $\mathcal{X}$ of MM has at most $O(n)$ sets.
- Partition Π of V with at most one "bad" part.
- IDEA: Detect groups of elements inside the same part and fuse them together.

Fusions

We will iteratively fuse elements together.

- Original system: $(V, f, \mathcal{I})$.
- Modified systems: $(V', f', \mathcal{I}')$.
- For $S \subseteq V'$, X_S is the set of original elements fused into S.
- $f'(S) = f(X_S)$ is a SSF.
- $\mathcal{I}' = \{S \colon X_S \in \mathcal{I}\}$ is hereditary.



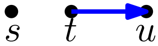
Pendant pairs

Definition

We say (t, u) is a Pendant Pair (PP) for f if $\{u\}$ has the minimum f-value among those sets separating t and u, i.e.

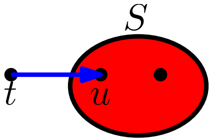
$$f(\{u\}) = \min\{f(U) \colon |U \cap \{t, u\}| = 1\}.$$

- [Queyranne 98]: every SSF f admits PP.
- [Nagamochi Ibaraki 98]: given $s \in V$, we can find a PP (t, u) with $s \notin \{t, u\}$.



A PP (t, u) and the partition Π

If S is a non-singleton MM of $(f, \mathcal{I})$ then we cannot have:



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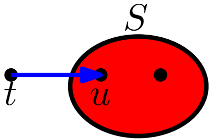
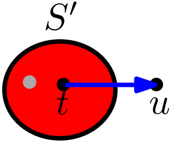


If t is in a MM S' and u is in the bad part then $f(\{u\}) \leq f(S')$. We conclude u is a loop (i.e. $\{u\} \notin \mathcal{I}$).



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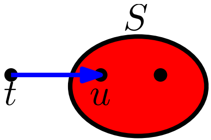

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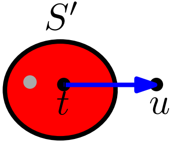

Theorem (One of the following holds:)

1. *u and t are in the same part of Π.*
2. *$\{u\}$ is a singleton MM.*
3. *u is a loop.*

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Warming up: Queyranne's algorithm

Algorithm to find one MM of a SSF in $2^V \setminus \{V, \emptyset\}$

- While $|V| \geq 2$,
  1. Find (t, u) pendant pair.
  2. Add $X_{\{u\}}$ as a candidate for minimum.
  3. Fuse t and u as one vertex.
- Return the (first) best of the $n - 1$ candidates.

Correctness

Cannot create loops!
We fuse pairs in the same part of Π until $\{u\}$ is a singleton MM (first best candidate).

Algorithm to find one MM in constrained version

Assume $\mathcal{I}$ has exactly one loop s. (If many, fuse them together)

Algorithm

- While $|V| \geq 3$,
  1. Find (t, u) pendant pair avoiding s.
  2. Add $X_{\{u\}}$ as a candidate for minimum.
  3. If $\{t, u\} \in \mathcal{I}$, Fuse t and u as one vertex.
     Else, Fuse s, t and u as one loop vertex (call it s).
- If $|V| = 2$, add the only non-loop as a candidate.
- Return the (first) best candidate.

Notes:

- u is never a loop!
- If no loop in $\mathcal{I}$, use any pendant pair in instruction 1.

Algorithm to find the family $\mathcal{X}$ of all the MM

- Find one MM S. Let $\mathrm{OPT} = f(S)$, $\mathcal{X} = \{S\}$.
- Add all singleton MM to $\mathcal{X}$.
- Fuse sets in $\mathcal{X}$ and loops together in a single element s.
- While $|V| \geq 3$,
  1. Find (t, u) pendant pair avoiding s. [$\{t, u\}$ is INSIDE a part.]
  2. If $\{t, u\} \notin \mathcal{I}$, Fuse s, t and u as one loop vertex as s.
  3. Else if $f'(\{t, u\}) = \mathrm{OPT}$, Add $X_{\{t,u\}}$ to $\mathcal{X}$ and Fuse s, t and u together as s.
  4. Else Fuse t and u as one vertex.
- If $|V| = 2$, check if the only non-loop is optimum.
- Return $\mathcal{X}$.

Conclusions.

- Can find all the MM of $(f, \mathcal{I})$ by using $\leq 2n$ calls to a PP finder procedure.
- Queyranne's PP procedure finds pendant pairs in $O(n^2)$ time/oracle calls.
- All together: $O(n^3)$-algorithm.

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Rizzi's Degree Function

Let f be a SSF on V with $f(\emptyset) = 0$.
Define the function $d(\cdot, :)$ on pairs of disjoint subsets of V as

$$d(A, B) = \frac{1}{2}\left(f(A) + f(B) - f(A \cup B)\right).$$

E.g., If $f(\cdot) = w(\delta(\cdot))$ is the cut function of a weighted graph, then

$$d(A, B) = w(A : B) = \sum_{uv: u \in A, v \in B} w(uv)$$

is the associated degree function.
Note: $f(A) = d(A, V \setminus A)$.

Maximum Adjacency (MA) order

The sequence $(v_1, \ldots, v_n)$ is a MA order of (V, f) if

$$d(v_i, \{v_1, \ldots, v_{i-1}\}) \geq d(v_j, \{v_1, \ldots, v_{i-1}\}).$$

We get a MA order by setting v_1 arbitrarily and selecting the next vertex as the one with MAX. ADJACENCY to the ones already selected.

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Lemma [Queyranne 98, Rizzi 00]

The last two elements (v_{n-1}, v_n) of a MA order are a pendant pair.

Remark:

If $|V| \geq 3$, we can always find a pendant pair **avoiding** one vertex.

MA order yields PP

S Symmetric: $d(A, B) = d(B, A)$.
M Monotone: $d(A, B) \leq d(A, B \cup C)$.
C Consistent: $d(A, C) \leq d(B, C) \Rightarrow d(A, B \cup C) \leq d(B, A \cup C)$.

Proof that MA yields PP

If $n = 2$, trivial.

If $n = 3$, the only sets separating v_2 and v_3 are $\{v_3\}$, $\{v_1, v_3\}$ and their complements.

MA implies $d(v_2, v_1) \geq d(v_3, v_1)$.

C implies $d(v_2, \{v_1, v_3\}) \geq d(v_3, \{v_1, v_2\})$,

i.e. $f(v_3) \leq f(\{v_1, v_3\})$.

MA order yields PP

- S Symmetric: $d(A, B) = d(B, A)$.
- M Monotone: $d(A, B) \leq d(A, B \cup C)$.
- C Consistent: $d(A, C) \leq d(B, C) \Rightarrow d(A, B \cup C) \leq d(B, A \cup C)$.

Proof that MA yields PP

If $n \geq 4$, let S be a set separating v_{n-1} and v_n.

Case 1: S does not separate v_1 and v_2.

Then: $(\{v_1, v_2\}, v_3, \ldots, v_{n-1}, v_n)$ is a MA order.

So: $f(v_n) \leq f(S)$.

MA order yields PP

S Symmetric: $d(A,B) = d(B,A)$.
M Monotone: $d(A,B) \leq d(A, B \cup C)$.
C Consistent: $d(A,C) \leq d(B,C) \Rightarrow d(A, B \cup C) \leq d(B, A \cup C)$.

Proof that MA yields PP

If $n \geq 4$, let S be a set separating v_{n-1} and v_n.

Case 2: S does not separate v_2 and v_3.

M implies $(v_1, \{v_2, v_3\}, \ldots, v_{n-1}, v_n)$ is a MA order.

$$(\ d(v_j, v_1) \leq d(v_2, v_1) \leq_M d(\{v_2, v_3\}, v_1) \ )$$

So: $f(v_n) \leq f(S)$.

MA order yields PP

S Symmetric: $d(A, B) = d(B, A)$.
M Monotone: $d(A, B) \leq d(A, B \cup C)$.
C Consistent: $d(A, C) \leq d(B, C) \Rightarrow d(A, B \cup C) \leq d(B, A \cup C)$.

Proof that MA yields PP

If $n \geq 4$, let S be a set separating v_{n-1} and v_n.

Case 3: S does not separate v_1 and v_3.

C+M implies $(v_2, \{v_1, v_3\}, \ldots, v_{n-1}, v_n)$ is a MA order.

If not: $\exists j, \quad d(\{v_1, v_3\}, v_2) < d(v_j, v_2) \leq_M d(v_j, \{v_1, v_2\})$,

by C: $\qquad d(v_2, \{v_1, v_3\}) \geq d(v_3, \{v_1, v_2\})$, then:

$$d(v_3, \{v_1, v_2\}) \geq d(v_j, \{v_1, v_2\}) > d(\{v_1, v_3\}, v_2) \geq d(v_3, \{v_1, v_2\}).$$