A simple PTAS for Weighted Matroid Matching on Strongly Base Orderable Matroids

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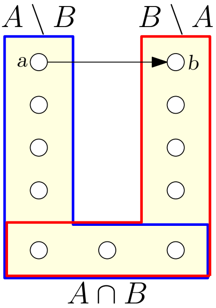
2011

Matroids.

$\emptyset \neq \mathcal{B} \subseteq 2^V$ is the basis system of a matroid if

- Every $B \in \mathcal{B}$ has the same size.
- basis exchange property: For all $A, B \in \mathcal{B}$ distinct there are $a \in A \setminus B$, $b \in B \setminus A$ s.t. $A - a + b \in \mathcal{B}$.



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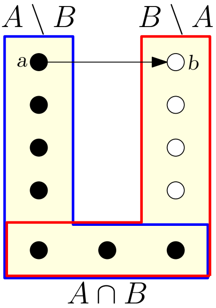

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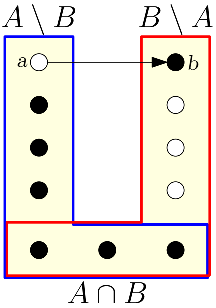

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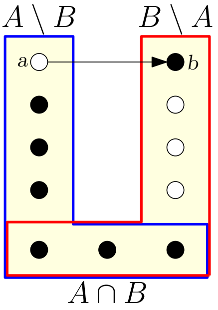

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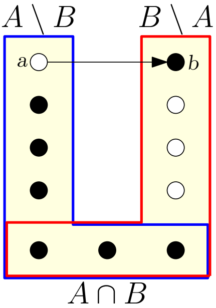

Independent sets $\mathcal{I}$.

Independent Sets = Basis subset.

Examples of matroid bases

- (Free) Only V.
- (Uniform) Sets of size k.
- (Graphic) Spanning forests.
- (Linear) Vector space bases.

Strongly Base Orderable (SBO) Matroids.

$\emptyset \neq \mathcal{B} \subseteq 2^V$ is the basis system of a SBO matroid

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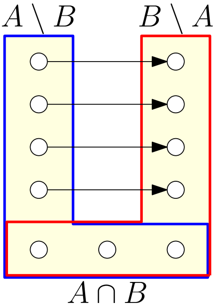

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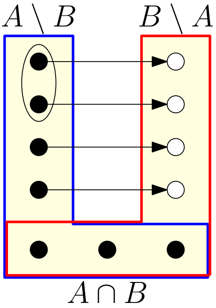

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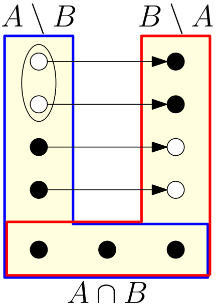

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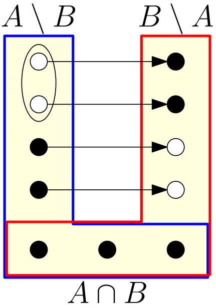
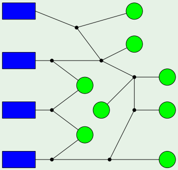


Examples of SBO matroid bases

- (Uniform) Sets of size k.
- (Gammoid)



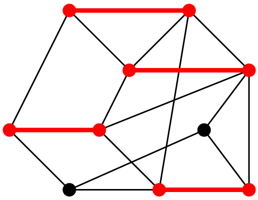
Maximum sets of clients connected to servers by edge-disjoint paths.

Weighted Matroid Matching Problem

Problem

- Weighted graph $G = (V, E)$, $w\colon E \to \mathbb{R}^+$.
- Matroid $\mathcal{M} = (V, \mathcal{I})$.

A matching $M \subseteq E$ is **feasible** for $\mathcal{M}$ if $V(M)$ is independent.
Goal: Find a maximum weight feasible matching.



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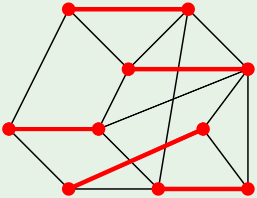
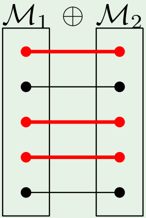
Weighted Matching

Free matroid.



Weighted Matroid Intersection



Complexity of WMM

- Not in **oracle coNP** even for unweighted case.
- **NP-hard** even for unweighted case.
- Special subproblems in **P**:
  Weighted matching / Weighted Matroid Intersection.
- (Lovász 1981) Unweighted case in **P** for linear matroids.
- (Tong et al. 1982) Weighted case in **P** for gammoids.
- (Camerini et al. 1992, Narayanan et al. 1994)
  Pseudopolynomial for weighted case in linear matroids.

Approximation algorithms

Unweighted

- **Greedy** gives 2-approximation.

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Weighted

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- (S. 2011) PTAS for SBO-matroids.

WMM on SBO-matroids

Hardness

Still outside **oracle coNP** and **NP-hard**.

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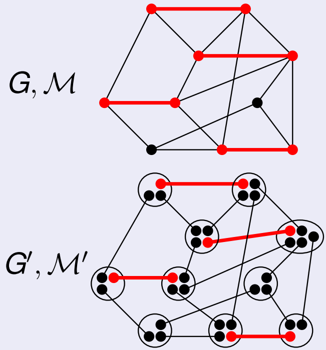

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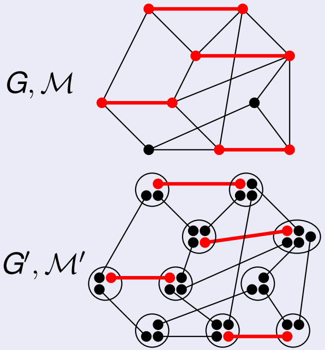
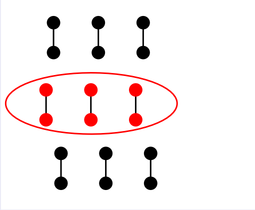


Weighted Parity on SBO-matroids

Find maximum weight set of pairs *feasible* for a SBO-matroid $\mathcal{M}$.



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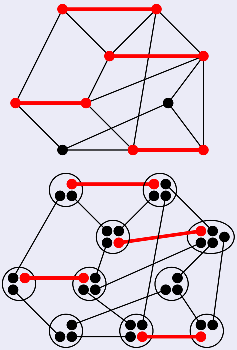
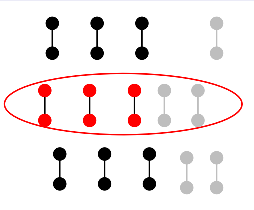
$G, \mathcal{M}$

$G', \mathcal{M}'$



Weighted Parity on SBO-matroids

Find maximum weight *paired basis* of a SBO-matroid $\mathcal{M}$. (with dummy pairs)



Local moves

t-swap: For a current paired basis A

Swap at most t pairs to obtain paired basis B.
Gain: $w(B) - w(A)$. **High gain**: $w(B) - w(A) \geq w(A)/n^2$.

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Algorithm: For constant $1 \leq t \leq n$,

- Start with greedy solution.
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At most $O(\log_{(1+1/n^2)} 1/2) = O(n^2)$ moves suffice.

Can find an t-local optimum in polynomial time ($O(n^{2t+2})$)

Main result for WMM on SBO-matroids.

Theorem

If paired basis A is a t-local optimum and $B = OPT$ then

$$w(B) \leq \left(1 + \frac{2}{t-1}\right) w(A).$$

PTAS: To get $(1+\varepsilon)$-approx set $t = 1 + 2/\varepsilon$.
Running time $n^{O(1/\varepsilon)}$.

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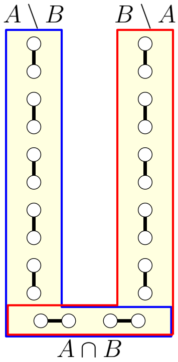
Auxiliar construction



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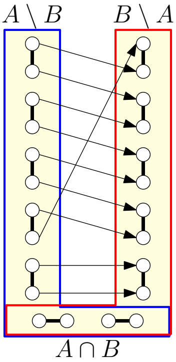

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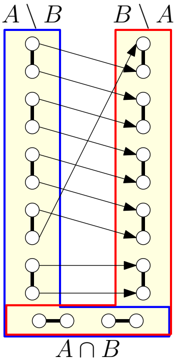
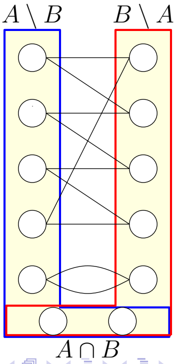
- *H*: Union of cycles.





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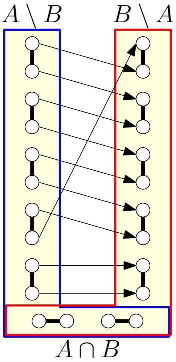
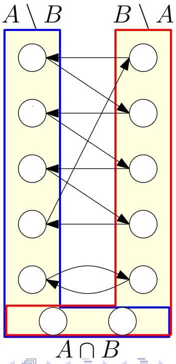
- $\vec{H}$: Union of directed cycles.

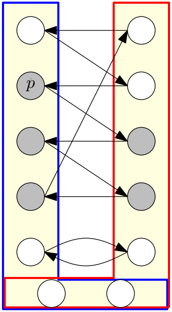




Proof of $w(B) \leq (1 + 2/(t-1))\, w(A)$.

For a pair p of A,

$\mathbf{H_p}$: Reachable from p using $\leq 2(t-1)$ edges.



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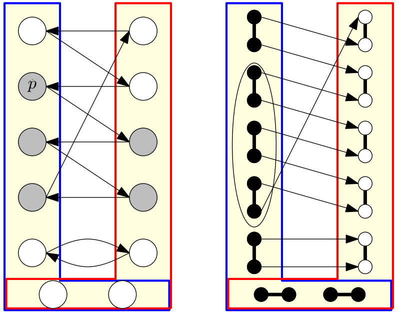

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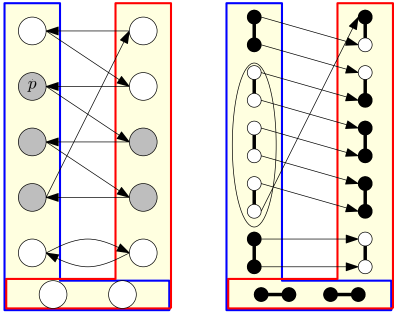

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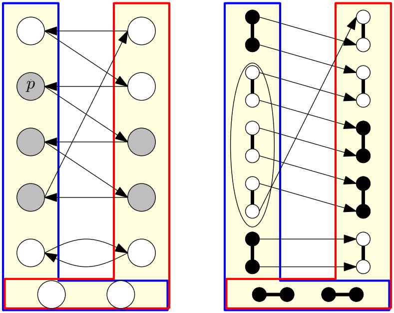

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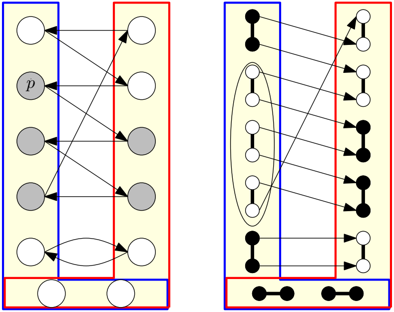

Since we swap at most t pairs

$$w(A)/n^2 > \text{Gain}(\mathbf{swap}(\mathbf{p})) = w(H_p \cap B) - w(H_p \cap A).$$

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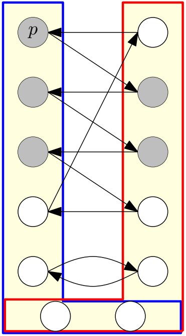

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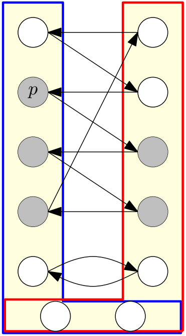

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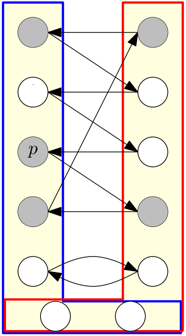

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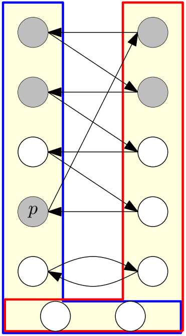

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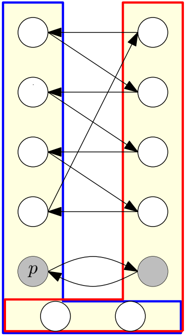

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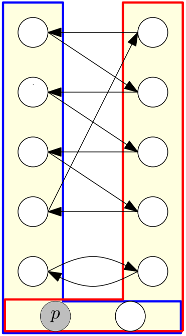

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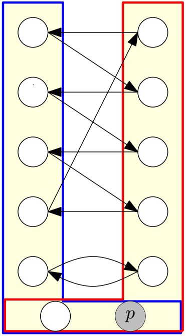

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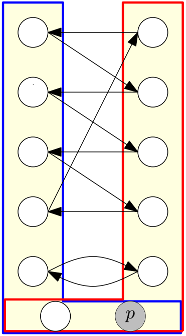
$$= w(B_{\text{rest}}) - w(A_{\text{rest}}).$$

Proof of $w(B) \le (1 + 2/(t-1))\, w(A)$. (cont.)



$$\sum_{p \in A_{\text{long}}} \frac{w(A)}{n^2} > \sum_{p \in A_{\text{long}}} w(H_p \cap B) - w(H_p \cap A)$$
$$= (t-1)\, w(B_{\text{long}}) - t\, w(A_{\text{long}}).$$

$$\sum_{p \in A_{\text{rest}}} \frac{w(A)}{n^2} > \sum_{p \in A_{\text{rest}}} w(H_p \cap B) - w(H_p \cap A)$$
$$= w(B_{\text{rest}}) - w(A_{\text{rest}}).$$

$$\begin{aligned}(t-1)\, w(B) - t\, w(A) &\le ((t-1)\, w(B_{\text{long}}) - t\, w(A_{\text{long}})) \\ &\quad + t\,(w(B_{\text{rest}}) - w(A_{\text{rest}})) \\ &< \frac{w(A)}{n^2}\left(|A_{\text{long}}| + t|A_{\text{rest}}|\right) \le w(A). \quad \square\end{aligned}$$

Summary

Conclusions

First PTAS for Weighted Matroid Matching on Strongly Orderable Matroids.

Open Problems

- Can we get a PTAS for general matroids?
- Can we get a FPTAS for this class?