Contributions on Secretary Problems, Independent Sets of Rectangles and Related Problems

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Doctoral Thesis Defense.
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Outline

1. Matroid Secretary Problem
2. Jump Number Problem and Independent Sets of Rectangles. (joint work with C. Telha)
3. Symmetric Submodular Function Minimization under Hereditary Constraints.

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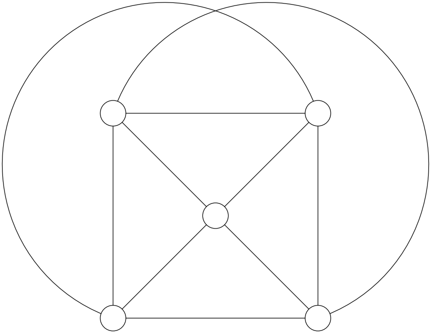
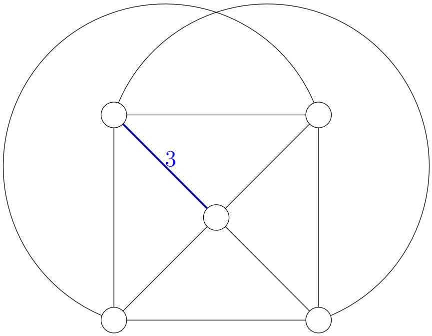


- Given a matroid.



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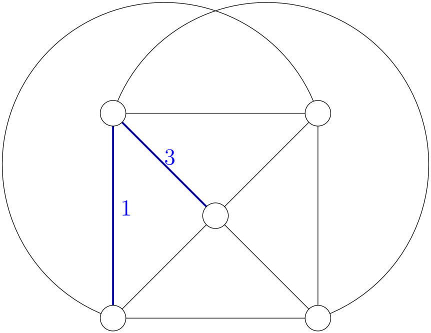

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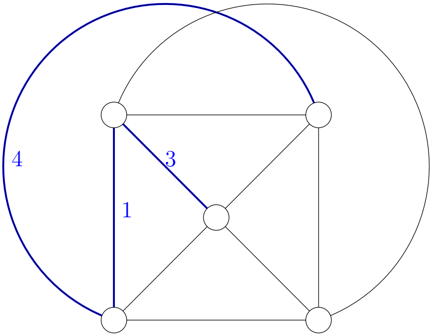

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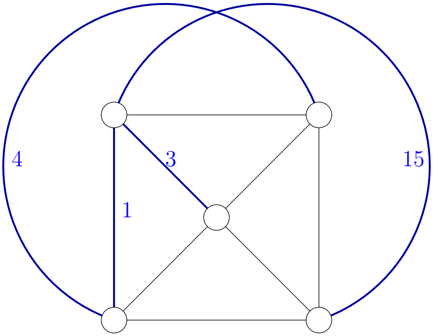

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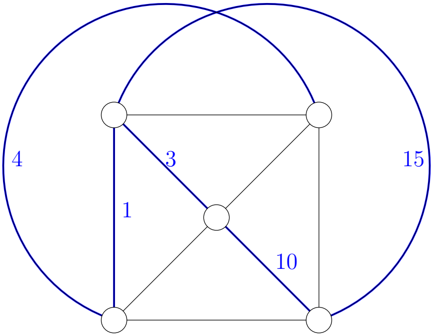

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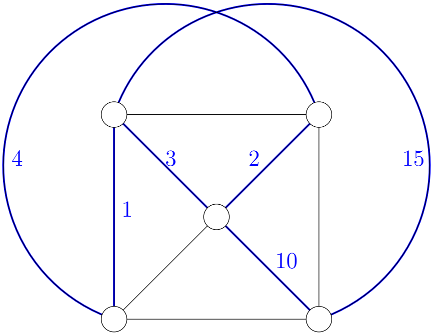

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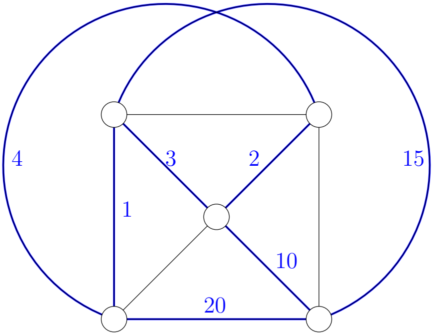

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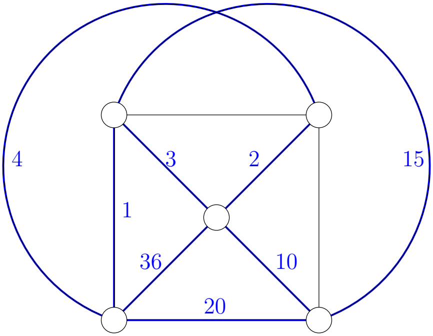

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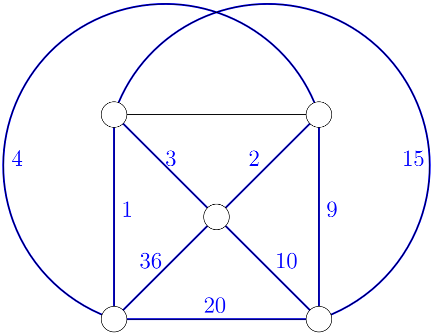

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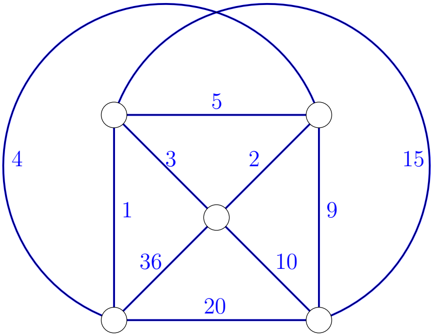

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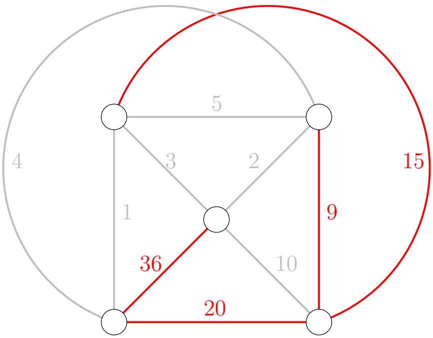
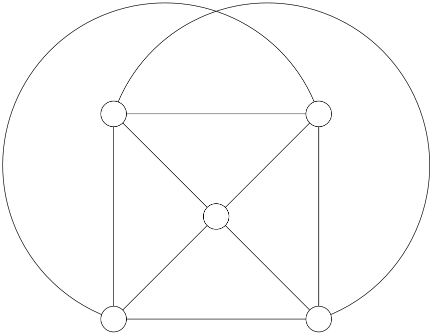


- Given a matroid.
- Elements' weights are revealed in certain (random) order.
- Want to select independent set of high weight.
  (In online way / secretary problem setting)



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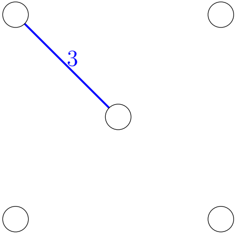

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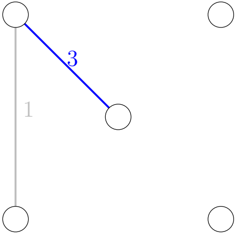

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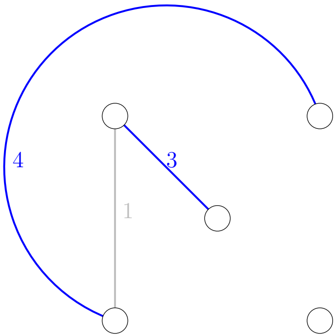

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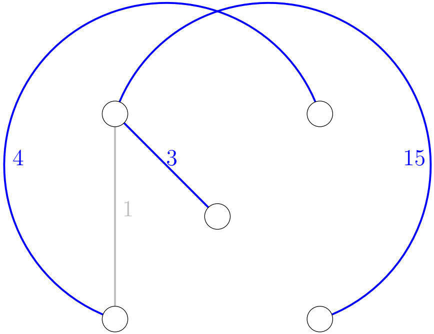

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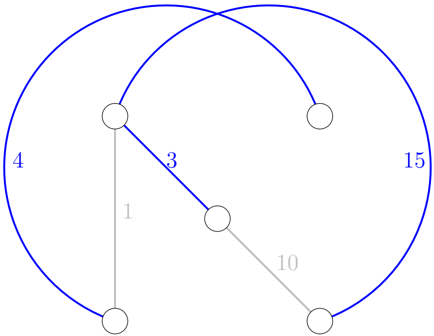

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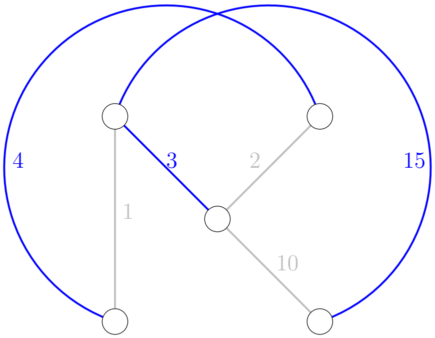

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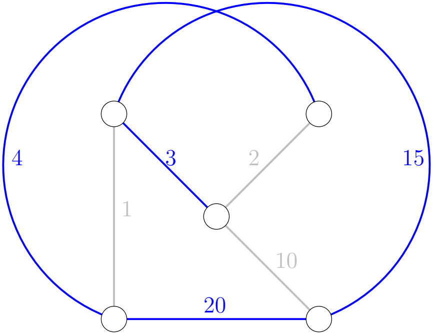

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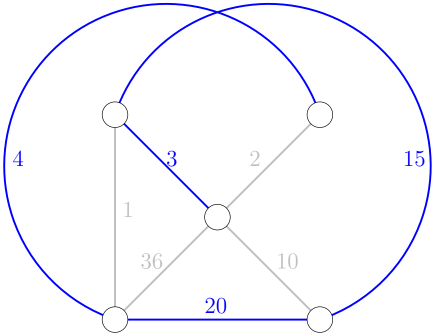

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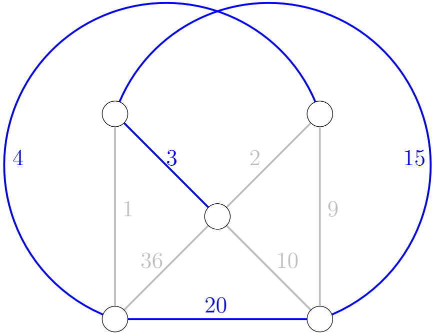

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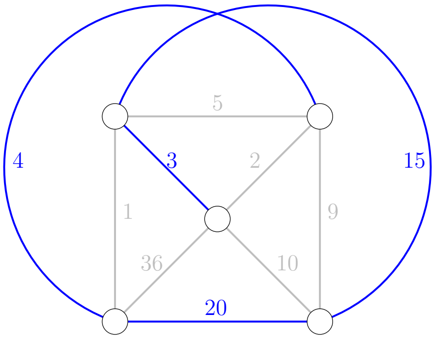

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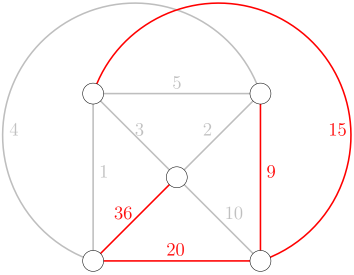
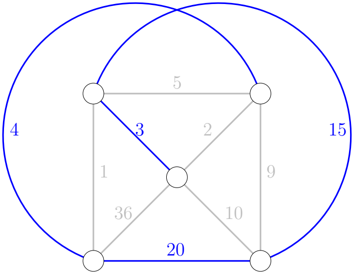


$w(\mathrm{OPT}) = 80$



$w(\mathrm{ALG}) = 42$

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Special Cases

Classical / Multiple choice

- Hire one person (or at most r).
- Sell one item to best bidder (or sell r identical items).

Models

Opponent selects n **weights.**
$w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$

then

The weights are assigned either:
adversarially or at **random.**

and independently

The presentation order is chosen:
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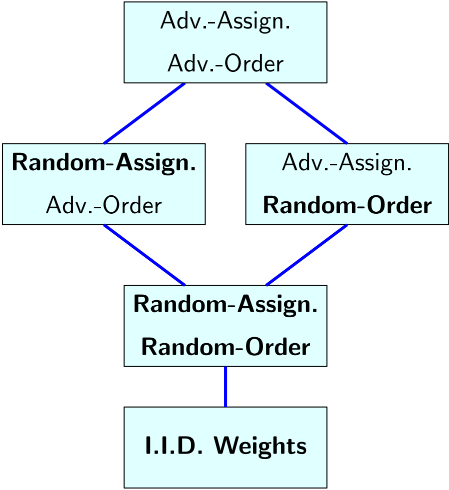
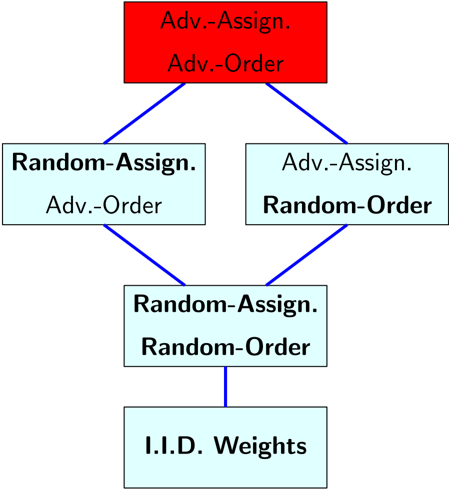


- (Adv.-Assign. Adv.-Order)
  Hard: n-competitive ratio

  [Babaioff, Immorlica, Kleinberg 07]
  Conjecture: O(1)-competitive algorithm for all other models.



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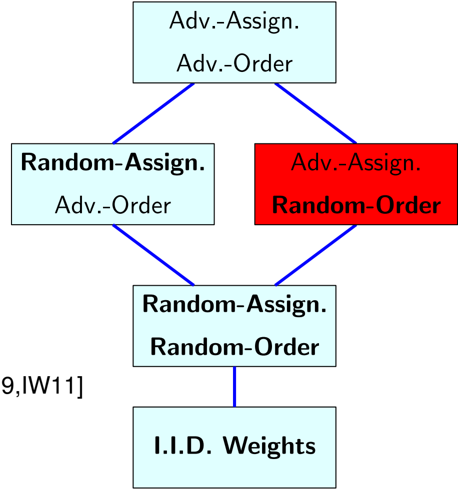
- (Adv.-Assign. Random-Order)
  $O(1)$ for partition, graphic, transversal, laminar.
  [L61,D63,K05,BIK07,DP08,KP09,BDGIT09,IW11]
  $O(\log \text{rk}(\mathcal{M}))$ for general matroids [BIK07].



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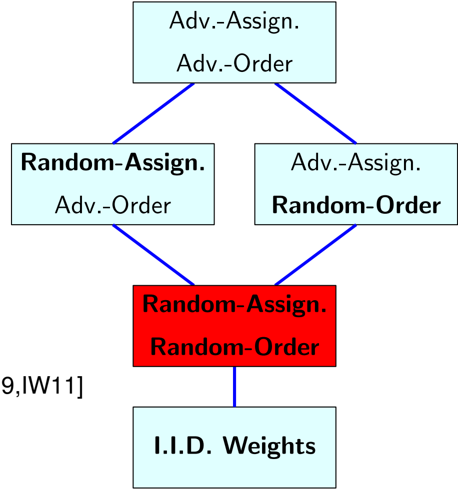
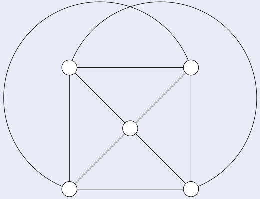
- (Random-Assign. Random-Order)
  [S11] $O(1)$ for general matroids.



Random-Assignment Random-Order.

Data



Known Matroid

$\xleftarrow[r.a.]{\sigma}$ $\mathbf{W: w_1 \geq w_2 \geq \cdots \geq w_n \geq 0.}$

Hidden weight list

Random assignment. $\sigma : [n] \to E$.
Random order. $\pi : E \to \{1, \ldots, n\}$.

Objective

Return an independent set $\mathrm{ALG} \in \mathcal{I}$ such that:

$$\mathbb{E}_{\pi,\sigma}[w(\mathrm{ALG})] \geq \Omega(1) \cdot \mathbb{E}_{\sigma}[w(\mathrm{OPT})], \text{ where}$$

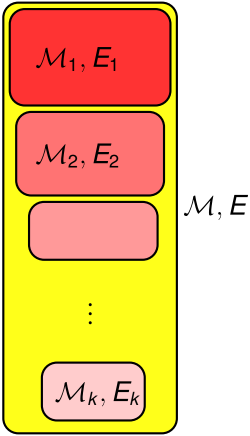
OPT is the optimum base of $\mathcal{M}$ under assignment σ. (Greedy)

Divide and Conquer to get $O(1)$-competitive algorithm.

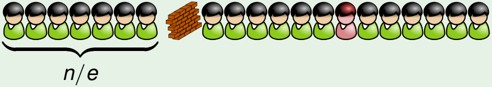
For a general matroid $\mathcal{M} = (E, \mathcal{I})$:

Find matroids $\mathcal{M}_i = (E_i, \mathcal{I}_i)$ with $E = \bigcup_{i=1}^{k} E_i$.

1. $\mathcal{M}_i$ admits $O(1)$-competitive algorithm (Easy parts).
2. Union of independent sets in each $\mathcal{M}_i$ is independent in $\mathcal{M}$. $\mathcal{I}(\bigoplus_{i=1}^{k} \mathcal{M}_i) \subseteq \mathcal{I}(\mathcal{M})$. (Combine nicely).
3. Optimum in $\bigoplus_{i=1}^{k} \mathcal{M}_i$ is comparable with Optimum in $\mathcal{M}$. (Don't lose much).



For $r = 1$: Dynkin's Algorithm



- Observe n/e objects. Accept the first record after that.

Top weight is selected w.p. $\geq 1/e$.

(Easiest matroids): Uniform. [Independent sets = Sets of size $\leq r$.]

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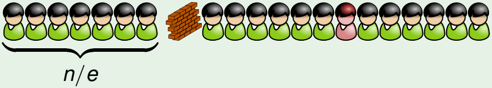

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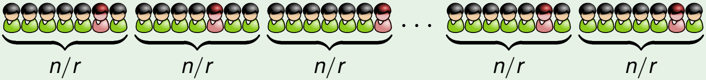

- Divide in r classes and apply Dynkin's algorithm in each class.

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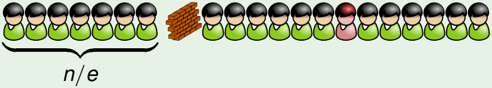

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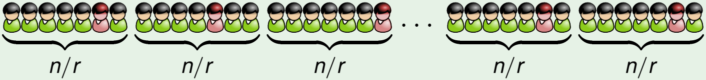

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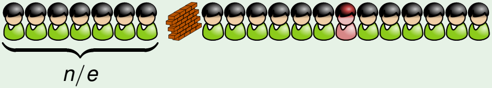

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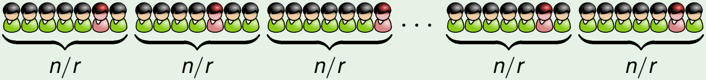

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- e/C (constant) competitive algorithm.

A loopless matroid is Uniformly dense if

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Property: Sets of $\mathrm{rk}(E)$ elements have almost full rank.

$$\mathbb{E}_{(X:|X|=\mathrm{rk}(E))}[\mathrm{rk}(X)] \geq \mathrm{rk}(E)(1-1/e).$$

(Easy matroids): Uniformly dense matroids are like Uniform

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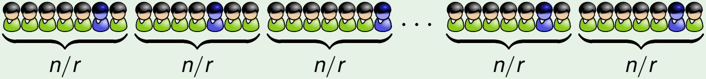
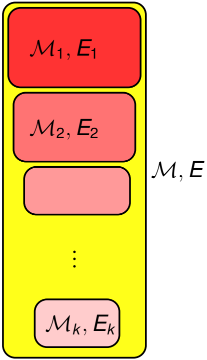
Algorithm: Simulate e/C-comp. alg. for Uniform Matroids.



- Try to add each selected weight to the independent set.
- Selected elements have expected rank $\geq r(1-1/e)$.
- We recover $(1-1/e) \cdot C/e$ fraction of the top r weights.

Uniformly Dense (sub)matroids That combine nicely



Want:

Matroids $\mathcal{M}_1, \ldots, \mathcal{M}_k$ such that:

1. Each $\mathcal{M}_i$ is uniformly dense.
2. If $I_i \in \mathcal{I}(\mathcal{M}_i)$, then $I_1 \cup I_2 \cup \cdots \cup I_k \in \mathcal{I}(\mathcal{M})$.

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Procedure.

- Let E_1 be the densest set of $\mathcal{M}$ of maximum cardinality.

$$\gamma(\mathcal{M}) := \max_{F \subseteq E} \frac{|F|}{\mathrm{rk}_{\mathcal{M}}(F)} = \frac{|E_1|}{\mathrm{rk}_{\mathcal{M}}(E_1)}.$$

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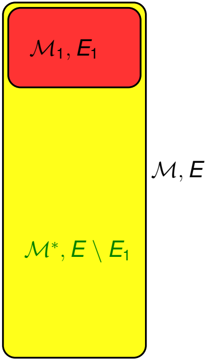
- $\mathcal{M}^* = \mathcal{M}/E_1$ has smaller density than $\mathcal{M}$.



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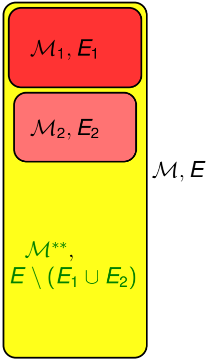

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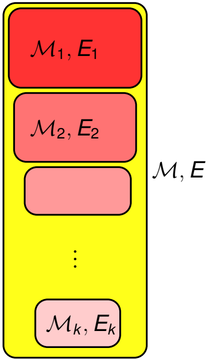


Theorem (Principal Partition) [Tomizawa, Narayanan]

There exists a partition $E = \bigcup_{i=1}^{k} E_i$ such that

1. Each principal minor $\mathcal{M}_i = (\mathcal{M}/E_{i-1})|_{E_i}$ is uniformly dense.
2. If $I_i \in \mathcal{I}(\mathcal{M}_i)$, then $I_1 \cup I_2 \cup \cdots \cup I_k \in \mathcal{I}(\mathcal{M})$.

Algorithm for a General Matroid $\mathcal{M}$

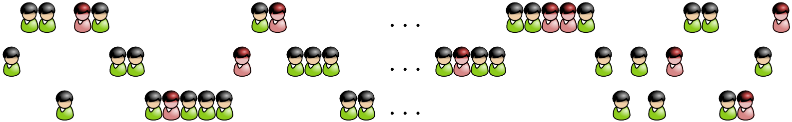
Algorithm

1. Let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ be the principal minors.
2. In each $\mathcal{M}_i$ use the $O(1)$-competitive algorithm for uniformly dense matroids to obtain an independent set I_i.
3. Return $\mathrm{ALG} = I_1 \cup I_2 \cup \cdots \cup I_k$.

Algorithm for a General Matroid $\mathcal{M}$

Algorithm

1. Let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ be the principal minors.
2. In each $\mathcal{M}_i$ use the $O(1)$-competitive algorithm for uniformly dense matroids to obtain an independent set I_i.
3. Return $\mathrm{ALG} = I_1 \cup I_2 \cup \cdots \cup I_k$.



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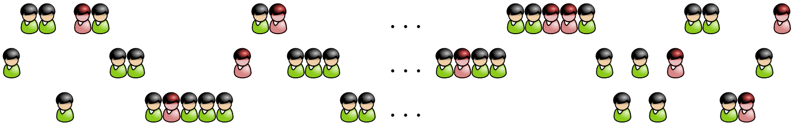


We have:

$$\mathbb{E}_{\pi,\sigma}[w(\mathrm{ALG})] \geq \Omega(1)\mathbb{E}_{\sigma}[w(\mathrm{OPT}_{\bigoplus \mathcal{M}_i})].$$

Also show $\mathbb{E}_{\pi,\sigma}[w(\mathrm{ALG})] \geq \Omega(1)/(1-1/e)\mathbb{E}_{\sigma}[w(\mathrm{OPT}_{\mathcal{M}})]$.

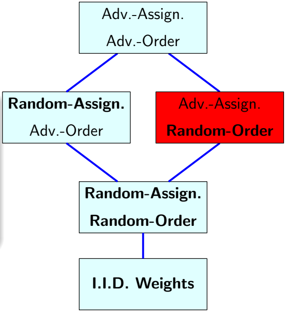
Conclusions and Open Problems.

Summary

- First constant competitive algorithm for Matroid Secretary Problem in Random-Assign. Random-Order Model.
- [OG-V] Can use same ideas for Random-Assign. Adv.-Order Model.
- Algorithm only makes comparisons.

Open

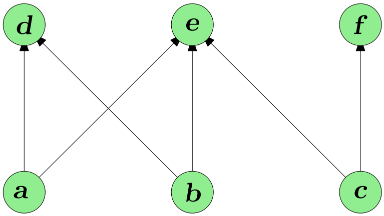
- Adv.-Assign. Random-Order Model
- Extend to independent systems beyond matroids.



Outline

1. Matroid Secretary Problem
2. Jump Number Problem and Independent Sets of Rectangles. (joint work with C. Telha)
3. Symmetric Submodular Function Minimization under Hereditary Constraints.

Jump Number Problem

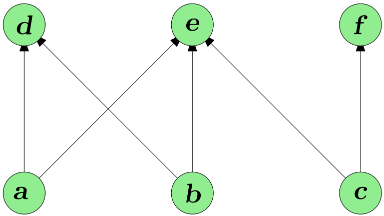


Jumps

$a \wr b \wr cf \wr d \wr e$ 4 jumps

$a \wr bd \wr cf \wr e$ 3 jumps

Jump Number Problem



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Jump number problem for a poset P

Find a linear extension (schedule) with minimum number of jumps $j(P)$.

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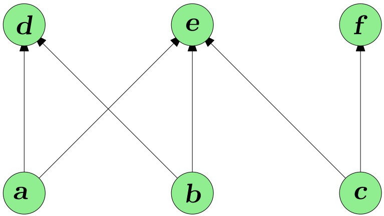

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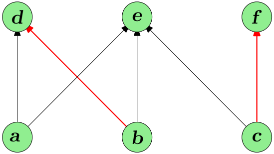
Properties

- Comparability invariant.
- NP-hard even for chordal bipartite graphs.
  (Every cycle of length ≥ 6 has a chord.)

Cross-free Matchings and Biclique Covers.

Cross-free matchings in a bipartite graph $G = (A \cup B, E)$

- Two edges ab and $a'b'$ cross if ab' and $a'b$ are also edges.
- $\alpha^*(G)$ = maximum size of a cross-free matching.



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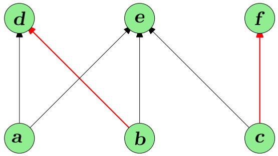

Fact: For G chordal bipartite.

$$\alpha^*(G) + j(P) = n - 1$$

Example: $a \wr bd \wr cf \wr e$

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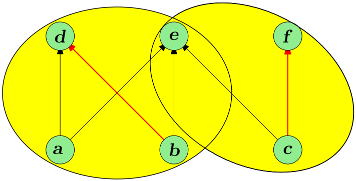

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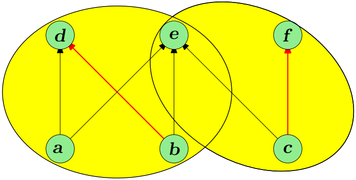

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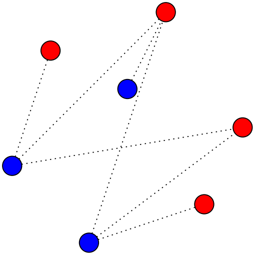
- $\kappa^*(G) =$ minimum size of a collection of complete bipartite graphs (bicliques) that covers E.

$\alpha^*(G) \leq \kappa^*(G)$.

Definition (Bicolored 2D-graphs or 2 d.o.r.g.)

Given two sets A and B of points in the plane.
$G(A, B)$ is the bipartite graph on $A \cup B$ where

$$ab \text{ is an edge if } a \in A,\ b \in B,\ a_x \le b_x \text{ and } a_y \le b_y$$



Special Chordal Bipartite Graphs .

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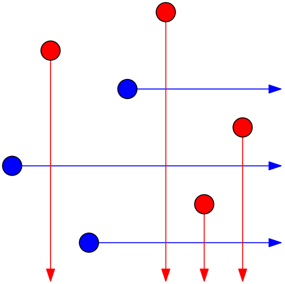

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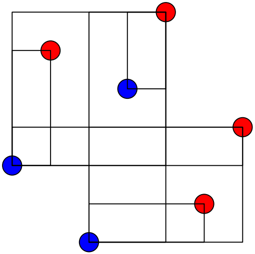
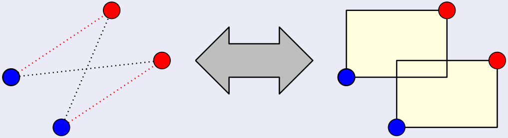
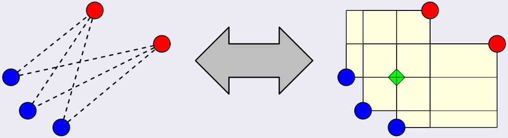


$\alpha^*(G(A, B))$ and $\kappa^*(G(A, B))$

Crossing edges = Overlapping Rectangles



Maximal Bicliques = Rectangle Hitting Sets



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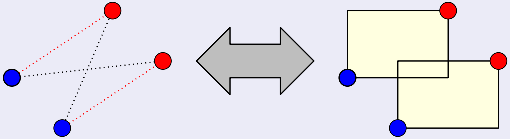

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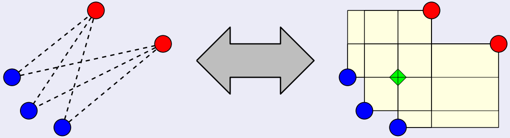

Theorem 1 [ST11] : In a 2 d.o.r.g. with rectangles $\mathcal{R}$

- $\alpha^* =$ max. cross-free matching = max. indep. set of $\mathcal{R}$ [MIS($\mathcal{R}$)].
- $\kappa^* =$ min. biclique cover = min. hitting set of $\mathcal{R}$ [MHS($\mathcal{R}$)].

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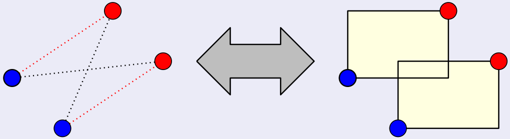

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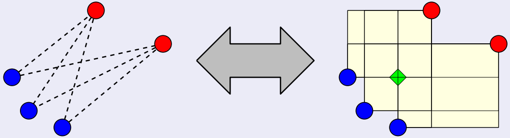

Can replace $\mathcal{R}$ by the inclusionwise minimal rectangles $\mathcal{R}_\downarrow$.

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Main results

Theorem 2 [ST11]: In a 2 d.o.r.g. with minimal rectangles $\mathcal{R}_\downarrow$

The fractional solution for the natural LP relaxation of MIS($\mathcal{R}_\downarrow$) having minimum **weighted area** is an integral solution: If

$$P = \left\{x \in (\mathbb{R}^+)^{\mathcal{R}_\downarrow}, \sum_{R \ni q} x_R \leq 1, q \in \text{Grid}\right\}, z^* = \max\left\{\mathbb{1}^T x,\ x \in P\right\}.$$

Then $\alpha^* = z^*$ and

$$\arg\min\left\{\sum_{\mathbf{R} \in \mathcal{R}_\downarrow} \text{area}(\mathbf{R})\mathbf{x_R} : \mathbb{1}^T x = z^*, x \in P\right\} \text{ is integral .}$$

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Theorem 3 [ST11]: For every 2 d.o.r.g.

$$\alpha^*(G(A, B)) = \kappa^*(G(A, B)).$$

(sketch) Theorem 3: $\alpha^*(G(A, B)) = \kappa^*(G(A, B))$.

H: Intersection graph of $\mathcal{R}_\downarrow$.

- $\alpha^*(G(A, B)) = \text{MIS}(\mathcal{R}_\downarrow) =$ **stability number** of H.
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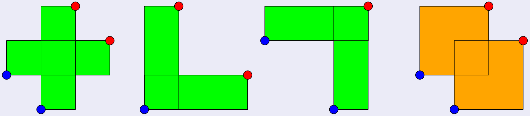
Intersections

The only possible intersections in H can be corner-free intersections or corner intersections.



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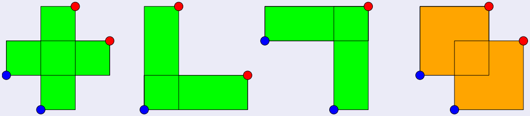

Perfect Case:

If $\mathcal{R}_\downarrow$ is such that the only intersections are corner-free-intersection, then its intersection graph H is a comparability graph (perfect).

Therefore $\alpha^*(G(A, B)) = \kappa^*(G(A, B))$.

(cont.) Theorem 3: MIS($\mathcal{R}_{\downarrow}$)=MHS($\mathcal{R}_{\downarrow}$)

General Case:

1. Construct a family $\mathcal{K} \subseteq \mathcal{R}_{\downarrow}$ by greedily including (in a certain order) rectangles in $\mathcal{K}$ if they do not form corner-intersection.

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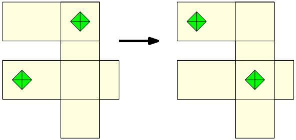
Swapping procedure.

If p, q in **P**, with $p_x < q_x$ and $p_y < q_y$ s.t.

$$\mathbf{P}' = \mathbf{P} \setminus \{p, q\} \cup \{(p_x, q_y), (p_y, q_x)\}$$

is a hitting set for $\mathcal{K}$ then set $\mathbf{P} \leftarrow \mathbf{P}'$.



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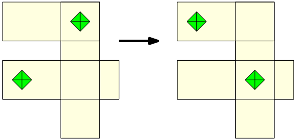
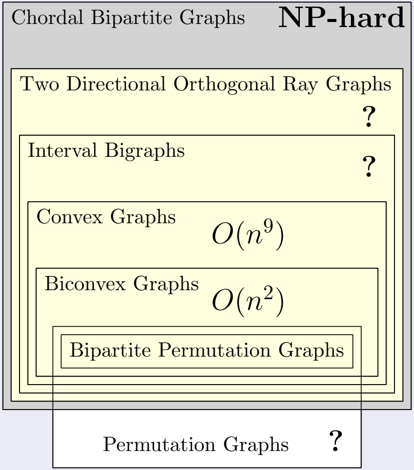


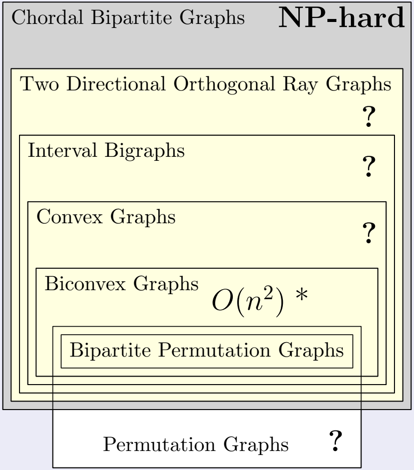
We can show that final **P** is also a hitting set for $\mathcal{R}_\downarrow$.

Results in Context

Jump Number
Max Cross-Free Matching



Min Biclique Cover



Results in Context (new results in red)

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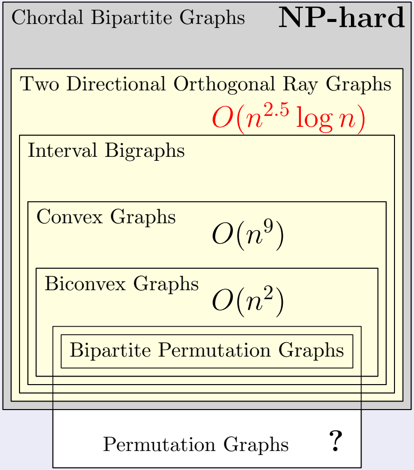

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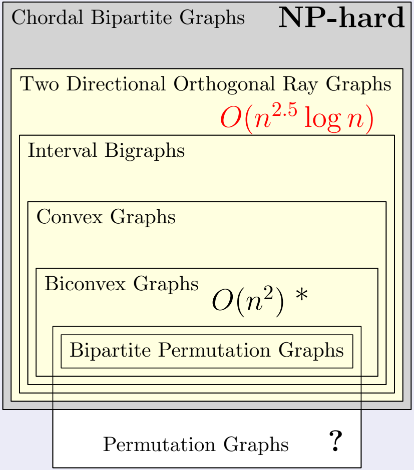

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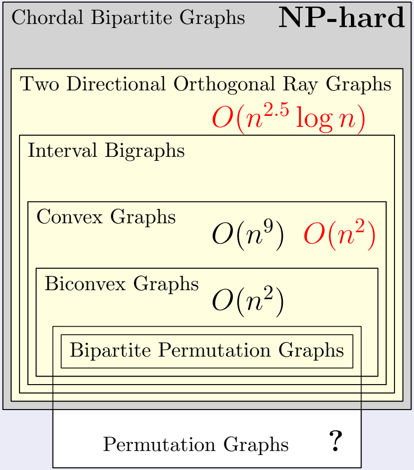

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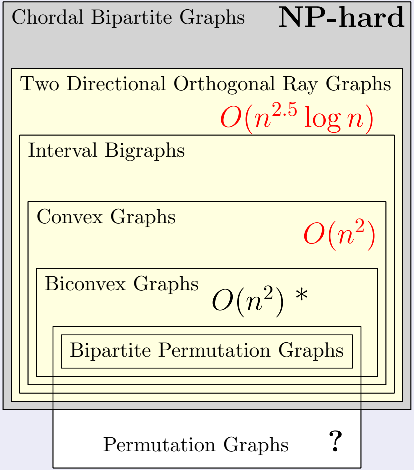
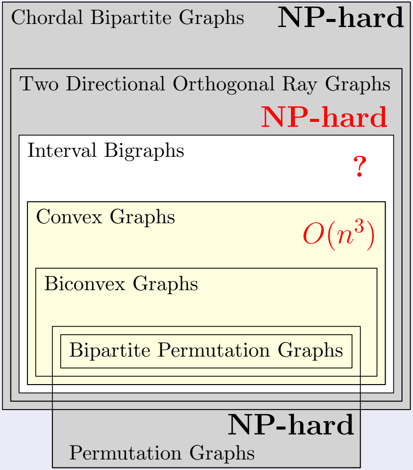


Additional Results

Weighted Jump Number
Max Weight Cross-Free Matching



- Show that maximum weight cross-free matching is NP-hard for 2 d.o.r.g.
- Give $O(n^3)$ algorithm for weighted problem in biconvex and convex graphs.

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SSF Minimization: Introduction

Definitions

$f : 2^V \to \mathbb{R}$ is submodular if

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B), \text{ for all } A, B \subseteq V$$

f is symmetric if

$$f(A) = f(V \setminus A), \text{ for all } A \subseteq V$$

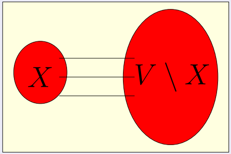
A family $\mathcal{I}$ of sets is an independent system if it is closed for inclusion.

Problem

Find $\emptyset \neq X^* \in \mathcal{I}$ that minimizes $f(X)$ over all $X \in \mathcal{I}$.

Examples

- Find a minimum unbalanced cut in a (weighted) graph.



$$\min\{|E(X;\overline{X})| : 0 \neq |X| \leq k\}.$$

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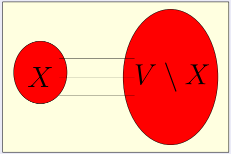

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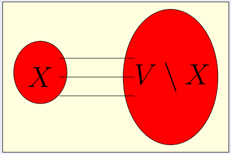

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- Find a nonempty subgraph satisfying an hereditary graph property (e.g. triangle-free, clique, stable-set, planar) minimizing the weights of the edges in its coboundary.
- Minimizing a SSF under any combination of upper cardinality / knapsack / matroid constraints.

Results (Old+New)

[Svitkina-Fleischer 08]

Minimizing a general submodular function under cardinality constraints is NP-hard to approximate within $o(\sqrt{|V|/\log |V|})$.

[GS10]

$O(n^3)$-algorithm for minimizing SSF on independent systems.

Let f be a SSF on V with $f(\emptyset) = 0$.
Define the function $d(\cdot, :)$ on pairs of disjoint subsets of V as

$$d(A, B) = \frac{1}{2}\left(f(A) + f(B) - f(A \cup B)\right).$$

Rizzi

A Rizzi bi-set function $d(\cdot, :)$ is any function satisfying

1. Symmetric: $d(A, B) = d(B, A)$.
2. Monotone: $d(A, B) \leq d(A, B \cup C)$.
3. Consistent: $d(A, C) \leq d(B, C) \Rightarrow d(A, B \cup C) \leq d(B, A \cup C)$.

E.g., $d(A, B) = |E(A : B)|$ is a Rizzi bi-set function associated to $|\delta(\cdot)|$.

Pendant Pairs and M.A. order

(s,t) is a pendant pair of d if

$d(\{t\}, V \setminus \{t\}) \leq d(S, V \setminus S)$, for all S separating s and t.

$v_1, \ldots, v_n$ is a M.A. order if

$$d(v_i, \{v_1, \ldots, v_{i-1}\}) \geq d(v_j, \{v_1, \ldots, v_{i-1}\}).$$

We get M.A. order by setting v_1 arbitrarily and selecting the next vertex as the one with MAX. ADJACENCY to the already selected.

Pendant Pairs and M.A. order

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Lemma [Queyranne, Rizzi]

The last two elements (v_{n-1}, v_n) of a M.A. order are a pendant pair.

Queyranne's algorithm

Algorithm to minimize SSF in $2^V \setminus \{V, \emptyset\}$

- While $|V| \geq 2$,
  1. Find (s, t) pendant pair.
  2. Add $\{t\}$ as a candidate for minimum.
  3. Fuse s and t as one vertex.
- Return the best of the $n - 1$ candidates.

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Remark:

If $|V| \geq 3$, we can always find a pendant pair **avoiding** one vertex.

Algorithm for constrained version

A loop of $\mathcal{I}$ is a singleton not in $\mathcal{I}$. (Assume $\mathcal{I}$ has exactly one loop ℓ).

Algorithm

- While $|V| \geq 3$,
  1. Find (s, t) pendant pair avoiding ℓ.
  2. Add $\{t\}$ as a candidate for minimum.
  3. If $\{s, t\} \in \mathcal{I}$, Fuse s and t as one vertex.
     Else, Fuse s, t and ℓ as one vertex (call it ℓ).
- If $|V| = 2$, add the only non-loop as a candidate.
- Return the best candidate.

Conclusions.

Results

- $O(n^3)$-algorithm for finding **all** inclusionwise minimal minimizers of a SSF of an independent system $\mathcal{I}$.
- An algorithm by Nagamochi also solves this problem (and more) in the same time.
  But our algorithm works for a wider class than Nagamochi's.

Open

Characterize functions admitting pendant pairs for all their fusions.

Thank you.