MULTIPARAMETER HOMOGENIZATION BY LOCALIZATION AND BLOW-UP

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Abstract. It is given an alternative self-contained proof of the homogenization theorem for periodic multiparameter integrals that was established by the authors in [4]. The proof in that paper relies on the so-called compactness method for Γ-convergence while the one presented here is by direct verification: the candidate to be the limit homogenized functional is first exhibited, the definition of Γ-convergence is then verified. This is done by extension of bounded gradient sequences using the Acerbi et al. extension theorem from connected sets [2], and by adaptation of some localization and blow-up techniques developed by Fonseca and Müller [14] together with De Giorgi's slicing method [11].

1. Introduction

In a recent paper we developped a framework to deal with some *multiparameter* homogenization problems by establishing a general Γ-convergence result for sequences of periodic integral functionals [4, Theorem 2.2]. We also gave applications to different "degenerate" homogenization processes (soft inclusions, iterated homogenization, thin inclusions), showing the versatility of this unified approach. The proof of the abstract result that we gave there is based on the so-called *compactness method* of the general theory for variational functionals due to Dal Maso and Modica [10]. Generally speaking, this method relies both on a compactness theorem in De Giorgi's Γ-convergence sense and on an integral representation theorem for variational functionals. In order to apply it to the multiparameter case it is necessary to adapt certain techniques from [6]. Therefore, this proof uses various particular results that are not easily accessible for a non-specialist reader.

In this article we give a different proof of [4, Theorem 2.2] (cf. Theorem 2.1) by *direct verification* of Γ-convergence. More precisely, we first exhibit the candidate to be the limit homogenized functional, we then verify the definition of Γ-convergence. The sketch of this alternative proof is the following. We first prove that the effective domain of the Γ-liminf of the sequence is equal to the effective domain of the candidate functional (cf. Proposition 3.1): to accomplish this we assume a connectness condition that permits us to extend bounded energy sequences thanks to the Acerbi et al. extension theorem [2]. The second step consists in showing that the candidate functional is a lower bound of the Γ-liminf on this domain (cf. Proposition 3.2): we adapt to this situation the *localization and blow-up* method developed by Fonseca and Müller [14, 15] to deal with similar problems, which has been already applied to nonlinear homogenization problems by Michaille et al. [1, 17] and uses

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the well-known De Giorgi *cut-off and slicing* method [11]. The proof is then completed by a density argument (cf. Proposition 3.4): following [18], we first prove that the candidate functional is the upper bound of the Γ-limsup on a subspace of piecewise affine continuous functions and we then extend this property to the whole Sobolev space by approximation. In contrast with the original proof, the new one is self-contained and no abstract result from Γ-convergence theory is required.

2. Multiparameter homogenization theorem

We begin this section by recalling the definition of Γ-convergence. Let $\{F_n\}$ be a sequence of functionals defined on $L^p(\Omega;\mathbb{R}^m)$ where $\Omega\subset\mathbb{R}^N$ is a bounded domain, and for each $u\in L^p(\Omega;\mathbb{R}^m)$ define

$$(\Gamma\text{-}\liminf_{n\to\infty} F_n)(u) := \inf\{\liminf_{n\to\infty} F_n(u_n) : u_n\to u \text{ in } L^p(\Omega;\mathbb{R}^m)\},$$
$$(\Gamma\text{-}\limsup_{n\to\infty} F_n)(u) := \inf\{\limsup_{n\to\infty} F_n(u_n) : u_n\to u \text{ in } L^p(\Omega;\mathbb{R}^m)\}.$$

Clearly, $\Gamma\text{-}\liminf_{n\to\infty} F_n \le \Gamma\text{-}\limsup_{n\to\infty} F_n$. We say that $\{F_n\}$, Γ-converges to $\overline{F}$ as $n\to\infty$ with respect to the strong topology of $L^p(\Omega;\mathbb{R}^m)$ and we write $\overline{F} = \Gamma\text{-}\lim_{n\to\infty} F_n$ whenever for every $u\in L^p(\Omega;\mathbb{R}^m)$, $\overline{F}(u) = (\Gamma\text{-}\liminf_{n\to\infty} F_n)(u) = (\Gamma\text{-}\limsup_{n\to\infty} F_n)(u)$. The following well-known result makes precise the variational nature of this notion of convergence; for deeper discussions of this theory we refer the reader to [5, 9, 7].

Theorem (De Giorgi-Franzoni [12]) *Let $G : L^p(\Omega;\mathbb{R}^m)\to\mathbb{R}$ be continuous and assume that $\overline{F} = \Gamma\text{-}\lim_{n\to\infty} F_n$. For each $n\in\mathbb{N}$, let $\widehat{u}_n\in L^p(\Omega;\mathbb{R}^m)$ be such that $F_n(\widehat{u}_n)+G(\widehat{u}_n)\le\inf\{F_n+G\}+\varepsilon_n$, with $\varepsilon_n\to 0$ as $n\to\infty$. Then $\limsup_{n\to\infty}(\inf\{F_n+G\})\le\inf\{\overline{F}+G\}$. If moreover $\{\widehat{u}_n\}$ is relatively compact in $L^p(\Omega;\mathbb{R}^m)$, then $\lim_{n\to\infty}(\inf\{F_n+G\}) = \inf\{\overline{F}+G\}$ and every cluster point $\widehat{u}$ of $\{\widehat{u}_n\}$ satisfies $\overline{F}(\widehat{u})+G(\widehat{u}) = \inf\{\overline{F}+G\}$.*

Let m, N and k be positive integers and write Y for the unit cell $[0,1[^N$. Let Λ be a nonempty subset of $\mathbb{R}^k$ such that $0\in\mathrm{cl}(\Lambda)$. Suppose that to every $\lambda\in\Lambda$, there corresponds a Carathéodory function $W_\lambda:\mathbb{R}^N\times\mathbb{R}^{mN}\to[0,+\infty[$ satisfying for each $\xi\in\mathbb{R}^{mN}$

(C_1) $W_\lambda(\cdot,\xi)$ is Y-periodic: $\forall (x,z)\in\mathbb{R}^N\times\mathbb{Z}^N$, $W_\lambda(x+z,\xi) = W_\lambda(x,\xi)$.

Consider a family of closed subsets $\{T_\lambda\}_{\lambda\in\Lambda}\subset Y$ and a function $r:\Lambda\to[0,\bar{r}]$ with $\bar{r}>0$. Define $E_\lambda := Y\setminus T_\lambda+\mathbb{Z}^N$, and $r_\lambda(x) := \bar{r}$ if $x\in E_\lambda$ and $r_\lambda(x) := r(\lambda)$ if $x\in\mathbb{R}^N\setminus E_\lambda = T_\lambda+\mathbb{Z}^N$. Assume that there exist $p\in]1,+\infty[$ and $c_0>0$ such that[1]

(C_2) $\forall\lambda\in\Lambda$, $\forall x\in\mathbb{R}^N$, $\forall\xi',\xi\in\mathbb{R}^{mN}$, $r_\lambda(x)|\xi|^p\le W_\lambda(x,\xi)\le c_0 r_\lambda(x)(1+|\xi|^p)$.

We also require the following "localization" condition: $\exists T\subset Y$ such that

(C_3) $\forall\lambda\in\Lambda$, $T_\lambda\subset T$ and $E := Y\setminus T+\mathbb{Z}^N$ is connected, open and ∂E is Lipschitz.

Let $\{\lambda_n\}\subset\Lambda$ be such that $\lambda_n\to 0$ as $n\to\infty$. For every $n\in\mathbb{N}$ and $\xi\in\mathbb{R}^{mN}$, we define

$$G_n^\xi(w;A) := \begin{cases} \displaystyle\int_A W_{\lambda_n}(x,\xi+\nabla w)dx & \text{if } w\lfloor_A\in W_0^{1,p}(A;\mathbb{R}^m) \\ +\infty & \text{otherwise,}\end{cases}$$

where $w\in L^p_{loc}(\mathbb{R}^N;\mathbb{R}^m)$ and A belongs to $\mathcal{U}_b(\mathbb{R}^N)$, the class of all bounded open subsets of $\mathbb{R}^N$. For every $\xi\in\mathbb{R}^{mN}$ and $n\in\mathbb{N}$, define $\overline{\mathcal{S}}^\xi$, $S_n^\xi:\mathcal{U}_b(\mathbb{R}^N)\to[0,\infty[$ by $\overline{\mathcal{S}}^\xi(A) := \inf\{\overline{G}^\xi(w;A) : w\in L^p(A;\mathbb{R}^m)\}$ and $\mathcal{S}_n^\xi(A) := \inf\{G_n^\xi(w;A) : w\in L^p(A;\mathbb{R}^m)\}$ respectively.

[1]This permits different types of singular behaviours: $r(\lambda)\to 0$ or $\mathrm{dist}(T_\lambda,\Sigma)\to 0$ as $\lambda\to 0$, where Σ is a submanifold of $\mathbb{R}^N$.

Theorem 2.1. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain and assume* (C_1), (C_2) *and* (C_3). *If*

(H_1) $\forall \xi \in \mathbb{R}^{mN}$, $\exists \overline{G}^{\xi} : L^p_{loc}(\mathbb{R}^N;\mathbb{R}^m) \times \mathcal{U}_b(\mathbb{R}^N) \to [0,+\infty]$ *such that* $\forall k \in \mathbb{N}^*$, $\forall v \in L^p_{loc}(\mathbb{R}^N;\mathbb{R}^m)$, $\overline{G}^{\xi}(v;]0,k[^N) = \Gamma\text{-}\lim_{n\to\infty} G^{\xi}_n(v;]0,k[^N)$

(H_2) $\forall k_n \to \infty$, $\forall \xi \in \mathbb{R}^{mN}$, $\lim_{n\to\infty} \frac{1}{k_n^N}\mathcal{S}^{\xi}_n(]0,k_n[^N) = \inf_{k\in\mathbb{N}^*}\{\frac{1}{k^N}\overline{\mathcal{S}}^{\xi}(]0,k[^N)\}$[2]

then, for every $\varepsilon_n \to 0$, *the functionals* $F_n : L^p(\Omega;\mathbb{R}^m) \to [0,+\infty]$ *defined by*

$$F_n(u) := \begin{cases} \displaystyle\int_\Omega W_{\lambda_n}\big(\frac{x}{\varepsilon_n},\nabla u\big)dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

satisfy $\Gamma\text{-}\lim_{n\to\infty} F_n = F^{hom}$, *where* $F^{hom} : L^p(\Omega;\mathbb{R}^m) \to [0,+\infty]$ *is given by*

$$F^{hom}(u) := \begin{cases} \displaystyle\int_\Omega W^{hom}(\nabla u)dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

with

$$W^{hom}(\xi) := \inf_{k\in\mathbb{N}^*}\inf_{v}\{\frac{1}{k^N}\overline{G}^{\xi}(v;]0,k[^N) : v \in L^p(]0,k[^N;\mathbb{R}^m)\}.$$

3. Proof of the theorem

3.1. Effective domain of Γ-liminf F_n.

The first step is to identify the effective domain of $\Gamma\text{-}\liminf_{n\to\infty} F_n$, which is defined by

$$\mathrm{dom}(\Gamma\text{-}\liminf_{n\to\infty} F_n) := \{u \in L^p(\Omega;\mathbb{R}^m) : (\Gamma\text{-}\liminf_{n\to\infty} F_n)(u) < \infty\}.$$

The arguments used in the proof of the following proposition are standard. For more details we refer the reader to [7].

Proposition 3.1. *Under* (C_1), (C_2) *and* (C_3), $\mathrm{dom}(\Gamma\text{-}\liminf_{n\to\infty} F_n) = W^{1,p}(\Omega;\mathbb{R}^m)$.

Proof. Let $u \in \mathrm{dom}\,(\Gamma\text{-}\liminf_{n\to\infty} F_n)$. By definition, there exists a sequence $u_n \to u$ in $L^p(\Omega;\mathbb{R}^m)$ such that, up to a subsequence, $\{F_n(u_n)\}$ is bounded. From the first inequality in (C_2), it follows that $\sup_{n\in\mathbb{N}}\int_{\Omega\cap\varepsilon_n E}|\nabla u_n|^p dx < \infty$. By (C_3), $E = Y \setminus T + \mathbb{Z}^N$ is a periodic, connected, open set with Lipschitz boundary. If $E = \emptyset$, then, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,p}$, hence $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. When $E \neq \emptyset$, we extend u_n from $\Omega \cap \varepsilon_n E$ to the whole of Ω, keeping the above uniform boundedness property. This extension is not difficult to construct when the complement of E is disconnected (see [16]), and it is no longer possible in the general case, where $\Omega\cap\varepsilon E$ may be disconnected so that we cannot expect to control the $W^{1,p}$ norm of the extended function. This extension problem is considered in [2].

Theorem (Acerbi et al. [2]) *Let* E *be a periodic, connected, open subset of* $\mathbb{R}^N$ *with Lipschitz boundary. There exist constants* $k_0, k_1, k_2 > 0$ *such that for every bounded open set* $\Omega \subset \mathbb{R}^N$ *and* $\varepsilon > 0$, *there exists a linear and continuous extension operator* $P_\varepsilon : W^{1,p}(\Omega\cap\varepsilon E;\mathbb{R}^m) \to W^{1,p}_{loc}(\Omega;\mathbb{R}^m)$ *with:*

(a) $P_\varepsilon u = u$ *a.e. in* $\Omega \cap \varepsilon E$,
(b) $\int_{\Omega(\varepsilon k_0)} |P_\varepsilon u|^p dx \le k_1 \int_{\Omega\cap\varepsilon E}|u|^p dx$,
(c) $\int_{\Omega(\varepsilon k_0)} |\nabla(P_\varepsilon u)|^p dx \le k_2 \int_{\Omega\cap\varepsilon E}|\nabla u|^p dx$,

[2]This hypothesis is the most difficult to verify in practice; see [4] for some examples.

for every $u \in W^{1,p}(\Omega \cap \varepsilon E; \mathbb{R}^m)$*, where* $\Omega(\alpha) := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \alpha\}$.

For each $n \in \mathbb{N}$, we define $v_n := P_{\varepsilon_n}(u_n|_{\Omega\cap\varepsilon_n E})$. We deduce that for every n', $\{v_n : n \geq n'\}$ is bounded in $W^{1,p}(\Omega'; \mathbb{R}^m)$ for every open set $\Omega' \subset \Omega$ with $\mathrm{dist}(\Omega', \partial\Omega) > \varepsilon_{n'} k_0$. Let us consider an increasing sequence $\{\Omega_i\}$ of open subsets of Ω with Lipschitz boundaries and such that in the limit we obtain Ω. Let Ω_i belong to this sequence. We assume moreover that $\mathrm{dist}(\Omega_i, \partial\Omega) > 0$. Thus, by the reflexivity of $W^{1,p}$ and the Rellich theorem, there exist $v \in W^{1,p}(\Omega_i; \mathbb{R}^m)$ and a subsequence of $\{v_n\}$ which converges to v strongly in $L^p(\Omega_i; \mathbb{R}^m)$ and weakly in $W^{1,p}(\Omega_i; \mathbb{R}^m)$. We can extract a diagonal subsequence, still denoted by $\{v_n\}$, which converges to a function $v \in W^{1,p}_{loc}(\Omega; \mathbb{R}^m)$ strongly in $L^p_{loc}(\Omega; \mathbb{R}^m)$ and weakly in $W^{1,p}_{loc}(\Omega; \mathbb{R}^m)$. Let now $\Omega' \subset\subset \Omega$ be arbitrary. For every n we have in particular $u_n = v_n$ a.e. in $\Omega' \cap \varepsilon_n E$. Since $1_{\Omega'\cap\varepsilon_n E} \rightharpoonup \mathcal{L}_N(Y \setminus T)$ weakly in $L^p(\Omega')$, we deduce that $\mathcal{L}_N(Y \setminus T)u = \mathcal{L}_N(Y \setminus T)v$. As $\mathcal{L}_N(Y \setminus T) > 0$, we have that $u = v$ a.e. in Ω', for every $\Omega' \subset\subset \Omega$. Hence, $u = v$ a.e. in Ω. Thus, $\|\nabla u\|_{p,\Omega'} \leq \liminf_{n\to\infty} \|\nabla v_n\|_{p,\Omega'} \leq c$, for every $\Omega' \subset\subset \Omega$, with the constant c being independent of Ω'. Consequently, $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. Hence $\mathrm{dom}(\Gamma\text{-}\liminf_{n\to\infty} F_n) \subset W^{1,p}(\Omega; \mathbb{R}^m)$. Finally, by the second inequality in (C_2) it follows easily that equality holds in the previous inclusion. □

3.2. **Lower bound on the Γ-liminf F_n.** We have to prove that

$$(\Gamma\text{-}\liminf_{n\to\infty} F_n)(u) \geq F^{hom}(u),$$

for every $u \in L^p(\Omega; \mathbb{R}^m)$. By Proposition 3.1, this is trivially satisfied when $u \notin W^{1,p}(\Omega; \mathbb{R}^m)$. Let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and consider a sequence $u_n \to u$ in $L^p(\Omega; \mathbb{R}^m)$. Without loss of generality, we can suppose that $\{F_n(u_n)\}$ is bounded. We are thus reduced to proving

$$\int_\Omega W^{hom}(\nabla u(x))dx \leq \liminf_{n\to\infty} \int_\Omega W_{\lambda_n}(\frac{x}{\varepsilon_n}, \nabla u_n(x))dx. \tag{3.1}$$

Proposition 3.2. *If* (C_1)*,* (C_2)*,* (H_1) *and* (H_2) *are satisfied, then* (3.1) *holds.*

Proof. Denote by $M(\Omega)$ the set of all Radon measures in Ω and define $M^+(\Omega) := \{\nu \in M(\Omega) : \nu \geq 0\}$. Consider the sequence $\{\mu_n\} \subset M^+(\Omega)$ defined by $\mu_n := W_{\lambda_n}(\frac{\cdot}{\varepsilon_n}, \nabla u_n)dx$. By assumption, $\{\mu_n\}$ is uniformly bounded in $M^+(\Omega)$, hence there exists $\mu \in M^+(\Omega)$ such that, up to a subsequence, $\mu_n \rightharpoonup \mu$ weakly in $M(\Omega)$. Let $\mu^{hom} \in M^+(\Omega)$ be defined by $\mu^{hom} := W^{hom}(\nabla u)dx$. The idea is to compare the limit measure μ with μ^{hom}. Since $\mu(\Omega) \leq \liminf_{n\to\infty} \mu_n(\Omega)$, it suffices to prove that $\mu^{hom}(\Omega) \leq \mu(\Omega)$.

a) *Localization.* We write $\mathcal{L}_N$ for the Lebesgue measure in $\mathbb{R}^N$ as well as for its restriction to Ω. Consider the Lebesgue decomposition of the limit measure $\mu = \mu^a + \mu^s$, where μ^a and μ^s are respectively the absolutely continuous and the singular part of μ with respect to $\mathcal{L}_N$. Thus there exists $f \in L^1(\Omega; \mathbb{R}_+)$ such that $\mu^a = f dx$ and the Besicovitch differentiation theorem ensures

$$f(x_0) = \lim_{\rho\to 0^+} \frac{\mu^a(\mathcal{Q}_\rho(x_0))}{\mathcal{L}_N(\mathcal{Q}_\rho(x_0))} = \lim_{\rho\to 0^+} \frac{\mu(\mathcal{Q}_\rho(x_0))}{\mathcal{L}_N(\mathcal{Q}_\rho(x_0))}$$

for $\mathcal{L}_N$-almost every $x_0 \in \Omega$. Here, $\mathcal{Q}_\rho(x_0)$ is the open cube centered at x_0 and of side ρ in all directions. Fix x_0 such that the previous equality holds. Since $\mu_n \rightharpoonup \mu$ in $M(\Omega)$, the Alexandroff theorem yields in particular that $\mu(\mathcal{Q}_\rho(x_0)) = \lim_{n\to\infty} \mu_n(\mathcal{Q}_\rho(x_0))$ whenever $\mu(\partial\mathcal{Q}_\rho(x_0)) = 0$. As $\mu(\Omega) < \infty$, the latter holds for every $\rho \in]0, \rho_0] \setminus D$, where D is a countable set. In the sequel, we will take ρ such that $\mu(\partial\mathcal{Q}_\rho(x_0)) = 0$. Consequently, it suffices to prove that

$$W^{hom}(\nabla u(x_0)) \leq \lim_{\rho\to 0^+} \lim_{n\to\infty} \frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(\frac{x}{\varepsilon_n}, \nabla u_n(x))dx. \tag{3.2}$$

Assume first that $u_n \in \bar{u} + W_0^{1,p}(\mathcal{Q}_\rho(x_0);\mathbb{R}^m)$, where $\bar{u} : \mathbb{R}^N \to \mathbb{R}^m$ is the affine function defined by $\bar{u}(x) := u(x_0) + \nabla u(x_0)\cdot(x - x_0)$. Then

$$\int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(\frac{x}{\varepsilon_n}, \nabla u_n)dx \quad \geq \quad \varepsilon_n^N \mathcal{S}_n^{\nabla u(x_0)}(\frac{1}{\varepsilon_n}\mathcal{Q}_\rho(x_0)).$$

Lemma 3.3. *Let $Cub(\mathbb{R}^N)$ be the class of all open cubes in $\mathbb{R}^N$. If* (C_1), (C_2), (H_1) *and* (H_2) *hold, then* $\forall \xi \in \mathbb{R}^{mN}$, $\forall \mathcal{Q} \in Cub(\mathbb{R}^N)$, $\lim_{n\to\infty} \frac{\mathcal{S}_n^\xi(\frac{1}{\varepsilon_n}\mathcal{Q})}{\mathcal{L}_N(\frac{1}{\varepsilon_n}\mathcal{Q})} = W^{hom}(\xi)$.

Proof. Fix $\xi \in \mathbb{R}^{mN}$ and $Q \in Cub(\mathbb{R}^N)$. Given $k \in \mathbb{N}^*$ and $n \in \mathbb{N}$ large enough, let $k_n \in \mathbb{N}^*$ be the largest integer such that $(k_n - 2)]0, k[^N + k(z_n + \hat{e}) \subset \frac{1}{\varepsilon_n}Q$ for an appropriate $z_n \in \mathbb{Z}^N$, where $\hat{e} := (1, 1, ..., 1)$. From ($\mathrm{C}_1$) and ($\mathrm{C}_2$), it follows that $\mathcal{S}_n^\xi$ is a subadditive and $\mathbb{Z}^N$-invariant set function satisfying $0 \leq \mathcal{S}_n^\xi(A) \leq c_0\bar{r}(1 + |\xi|^p)\mathcal{L}_N(A)$ for all $A \in \mathcal{U}_b(\mathbb{R}^N)$. Therefore,

$$\mathcal{S}_n^\xi(\frac{1}{\varepsilon_n}Q) \leq (k_n - 2)^N \mathcal{S}_n^\xi(]0, k[^N) + \mathcal{S}_n^\xi(\frac{1}{\varepsilon_n}Q \setminus [(k_n - 2)[0, k]^N + k(z_n + \hat{e})]).$$

Since, up to a set of zero Lebesgue measure, the set $1/\varepsilon_n Q\setminus[(k_n-2)[0, k]^N+k(z_n+\hat{e})]$ may be written as the disjoint union of $k_n^N - (k_n - 2)^N$ integer translations of open sets contained in $]0, k[^N$, we deduce that $\mathcal{S}_n^\xi(\frac{1}{\varepsilon_n}Q) \leq (k_n - 2)^N\mathcal{S}_n^\xi(]0, k[^N) + (k_n^N - (k_n - 2)^N)ck^N$, where $c = c_0\bar{r}(1 + |\xi|^p)$. We thus obtain the estimate

$$\frac{\mathcal{S}_n^\xi(\frac{1}{\varepsilon_n}Q)}{\mathcal{L}_N(\frac{1}{\varepsilon_n}Q)} \leq \frac{\mathcal{S}_n^\xi(]0, k[^N)}{k^N} + \frac{k_n^N - (k_n - 2)^N}{(k_n - 2)^N}c.$$

From (H_1) we have that $\limsup_{n\to\infty} \mathcal{S}_n^\xi(]0, k[^N) \leq \overline{\mathcal{S}}^\xi(]0, k[^N)$ for every $k \in \mathbb{N}^*$. Since $k_n \to \infty$ as $n \to \infty$, $\limsup_{n\to\infty} \frac{\mathcal{S}_n^\xi(\frac{1}{\varepsilon_n}Q)}{\mathcal{L}_N(\frac{1}{\varepsilon_n}Q)} \leq \inf_{k\in\mathbb{N}^*}\{\frac{\overline{\mathcal{S}}^\xi(]0,k[^N)}{k^N}\} = W^{hom}(\xi)$.

Similarly, for every $n \in \mathbb{N}$, let $k_n \in \mathbb{N}^*$ be such that $\frac{1}{\varepsilon_n}Q \subset]0, k_n[^N + z_n$ for a suitable $z_n \in \mathbb{Z}^N$. We then have $\mathcal{S}_n^\xi(]0, k_n[^N) \leq \mathcal{S}_n^\xi(\frac{1}{\varepsilon_n}Q) + \mathcal{S}_n^\xi((]0, k_n[^N + z_n)\setminus\frac{1}{\varepsilon_n}\overline{Q})$, and so

$$\frac{\mathcal{S}_n^\xi(]0, k_n[^N)}{k_n^N} \leq \frac{\mathcal{S}_n^\xi(\frac{1}{\varepsilon_n}Q)}{\mathcal{L}_N(\frac{1}{\varepsilon_n}Q)} + \frac{k_n^N - (k_n - 2)^N}{(k_n)^N}c.$$

¿From (H_2) we see that $W^{hom}(\xi) = \lim_{n\to\infty} \frac{1}{k_n^N}\mathcal{S}_n^\xi(]0, k_n[^N) \leq \liminf_{n\to\infty} \frac{\mathcal{S}_n^\xi(\frac{1}{\varepsilon_n}Q)}{\mathcal{L}_N(\frac{1}{\varepsilon_n}Q)}$, which completes the proof. □

By Lemma 3.3, we have $W^{hom}(\nabla u(x_0)) \leq \lim_{n\to\infty} \frac{1}{\rho^N}\int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(\frac{x}{\varepsilon_n}, \nabla u_n)dx$, and we get (3.2). We next indicate how to remove the restriction $u_n \in \bar{u} + W_0^{1,p}(\mathcal{Q}_\rho(x_0);\mathbb{R}^m)$ by the application of a well-known technique introduced by De Giorgi in [11].

b) *Cut-off and slicing method of De Giorgi.* We say that a function φ is a cut-off function between A' and A, with $A' \subset\subset A \in \mathcal{U}_b(\mathbb{R}^N)$, if $\varphi \in \mathcal{D}(A)$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on A'. Let $\alpha \in]0, 1]$ and $l \in \mathbb{N}^*$. For each $i \in \{0, ..., l\}$, define $\mathcal{Q}_i := \mathcal{Q}_{(1-\alpha+i\alpha/l)\rho}(x_0)$ and consider a cut-off function φ_i between $\mathcal{Q}_{i-1}$ and $\mathcal{Q}_i$ ($i \geq 1$) such that $\|\nabla\varphi_i\|_\infty \leq 2l/\alpha\rho$. Setting $u_n^i(x) := \bar{u}(x) + \varphi_i(x)(u_n(x) - \bar{u}(x))$, we obtain $u_n^i \in \bar{u} + W_0^{1,p}(\mathcal{Q}_\rho(x_0);\mathbb{R}^m)$ with

$$\nabla u_n^i = \begin{cases} \nabla u_n & \text{on } \mathcal{Q}_{i-1} \\ \nabla u(x_0) + (u_n - \bar{u}) \otimes \nabla\varphi_i + \varphi_i(\nabla u_n - \nabla u(x_0)) & \text{on } \mathcal{Q}_i \setminus \mathcal{Q}_{i-1} \\ \nabla u(x_0) & \text{on } Q_\rho(x_0) \setminus \mathcal{Q}_i \end{cases}$$

We have the following estimates:

$$\frac{1}{\rho^N}\int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(\frac{x}{\varepsilon_n}, \nabla u_n^i)dx \leq \frac{1}{\rho^N}\int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(\frac{x}{\varepsilon_n}, \nabla u_n)dx + E_{l,\alpha}^i(\rho, n),$$

where

$$E^i_{l,\alpha}(\rho,n) := \frac{1}{\rho^N}\int_{\mathcal{Q}_i\setminus\mathcal{Q}_{i-1}} W_{\lambda_n}(\frac{x}{\varepsilon_n},\nabla u^i_n)dx + c_0\bar{r}(1+|\nabla u(x_0)|^p)(1-(1-\alpha)^N).$$

Noticing that

$$\int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}\left(x/\varepsilon_n,\nabla u^i_n\right)dx = \varepsilon_n^N\int_{1/\varepsilon_n\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(x,\nabla u(x_0)+\nabla w^i_n)dx,$$

with $w^i_n \in W^{1,p}_0(1/\varepsilon_n\mathcal{Q}_\rho(x_0);\mathbb{R}^m)$, we conclude that for every $i\in\{1,...,l\}$

$$\frac{\mathcal{S}^{\nabla u(x_0)}_n(\frac{1}{\varepsilon_n}\mathcal{Q}_\rho(x_0))}{\mathcal{L}_N(\frac{1}{\varepsilon_n}\mathcal{Q}_\rho(x_0))} \le E^i_{l,\alpha}(\rho,n) + \frac{1}{\rho^N}\int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(\frac{x}{\varepsilon_n},\nabla u_n)dx.$$

Consequently, averaging these inequalities over the layers $\mathcal{Q}_i\setminus\mathcal{Q}_{i-1}$, we obtain

$$\frac{\mathcal{S}^{\nabla u(x_0)}_n(\frac{1}{\varepsilon_n}\mathcal{Q}_\rho(x_0))}{\mathcal{L}_N(\frac{1}{\varepsilon_n}\mathcal{Q}_\rho(x_0))} \le \overline{E}_{l,\alpha}(\rho,n) + \frac{1}{\rho^N}\int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(\frac{x}{\varepsilon_n},\nabla u_n)dx, \tag{3.3}$$

where $\overline{E}_{l,\alpha}(\rho,n) := 1/l\sum_{i=1}^l E^i_{l,\alpha}(\rho,n)$. ¿From (C_2) and the definition of u^i_n, it follows that there exists a constant $c>0$ such that $W_{\lambda_n}(\cdot,\nabla u^i_n)\le c[1+|\nabla u(x_0)|^p+(2l/\alpha\rho)^p|u_n-\bar{u}|^p+r_{\lambda_n}(\cdot)|\nabla u_n|^p]$. Then, we deduce that

$$\overline{E}_{l,\alpha}(\rho,n)\le c[R_{l,\alpha}+(\frac{2l}{\alpha\rho})^p\frac{1}{\rho^N}\int_{\mathcal{Q}_\rho(x_0)}|u_n-\bar{u}|^pdx+\frac{1}{l\rho^N}\int_{\mathcal{Q}_\rho(x_0)} r_{\lambda_n}(\frac{x}{\varepsilon_n})|\nabla u_n|^pdx],$$

where $R_{l,\alpha} := (1-(1-\alpha)^N)+1/l$. By the coercivity condition $r_{\lambda_n}(x/\varepsilon_n)|\nabla u_n(x)|^p \le W_{\lambda_n}(x/\varepsilon_n,\nabla u_n(x))$, and since $1/\rho^N\int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(x/\varepsilon_n,\nabla u_n)dx\le K$ with K being a constant independent of ρ and n, we deduce that for a suitable constant $c'>0$ we have $\overline{E}_{l,\alpha}(\rho,n)\le c'[R_{l,\alpha}+(2l/\alpha\rho)^p 1/\rho^N\int_{\mathcal{Q}_\rho(x_0)}|u_n-\bar{u}|^pdx]$. Hence,

$$\limsup_{n\to\infty}\overline{E}_{l,\alpha}(\rho,n)\le c'[R_{l,\alpha}+(\frac{2l}{\alpha\rho})^p\frac{1}{\rho^N}\int_{\mathcal{Q}_\rho(x_0)}|u-\bar{u}|^pdx].$$

Let us recall that every function $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ satisfies the following weak differentiability property: $\lim_{\rho\to 0^+}\frac{1}{\rho^N}\int_{\mathcal{Q}_\rho(x_0)}\frac{1}{\rho^p}|u(x)-u(x_0)-\nabla u(x_0)\cdot(x-x_0)|^pdx = 0$ for $\mathcal{L}_N$-almost every $x_0\in\Omega$ (see [19, theorem 3.4.2]). Thus, letting $\rho\to 0$ we have that $\limsup_{\rho\to 0}\limsup_{n\to\infty}\overline{E}_{l,\alpha}(\rho,n)\le c'R_{l,\alpha}$. We conclude from (3.3) and Lemma 3.3 that

$$W^{hom}(\nabla u(x_0))\le c'R_{l,\alpha}+\lim_{\rho\to 0^+}\lim_{n\to\infty}\frac{1}{\rho^N}\int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(\frac{x}{\varepsilon_n},\nabla u_n)dx.$$

Finally, we let $l\to\infty$ and $\alpha\to 0$ to prove our claim. □

3.3. **Upper bound on the Γ-limsup F_n.** We prove that for every $u\in L^p(\Omega;\mathbb{R}^m)$

$$F^{hom}(u)\ge(\Gamma\text{-}\limsup_{n\to\infty}F_n)(u).$$

By definition of F^{hom}, this is trivially satisfied when $u\notin W^{1,p}(\Omega;\mathbb{R}^m)$.

Proposition 3.4. *If* (C_1), (C_2) *and* (H_1) *hold, then* $\forall u\in W^{1,p}(\Omega;\mathbb{R}^m)$, $\exists u_n\to u$ *in* $L^p(\Omega;\mathbb{R}^m)$ *such that* $\lim_{n\to\infty}F_n(u_n)=F^{hom}(u)$.

Proof. We divide the proof into two parts.
a) *Piecewise affine continuous functions.* Let us denote by $\mathrm{Aff}(\Omega;\mathbb{R}^m)$ the subspace of piecewise affine continuous functions.

Lemma 3.5. *If* (C_1), (C_2) *and* (H_1) *hold, then* $\forall u\in\mathrm{Aff}(\Omega;\mathbb{R}^m)$, $\exists u_n\to u$ *in* $L^p(\Omega;\mathbb{R}^m)$ *with* $u_n\in u+W^{1,p}_0(\Omega;\mathbb{R}^m)$ *such that* $\lim_{n\to\infty}F_n(u_n)=F^{hom}(u)$.

Proof. We begin by proving the lemma for an arbitrary linear function. The proof is adapted from [18, lemma 2.1(a)]. Let $\xi \in \mathbb{R}^{mN}$. By definition of W^{hom}, for every $\delta > 0$ there exist $k \in \mathbb{N}^*$ and $\psi^\delta \in L^p(]0,k[^N;\mathbb{R}^m)$ such that

$$W^{hom}(\xi) \leq \frac{1}{k^N}\overline{G}^\xi(\psi^\delta;]0,k[^N) < W^{hom}(\xi) + \delta.$$

Fix $\delta > 0$. According to (H_1), there exists a sequence $\{\psi_n^\delta\} \subset W_0^{1,p}(]0,k[^N;\mathbb{R}^m)$ such that $\lim_{n\to\infty} \|\psi_n^\delta - \psi^\delta\|_{p,]0,k[^N} = 0$ and

$$\lim_{n\to\infty} G_n^\xi(\psi_n^\delta;]0,k[^N) = \overline{G}^\xi(\psi^\delta;]0,k[^N). \tag{3.4}$$

We extend ψ_n^δ from $]0,k[^N$ to $\mathbb{R}^N$ by kY-periodicity, and for each $n \in \mathbb{N}$, we define

$$u_n^\delta(x) := \begin{cases} \xi\cdot x + \varepsilon_n\psi_n^\delta(\frac{x}{\varepsilon_n}) & \text{if } x \in \Omega^{\varepsilon_n k}, \\ \xi\cdot x & \text{if } x \in \Omega\setminus\Omega^{\varepsilon_n k}, \end{cases}$$

where $\Omega^{\varepsilon_n k}$ is the union of all the cubes of side $\varepsilon_n k$ which are contained in Ω. Of course, $u_n^\delta - \xi\cdot x \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. Since $\|u_n^\delta - \xi\cdot x\|_{p,\Omega} \leq \varepsilon_n\mathcal{L}_N(\Omega)/k^N\|\psi_n^\delta\|_{p,]0,k[^N}$, we have that $\lim_{n\to\infty}\|u_n^\delta - \xi\cdot x\|_{p,\Omega} = 0$. By definition of F_n and u_n^δ

$$F_n(u_n^\delta) = \int_{\Omega^{\varepsilon_n k}} W_{\lambda_n}(\frac{x}{\varepsilon_n},\xi + \nabla\psi_n^\delta(\frac{x}{\varepsilon_n}))dx + \int_{\Omega\setminus\Omega^{\varepsilon_n k}} W_{\lambda_n}(\frac{x}{\varepsilon_n},\xi)dx.$$

By kY-periodicity, we obtain

$$\int_{\Omega^{\varepsilon_n k}} W_{\lambda_n}(\frac{x}{\varepsilon_n},\xi + \nabla\psi_n^\delta(\frac{x}{\varepsilon_n}))dx = \frac{\mathcal{L}_N(\Omega^{\varepsilon_n k})}{k^N}\int_{]0,k[^N} W_{\lambda_n}(y,\xi+\nabla\psi_n^\delta(y))dy.$$

By (3.4), we deduce that there exists $n_0 \in \mathbb{N}$ such that

$$W^{hom}(\xi) - \delta < 1/k^N\int_{]0,k[^N} W_{\lambda_n}(y,\xi+\nabla\psi_n^\delta)dy < W^{hom}(\xi)+\delta,$$

for every $n \geq n_0$. We thus have the following estimates:

$$\mathcal{L}_N(\Omega^{\varepsilon_n k})[W^{hom}(\xi)-\delta] \leq F_n(u_n^\delta) \leq \mathcal{L}_N(\Omega^{\varepsilon_n k})[W^{hom}(\xi)+\delta] + c'\mathcal{L}_N(\Omega\setminus\Omega^{\varepsilon_n k})$$

for every $n \geq n_0$, where $c' = c_0\bar{r}(1+|\xi|^p)$. Consequently, for every $\delta > 0$

$$F^{hom}(\xi\cdot x) - \delta\mathcal{L}_N(\Omega) \leq \liminf_{n\to\infty} F_n(u_n^\delta) \leq \limsup_{n\to\infty} F_n(u_n^\delta) \leq F^{hom}(\xi\cdot x) + \delta\mathcal{L}_N(\Omega).$$

By a standard diagonalization argument [5, corollary 1.16], we obtain a mapping $n \mapsto \delta_n$ such that $\delta_n \to 0$ as $n \to \infty$, $\lim_{n\to\infty}\|u_n^{\delta_n} - \xi\cdot x\|_{p,\Omega} = 0$ and $\lim_{n\to\infty} F_n(u_n^{\delta_n}) = F^{hom}(\xi\cdot x)$. Finally, setting $u_n := u_n^{\delta_n}$ we obtain the required sequence. The case of an arbitrary $u \in \mathrm{Aff}(\Omega;\mathbb{R}^m)$ follows by a straightforward generalization of the above construction. □

b) *Density argument.* Before dealing with a general $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, we establish the following properties of the homogenized integrand.

Lemma 3.6. *Under* (C_1), (C_2) *and* (C_3),
(i) *if* (H_1) *holds then* $\exists c_1 > 0$ *such that* $\forall\xi \in \mathbb{R}^{mN}$

$$c_1|\xi|^p \leq W^{hom}(\xi) \leq c_0\bar{r}(1+|\xi|^p),$$

(ii) *if* (H_1) *and* (H_2) *hold then* $\exists c_2 > 0$ *such that* $\forall\xi',\xi \in \mathbb{R}^{mN}$

$$|W^{hom}(\xi') - W^{hom}(\xi)| \leq c_2(1+|\xi'|^{p-1}+|\xi|^{p-1})|\xi'-\xi|.$$

Proof. (i) From (H_1) it follows easily that $\overline{G}^{\xi}(0;]0,k[^N) \leq k^N c_0 \bar{r}(1+|\xi|^p)$, for every $\xi \in \mathbb{R}^{mN}$. Hence, the upper estimate for W^{hom} follows. For the coercivity condition we may argue as in [2, proposition 3.3]. By lemma 3.5, there exists a sequence $u_n \to \xi \cdot x$ in L^p with $u_n \in \xi \cdot x + W_0^{1,p}(\Omega;\mathbb{R}^m)$ and such that $\lim_{n\to\infty} F_n(u_n) = F^{hom}(\xi \cdot x) = W^{hom}(\xi)\mathcal{L}_N(\Omega)$. Let $\Omega' \subset \mathbb{R}^N$ be an open set with $\Omega \subset\subset \Omega'$. Letting $u_n = \xi \cdot x$ outside of Ω, we extend it to Ω'. Consider the extension operator $P_{\varepsilon_n} : W^{1,p}(\Omega' \cap \varepsilon_n E;\mathbb{R}^m) \to W^{1,p}_{loc}(\Omega';\mathbb{R}^m)$ given by the Acerbi et al. theorem [2]. For every $n \in \mathbb{N}$ with ε_n small enough such that $\Omega \subset \Omega'(\varepsilon_n k_0)$ we have $\|P_{\varepsilon_n} u_n\|^p_{p,\Omega} \leq k_1 \|u_n\|^p_{p,\Omega\cap\varepsilon_n E} + k_1\|\xi \cdot x\|^p_{\Omega'\setminus\Omega}$ and $\|\nabla(P_{\varepsilon_n} u_n)\|^p_{p,\Omega} \leq k_2 \|\nabla u_n\|^p_{p,\Omega\cap\varepsilon E} + k_2|\xi|^p\mathcal{L}_N(\Omega'\setminus\Omega)$. Using the inequality $\bar{r}\|\nabla u_n\|^p_{p,\Omega\cap\varepsilon_n E} \leq F_n(u_n)$, together with similar arguments to the proof of Proposition 3.1, we deduce that, up to a subsequence $P_{\varepsilon_n} u_n \rightharpoonup \xi \cdot x$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. Hence, by weak lower-semicontinuity we obtain $\liminf_{n\to\infty} \|\nabla(P_{\varepsilon_n} u_n)\|^p_{p,\Omega} \geq |\xi|^p \mathcal{L}_N(\Omega)$, and from

$$\liminf_{n\to\infty} \|\nabla(P_{\varepsilon_n} u_n)\|^p_{p,\Omega} \leq \frac{k_2}{\bar{r}} W^{hom}(\xi)\mathcal{L}_N(\Omega) + k_2|\xi|^p\mathcal{L}_N(\Omega'\setminus\Omega),$$

it follows that $|\xi|^p\mathcal{L}_N(\Omega) \leq \frac{k_2}{\bar{r}} W^{hom}(\xi)\mathcal{L}_N(\Omega) + k_2|\xi|^p\mathcal{L}_N(\Omega'\setminus\Omega)$. Since $\Omega' \supset\supset \Omega$ is arbitrary, the lower estimate for $W^{hom}(\xi)$ follows.

(ii) First, observe that for every $A \in \mathcal{U}_b(\mathbb{R}^N)$

$$\mathcal{S}^{\xi}_n(A) = \inf\{\int_A \mathcal{Q}W_{\lambda_n}(x,\nabla w)dx : w \in \xi \cdot x + W_0^{1,p}(A;\mathbb{R}^m)\},$$

where $\mathcal{Q}W_{\lambda_n}$ is the quasiconvexification of W_{λ_n} (see [3, 8]). Fix $\xi', \xi \in \mathbb{R}^{mN}$. For every $n \in \mathbb{N}$, consider a function $w_n \in \xi \cdot x + W_0^{1,p}(\frac{1}{\varepsilon_n}]0,1[^N;\mathbb{R}^m)$ such that

$$\int_{A_n} \mathcal{Q}W_{\lambda_n}(x,\nabla w_n)dx \leq \mathcal{S}^{\xi}_n(A_n) + \delta_n,$$

with $A_n := \frac{1}{\varepsilon_n}]0,1[^N$ and $\delta_n := \varepsilon_n r(\lambda_n) \to 0$ as $n \to \infty$. We have

$$\mathcal{S}^{\xi'}_n(A_n) - \mathcal{S}^{\xi}_n(A_n) \leq \int_{A_n} |\mathcal{Q}W_{\lambda_n}(x,\xi'-\xi+\nabla w_n) - \mathcal{Q}W_{\lambda_n}(x,\nabla w_n)|dx + \delta_n.$$

By [8, ch.4, lemma 2.2], it follows from (C_2) that for a suitable constant $c > 0$

$$|\mathcal{Q}W_{\lambda_n}(\cdot,\xi'-\xi+\nabla w_n) - \mathcal{Q}W_{\lambda_n}(\cdot,\nabla w_n)| \leq c r_{\lambda_n}(1+|\xi'|^{p-1}+|\xi|^{p-1}+|\nabla w_n|^{p-1})|\xi'-\xi|.$$

Then, we have to estimate the integral

$$\int_{A_n} r_{\lambda_n}(x)|\nabla w_n|^{p-1}dx = \bar{r}\int_{A_n\cap E_{\lambda_n}} |\nabla w_n|^{p-1}dx + r(\lambda_n)\int_{A_n\setminus E_{\lambda_n}} |\nabla w_n|^{p-1}dx$$

On the one hand, Hölder's inequality yields

$$\begin{aligned}\int_{A_n\cap E_{\lambda_n}} |\nabla w_n|^{p-1}dx &\leq \mathcal{L}_N(A_n\cap E_{\lambda_n})^{1/p}(\int_{A_n\cap E_{\lambda_n}} |\nabla w_n|^p dx)^{(p-1)/p} \\ &\leq \frac{1}{\varepsilon_n^{N/p}}\|\nabla w_n\|^{p-1}_{p,A_n\cap E_{\lambda_n}}.\end{aligned}$$

On the other hand, using the coercivity condition in (C_2) we can deduce that

$$\int_{A_n} r_{\lambda_n}(x)|\nabla w_n|^p dx \leq \mathcal{S}^{\xi}_n(A_n) + \varepsilon_n r(\lambda_n) \leq c_0\bar{r}(1+|\xi|^p)\frac{1}{\varepsilon_n^N} + \delta_n,$$

which gives in particular $\|\nabla w_n\|^{p-1}_{p,A_n\cap E_{\lambda_n}} \leq [c_0(1+|\xi|^p)\frac{1}{\varepsilon_n^N} + \delta_n/\bar{r}]^{(p-1)/p}$. Consequently, there exists a constant c such that

$$\int_{A_n\cap E_{\lambda_n}} |\nabla w_n|^{p-1}dx \leq \frac{c}{\varepsilon_n^N}[(1+|\xi|^{p-1}) + \varepsilon_n^{(N+1)(p-1)/p}].$$

By similar arguments, we obtain

$$\int_{A_n\setminus E_{\lambda_n}} |\nabla w_n|^{p-1}dx \leq \frac{c}{\varepsilon_n^N}[(1+|\xi|^{p-1})\frac{1}{r(\lambda_n)^{(p-1)/p}}+\varepsilon_n^{(N+1)(p-1)/p}].$$

We thus deduce that

$$\int_{A_n} |\nabla w_n|^{p-1}dx \leq \frac{c}{\varepsilon_n^N}[(\bar{r}+\bar{r}^{1/p})(1+|\xi|^{p-1})+2\bar{r}\varepsilon_n^{(N+1)(p-1)/p}].$$

Therefore, there exists a constant c such that

$$\varepsilon_n^N\mathcal{S}_n^{\xi'}(A_n)-\varepsilon_n^N\mathcal{S}_n^{\xi}(A_n)\leq c(1+|\xi'|^{p-1}+|\xi|^{p-1}+\varepsilon_n^{(N+1)(p-1)/p})|\xi'-\xi|+\varepsilon_n^N\delta_n.$$

Letting $n\to\infty$, we get $W^{hom}(\xi')-W^{hom}(\xi)\leq c(1+|\xi'|^{p-1}+|\xi|^{p-1})|\xi'-\xi|$. □

Now, we can complete the proof by a standard density argument. First, note that F^{hom} is a continuous function on $W^{1,p}(\Omega;\mathbb{R}^m)$. In fact, from lemma 3.6(ii) it follows that

$$|F^{hom}(u)-F^{hom}(v)|\leq c(1+\|\nabla u\|_{p,\Omega}^p+\|\nabla v\|_{p,\Omega}^p)^{\frac{p-1}{p}}\|\nabla u-\nabla v\|_{p,\Omega},$$

for every $u,v\in W^{1,p}(\Omega;\mathbb{R}^m)$. Since Ω has Lipschitz boundary, the space $\mathrm{Aff}(\Omega;\mathbb{R}^m)$ is dense in $W^{1,p}(\Omega;\mathbb{R}^m)$ for the strong topology (see [13]). Let $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and consider $\{u^k\}\subset\mathrm{Aff}(\Omega;\mathbb{R}^m)$ such that $u^k\to u$ as $k\to\infty$ strongly in $W^{1,p}(\Omega;\mathbb{R}^m)$. Then $\lim_{k\to\infty}F^{hom}(u^k)=F^{hom}(u)$. By Lemma 3.5, $\forall k\in\mathbb{N}$, $\exists\{u_n^k\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u_n^k\to u^k$ in $L^p(\Omega;\mathbb{R}^m)$ as $n\to\infty$ and $\lim_{n\to\infty}F_n(u_n^k)=F^{hom}(u^k)$. Setting $f(k,n):=|F_n(u_n^k)-F^{hom}(u)|+\|u_n^k-u\|_{p,\Omega}$, we have $\lim_{k\to\infty}\lim_{n\to\infty}f(k,n)=0$. By diagonalization (see [5, corollary 1.16]), there exists a mapping $n\to k_n$ increasing to ∞ as $n\to\infty$ such that $\lim_{n\to\infty}f(k_n,n)=0$. Defining $u_n:=u_n^{k_n}$, we have thus proved the result. □

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