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Homogenization of multiparameter integrals☆

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1. Introduction

In this paper we are concerned with some multiparameter integral functionals of the form

$$\int_{\Omega} W_{\lambda}\left(\frac{x}{\varepsilon}, \nabla u(x)\right)\,\mathrm{d}x, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $u:\Omega \to \mathbb{R}^m$, $\varepsilon > 0$ and $W_\lambda:\mathbb{R}^N \times \mathbb{R}^{mN} \to [0,+\infty[$ is supposed to be $[0,1[^N$-periodic with respect to the first variable $x \in \mathbb{R}^N$. The distinguishing feature of (1) here is that the integrand is permitted to depend on a vector of parameters $\lambda = (\lambda_1,\ldots,\lambda_k) \in \mathbb{R}^k$ with $k \geqslant 1$. We are interested in the asymptotic behavior of (1) as $\varepsilon \to 0$ and $\lambda \to 0$.

Integral functionals of this type appear in the mathematical modeling of cellular composite materials. When $N = m = 3$, (1) can be interpreted as the stored strain energy of an elastic and heterogeneous material, u being a deformation or displacement field. When $m = 1$, u may be a difference of potential in a condenser. In any case, the medium under consideration is composed of several materials, which are periodically distributed at the microscopic scale given by ε. In the applications, we often deal with

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two materials, one periodically included into the other one. In this case, the vector of parameters λ is used to describe certain properties of the inclusions like size, thickness, stiffness or conductivity; we refer the reader to Sections 3 and 4 for some examples.

When ε is very small, then the microscopic structure of such a composite material becomes complicated. Moreover, in some situations the parameter λ is also very small. The question is how to describe approximately the macroscopic behavior of the material. Passing from the microscopic level to the macroscopic one corresponds to letting $\varepsilon \to 0$ and $\lambda \to 0$ in (1). Assume that for each fixed microscopic scale $\varepsilon > 0$ we have a model of the limit case "$\lambda = 0$". An alternative to derive a limit model for "$\lambda = 0$ and $\varepsilon = 0$" would be to perform an iterate limit process: first let $\lambda \to 0$, then let $\varepsilon \to 0$. However, without further justifications, this may appear to be arbitrary and ambiguous. Furthermore, in many situations there are intrinsic relations between ε and λ, which prevent us from letting $\lambda \to 0$ without letting $\varepsilon \to 0$ at the same time.

The present work is an attempt to develop general techniques for the asymptotic analysis of functionals like (1) when all the parameters tend to zero (possibly following a particular *path* in the set of parameters). We restrict our attention to certain parametric integrands for which one may expect that *homogenization* occurs. In physical terms, this means that the heterogeneous medium behaves at the macroscopic scale as an ideal homogeneous one, so that the limit energy is of the form

$$\int_{\Omega} W^{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x.$$

In absence of λ, i.e. when $W_\lambda \equiv W$, this kind of results have been obtained by applying suitable variational methods. For scalar u and convex W see [20], the books [3,9] and references therein; for vector-valued u and nonconvex W see [5–7,22] and the book [8]. The notion of Γ-convergence for sequences of functions is used in all these works. This convergence is variational in the sense that under some conditions, it ensures the convergence of minimizers and minimum values, and, moreover, it is stable under continuous perturbations. The accomplishment of this asymptotic analysis for actual parametric integrands W_λ requires to overcome additional technical difficulties. On the other hand, an important advantage of this approach is that many "degenerate" homogenization problems can be interpreted as limits of this kind of multiparameter integral functionals.

This paper is organized as follows. Section 2 is devoted to a general nonlinear homogenization result for a class of multiparameter variational functionals. More precisely, in Section 2.1 we give some natural conditions on the integrands (cf. (C_1)–(C_3)), which define the type of multiparameter functionals that we study in this paper. After a brief exposition of Γ-convergence theory in Section 2.2, we state and prove a general homogenization theorem in Section 2.3. In fact, we introduce an *unconstrained* family of functionals $\{F_{\varepsilon,\lambda}\}$ for which we establish Γ-convergence towards a homogeneous functional F^{hom}. We precise the meaning of "having a limit model for $\lambda = 0$" in (H_1) and we introduce a condition (H_2), which allows us to identify the limit density of the homogenized functional. Then, we prove in Section 2.4 that the Γ-limit is not affected by Dirichlet boundary conditions and we give a sufficient condition for the relative compactness of minimizing sequences. This condition relates the behavior of ε

and λ as they tend to zero. In Sections 3 and 4 some applications are indicated, with a particular attention to convex integrands. A variety of techniques that may be useful in the applications are described there. We recall in Appendix A an extension theorem of [2], which is useful to overcome some technical difficulties due to an eventual lack of coerciveness. In Appendix B we prove an asymptotic formula for parametric subadditive set functions, which is used to verify (H_2) in the applications. Finally, in Appendix C we prove a technical result used in Section 4.

In the multiparameter setting of this work, some situations have been considered in the literature. Usually, a homogenized Γ-limit functional is obtained for a path of the form $(\varepsilon, \lambda(\varepsilon))$ with $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$ (see, for instance, the iterated homogenization theorem in [8, Chapter 22]). In some simpler cases, this is done by comparison with other functionals (see [3, Remark 1.24]). Note that in our case, the representation formula for W^{hom} may depend on a relative behavior between the parameters.

2. Multiparameter homogenization

2.1. A class of multiparameter variational problems

Let m, N and k be positive integers. Here and subsequently, Y denotes the unit cell $[0,1[^N$ and $\Lambda \subset \mathbb{R}^k$ is a nonempty set of parameters such that $\lambda_n \to 0$ for at least one sequence $\{\lambda_n\} \subset \Lambda$. Let us suppose that to every $\lambda \in \Lambda$, there corresponds a Carathéodory function

$$W_\lambda : \mathbb{R}^N \times \mathbb{R}^{mN} \to [0, +\infty[$$

satisfying for each $\xi \in \mathbb{R}^{mN}$

(C_1) $W_\lambda(\cdot, \xi)$ is Y-periodic: for every $(x, z) \in \mathbb{R}^N \times \mathbb{Z}^N$, $W_\lambda(x + z, \xi) = W_\lambda(x, \xi)$.

Consider a family of closed subsets $\{T_\lambda\}_{\lambda \in \Lambda} \subset Y$ and a function $r : \Lambda \to [0, \bar{r}]$ with $\bar{r} > 0$.

Let us define $E_\lambda := Y \setminus T_\lambda + \mathbb{Z}^N$ and

$$r_\lambda(x) := \begin{cases} \bar{r} & \text{if } x \in E_\lambda, \\ r(\lambda) & \text{if } x \in \mathbb{R}^N \setminus E_\lambda = T_\lambda + \mathbb{Z}^N. \end{cases}$$

Assume that there exist $p \in]1, +\infty[$ and $c_0 > 0$ such that

(C_2) for every $\lambda \in \Lambda$ and for every $x \in \mathbb{R}^N$, $\xi', \xi \in \mathbb{R}^{mN}$

$$r_\lambda(x)|\xi|^p \leqslant W_\lambda(x, \xi) \leqslant c_0 r_\lambda(x)(1 + |\xi|^p).$$

Note that $r(\lambda)$ may not be bounded away from 0; in this case, there are some technical difficulties due to a lack of equi-coerciveness. In order to apply the extension techniques recalled in Appendix A, it is required that

(C_3) there exists $T \subset Y$ such that for all $\lambda \in \Lambda$, $T_\lambda \subset T$ and the set $E := Y \setminus T + \mathbb{Z}^N$ is connected, open and has Lipschitz boundary.

Let Ω be a bounded open set in $\mathbb{R}^N$. Let us consider the following variational problem:

$$(\mathscr{P}_{\varepsilon,\lambda}; g, \phi) \qquad \inf\left\{\int_\Omega W_\lambda\left(\frac{x}{\varepsilon}, \nabla u\right) \mathrm{d}x + \int_\Omega gu \, \mathrm{d}x \colon u \in \phi + W_0^{1,p}(\Omega; \mathbb{R}^m)\right\},$$

where $(\varepsilon, \lambda) \in]0, \varepsilon_0] \times \Lambda$, $g \in L^{p'}(\Omega; \mathbb{R}^m)$ and $\phi \in W^{1,p}(\Omega; \mathbb{R}^m)$ is a boundary condition on the displacement. The first integral corresponds to the stored strain energy of a cellular elastic material, which is subject to external body forces given by g. Roughly speaking, each cell contains an inclusion, which is characterized by the value of the energy density in the set εT_λ. Observe that, since we only assume that E is connected, the inclusions may be connected throughout the whole microstructure. Solving $(\mathscr{P}_{\varepsilon,\lambda}; g, \phi)$ amounts to finding the stable equilibria (i.e. minimal energy configurations) of the structure. We are interested in the asymptotic behavior of $(\mathscr{P}_{\varepsilon,\lambda}; g, \phi)$ as $(\varepsilon, \lambda) \to (0, 0)$. The strategy will be to compute a suitable variational limit of a parametrized sequence of integral functionals. This variational convergence will be De Giorgi's Γ-convergence.

2.2. Γ-convergence

Motivated by certain lower semi-continuity and perturbation problems in the calculus of variations, De Giorgi introduced in [12] (see also [13]) the notion of Γ-convergence for sequences of functions. For the convenience of the reader, we are going to recall the definition and some properties of Γ-convergence.

Let (X, d) be a metric space and consider a family $\{F_s\}$ of functionals from X into $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, where s is taken in a subset of $\mathbb{R}^q$ (s plays the role of a vector of parameters). Given a sequence $\{s_n\}$ with $s_n \to 0 \in \mathbb{R}^q$ as $n \to \infty$, define for every $u \in X$

$$\left(\Gamma(d)\text{-}\liminf_{n\to\infty} F_{s_n}\right)(u) := \inf\left\{\liminf_{n\to\infty} F_{s_n}(u_n) \colon u_n \xrightarrow{\mathrm{d}} u\right\}$$

and

$$\left(\Gamma(d)\text{-}\limsup_{n\to\infty} F_{s_n}\right)(u) := \inf\left\{\limsup_{n\to\infty} F_{s_n}(u_n) \colon u_n \xrightarrow{\mathrm{d}} u\right\}.$$

Clearly

$$\Gamma(d)\text{-}\liminf_{n\to\infty} F_{s_n} \leqslant \Gamma(d)\text{-}\limsup_{n\to\infty} F_{s_n}.$$

The sequence $\{F_{s_n}\}$ is said to be $\Gamma(d)$-convergent to $\bar{F}(u)$ in u as $n \to \infty$ whenever

$$\bar{F}(u) = \left(\Gamma(d)\text{-}\liminf_{n\to\infty} F_{s_n}\right)(u) = \left(\Gamma(d)\text{-}\limsup_{n\to\infty} F_{s_n}\right)(u).$$

If the latter holds for every $u \in X$ then we write $\bar{F} = \Gamma(d)\text{-}\lim_{n\to\infty} F_{s_n}$.

Let $\tau_0 \subset (\mathbb{R}^q)^{\mathbb{N}}$ be a set consisting of sequences $\{s_n\} \subset \mathbb{R}^q$ such that $s_n \to 0$. If for each sequence $\{s_n\} \in \tau_0$ we have $\bar{F} = \Gamma(d)\text{-}\lim_{n\to\infty} F_{s_n}$, then we say that the functional $\bar{F}$ is the $\Gamma(d)$-limit of $\{F_s\}$ as $s \to 0$ in τ_0 and we write $\bar{F} = \Gamma(d)\text{-}\lim_{s \xrightarrow{\tau_0} 0} F_s$. Note that this definition is equivalent to:

(i) for all $u \in X$ and for all sequences $s_n \to 0$ with $\{s_n\} \in \tau_0$ and $u_n \xrightarrow{\mathrm{d}} u$

$$\bar{F}(u) \leqslant \liminf_{n\to\infty} F_{s_n}(u_n),$$

(ii) for all $u \in X$ and for all $s_n \to 0$ with $\{s_n\} \in \tau_0$ there exists a *recovery sequence*, that is, a sequence $\{u_n\} \in X$ such that $u_n \xrightarrow{\mathrm{d}} u$ and

$$\bar{F}(u) = \lim_{n\to\infty} F_{s_n}(u_n).$$

Observe that the constant sequence $F_n \equiv F$, $\Gamma(d)$-converges to the lower closure, also called lower semi-continuous envelope, of F, which we denoted by $\mathrm{cl}(F)$. In fact, we have

$$(\Gamma(d)\text{-}\lim_{n\to\infty} F)(u) = \mathrm{cl}(F)(u) = \inf\left\{\liminf_{n\to\infty} F(u_n):\ u_n \xrightarrow{\mathrm{d}} u\right\}.$$

More generally, if $\bar{F} = \Gamma(d)\text{-}\lim_{n\to\infty} F_n$ for some sequence $\{F_n\}$ then $\bar{F}$ is lower semi-continuous in (X,d). Another simple situation where Γ-convergence holds is the case of nonincreasing sequences: if for all $n \in \mathbb{N}$, $F_n \geqslant F_{n+1}$ then $\Gamma(d)\text{-}\lim_{n\to\infty} F_n = \mathrm{cl}(\inf_{n\in\mathbb{N}}\{F_n\})$. Nevertheless, in general the pointwise convergence and the Γ-convergence are not comparable.

The following theorem is a well-known result (see [13]) that makes precise the variational nature of Γ-convergence:

Theorem 2.1. *Let $G : X \to \mathbb{R}$ be a continuous function and assume that*

$$\bar{F} = \Gamma(d)\text{-}\lim_{s \xrightarrow{\tau_0} 0} F_s.$$

Then

$$\limsup_{s \xrightarrow{\tau_0} 0} (\inf\{F_s + G\}) \leqslant \inf\{\bar{F} + G\}.$$

Moreover, if for each $\{s_n\} \in \tau_0$ there exists a relative d-compact sequence $\{\hat{u}_n\} \subset X$ such that

$$F_{s_n}(\hat{u}_n) + G(\hat{u}_n) \leqslant \inf\{F_{s_n} + G\} + \varepsilon_n \tag{2}$$

with $\varepsilon_n \to 0$ as $n \to \infty$, then

$$\lim_{s \xrightarrow{\tau_0} 0} (\inf\{F_s + G\}) = \inf\{\bar{F} + G\}$$

and every d-cluster point $\hat{u} \in X$ of *$\{\hat{u}_n\}$ satisfies*

$$\bar{F}(\hat{u}) + G(\hat{u}) = \inf\{\bar{F} + G\}.$$

Remark 2.1. A sequence $\{\hat{u}_n\}$ that satisfies (2) is usually referred to as an ε_n*-minimizing sequence* for $\{F_{s_n} + G\}$.

For a proof of this result and deeper discussions of the Γ-convergence theory we refer the reader to the books [3,8,9].

Let us return to the setting of Section 2.1. Define $G: L^p(\Omega;\mathbb{R}^m) \to \mathbb{R}$ by

$$G(u) := \int_\Omega gu \,\mathrm{d}x$$

and let $F^{\phi}_{\varepsilon,\lambda}: L^p(\Omega;\mathbb{R}^m) \to [0,+\infty]$ be defined by

$$F^{\phi}_{\varepsilon,\lambda}(u) := \begin{cases} \int_\Omega W_\lambda(\frac{x}{\varepsilon}, \nabla u)\,\mathrm{d}x & \text{if } u \in \phi + W^{1,p}_0(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

The variational problem $(\mathscr{P}_{\varepsilon,\lambda}; g, \phi)$ may be written

$$\inf\{F^{\phi}_{\varepsilon,\lambda}(u) + G(u) \colon u \in L^p(\Omega;\mathbb{R}^m)\}.$$

According to Theorem 2.1, to describe the "limit" of $(\mathscr{P}_{\varepsilon,\lambda}; g, \phi)$ as $(\varepsilon,\lambda) \to (0,0)$, it suffices to compute the Γ-limit of $\{F^{\phi}_{\varepsilon,\lambda}\}$ for an appropriate topology. Because of the highly oscillating nature of the density involved in the definition of $F^{\phi}_{\varepsilon,\lambda}$, the best we can expect to obtain is a weak convergence of the gradient of the minimizers. On the other hand, under some conditions it is possible to establish a compactness property of minimizing sequences for the strong topology of $L^p(\Omega;\mathbb{R}^m)$. Therefore, the Γ-convergence will be taken with respect to the strong topology of $L^p(\Omega;\mathbb{R}^m)$. Taking into account previous works in absence of parameter λ, we may conjecture that there exists a suitable density $W^{\hom}: \mathbb{R}^{mN} \to [0,+\infty[$ such that $\{F^{\phi}_{\varepsilon,\lambda}\}$ Γ-converges to a *homogenized* functional of the form

$$F^{\hom,\phi}(u) = \begin{cases} \int_\Omega W^{\hom}(\nabla u)\,\mathrm{d}x & \text{if } u \in \phi + W^{1,p}_0(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

In general, this is not true due to eventual *irregular* behaviors with respect to λ. In fact, we may have two or more subsequences Γ-converging to different limits. Nevertheless, we claim that under some additional hypotheses it is possible to obtain a homogenization theorem of this type, giving a representation formula for $W^{\hom}$. This is the aim of Section 2.3.

2.3. General homogenization theorem

Given a sequence $\{F_n\}$ of functionals defined on $L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m)$, we denote by $\Gamma(L^p)\text{-}\lim_{n\to\infty} F_n$ the Γ-limit of $\{F_n\}$ with respect to the topology of $L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m)$ induced by the extended distance $d(u,v) = (\int_{\mathbb{R}^N} |u-v|^p\,\mathrm{d}x)^{1/p}$. Let $\mathscr{U}_{\mathrm{b}}(\mathbb{R}^N)$ denote the class of all bounded open subsets of $\mathbb{R}^N$. For each $(\xi,\lambda) \in \mathbb{R}^{mN} \times \Lambda$, we define $G^{\xi}_{\lambda}: L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m) \times \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N) \to [0,+\infty]$ by

$$G^{\xi}_{\lambda}(v; A) := \begin{cases} \int_A W_\lambda(x, \xi + \nabla v)\,\mathrm{d}x & \text{if } v\lfloor_A \in W^{1,p}_0(A;\mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

Let $\tau_0 \subset (\Lambda)^{\mathbb{N}}$ be a set consisting of sequences $\{\lambda_n\} \subset \mathbb{R}^k$ such that $\lambda_n \to 0$. For simplicity of notation, we denote by $\lambda \xrightarrow{\tau_0} 0$ an arbitrary sequence $\{\lambda_n\} \in \tau_0$. We first require:

(H_1) for every $\xi \in \mathbb{R}^{mN}$, there exists $\bar{G}^{\xi}_{\tau_0} : L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m) \times \mathcal{U}_b(\mathbb{R}^N) \to [0,+\infty]$ such that for all $k \in \mathbb{N}^*$ and $v \in L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m)$,

$$\bar{G}^{\xi}_{\tau_0}(v;]0,k[^N) = \Gamma(L^p)\text{-}\lim_{\lambda \xrightarrow{\tau_0} 0} G^{\xi}_{\lambda}(v;]0,k[^N).$$

This hypothesis may be interpreted by saying that one has to know the behavior of the periodic structure over an ensemble of k^N cells and at the unit scale. When (H_1) holds we say that we have a "limit model for the case $\lambda = 0$ at the unit scale".

In order to characterize the limit homogenized density, we consider the optimal value function $\mathcal{S}^{\xi}_{\lambda} : \mathcal{U}_b(\mathbb{R}^N) \to [0,+\infty[$ defined by

$$\begin{aligned}\mathcal{S}^{\xi}_{\lambda}(A) &:= \inf\{G^{\xi}_{\lambda}(v;A) \colon v \in L^p(A;\mathbb{R}^m)\}\\ &= \inf\left\{\int_A W_\lambda(x,\xi+\nabla v)\,\mathrm{d}x \colon v \in W^{1,p}_0(A;\mathbb{R}^m)\right\}\end{aligned}$$

for every $(\xi,\lambda) \in \mathbb{R}^{mN} \times \Lambda$. Similarly, we define

$$\bar{\mathcal{S}}^{\xi}_{\tau_0}(]0,k[^N) := \inf\{\bar{G}^{\xi}_{\tau_0}(v;]0,k[^N) \colon v \in L^p(]0,k[^N;\mathbb{R}^m)\}.$$

Let us denote by $(\varepsilon,\lambda) \to (0,0)$ with $\lambda \in \tau_0$ an arbitrary sequence $\{(\varepsilon_n,\lambda_n)\} \subset]0,\varepsilon_0] \times \Lambda$ such that $(\varepsilon_n,\lambda_n) \to (0,0)$ and $\{\lambda_n\} \in \tau_0$. We suppose:

(H_2) for every $\xi \in \mathbb{R}^{mN}$,

$$\lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda \in \tau_0}} \varepsilon^N \mathcal{S}^{\xi}_{\lambda}\left(\frac{1}{\varepsilon}]0,1[^N\right) = \gamma(\bar{\mathcal{S}}^{\xi}_{\tau_0}),$$

where

$$\gamma(\bar{\mathcal{S}}^{\xi}_{\tau_0}) := \inf_{k \in \mathbb{N}^*}\left\{\frac{1}{k^N}\bar{\mathcal{S}}^{\xi}_{\tau_0}(]0,k[^N)\right\}.$$

For a discussion of (H_2), we refer the reader to Sections 3 and 4.

Theorem 2.2. *Suppose that* (C_1)–(C_3), (H_1) *and* (H_2) *hold. Given* $\varepsilon > 0$ *and* $\lambda \in \Lambda$, *let us consider the variational functional* $F_{\varepsilon,\lambda} : L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m) \times \mathcal{U}_b(\mathbb{R}^N) \to [0,+\infty]$ *defined by*

$$F_{\varepsilon,\lambda}(u;A) := \begin{cases} \displaystyle\int_A W_\lambda\left(\tfrac{x}{\varepsilon},\nabla u\right)\mathrm{d}x & \text{if } u|_A \in W^{1,p}(A;\mathbb{R}^m),\\ +\infty & \text{otherwise}\end{cases}$$

and the homogeneous variational functional $F^{\mathrm{hom}}_{\tau_0} : L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m) \times \mathcal{U}_b(\mathbb{R}^N) \to [0,+\infty]$ *defined by*

$$F^{\mathrm{hom}}_{\tau_0}(u;A) := \begin{cases} \displaystyle\int_A W^{\mathrm{hom}}_{\tau_0}(\nabla u)\,\mathrm{d}x & \text{if } u|_A \in W^{1,p}(A;\mathbb{R}^m),\\ +\infty & \text{otherwise,}\end{cases}$$

where $W^{\hom}_{\tau_0}: \mathbb{R}^{mN} \to [0, +\infty[$ *is given by*

$$W^{\hom}_{\tau_0}(\xi) := \gamma(\bar{S}^{\xi}_{\tau_0}) = \inf_{k\in\mathbb{N}^*} \inf_{v} \left\{ \frac{1}{k^N} \bar{G}^{\xi}_{\tau_0}(v;]0,k[^N): v \in L^p(]0,k[^N;\mathbb{R}^m) \right\}.$$

Then

$$F^{\hom}_{\tau_0}(u;A) = \Gamma(L^p) - \lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} F_{\varepsilon,\lambda}(u;A)$$

for every $A \in \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N)$ *and* $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m)$.

Proof. For simplicity of notation, throughout the proof we write $(\varepsilon,\lambda)\to(0,0)$ with $\lambda\in\tau_0$ for an arbitrary sequence $(\varepsilon_n,\lambda_n)\to(0,0)$ with $\{\lambda_n\}\in\tau_0$. Similarly, $\lambda\xrightarrow{\tau_0}0$ with $\lambda\in\Lambda$ stands for $\lambda_n\to 0$ with $\{\lambda_n\}\in\tau_0$. Finally, the expression $u_{\varepsilon,\lambda}\to u$ stands for a sequence $\{u_n\}$ converging to u for a L^p-norm.

We begin by noticing that (C_1) and (C_2) ensure the application of the theory of variational functionals developed by Dal Maso and Modica in [10] (see also [8,9]) to obtain that, upon extracting a subsequence, there exists $\varphi: \mathbb{R}^{mN}\to [0,+\infty[$ such that the sequence $\{F_{\varepsilon,\lambda}\}$ $\Gamma(L^p)$-converges on each $A\in\mathscr{U}_{\mathrm{b}}(\mathbb{R}^N)$ and for every $u\in L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m)$ to a variational functional $F_0(u;A)$ with $F_0(u;A)=\int_A \varphi(\nabla u)\,\mathrm{d}x$ whenever $u|_A\in W^{1,p}(A;\mathbb{R}^m)$. This follows by using some well-known direct methods of Γ-convergence theory. Indeed, it is possible to apply the same arguments as in the proof of Proposition 3.1 in [7] to this situation, with $g(x,\xi)$ replaced by $g_\lambda(x,\xi)=r_\lambda(x)|\xi|^p$ therein (see also Remark 12.4 and Example 11.4 in [8] and Theorems 4.8 and 6.1 in [10]); we omit the details. Moreover, using (C_3) and arguing as in [2, Proposition 3.6], we can deduce that $F_0(u;A) = +\infty$ if $u\in L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m)\setminus W^{1,p}(A;\mathbb{R}^m)$. Hence

$$F_0(u;A) = \begin{cases} \displaystyle\int_A \varphi(\nabla u)\,\mathrm{d}x & \text{if } u\in W^{1,p}(A;\mathbb{R}^m),\\ +\infty & \text{otherwise.}\end{cases}$$

It remains to prove that $\varphi(\xi) = W^{\hom}_{\tau_0}(\xi)$ for every $\xi\in\mathbb{R}^{mN}$. The inequality $\varphi(\xi)\leqslant W^{\hom}_{\tau_0}(\xi)$ under (H_1) follows immediately from

Lemma 2.1. *If* (H_1) *holds then for every linear function* $u(x)=\xi x$, *there exists a sequence* $u_{\varepsilon,\lambda}\to\xi x$ *in* $L^p(]0,1[^N;\mathbb{R}^m)$ *with* $u_{\varepsilon,\lambda}\in \xi x + W^{1,p}_0(]0,1[^N;\mathbb{R}^m)$ *and such that*

$$\lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} F_{\varepsilon,\lambda}(u_{\varepsilon,\lambda};]0,1[^N) = F^{\hom}_{\tau_0}(\xi x;]0,1[^N) = W^{\hom}_{\tau_0}(\xi).$$

Proof. The proof is adapted from [22, Lemma 2.1(a)]. Let $\xi\in\mathbb{R}^{mN}$. By definition of $W^{\hom}_{\tau_0}$, for every $\delta>0$ there exist $k\in\mathbb{N}^*$ and $\psi^\delta\in L^p(]0,k[^N;\mathbb{R}^m)$ such that

$$W^{\hom}_{\tau_0}(\xi) \leqslant \frac{1}{k^N}\bar{G}^{\xi}_{\tau_0}(\psi^\delta;]0,k[^N) < W^{\hom}_{\tau_0}(\xi) + \delta.$$

Fix $\delta > 0$. According to (H_1), there exists a sequence $\{\psi_\lambda^\delta\}_{\lambda\in\Lambda} \subset W_0^{1,p}(]0,k[^N;\mathbb{R}^m)$ such that $\lim_{\lambda \xrightarrow{\tau_0} 0} ||\psi_\lambda^\delta - \psi^\delta||_{p,]0,k[^N} = 0$ and

$$\lim_{\lambda \xrightarrow{\tau_0} 0} G_\lambda^\xi(\psi_\lambda^\delta;]0,k[^N) = \bar{G}_{\tau_0}^\xi(\psi^\delta;]0,k[^N). \tag{5}$$

We extend ψ_λ^δ from $]0,k[^N$ to $\mathbb{R}^N$ by kY-periodicity, and for given (ε,λ) we define

$$u_{\varepsilon,\lambda}^\delta(x) := \begin{cases} \xi x + \varepsilon\psi_\lambda^\delta(\frac{x}{\varepsilon}) & \text{if } x \in (]0,1[^N)^{\varepsilon k}, \\ \xi x & \text{if } x \in]0,1[^N \setminus (]0,1[^N)^{\varepsilon k}, \end{cases}$$

where $(]0,1[^N)^{\varepsilon k}$ is the union of all the cubes of side εk which are contained in $]0,1[^N$. Of course, $u_{\varepsilon,\lambda}^\delta - \xi x \in W_0^{1,p}(]0,1[^N;\mathbb{R}^m)$. Moreover, since

$$\|u_{\varepsilon,\lambda}^\delta - \xi x\|_{p,]0,1[^N} = \varepsilon \left|\left|\psi_\lambda^\delta\left(\frac{x}{\varepsilon}\right)\right|\right|_{p,(]0,1[^N)^{\varepsilon k}} \leqslant \varepsilon \frac{\mathscr{L}_N(]0,1[^N)}{k^N}\|\psi_\lambda^\delta\|_{p,]0,k[^N}$$

we have that $\lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \|u_{\varepsilon,\lambda}^\delta - \xi x\|_{p,]0,1[^N} = 0$. By definition of $F_{\varepsilon,\lambda}$ and $u_{\varepsilon,\lambda}^\delta$

$$F_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}^\delta;]0,1[^N) = \int_{(]0,1[^N)^{\varepsilon k}} W_\lambda\left(\frac{x}{\varepsilon}, \xi + \nabla\psi_\lambda^\delta\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x + \int_{]0,1[^N\setminus(]0,1[^N)^{\varepsilon k}} W_\lambda\left(\frac{x}{\varepsilon},\xi\right) \mathrm{d}x.$$

By kY-periodicity, we obtain

$$\int_{(]0,1[^N)^{\varepsilon k}} W_\lambda\left(\frac{x}{\varepsilon}, \xi + \nabla\psi_\lambda^\delta\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x = \frac{\mathscr{L}_N((]0,1[^N)^{\varepsilon k})}{k^N}\int_{]0,k[^N} W_\lambda(y,\xi + \nabla\psi_\lambda^\delta(y))\,\mathrm{d}y.$$

From (5), we deduce that there exists $\rho_0 > 0$ such that for all $\lambda \in B(0,\rho_0)\cap\Lambda$

$$W_{\tau_0}^{\mathrm{hom}}(\xi) - \delta < \frac{1}{k^N}\int_{]0,k[^N} W_\lambda(y,\xi+\nabla\psi_\lambda^\delta)\,\mathrm{d}y < W_{\tau_0}^{\mathrm{hom}}(\xi) + \delta.$$

We thus have the following estimates:

$$F_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}^\delta;]0,1[^N) \geqslant \mathscr{L}_N((]0,1[^N)^{\varepsilon k})[W_{\tau_0}^{\mathrm{hom}}(\xi) - \delta]$$

and

$$F_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}^\delta;]0,1[^N) \leqslant \mathscr{L}_N((]0,1[^N)^{\varepsilon k})[W_{\tau_0}^{\mathrm{hom}}(\xi) + \delta] + c_0\bar{r}(1+|\xi|^p)\mathscr{L}_N(]0,1[^N \setminus (]0,1[^N)^{\varepsilon k})$$

for every (ε,λ) with $\lambda\in B(0,\rho_0)\cap\Lambda$. Consequently, for every $\delta>0$

$$\begin{aligned}W^{\rm hom}_{\tau_0}(\xi)-\delta &\leqslant \liminf_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} F_{\varepsilon,\lambda}(u^\delta_{\varepsilon,\lambda};]0,1[^N)\\ &\leqslant \limsup_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} F_{\varepsilon,\lambda}(u^\delta_{\varepsilon,\lambda};]0,1[^N)\leqslant W^{\rm hom}_{\tau_0}(\xi)+\delta.\end{aligned}$$

By a standard diagonalization argument (see for instance [3, Corollary 1.16]), we obtain a mapping $(\varepsilon,\lambda)\mapsto\delta(\varepsilon,\lambda)$ with $\delta(\varepsilon,\lambda)\to 0$ as $(\varepsilon,\lambda)\to(0,0)$ such that

$$\lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}}\|u^{\delta(\varepsilon,\lambda)}_{\varepsilon,\lambda}-\xi x\|_{p,]0,1[^N}=0$$

and

$$\lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} F_{\varepsilon,\lambda}(u^{\delta(\varepsilon,\lambda)}_{\varepsilon,\lambda};]0,1[^N)=W^{\rm hom}_{\tau_0}(\xi).$$

Finally, setting $u_{\varepsilon,\lambda}:=u^{\delta(\varepsilon,\lambda)}_{\varepsilon,\lambda}$ we obtain the required sequence. □

To prove the converse inequality, i.e. $\varphi(\xi)\geqslant W^{\rm hom}_{\tau_0}(\xi)$, note that by the usual cut-off and slicing De Giorgi trick, recovery sequences can be chosen with the same boundary values as their limit (see [14] and Remark 3.2 in [7]). Hence, when $u_{\varepsilon,\lambda}\in\xi x+W^{1,p}_0(]0,1[^N;\mathbb{R}^m)$ is a recovery sequence for $F_0(\xi x;]0,1[^N)$, by (H_2) we have that

$$\begin{aligned}\varphi(\xi)&=F_0(\xi x;]0,1[^N)\\ &=\lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} F_{\varepsilon,\lambda}(u_{\varepsilon,\lambda};]0,1[^N)\\ &\geqslant \limsup_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \inf\{F_{\varepsilon,\lambda}(v;]0,1[^N)\colon v\in\xi x+W^{1,p}_0(]0,1[^N;\mathbb{R}^m)\}\\ &=\limsup_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \inf\left\{\int_{]0,1[^N} W_\lambda\left(\frac{x}{\varepsilon},\xi+\nabla v\right){\rm d}x\colon v\in W^{1,p}_0(]0,1[^N;\mathbb{R}^m)\right\}\\ &=\limsup_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \varepsilon^N\mathscr{S}^\xi_\lambda\left(\frac{1}{\varepsilon}]0,1[^N\right)\\ &=\gamma(\bar{\mathscr{S}}^\xi)\end{aligned}$$

and the proof is complete. □

Remark 2.2. Arguing as in [2, Proposition 3.6] and [7, Proposition 3.3], it is possible to prove that there exist some constants $R, r > 0$ such that

$$r|\xi|^p \leqslant W_{\tau_0}^{\text{hom}}(\xi) \leqslant R(1 + |\xi|^p)$$

for every $\xi \in \mathbb{R}^{mN}$, so that the limit homogenized functional is coercive.

2.4. Dirichlet boundary value problems and compactness

Theorem 2.2 may be useful for the asymptotic analysis of constrained variational problems involving the functionals $F_{\varepsilon,\lambda}$. In this section we consider the particular case of Dirichlet boundary conditions.

Theorem 2.3. *Under the hypotheses of Theorem 2.2, let $\Omega \subset \mathbb{R}^N$ be a Lipschitz bounded open set and consider the variational problem*

$$m_{\varepsilon,\lambda} = \inf \left\{ \int_\Omega W_\lambda \left(\frac{x}{\varepsilon}, \nabla u \right) \mathrm{d}x \colon u \in \phi + W_0^{1,p}(\Omega; \mathbb{R}^m) \right\},$$

where $\varepsilon > 0$, $\lambda \in \Lambda$ and $\phi \in W^{1,p}(\Omega; \mathbb{R}^m)$ is given. If we assume in addition that $\varepsilon^p / r(\lambda) \to 0$ as $(\varepsilon, \lambda) \to (0,0)$ with $\lambda \in \tau_0$ then

$$\lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} m_{\varepsilon,\lambda} = m,$$

where

$$m = \min \left\{ \int_\Omega W_{\tau_0}^{\text{hom}}(\nabla u)\, \mathrm{d}x \colon u \in \phi + W_o^{1,p}(\Omega; \mathbb{R}^m) \right\}.$$

Moreover, minimizing sequences for the problems $m_{\varepsilon,\lambda}$ converge as $(\varepsilon, \lambda) \to (0,0)$ with $\lambda \in \tau_0$, upon extracting a subsequence, to minimizers for the problem m.

Proof. *Let* $u \in \phi + W_0^{1,p}(\Omega; \mathbb{R}^m)$. Theorem 2.2 ensures the existence of a sequence $\{u_{\varepsilon,\lambda}\} \subset W^{1,p}(\Omega; \mathbb{R}^m)$ such that $u_{\varepsilon,\lambda} \to u$ in $L^p(\Omega; \mathbb{R}^m)$ and

$$F_{\tau_0}^{\text{hom}}(u) = \lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} F_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}).$$

Moreover $\{u_{\varepsilon,\lambda}\}$ can be chosen such that $u_{\varepsilon,\lambda} \in \phi + W_0^{1,p}(\Omega; \mathbb{R}^m)$ (see [14]). Therefore $F_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) \geqslant m_{\varepsilon,\lambda}$ and

$$F_{\tau_0}^{\text{hom}}(u) \geqslant \limsup_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} m_{\varepsilon,\lambda}.$$

Since $u \in \phi + W_0^{1,p}(\Omega; \mathbb{R}^m)$ is arbitrary, we deduce that

$$m \geqslant \limsup_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} m_{\varepsilon,\lambda}.$$

On the other hand, let $u_{\varepsilon,\lambda} \in \phi + W_0^{1,p}(\Omega;\mathbb{R}^m)$ be such that $F_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) \leqslant m_{\varepsilon,\lambda} + \varepsilon$. Suppose that there exists $u \in L^p(\Omega;\mathbb{R}^m)$ such that $u_{\varepsilon,\lambda} \to u$ in $L^p(\Omega;\mathbb{R}^m)$. Then, by Theorem 2.2 we have

$$F_{\tau_0}^{\hom}(u) \leqslant \liminf_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} F_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}).$$

Arguing as in [2] we obtain that indeed $u \in \phi + W_0^{1,p}(\Omega;\mathbb{R}^m)$, which yields

$$m = F_{\tau_0}^{\hom}(u) \leqslant \liminf_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} F_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) \leqslant \limsup_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} m_{\varepsilon,\lambda} \leqslant m.$$

Thus, we are reduced to proving $u_{\varepsilon,\lambda} \to u$ in $L^p(\Omega;\mathbb{R}^m)$ for some $u \in L^p(\Omega;\mathbb{R}^m)$. Let Ω' be a bounded open subset of $\mathbb{R}^N$ such that $\Omega \subset\subset \Omega'$, and let us consider the functions

$$\bar{u}_{\varepsilon,\lambda} := u_{\varepsilon,\lambda} - \phi \in W_0^{1,p}(\Omega;\mathbb{R}^m) \subset W_0^{1,p}(\Omega';\mathbb{R}^m)$$

extended to 0 in $\Omega' \setminus \Omega$. If we set $v_{\varepsilon,\lambda} := \bar{u}_{\varepsilon,\lambda}|_{\Omega'\cap\varepsilon E}$, then from the growth property (C_2) it follows easily that

$$\|v_{\varepsilon,\lambda}\|_{1,p,\Omega'\cap\varepsilon E} = \|v_{\varepsilon,\lambda}\|_{1,p,\Omega\cap\varepsilon E} \leqslant c,$$

where c is a nonnegative constant which only depends on p and Ω. Let us take $\delta > 0$ such that $\Omega \subset\subset \Omega'(\delta)$. There exists $\varepsilon' > 0$ such that for each $0 < \varepsilon < \varepsilon'$, $\Omega'(\delta) \subset \Omega'(\varepsilon k_0)$ (with k_0 given by Theorem A.1 in Appendix A) and there exists $\{z_i : i \in I_\varepsilon\} \subset \mathbb{Z}^N$ with

$$\Omega \setminus \varepsilon E \subset \bigcup_{i\in I_\varepsilon} \varepsilon(T + z_i) \subset \Omega'(\delta) \setminus \varepsilon E,$$

where I_ε is a finite index set. By Theorem A.1 in Appendix A, we have $P_\varepsilon v_{\varepsilon,\lambda} = v_{\varepsilon,\lambda}$ on $\Omega' \cap \varepsilon E$ and $\|P_\varepsilon v_{\varepsilon,\lambda}\|_{1,p,\Omega'(\delta)} \leqslant c'$, where c' is a non-negative constant which only depends on E, N and p. We can suppose, upon passing to a subsequence, that there exists $\bar{u} \in W^{1,p}(\Omega'(\delta);\mathbb{R}^m)$ such that $P_\varepsilon v_{\varepsilon,\lambda} \to \bar{u}$ in $L^p(\Omega'(\delta);\mathbb{R}^m)$. We claim that under the assumption $\varepsilon^p/r(\lambda) \to 0$,

$$\lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \|P_\varepsilon v_{\varepsilon,\lambda} - \bar{u}_{\varepsilon,\lambda}\|_{p,\Omega} = 0$$

and consequently $u_{\varepsilon,\lambda} \to \bar{u} + \phi$ in $L^p(\Omega'(\delta);\mathbb{R}^m)$. Indeed, we have that

$$\begin{aligned}\int_\Omega |P_\varepsilon v_{\varepsilon,\lambda} - \bar{u}_{\varepsilon,\lambda}|^p \,\mathrm{d}x &= \int_{\Omega\setminus\varepsilon E} |P_\varepsilon v_{\varepsilon,\lambda} - \bar{u}_{\varepsilon,\lambda}|^p \,\mathrm{d}x \\ &\leqslant \sum_{i\in I_\varepsilon} \int_{\varepsilon(T+z_i)} |P_\varepsilon v_{\varepsilon,\lambda} - \bar{u}_{\varepsilon,\lambda}|^p \,\mathrm{d}x \\ &= \sum_{i\in I_\varepsilon} \varepsilon^N \int_T |P_\varepsilon v_{\varepsilon,\lambda}(\varepsilon y + z_i) - \bar{u}_{\varepsilon,\lambda}(\varepsilon y + z_i)|^p \,\mathrm{d}y.\end{aligned}$$

But $P_\varepsilon v_{\varepsilon,\lambda}(\varepsilon y + z_i) = \bar{u}_{\varepsilon,\lambda}(\varepsilon y + z_i)$ for a.e. y in a subset of ∂T of strictly positive measure (in fact, this is true for every connected component of T). Then, by Poincaré's inequality we obtain

$$\int_\Omega |P_\varepsilon v_{\varepsilon,\lambda} - \bar{u}_{\varepsilon,\lambda}|^p \,\mathrm{d}x \leqslant c''\varepsilon^p \sum_{i\in I_\varepsilon} \varepsilon^N \int_T |\nabla P_\varepsilon v_{\varepsilon,\lambda}(\varepsilon y + z_i) - \nabla \bar{u}_{\varepsilon,\lambda}(\varepsilon y + z_i)|^p \,\mathrm{d}y$$

$$\leqslant c''\varepsilon^p \int_{\Omega'(\delta)\setminus \varepsilon E} |\nabla P_\varepsilon v_{\varepsilon,\lambda} - \nabla \bar{u}_{\varepsilon,\lambda}|^p \,\mathrm{d}x,$$

where c'' is the Poincaré constant which only depends on p and T. But

$$\int_{\Omega'(\delta)\setminus \varepsilon E} |\nabla P_\varepsilon v_{\varepsilon,\lambda} - \nabla \bar{u}_{\varepsilon,\lambda}|^p \,\mathrm{d}x$$

$$\leqslant 2^p \left(\int_{\Omega'(\delta)\setminus \varepsilon E} |\nabla P_\varepsilon v_{\varepsilon,\lambda}|^p \,\mathrm{d}x + \int_{\Omega\setminus \varepsilon E} |\nabla \bar{u}_{\varepsilon,\lambda}|^p \,\mathrm{d}x \right)$$

because $\bar{u}_{\varepsilon,\lambda} = 0$ on $\Omega'(\delta) \setminus \Omega$. Moreover

$$\int_{\Omega\setminus \varepsilon E} |\nabla \bar{u}_{\varepsilon,\lambda}|^p \,\mathrm{d}x \leqslant 2^p \left(\int_{\Omega\setminus \varepsilon E} |\nabla \phi|^p \,\mathrm{d}x + \int_{\Omega\setminus \varepsilon E} |\nabla u_{\varepsilon,\lambda}|^p \,\mathrm{d}x \right)$$

$$\leqslant 2^p \left(\int_\Omega |\nabla \phi|^p \,\mathrm{d}x + \frac{1}{r(\lambda)} \int_{\Omega\setminus \varepsilon E} W_\lambda \left(\frac{x}{\varepsilon}, \nabla u_{\varepsilon,\lambda} \right) \,\mathrm{d}x \right)$$

$$\leqslant 2^p \left(\int_\Omega |\nabla \phi|^p \,\mathrm{d}x + \frac{1}{r(\lambda)} F_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}) \right).$$

Therefore

$$\int_\Omega |P_\varepsilon v_{\varepsilon,\lambda} - \bar{u}_{\varepsilon,\lambda}|^p \,\mathrm{d}x \leqslant c''' \left(\varepsilon^p + \varepsilon^p \int_\Omega |\nabla \phi|^p \,\mathrm{d}x + \frac{\varepsilon^p}{r(\lambda)} (m_{\varepsilon,\lambda} + \varepsilon) \right)$$

hence

$$\limsup_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \int_\Omega |P_\varepsilon v_{\varepsilon,\lambda} - \bar{u}_{\varepsilon,\lambda}|^p \,\mathrm{d}x \leqslant m \lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \frac{\varepsilon^p}{r(\lambda)} = 0$$

which finishes the proof. □

Remark 2.3. In the case $T = \emptyset$ or $r(\lambda) \geqslant \alpha > 0$, the condition $\varepsilon^p / r(\lambda) \to 0$ is automatically satisfied. When $T_\lambda \equiv T \neq \emptyset$ and $r(\lambda) \equiv 0$, we can argue as in the proof of [7, Proposition 4.1] to establish the convergence of $m_{\varepsilon,\lambda}$ to m and the relative compactness of P_ε-extensions of the restrictions to $\Omega \cap \varepsilon E$ of ε-minimizing sequences for $m_{\varepsilon,\lambda}$.

3. Applications I: nonconvex integrands

3.1. Preliminaries

We are going to discuss some aspects and techniques related to the verification of (H_1) and (H_2) in the applications. Let us begin by considering the *nonparametric* case $W_\lambda \equiv W$, where $W : \mathbb{R}^N \times \mathbb{R}^{mN} \to [0,+\infty[$ is a nonconvex Carathéodory function. Under suitable periodicity and growth requirements on W, conditions (C_1)–(C_3) are satisfied with $T_\lambda \equiv T$, where either (i) $T=\emptyset$ and $r_\lambda(x) \equiv \bar{r} > 0$ or (ii) $\mathscr{L}_N(T) > 0$ and $r(\lambda) \equiv 0$. It is well-known that in this case homogenization occurs with homogenized density given by

$$W^{\mathrm{hom}}(\xi) = \inf_{k\in\mathbb{N}^*} \left\{ \frac{1}{k^N} \mathscr{S}^\xi(]0,k[^N) \right\}$$

with

$$\mathscr{S}^\xi(]0,k[^N) := \inf_w \left\{ \int_{]0,k[^N \cap E} W(x,\xi+\nabla w)\,\mathrm{d}x \colon w \in W_0^{1,p}(]0,k[^N;\mathbb{R}^m) \right\},$$

where $E = Y \setminus T + \mathbb{Z}^N$ (of course, $E = \mathbb{R}^N$ when $T = \emptyset$). We refer the reader to [22, Theorem 1.3] for the coercive case, that is, when $E=\mathbb{R}^N$. For the case $\mathscr{L}_N(T) > 0$, see [2,7]. Note that in this nonparametric setting, (H_1) trivially holds with limit functional $\bar{G}^\xi$ equals to the lower semi-continuous envelope $\mathrm{cl}(G^\xi)$ of $G^\xi_\lambda \equiv G^\xi$ with respect to the strong topology of L^p. Since the infimum values of a function and its l.s.c. envelope are equal, it is possible to show that (H_2) holds as a consequence of a classical result for subadditive and $\mathbb{Z}^N$-invariant set functions (see Appendix B). Consequently, Theorem 2.2 recovers the nonparametric case.

On the other hand, general properties of Γ-convergence (cf. Theorem 2.1) ensure that in the parametric setting of Theorem 2.2 and under (H_1), we have that

$$\limsup_{\lambda \xrightarrow{\tau_0} 0} \mathscr{S}^\xi_\lambda(]0,k[^N) \leqslant \bar{\mathscr{S}}^\xi(]0,k[^N).$$

This allows us to apply Lemma B.1 in Appendix B to the parametric family $\{\mathscr{S}^\xi_\lambda\}_{\lambda\in\Lambda}$ of subadditive and $\mathbb{Z}^N$-invariant set functions and with $a_k := \bar{\mathscr{S}}^\xi(]0,k[^N)$. We deduce that for every open cube $Q \in Cub(\mathbb{R}^N)$

$$\liminf_{\lambda \xrightarrow{\tau_0} 0} \gamma(\mathscr{S}^\xi_\lambda) \leqslant \liminf_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \frac{\mathscr{S}^\xi_\lambda((1/\varepsilon)Q)}{\mathscr{L}_N((1/\varepsilon)Q)}$$

$$\leqslant \limsup_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \frac{\mathscr{S}^\xi_\lambda((1/\varepsilon)Q)}{\mathscr{L}_N((1/\varepsilon)Q)} \leqslant \gamma(\bar{\mathscr{S}}^\xi).$$

Let us consider the following stronger condition:

$(\hat{H}_2)$ For every $\xi \in \mathbb{R}^{mN}$, $\liminf_{\lambda \xrightarrow{\tau_0} 0} \gamma(\mathscr{S}^\xi_\lambda) \geqslant \gamma(\bar{\mathscr{S}}^\xi)$.

Therefore, under (H_1), it suffices to verify $(\hat{H}_2)$ to ensure that (H_2) holds. In the next section it is given a simple situation where it is easy to accomplish this.

3.2. Elastic material with soft inclusions: monotone family of integrands

Let $T \subset Y$ be a closed subset of the unit cell and suppose that $E := Y \setminus T + \mathbb{Z}^N$ is a connected open subset of $\mathbb{R}^N$ with Lipschitz boundary. Let $p \in]1, +\infty[$ and consider two Carathéodory functions $f_1, f_2 : \mathbb{R}^N \times \mathbb{R}^{mN} \to [0, +\infty[$ satisfying the following conditions: $f_i(\cdot, \xi)$ is Y-periodic and there exist $R, r > 0$ such that $r|\xi|^p \leqslant f_i(x, \xi) \leqslant R(1 + |\xi|^p)$ for each $i = 1, 2$. Given $\delta \in [0, \delta_0]$, define $W_\delta : \mathbb{R}^N \times \mathbb{R}^{mN} \to [0, +\infty[$ by

$$W_\delta(x, \xi) := \begin{cases} f_1(x, \xi) & \text{if } x \in E, \\ \delta f_2(x, \xi) & \text{otherwise.} \end{cases}$$

Proposition 3.1. *Let $\Omega \in \mathscr{U}(\mathbb{R}^N)$. The functional $F_{\varepsilon,\delta} : L^p(\Omega; \mathbb{R}^m) \to [0, +\infty]$ defined by*

$$F_{\varepsilon,\delta}(u) = \begin{cases} \displaystyle\int_\Omega W_\delta\left(\frac{x}{\varepsilon}, \nabla u\right) \mathrm{d}x & \textit{if } u \in W^{1,p}(\Omega; \mathbb{R}^m), \\ +\infty & \textit{otherwise,} \end{cases}$$

$\Gamma(L^p)$-converges as $(\varepsilon, \delta) \to (0, 0)$ towards the homogenized functional

$$F^{\mathrm{hom}}(u) = \begin{cases} \displaystyle\int_\Omega W^{\mathrm{hom}}(\nabla u)\, \mathrm{d}x & \textit{if } u \in W^{1,p}(\Omega; \mathbb{R}^m), \\ +\infty & \textit{otherwise} \end{cases}$$

with density given by

$$W^{\mathrm{hom}}(\xi) = \inf_{k \in \mathbb{N}^*} \inf_w \left\{ \frac{1}{k^N} \int_{]0,k[^N \cap E} f_1(x, \xi + \nabla w)\, \mathrm{d}x \colon w \in W_0^{1,p}(]0, k[^N; \mathbb{R}^m) \right\}.$$

Proof. In order to apply Theorem 2.2 to this situation, set $\Lambda := [0, \delta_0]$, $s := (\varepsilon, \delta)$ and identify λ with δ. Conditions (C_1)–(C_3) are trivially satisfied with $T_\delta \equiv T$, $r(\delta) = r\delta$ and $c_0 = R/r$. As for every $A \in \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N)$, the corresponding sequence $\{G_\delta^\xi(\cdot; A)\}_{\delta \in [0, \delta_0]}$ is nonincreasing, we have (cf. Section 2.2):

$$\Gamma(L^p)\text{-}\lim_{\delta \to 0} G_\delta^\xi(\cdot; A) = \mathrm{cl}\left(\inf_{\delta > 0} G_\delta^\xi(\cdot; A)\right) = \mathrm{cl}(G_0^\xi(\cdot; A))$$

so that (H_1) holds. Furthermore

$$\lim_{\delta \to 0} \gamma(\mathscr{S}_\delta^\xi) = \inf_{\delta > 0} \inf_{k \in \mathbb{N}^*} \left\{ \frac{1}{k^N} \mathscr{S}_\delta^\xi(]0, k[^N) \right\} = \inf_{k \in \mathbb{N}^*} \left\{ \frac{1}{k^N} \mathscr{S}_0^\xi(]0, k[^N) \right\} = \gamma(\mathscr{S}_0^\xi)$$

with

$$\mathscr{S}_\delta^\xi(A) = \inf_w \left\{ \int_{A \cap E} f_1(x, \xi + \nabla w)\, \mathrm{d}x \right.$$
$$\left. + \delta \int_{A \setminus E} f_2(x, \xi + \nabla w)\, \mathrm{d}x \colon w \in W_0^{1,p}(A; \mathbb{R}^m) \right\}.$$

Since the infimum values of a function and its l.s.c. envelope are equal, we have

$$\mathscr{S}_0^\xi(A) = \inf_w \{\mathrm{cl}(G_0^\xi(\cdot; A))(w) \colon w \in L^p(A; \mathbb{R}^m)\}.$$

Consequently, hypothesis $(\hat{\mathrm{H}}_2)$ follows, which finishes the proof. □

3.3. Some remarks and open problems

In the general multiparameter setting under (C_1)–(C_3), homogenization always occurs in the sense of $\Gamma(L^p)$-convergence to an x-independent integral functional of the displacements gradient, upon passing to a subsequence and with limit energy density eventually depending on the corresponding subsequence. However, it is necessary to assume additional hypotheses in order to characterize the limit functional as in the conclusion of Theorem 2.2. It seems that (H_1) alone does not suffice to do this. On the other hand, since it is apparent that (H_2) is very difficult to verify for general nonconvex integrands, one may ask whether this hypothesis is actually essential for the $\Gamma(L^p)$-convergence analysis. This question has an affirmative answer under some natural conditions and in a sense that we are going to make precise.

From now on, assume that the family $\{F_{\varepsilon,\lambda}\}$ is equicoercive under (linear) Dirichlet boundary value conditions, that is, there exists a L^p-relatively compact ε-minimizing sequence associated with the following multiparameter family of problems:

$$\varepsilon^N \mathscr{S}^{\xi}_{\lambda}\left(\frac{1}{\varepsilon}]0,1[^N\right) = \inf\left\{\int_{]0,1[^N} W_\lambda\left(\tfrac{x}{\varepsilon},\nabla u\right)\,\mathrm{d}x\colon u \in \xi x + W^{1,p}_0(]0,1[^N;\mathbb{R}^m)\right\}.$$

For instance, by Theorem 2.3, this compactness property holds whenever $\varepsilon^p/r(\lambda) \to 0$ as $(\varepsilon,\lambda)\xrightarrow{\tau_0}(0,0)$. Then, by the property of convergence of minimum problems for Γ-convergence, it follows that there exists $W^{\mathrm{hom}}_{\tau_0}\colon \mathbb{R}^{mN} \to [0,+\infty[$ such that for every $A \in \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N)$ and $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m)$

$$\Gamma(L^p) - \lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} F_{\varepsilon,\lambda}(u;A) = \begin{cases} \displaystyle\int_A W^{\mathrm{hom}}_{\tau_0}(\nabla u(x))\,\mathrm{d}x & \text{if } u|_A \in W^{1,p}(A;\mathbb{R}^m),\\ +\infty & \text{otherwise}\end{cases}$$

if and only if for every $\xi \in \mathbb{R}^{mN}$ the following limit exists:

$$\lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \varepsilon^N \mathscr{S}^{\xi}_{\lambda}\left(\frac{1}{\varepsilon}]0,1[^N\right)$$

and is equal to $W^{\mathrm{hom}}_{\tau_0}(\xi)$ (we leave the details to the reader). Let us now suppose:

$(\hat{H}_1)$ For every $\xi \in \mathbb{R}^{mN}$, there exists $\bar{G}^{\xi}_{\tau_0}\colon L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m)\times\mathscr{U}_{\mathrm{b}}(\mathbb{R}^N) \to [0,+\infty]$ such that for all $A \in \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N)$ and $v \in L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m)$,

$$\bar{G}^{\xi}_{\tau_0}(v;A) = \Gamma(L^p)\text{-}\lim_{\lambda\xrightarrow{\tau_0}0} G^{\xi}_{\lambda}(v;A) \quad\text{and}\quad \lim_{\lambda\xrightarrow{\tau_0}0} \mathscr{S}^{\xi}_{\lambda}(A) = \bar{\mathscr{S}}^{\xi}_{\tau_0}(A).$$

Of course $(\hat{H}_1)$ is stronger than (H_1): we need to have $\Gamma(L^p)$-convergence for every $A \in \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N)$ together with relative compactness of minimizing sequences. However $(\hat{H}_1)$ holds in many applications (for instance, the examples given in this paper). Under this hypothesis, by a well-known asymptotic formula for subadditive and $\mathbb{Z}^N$-invariant set functions (cf. Lemma B.1 in Appendix B), we have that

$$\lim_{\varepsilon\to 0} \varepsilon^N \bar{\mathscr{S}}^{\xi}_{\tau_0}\left(\frac{1}{\varepsilon}]0,1[^N\right) = \gamma(\bar{\mathscr{S}}^{\xi}_{\tau_0})$$

hence

$$\lim_{\varepsilon\to 0}\lim_{\lambda\xrightarrow{\tau_0}0}\varepsilon^N\mathscr{S}_\lambda^\xi\left(\frac{1}{\varepsilon}]0,1[^N\right)=\gamma(\bar{\mathscr{S}}_{\tau_0}^\xi).$$

Therefore, a standard diagonalization argument yields

$$\limsup_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}}\varepsilon^N\mathscr{S}_\lambda^\xi\left(\frac{1}{\varepsilon}]0,1[^N\right)=\gamma(\bar{\mathscr{S}}_{\tau_0}^\xi).$$

Thus, under the above conditions, there is homogenization with a unique limit functional if and only if (H_2) holds and the limit energy density $W_{\tau_0}^{\hom}(\xi)$ is equal to $\gamma(\bar{\mathscr{S}}_{\tau_0}^\xi)$.

Observe that under ($\hat{H}_1$), by the classical subadditive theorem, the hypothesis ($\hat{H}_2$) becomes

$$\lim_{\lambda\xrightarrow{\tau_0}0}\lim_{k\to\infty}\frac{\mathscr{S}_\lambda^\xi(]0,k[^N)}{k^N}=\lim_{k\to\infty}\lim_{\lambda\xrightarrow{\tau_0}0}\frac{\mathscr{S}_\lambda^\xi(]0,k[^N)}{k^N}.$$

An easy property ensuring this hypothesis is the continuity and uniform convergence as $k\to\infty$ of $f_k(\lambda):=(1/k^N)\mathscr{S}_\lambda^\xi(]0,k[^N)$ on a compact set for λ. This will hold under a sort of equicontinuity requirement on W_λ with respect to λ, a very restrictive condition (we leave the details to the reader). To our best knowledge, it is still unanswered whether additional compactness conditions allow us to identify the limit energy density in the general nonlinear setting. One may conjecture that the equicoerciveness of $\{F_{\varepsilon,\lambda}\}$ together with ($\hat{H}_1$) are sufficient to this end, but we do not have general results in this direction for the nonconvex case. Nevertheless, we shall see in Section 4 that in the case of convex integrands it is possible to exploit ($\hat{H}_1$) together with some estimates for minimizing sequences in order to prove that (H_2) holds, obtaining thus the desired characterization of the limit.

4. Applications II: convex integrands

4.1. General procedure

The aim of this section is to show that in the case of convex integrands it is possible to exploit the existence of a relatively compact minimizing sequence associated with $\{G_\lambda^\xi(\cdot;]0,k[^N)\}_{\lambda\in\Lambda}$ in order to prove that ($\hat{H}_2$) holds. From now on, we suppose that (C_4) for every $\lambda\in\Lambda$ and $x\in\mathbb{R}^N$, the function $\xi\to W_\lambda(x,\xi)$ is convex.

It is well-known (see [8,9,22]) that (C_4) yields the so-called "cell-problem formula"

$$\inf_{k\in\mathbb{N}^*}\left\{\frac{\mathscr{S}_\lambda^\xi(]0,k[^N)}{k^N}\right\}=\mathscr{S}_\lambda^{\xi,\#}, \tag{6}$$

where

$$\mathscr{S}_\lambda^{\xi,\#}:=\inf_w\left\{\int_{]0,1[^N}W_\lambda(y,\xi+\nabla w)\,\mathrm{d}y:\ w\in W_\#^{1,p}(]0,1[^N;\mathbb{R}^m)\right\}.$$

Here, $W_{\sharp}^{1,p}(]0,1[^N;\mathbb{R}^m)$ is the space consisting of all the $[0,1[^N$-periodic functions that belong to $W_{\mathrm{loc}}^{1,p}(\mathbb{R}^N;\mathbb{R}^m)$. The interest of $\mathscr{S}_\lambda^{\xi,\sharp}$ is that it takes into account only one minimization problem. Thus, $(\hat{\mathrm{H}}_2)$ is equivalent to

$$\liminf_{\lambda \xrightarrow{\tau_0} 0} \mathscr{S}_\lambda^{\xi,\sharp} \geqslant \gamma(\bar{\mathscr{S}}^\xi). \tag{7}$$

In the sequel, we say that $\{w_\lambda\} \subset W_{\sharp}^{1,p}(]0,1[^N;\mathbb{R}^m)$ is an α_λ-minimizing sequence for $\{\mathscr{S}_\lambda^{\xi,\sharp}\}$ if for every $\lambda \in \Lambda$

$$\mathscr{S}_\lambda^{\xi,\sharp} \leqslant \int_{]0,1[^N} W_\lambda(y,\xi+\nabla w_\lambda)\,\mathrm{d}y \leqslant \mathscr{S}_\lambda^{\xi,\sharp} + \alpha_\lambda$$

with $\alpha_\lambda \to 0$ as $\lambda \xrightarrow{\tau_0} 0$ so that

$$\liminf_{\lambda \xrightarrow{\tau_0} 0} \mathscr{S}_\lambda^{\xi,\sharp} = \liminf_{\lambda \xrightarrow{\tau_0} 0} \left\{ \int_{]0,1[^N} W_\lambda(y,\xi+\nabla w_\lambda)\,\mathrm{d}y \right\}.$$

The next result is useful to verify (7) in the applications.

Lemma 4.1. *Suppose that* (C_1)–(C_3) *hold and let* $\xi \in \mathbb{R}^{mN}$. *If there exists an* α_λ-*minimizing sequence* $\{w_\lambda\} \subset W_{\sharp}^{1,p}(]0,1[^N;\mathbb{R}^m)$ *for* $\{\mathscr{S}_\lambda^{\xi,\sharp}\}$ *such that*

$$M(\{w_\lambda\}) := \sup_{\lambda \in \Lambda} \left\{ \int_{]0,1[^N} r_\lambda(y)|w_\lambda(y)|^p\,\mathrm{d}y \right\} < +\infty$$

then

$$\liminf_{\lambda \xrightarrow{\tau_0} 0} \mathscr{S}_\lambda^{\xi,\sharp} \geqslant \inf_{k \in \mathbb{N}^*} \left\{ \frac{1}{k^N} \liminf_{\lambda \xrightarrow{\tau_0} 0} \mathscr{S}_\lambda^{\xi}(]0,k[^N) \right\}. \tag{8}$$

Proof. Fix $k \in \mathbb{N}^*$. Let us consider a cut-off function φ between $]0,k[^N$ and $R_k :=]0,k-2[^N + \hat{e}$ with $\hat{e} = (1,\ldots,1)$, that is, $\varphi \in \mathscr{D}(]0,k[^N)$, $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ on R_k. We can suppose that $\|\nabla\varphi\|_\infty \leqslant 2$. Setting $u_\lambda := \varphi w_\lambda$, we have $u_\lambda \in W_0^{1,p}(]0,k[^N;\mathbb{R}^m)$ and $\nabla u_\lambda = \varphi\nabla w_\lambda + w_\lambda \otimes \nabla\varphi$. Hence

$$\begin{aligned}
\mathscr{S}_\lambda^{\xi}(]0,k[^N) &\leqslant \int_{]0,k[^N} W_\lambda(x,\xi+\nabla u_\lambda)\,\mathrm{d}x \\
&= \int_{R_k} W_\lambda(x,\xi+\nabla w_\lambda)\,\mathrm{d}x + \int_{]0,k[^N \setminus R_k} W_\lambda(x,\xi+\nabla u_\lambda)\,\mathrm{d}x \\
&\leqslant \int_{]0,k[^N} W_\lambda(x,\xi+\nabla w_\lambda)\,\mathrm{d}x \\
&\quad + c\int_{]0,k[^N \setminus R_k} r_\lambda(x)(1+|w_\lambda|^p+|\xi+\nabla w_\lambda|^p)\,\mathrm{d}x
\end{aligned}$$

for a suitable constant $c > 0$. Using the Y-periodicity of w_λ and r_λ, we deduce that

$$\begin{aligned}
\mathscr{S}^\xi_\lambda(]0,k[^N) &\leqslant k^N \int_{]0,1[^N} W_\lambda(y,\xi+\nabla w_\lambda)\,\mathrm{d}y \\
&\quad + c(k^N-(k-2)^N)\int_{]0,1[^N} r_\lambda(y)(1+|w_\lambda|^p+|\xi+\nabla w_\lambda|^p)\,\mathrm{d}y \\
&\leqslant k^N \int_{]0,1[^N} W_\lambda(y,\xi+\nabla w_\lambda)\,\mathrm{d}y \\
&\quad + c(k^N-(k-2)^N)\left(\bar{r}+M(\{w_\lambda\})+\int_{]0,1[^N} r_\lambda(y)|\xi+\nabla w_\lambda|^p\,\mathrm{d}y\right) \\
&\leqslant [k^N+c(k^N-(k-2)^N)]\int_{]0,1[^N} W_\lambda(y,\xi+\nabla w_\lambda)\,\mathrm{d}y \\
&\quad + c(k^N-(k-2)^N)(\bar{r}+M(\{w_\lambda\})).
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\inf_{k\in\mathbb{N}^*}\left\{\frac{1}{k^N}\liminf_{\lambda\xrightarrow{\tau_0}0}\mathscr{S}^\xi_\lambda(]0,k[^N)\right\} &\leqslant \liminf_{\lambda\xrightarrow{\tau_0}0}\frac{\mathscr{S}^\xi_\lambda(]0,k[^N)}{k^N} \\
&\leqslant (1+c-c(k-2)^N/k^N)\liminf_{\lambda\xrightarrow{\tau_0}0}\mathscr{S}^{\xi,\sharp}_\lambda \\
&\quad + c(1-(k-2)^N/k^N)(\bar{r}+M(\{w_\lambda\})).
\end{aligned}$$

Letting $k \to +\infty$, we obtain (8). $\square$

We can now describe a general procedure in the convex case. First, one establishes (H_1), that is, the $\Gamma(L^p)$-convergence of $G^\xi_\lambda(\cdot;]0,k[^N)$ to some functional $\bar{G}^\xi_{\tau_0}(\cdot;]0,k[^N)$ when $\lambda\xrightarrow{\tau_0}0$. Then, we study the relative compactness in $L^p(]0,k[^N;\mathbb{R}^m)$ of minimizing sequences for $\{\mathscr{S}^\xi_\lambda(]0,k[^N)\}$. When such compactness property holds, we conclude that $\mathscr{S}^\xi_\lambda(]0,k[^N) \to \bar{\mathscr{S}}^\xi_{\tau_0}(]0,k[^N)$ as $\lambda\xrightarrow{\tau_0}0$. Next, we prove the existence of a minimizing sequence $\{\bar{w}_\lambda\}$ for $\{\mathscr{S}^{\xi,\sharp}_\lambda\}$ such that $M(\{\bar{w}_\lambda\}) < +\infty$ so that, as a consequence of Lemma 4.1, we obtain

$$\liminf_{\lambda\xrightarrow{\tau_0}0}\mathscr{S}^{\xi,\sharp}_\lambda \geqslant \inf_{k\in\mathbb{N}^*}\left\{\frac{1}{k^N}\lim_{\lambda\xrightarrow{\tau_0}0}\mathscr{S}^\xi_\lambda(]0,k[^N)\right\} = \gamma(\bar{\mathscr{S}}^\xi_{\tau_0}).$$

Finally, we apply Theorem 2.2 to obtain a homogenization result with $W^{\text{hom}}_{\tau_0}(\xi)=\gamma(\bar{\mathscr{S}}^\xi_{\tau_0})$. Furthermore, we shall see how a convexity argument permits to show that it is possible to consider only one minimization problem over a set of periodic functions, obtaining, at least formally, a formula of the type $W^{\text{hom}}_{\tau_0}(\xi)=\bar{\mathscr{S}}^{\xi,\sharp}_{\tau_0}$.

Remark 4.1. This procedure is also valid for the scalar case $m=1$ without any convexity condition because

$$\mathscr{S}_\lambda^\xi(A)=\inf\left\{\int_A W_\lambda^{**}(x,\xi+\nabla v)\,\mathrm{d}x\colon v\in W_0^{1,p}(A;\mathbb{R}^m)\right\},$$

where $W_\lambda^{**}(x,\cdot)$ denotes the lower convex envelope of $W_\lambda(x,\cdot)$, so that the cell-problem formula (6) holds (see [22]).

4.2. Iterated homogenization: equicoercive family of integrands

Let $H\subset\subset Y{:=}[0,1[^N$ be a nonempty closed subset of the unit cell with Lipschitz boundary, and let us consider two Carathéodory convex functions $f_1,f_2:\mathbb{R}^N\times\mathbb{R}^{mN}\to[0,+\infty[$ such that $r|\xi|^p\leqslant f_i(x,\xi)\leqslant R(1+|\xi|^p)$ for $i=1,2$ and for some constants $R,r>0$.

We suppose that each $f_i(\cdot,\xi)$ is Y-periodic. For all $\sigma\in]0,\sigma_0]$ and $\xi\in\mathbb{R}^{mN}$ we define

$$W_\sigma(y,\xi){:=}\begin{cases} f_1(y,\xi) & \text{if } y\in Y\setminus H,\\ f_2(\frac{y}{\sigma},\xi) & \text{if } y\in H\end{cases}\tag{9}$$

and we extend it from Y to $\mathbb{R}^N$ by Y-periodicity, obtaining $W_\sigma:\mathbb{R}^N\times\mathbb{R}^{mN}\to[0,+\infty[$. Observe that the integrand W_σ is not simply obtained by considering

$$\begin{array}{l} f_1(y,\xi) \text{ if } y\in Y\setminus H+\mathbb{Z}^N,\\ f_2(\frac{y}{\sigma},\xi) \text{ if } y\in H+\mathbb{Z}^N\end{array}$$

because this function is not periodic. In this example, the unit cell Y contains a heterogeneous inclusion H of periodic structure. Thus, in the density $W_\sigma(\frac{\cdot}{\varepsilon},\xi)$ we consider two periodicity scales: ε and $\varepsilon\sigma$. By identifying λ with σ, it is easy to see that $\{W_\sigma\}_{\sigma\in]0,\sigma_0]}$ satisfies (C_1)–(C_4) with $T=\emptyset$, $r_\sigma(x)\equiv r$ and $c_0=R/r$ (there is no lack of coercivity).

Proposition 4.1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded open set. The functional $F_{\varepsilon,\sigma}:L^p(\Omega;\mathbb{R}^m)\to[0,+\infty]$ defined by*

$$F_{\varepsilon,\sigma}(u){:=}\begin{cases}\displaystyle\int_\Omega W_\sigma(\tfrac{x}{\varepsilon},\nabla u)\,\mathrm{d}x & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^m),\\ +\infty & \text{otherwise},\end{cases}$$

$\Gamma(L^p)$-converges as $(\varepsilon,\sigma)\to(0,0)$ towards the homogenized functional

$$F^{\text{hom}}(u)=\begin{cases}\displaystyle\int_\Omega W^{\text{hom}}(\nabla u)\,\mathrm{d}x & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^m),\\ +\infty & \text{otherwise}\end{cases}$$

with convex integrand given by

$$W^{\text{hom}}(\xi)=\inf_w\left\{\int_{]0,1[^N\setminus H} f_1(x,\xi+\nabla w)\,\mathrm{d}x + \int_H f_2^{\text{hom}}(\xi+\nabla w)\,\mathrm{d}x\colon w\in W_\#^{1,p}(]0,1[^N;\mathbb{R}^m)\right\},$$

where

$$f_2^{\rm hom}(\xi'):=\inf\left\{\int_{]0,1[^N} f_2(y,\xi'+\nabla w')\,\mathrm{d}y:\ w'\in W^{1,p}_{\sharp}(]0,1[^N;\mathbb{R}^m)\right\}.$$

Proof. In order to apply Theorem 2.2, we begin by noticing that it is possible to adapt to this situation standard homogenization and lower semi-continuity methods to prove that for every $k\in\mathbb{N}^*$, the functional given by

$$G^{\xi}_{\sigma}(v;]0,k[^N):=\begin{cases}\displaystyle\int_{]0,k[^N} W_\sigma(x,\xi+\nabla v)\,\mathrm{d}x & \text{if } v\in W^{1,p}_0(]0,k[^N;\mathbb{R}^m),\\ +\infty & \text{otherwise}\end{cases}$$

$\Gamma(L^p)$-converges as $\sigma\to0^+$ towards

$$\bar{G}^{\xi}(v;]0,k[^N):=\begin{cases} I_k(\xi x+v) & \text{if } v\in W^{1,p}_0(]0,k[^N;\mathbb{R}^m),\\ +\infty & \text{otherwise,}\end{cases}$$

where

$$I_k(g)=\int_{]0,k[^N\setminus(H+\mathbb{Z}^N)} f_1(x,\nabla g)\,\mathrm{d}x+\int_{]0,k[^N\cap(H+\mathbb{Z}^N)} f_2^{\rm hom}(\nabla g)\,\mathrm{d}x.$$

This means that (H_1) holds. Moreover, if $\{v_\sigma\}_{\sigma>0}\subset W^{1,p}_0(]0,k[^N;\mathbb{R}^m)$ is a σ-minimizing sequence for $\{\mathscr{S}^{\xi}_{\sigma}(]0,k[^N)\}_{\sigma>0}$, that is

$$\int_{]0,k[^N} W_\sigma(y,\xi+\nabla v_\sigma)\,\mathrm{d}y=G^{\xi}_{\sigma}(v_\sigma;]0,k[^N)\leqslant\mathscr{S}^{\xi}_{\sigma}(]0,k[^N)+\sigma$$

then, by coerciveness, $\{v_\sigma\}_{\sigma>0}$ is relatively compact in $L^p(]0,k[^N;\mathbb{R}^m)$, and consequently

$$\lim_{\sigma\to0^+}\mathscr{S}^{\xi}_{\sigma}(]0,k[^N)=\bar{\mathscr{S}}^{\xi}(]0,k[^N).$$

On the other hand, let $\{w_\sigma\}_{\sigma>0}\subset W^{1,p}_{\sharp}(]0,1[^N;\mathbb{R}^m)$ be a σ-minimizing sequence for $\{\mathscr{S}^{\xi,\sharp}_{\sigma}\}$.

From the growth condition (C_2) with $r_\sigma(x)\equiv r>0$ and $c_0=R/r$, we obtain

$$\int_{]0,1[^N}|\nabla w_\sigma(y)|^p\,\mathrm{d}y\leqslant c_0(1+|\xi|^p)+\sigma.$$

Setting

$$\bar{w}_\sigma:=w_\sigma-\int_{]0,1[^N} w_\sigma\,\mathrm{d}y$$

we have that $\{\bar{w}_\sigma\}_{\sigma>0}$ is also a σ-minimizing sequence for $\{\mathscr{S}^{\xi,\sharp}_{\sigma}\}$ and, moreover, from the Poincaré–Wirtinger inequality we deduce that, for a suitable constant $c>0$,

$$\int_{]0,1[^N}|\bar{w}_\sigma(y)|^p\,\mathrm{d}y\leqslant c\int_{]0,1[^N}|\nabla w_\sigma(y)|^p\,\mathrm{d}y.$$

Hence

$$M(\{\bar{w}_\sigma\})=\sup_{\sigma\in]0,\sigma_0]}\left\{\int_{]0,1[^N} r|\bar{w}_\sigma(y)|^p\,\mathrm{d}y\right\}\leqslant cR(1+|\xi|^p)+cr\sigma_0.$$

Therefore, by Lemma 4.1 we obtain

$$\liminf_{\sigma\to 0^+} \mathscr{S}^{\sharp,\xi}_{\sigma} \geqslant \inf_{k\in\mathbb{N}^*} \left\{ \frac{1}{k^N} \lim_{\sigma\to 0^+} \mathscr{S}^{\xi}_{\sigma}(]0,k[^N) \right\} = \gamma(\bar{\mathscr{S}}^{\xi}).$$

Thus, (H_2) holds. Applying Theorem 2.2 yields as homogenized density

$$W^{\mathrm{hom}}(\xi) = \gamma(\bar{\mathscr{S}}^{\xi}) = \inf_{k\in\mathbb{N}^*} \inf_{w} \left\{ \frac{1}{k^N} I_k(\xi + \nabla w) \colon w \in W^{1,p}_0(]0,k[^N;\mathbb{R}^m) \right\}.$$

Since $f_1(x,\cdot)$ and f_2^{hom} are convex, we deduce [22] that indeed

$$W^{\mathrm{hom}}(\xi) = \inf_{w} \left\{ I_1(\xi + \nabla w) \colon w \in W^{1,p}_{\sharp}(]0,1[^N;\mathbb{R}^m) \right\},$$

which finishes the proof. $\square$

For similar iterated homogenization formulae see [4] and [8, Ch. 22].

4.3. *Thin and soft inclusions: nonequicoercive family of integrands*

For simplicity of exposition, we are going to consider a particular configuration. However, the following analysis remains valid for more general situations. In the sequel, we denote by $\hat{Y}$ the $(N-1)$-dimensional unit cell $[0,1[^{N-1}$ so that any element $y \in Y$ may be written $(\hat{y}, y_N)$ where $\hat{y}\in\hat{Y}$ and $y_N \in [0,1[$. Let $\Sigma \subset\subset Y$ be given by $\Sigma := \omega \times \{1/2\}$ with $\omega \subset\subset \hat{Y}$ being a smooth closed subset. Thus Σ is a closed subset of the hyperplane $\{y\in\mathbb{R}^N \colon y_N = 1/2\}$, whose constant normal vector e_N is $(0,\dots,0,1)\in\mathbb{R}^N$, the Nth vector of the canonical basis of $\mathbb{R}^N$. Setting

$$T_\eta := \{(\hat{y}, y_N) \in Y \colon \hat{y}\in\omega,\ |y_N - 1/2| \leqslant \eta/2\}$$

for every $\eta \in]0,1/2]$, and

$$T := T_{1/2} = \omega \times [1/4, 3/4]$$

we have that $\Sigma \subset T_\eta \subset T$ and, moreover, $\mathrm{dist}(T_\eta, \Sigma) = \eta/2$. Hence, T_η *shrinks* to Σ as $\eta \to 0$. Note also that $\mathscr{L}_N(T_\eta) = \eta\mathscr{H}_{N-1}(\Sigma)$, where $\mathscr{H}_{N-1}$ is the $(N-1)$-Hausdorff measure. Let us consider two convex functions $f_1 : \mathbb{R}^N\times\mathbb{R}^{mN} \to [0,+\infty[$ and $f_2 : \mathbb{R}^{mN} \to [0,+\infty[$ such that $r|\xi|^p \leqslant f_i(x,\xi) \leqslant R(1+|\xi|^p)$ for $i=1,2$ and for some constants $R, r > 0$. For each $(\delta,\eta)\in]0,+\infty[\times]0,\frac{1}{2}]$ we define $W_{\delta,\eta} : \mathbb{R}^N\times\mathbb{R}^{mN} \to [0,+\infty[$ by

$$W_{\delta,\eta}(x,\xi) := \begin{cases} f_1(x,\xi) & \text{if } x \in Y\setminus T_\eta + \mathbb{Z}^N, \\ \delta f_2(\xi) & \text{if } x\in T_\eta + \mathbb{Z}^N. \end{cases}$$

Intuitively, δ is a small parameter taking into account the relatively low stiffness of the inclusion in T_η with respect to the material in $Y\setminus T_\eta$. From now on, we write $(\delta,\eta)\xrightarrow{\mu_0}(0,0)$ for an arbitrary sequence $(\delta_n,\eta_n)\to(0,0)$ such that $\lim_{n\to\infty}\delta_n/(2\eta_n)^{p-1} = \mu_0 \in [0,+\infty]$. If $\mu_0 = 0$ then we assume in addition that $\lim_{n\to\infty}\eta_n^p/\delta_n = 0$, which clearly holds when $\mu_0 > 0$. When $\mu_0 = +\infty$, we use the convention $+\infty\times 0 = 0$.

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The sequence of functionals $F_{\varepsilon,\delta,\eta} : L^p(\Omega;\mathbb{R}^m) \to [0,+\infty]$ defined by*

$$F_{\varepsilon,\delta,\eta}(u) := \begin{cases} \int_\Omega W_{\delta,\eta}(\frac{x}{\varepsilon}, \nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise} \end{cases}$$

$\Gamma(L^p)$-converges as $(\varepsilon,\delta,\eta) \to (0,0,0)$ with $(\delta,\eta) \xrightarrow{\mu_0} (0,0)$ towards the homogenized functional

$$F^{\mathrm{hom}}(u) = \begin{cases} \int_\Omega W^{\mathrm{hom}}_{\mu_0}(\nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise} \end{cases}$$

with convex density given by

$$W^{\mathrm{hom}}_{\mu_0}(\xi) = \inf_{w \in W^{1,p}_\#(]0,1[^N \setminus \Sigma;\mathbb{R}^m)} \left\{ \int_{]0,1[^N \setminus \Sigma} f_1(y, \xi + \nabla w)\,\mathrm{d}y + \mu_0 \int_\Sigma f_2^{\infty,p}([w] \otimes e_N)\,\mathrm{d}\hat{y} \right\},$$

where $f_2^{\infty,p}$ is the recession function of order p of f_2, which is defined by

$$f_2^{\infty,p}(\xi') := \lim_{t\to\infty} \frac{1}{t^p} f_2(t\xi')$$

and $[w]$ is the jump of w on Σ, that is, $[w] := w^+|_\Sigma - w^-|_\Sigma$, where $w^\pm|_\Sigma$ denotes the trace of $w^\pm$ on Σ with $w^+ = w|_{\{y_N > 1/2\}}$ and $w^- = w|_{\{y_N < 1/2\}}$.

Proof. In this example we take $\lambda = (\delta,\eta) \in]0,1] \times]0,\frac{1}{2}]$. If we set

$$r_{\delta,\eta}(x) := \begin{cases} r & \text{if } x \in Y \setminus T_\eta + \mathbb{Z}^N, \\ r\delta & \text{if } x \in T_\eta + \mathbb{Z}^N \end{cases}$$

then it is direct to verify that $\{W_{\delta,\eta}\}$ satisfies (C_1)–(C_4). By direct application of [17, Propositions 4.1–4.3], we obtain that the functional

$$G^\xi_{\delta,\eta}(v;]0,1[^N) := \begin{cases} \int_{]0,1[^N} W_{\delta,\eta}(x, \xi + \nabla v)\,\mathrm{d}x & \text{if } v \in W^{1,p}_0(]0,1[^N;\mathbb{R}^m), \\ +\infty & \text{otherwise} \end{cases}$$

$\Gamma(L^p)$-converges as $(\delta,\eta) \xrightarrow{\mu_0} (0,0)$ towards

$$\bar{G}^\xi_{\mu_0}(v;]0,1[^N) := \begin{cases} I_{\mu_0}(\xi x + v) & \text{if } v \in W^{1,p}_0(]0,1[^N \setminus \Sigma;\mathbb{R}^m), \\ +\infty & \text{otherwise} \end{cases}$$

with

$$I_{\mu_0}(g) := \int_{]0,1[^N \setminus \Sigma} f_1(y, \nabla g)\,\mathrm{d}y + \mu_0 \int_\Sigma f_2^{\infty,p}([g] \otimes e_N)\,\mathrm{d}\hat{y}.$$

Moreover, if $\{v_{\delta,\eta}\} \subset W_0^{1,p}(]0,1[^N;\mathbb{R}^m)$ is a δ-minimizing sequence associated with the minimization of $G^{\xi}_{\delta,\eta}(\cdot;]0,1[^N)$, then $\{v_{\delta,\eta}\}$ is relatively compact in $L^p(]0,1[^N;\mathbb{R}^m)$. The $\Gamma(L^p)$-convergence of $G^{\xi}_{\delta,\eta}(v;]0,k[^N)$ for an arbitrary $k\in\mathbb{N}^*$ can be handled in much the same way. Indeed, it is possible to prove that for every $k\in\mathbb{N}^*$, the corresponding functional $G^{\xi}_{\delta,\eta}(v;]0,k[^N)$, $\Gamma(L^p)$-converges as $(\delta,\eta)\overset{\mu_0}{\to}(0,0)$ towards

$$\bar{G}^{\xi}_{\mu_0}(v;]0,k[^N):=\begin{cases} I_{\mu_0,k}(\xi x+v) & \text{if } v\in W_0^{1,p}(]0,k[^N\setminus(\Sigma+\mathbb{Z}^N);\mathbb{R}^m),\\ +\infty & \text{otherwise,}\end{cases}$$

where

$$I_{\mu_0,k}(g):=\int_{]0,k[^N\setminus(\Sigma+\mathbb{Z}^N)} f_1(x,\nabla g)\,\mathrm{d}x+\mu_0\int_{]0,k[^N\cap(\Sigma+\mathbb{Z}^N)} f_2^{\infty,p}([g]\otimes e_N)\,\mathrm{d}\hat{x}$$

and if $\{v_{\delta,\eta}\}\subset W_0^{1,p}(]0,k[^N;\mathbb{R}^m)$ is a δ-minimizing sequence for $\{\mathcal{S}^{\xi}_{\delta,\eta}(]0,k[^N)\}$, then $\{v_{\delta,\eta}\}$ is relatively compact in $L^p(]0,k[^N;\mathbb{R}^m)$. This follows by the same method as in [18]; we omit the details. Consequently, (H_1) holds and, moreover, the relative compactness of minimizing sequences ensures

$$\lim(\delta,\eta)\overset{\mu_0}{\to}(0,0)\mathcal{S}^{\xi}_{\delta,\eta}(]0,k[^N)=\bar{\mathcal{S}}^{\xi}_{\mu_0}(]0,k[^N),$$

where

$$\bar{\mathcal{S}}^{\xi}_{\mu_0}(]0,k[^N):=\inf\left\{I_{\mu_0,k}(\xi x+v)\colon v\in W_0^{1,p}(]0,k[^N\setminus(\Sigma+\mathbb{Z}^N);\mathbb{R}^m)\right\}$$

for every $k\in\mathbb{N}^*$. In order to apply Lemma 4.1, the only point remaining concerns the existence of a δ-minimizing sequence $\{\bar{w}_{\delta,\eta}\}\subset W_\#^{1,p}(]0,1[^N;\mathbb{R}^m)$ for $\{\mathcal{S}^{\xi,\#}_{\delta,\eta}\}$ such that $M(\{\bar{w}_{\delta,\eta}\})<+\infty$. Let $\{w_{\delta,\eta}\}\subset W_\#^{1,p}(]0,1[^N;\mathbb{R}^m)$ be a δ-minimizing sequence for $\{\mathcal{S}^{\xi,\#}_{\delta,\eta}\}$, and set

$$\bar{w}_{\delta,\eta}:=w_{\delta,\eta}-\int_{]0,1[^N\setminus T} w_{\delta,\eta}\,\mathrm{d}y,$$

which is also a δ-minimizing sequence for $\{\mathcal{S}^{\xi,\#}_{\delta,\eta}\}$. We can apply Lemma C.1 of Appendix C to deduce that

$$\int_{]0,1[^N\setminus T_\eta}|\bar{w}_{\delta,\eta}|^p\,\mathrm{d}y\leqslant c'\int_{]0,1[^N\setminus T_\eta}|\nabla w_{\delta,\eta}|^p\,\mathrm{d}y$$

and

$$\int_{T_\eta}|\bar{w}_{\delta,\eta}|^p\,\mathrm{d}y\leqslant c'\int_{]0,1[^N}|\nabla w_{\delta,\eta}|^p\,\mathrm{d}y,$$

where the constant $c'>0$ does not depend on η. Hence

$$\begin{aligned}\int_{]0,1[^N} r_{\delta,\eta}(y)|\bar{w}_{\delta,\eta}(y)|^p\,\mathrm{d}y &= r\int_{]0,1[^N\setminus T_\eta}|\bar{w}_{\delta,\eta}(y)|^p\,\mathrm{d}y+r\delta\int_{T_\eta}|\bar{w}_{\delta,\eta}(y)|^p\,\mathrm{d}y\\ &\leqslant c''\int_{]0,1[^N} r_{\delta,\eta}(y)|\nabla w_{\delta,\eta}(y)|^p\,\mathrm{d}y\end{aligned}$$

which together with (C_2) yield $M(\{\bar{w}_{\delta,\eta}\}) < +\infty$. Therefore we can apply Theorem 2.2 to this situation, and we obtain a homogenization result with limit density given by

$$\gamma(\bar{\mathscr{S}}^{\xi}_{\mu_0}) = \inf_{k\in\mathbb{N}^*} \inf_{w} \left\{ \frac{1}{k^N} I_{\mu_0,k}(\xi x + v)\colon v \in W_0^{1,p}(]0,k[^N \setminus (\Sigma + \mathbb{Z}^N); \mathbb{R}^m) \right\}.$$

But $f_1(x,\cdot)$ and $f_2^{\infty,p}$ are convex, which allows us to consider only periodic functions in a single minimization problem. This completes the proof. □

Remark 4.2. We may distinguish three different qualitative behaviors at the microscopic level depending on the value of μ_0. If $\mu_0 = 0$ then the stiffness of the inclusion is too low to maintain adherence and at the limit the material is free to separate and present fissures along Σ. When $\mu_0 \in]0,+\infty[$, there is an elastic restoring potential on Σ, which does not forbid fissures to appear but penalizes them. If $\mu_0 = +\infty$ then the stiffness of the inclusion is strong enough to prevent fissures; in fact, since $+\infty \times 0 = 0$ and $f_2^{\infty,p}(\xi) \geqslant r|\xi|^p$, the minimization problem defining $W^{\mathrm{hom}}_{\mu_0}(\xi)$ has $[v] = 0$ as an implicit constraint.

Remark 4.3. Taking into account [17, Proposition 5.2], it is possible to consider the case

$$W_{\delta,\eta,\sigma}(y,\xi) := \begin{cases} f_1(y,\xi) & \text{if } y \in Y \setminus T_\eta, \\ \delta f_2(\frac{y}{\sigma},\xi) & \text{if } y \in T_\eta \end{cases}$$

under the additional conditions $\sigma \approx \eta$ or $\eta/\sigma \to +\infty$ and $\mu_0 \in [0,+\infty[$, obtaining a limit density $W^{\mathrm{hom}}_{\mu_0}(\xi)$ of the form

$$\inf_{w\in W_\sharp^{1,p}(]0,1[^N\setminus\Sigma;\mathbb{R}^m)} \left\{ \int_{]0,1[^N\setminus\Sigma} f_1(y,\xi+\nabla w)\,\mathrm{d}y + \mu_0 \int_\Sigma (f_2^{\mathrm{hom}})^{\infty,p}([w]\otimes e_N)\,\mathrm{d}\hat{y} \right\}.$$

On the other hand, under suitable connectness conditions, these homogenization results are valid when $\Sigma \cap \partial Y \neq \emptyset$, that is, when in some directions the inclusions are connected throughout the whole periodic structure.

Acknowledgements

The particular homogenization problem that motivated this work (cf. Section 4.3) was suggested to the second author by Professor G. Michaille of Université de Montpellier as DEA memoir subject [19]. This was continued by the authors as part of their Ph.D.

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Appendix A

A.1. The Acerbi et al. extension theorem from periodic connected sets

In this paper we consider families of functionals $\{F_{\varepsilon,\lambda}\}$ with $\varepsilon > 0$ and $\lambda \in \mathbb{R}^k$. For abbreviation, we write s instead of (ε, λ) when no confusion can arise. Let $s_n \to 0$ and $u \in L^p(\Omega; \mathbb{R}^m)$ be such that

$$\left(\Gamma\text{-}\liminf_{n\to\infty} F_{s_n}\right)(u) < +\infty.$$

By definition, there exists a sequence $u_n \to u$ in $L^p(\Omega; \mathbb{R}^m)$ such that $\liminf_{n\to\infty} F_{s_n}(u_n) < +\infty$, hence, up to a subsequence $\sup_{n\in\mathbb{N}} \{F_{s_n}(u_n)\} < +\infty$. Suppose that $p \in]1, +\infty[$ and that the family $\{F_{\varepsilon,\lambda}\}$ satisfies the following growth condition: there exist constants $r, R > 0$ and a connected open set $E \subset \mathbb{R}^N$ such that for every $v \in W^{1,p}(\Omega; \mathbb{R}^m)$

$$r\|\nabla v\|^p_{p,\Omega\cap\varepsilon E} \leqslant F_{\varepsilon,\lambda}(v) \leqslant R(1 + \|\nabla v\|^p_{p,\Omega}). \tag{A.1}$$

Since we deal with periodic structures, we assume moreover that E is *periodic*, i.e. for all $z \in \mathbb{Z}^N$, $E = z + E$. We deduce that

$$\sup_{n\in\mathbb{N}} \left\{ \int_{\Omega\cap\varepsilon_n E} |\nabla u_n|^p \,\mathrm{d}x \right\} < +\infty.$$

If $E = \emptyset$, then a standard argument shows that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,p}$, so that u belongs to $W^{1,p}(\Omega; \mathbb{R}^m)$. When $E \neq \emptyset$, the idea is to extend u_n from $\Omega \cap \varepsilon_n E$ to the whole of Ω, keeping the above uniform boundedness property. This extension is not difficult to construct when the complement of E is disconnected (see [15]), and it is no longer possible in the general case. In fact, $\Omega \cap \varepsilon E$ may be disconnected so that we cannot expect to control the $W^{1,p}$ norm of the extended function on the whole of Ω. In Acerbi et al. [2] have considered this extension problem, and they showed that all the difficulty lies in the behavior near $\partial\Omega$. For the reader's convenience, we state without proof their precise result.

Theorem A.1 (Acerbi et al. [2]). *Let E be a periodic, connected, open subset of $\mathbb{R}^N$, with Lipschitz boundary. There exist constants $k_0, k_1, k_2 > 0$ such that for every bounded open set $\Omega \subset \mathbb{R}^N$ and $\varepsilon > 0$, there exists a linear and continuous extension operator $P_\varepsilon : W^{1,p}(\Omega \cap \varepsilon E; \mathbb{R}^m) \to W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ with:*
(i) $P_\varepsilon u = u$ *a.e. in* $\Omega \cap \varepsilon E$,
(ii) $\int_{\Omega(\varepsilon k_0)} |P_\varepsilon u|^p \,\mathrm{d}x \leqslant k_1 \int_{\Omega\cap\varepsilon E} |u|^p \,\mathrm{d}x$,
(iii) $\int_{\Omega(\varepsilon k_0)} |\nabla(P_\varepsilon u)|^p \,\mathrm{d}x \leqslant k_2 \int_{\Omega\cap\varepsilon E} |\nabla u|^p \,\mathrm{d}x$,
for every $u \in W^{1,p}(\Omega \cap \varepsilon E; \mathbb{R}^m)$, *where* $\Omega(\alpha) := \{x \in \Omega :\ \mathrm{dist}(x, \partial\Omega) > \alpha\}$.

Appendix B

B.1. An asymptotic formula for parametric subadditive set functions

Let us consider a Carathéodory function $W\colon \mathbb{R}^N \times \mathbb{R}^{mN} \to \mathbb{R}$ that satisfies

$$\forall x \in \mathbb{R}^N \quad \forall \xi \in \mathbb{R}^{mN},\ 0 \leqslant W(x,\xi) \leqslant c(1+|\xi|^p)$$

for some constants $c > 0$ and $p > 1$. For each bounded open set $A \in \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N)$, we set

$$\mathscr{S}^{\xi}(A):=\inf\left\{\int_A W(x,\xi+\nabla u)\,\mathrm{d}x\colon u \in W_0^{1,p}(A;\mathbb{R}^m)\right\}, \tag{B.1}$$

where $\xi \in \mathbb{R}^{mN}$ is fixed. We have thus defined a set function $\mathscr{S}^{\xi}\colon \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N) \to [0,+\infty[$, which satisfies

$$0 \leqslant \mathscr{S}^{\xi}(A) \leqslant c(1+|\xi|^p)\mathscr{L}_N(A).$$

Definition B.1. Let $\mathscr{S}: \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N) \to [0,+\infty[$ be a set function.

(i) $\mathscr{S}$ is called *subadditive* if for every $A,B,C \in \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N)$ with $B \subset A$, $C \subset A$, $B \cap C = \emptyset$ and $\mathscr{L}_N(A \setminus (B \cup C)) = 0$, then $\mathscr{S}(A) \leqslant \mathscr{S}(B) + \mathscr{S}(C)$.

(ii) If for every $z \in \mathbb{Z}^N$ and $A \in \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N)$, $\mathscr{S}(z+A) = \mathscr{S}(A)$, then $\mathscr{S}$ is said to be $\mathbb{Z}^N$*-invariant*.

It is easy to see that the set function $\mathscr{S}^{\xi}$ defined by (B.1) is subadditive. If we suppose in addition that W is $[0,1[^N$-periodic with respect to the first variable, then $\mathscr{S}^{\xi}$ is $\mathbb{Z}^N$-invariant.

As a basic tool of some *localization methods*, the function $\mathscr{S}^{\xi}$ have been used to characterize the limit densities in Γ-convergence problems. In the case of homogenization, one is led to study the asymptotic behavior as $\varepsilon \to 0$ of

$$\frac{1}{\mathscr{L}_N(Q)}\inf\left\{\int_Q W\left(\frac{x}{\varepsilon},\nabla u\right)\,\mathrm{d}x\colon u \in \xi x + W_0^{1,p}(Q;\mathbb{R}^m)\right\} = \frac{\mathscr{S}^{\xi}((1/\varepsilon)Q)}{\mathscr{L}_N((1/\varepsilon)Q)},$$

where Q is an open cube of $\mathbb{R}^N$ and we denote by $\mathscr{L}_N(Q)$ the Lebesgue measure of Q.

Given a subadditive and $\mathbb{Z}^N$-invariant set function $\mathscr{S}: \mathscr{U}_{\mathrm{b}}(\mathbb{R}^N) \to [0,+\infty[$, define

$$\gamma(\mathscr{S}):=\inf_{k\in\mathbb{N}^*}\left\{\frac{1}{k^N}\mathscr{S}(]0,k[^N)\right\}.$$

It is not difficult to show that

$$\lim_{\varepsilon\to 0+}\frac{\mathscr{S}((1/\varepsilon)Q)}{\mathscr{L}_N((1/\varepsilon)Q)} = \gamma(\mathscr{S}), \tag{B.2}$$

for every $Q \in \mathit{Cub}(\mathbb{R}^N)$ where we denote by $\mathit{Cub}(\mathbb{R}^N)$ the class of all open cubes in $\mathbb{R}^N$. This fact has been used in some homogenization problems to identify the limit homogenized density $W^{\mathrm{hom}}(\xi)$ with $\gamma(\mathscr{S}^{\xi})$. A generalization of this result to the stochastic setting (see [16] and references therein) has been used to deal with the homogenization of stochastic functionals; see, for instance [1,11,21]. For other results in this connection see [18].

In the parametric case that is considered in this paper, one deals with a *parametric family* $\{\mathscr{S}_\lambda^\xi\}$ of subadditive and $\mathbb{Z}^N$-invariant set functions, which is associated with $\{W_\lambda\}$. We are going to give a simple generalization of the asymptotic formula (12) for such parametric set functions. More precisely, we consider a family $\{\mathscr{S}_\lambda\}$ of set functions defined on $\mathscr{U}_\mathrm{b}(\mathbb{R}^N)$ that satisfies:

(S_1) For each $\lambda \in \Lambda$, $\mathscr{S}_\lambda$ is subadditive and $\mathbb{Z}^N$-invariant.

(S_2) There exists $c > 0$ such that $0 \leqslant \mathscr{S}_\lambda(A) \leqslant c\mathscr{L}_N(A)$, for every $A \in \mathscr{U}_\mathrm{b}(\mathbb{R}^N)$ and for every $\lambda \in \Lambda$.

(S_3) Let $\{a_k\} \subset \mathbb{R}$ be such that for each $k \in \mathbb{N}^*$, $\limsup_{\lambda \xrightarrow{\tau_0} 0} \mathscr{S}_\lambda(]0,k[^N) \leqslant a_k$.

Lemma B.1. *If* (S_1)–(S_3) *hold then for each* $Q \in cub(\mathbb{R}^N)$ *we have that*:

(a) $\limsup\limits_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \frac{\mathscr{S}_\lambda((1/\varepsilon)Q)}{\mathscr{L}_N((1/\varepsilon)Q)} \leqslant \inf\limits_{k\in\mathbb{N}^*} \left\{\frac{a_k}{k^N}\right\}$;

(b) $\liminf\limits_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \frac{\mathscr{S}_\lambda((1/\varepsilon)Q)}{\mathscr{L}_N((1/\varepsilon)Q)} \geqslant \liminf\limits_{\lambda \xrightarrow{\tau_0} 0} \gamma(\mathscr{S}_\lambda)$.

In particular, *if*

$$\liminf_{\lambda \xrightarrow{\tau_0} 0} \gamma(\mathscr{S}_\lambda) \geqslant \inf_{k\in\mathbb{N}^*} \left\{\frac{a_k}{k^N}\right\} \tag{B.3}$$

then

$$\lim_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}} \frac{\mathscr{S}_\lambda((1/\varepsilon)Q)}{\mathscr{L}_N((1/\varepsilon)Q)} = \inf_{k\in\mathbb{N}^*} \left\{\frac{a_k}{k^N}\right\}.$$

Remark B.1. In the nonparametric case $\mathscr{S}_\lambda \equiv \mathscr{S}$, we can take $a_k = \mathscr{S}(]0,k[^N)$, for which (B.3) holds so that we obtain (B.2) as a corollary.

Remark B.2. It is easy to see that under (S_3)

$$\limsup_{\lambda \xrightarrow{\tau_0} 0} \gamma(\mathscr{S}_\lambda) \leqslant \inf_{k\in\mathbb{N}^*} \left\{\frac{a_k}{k^N}\right\}.$$

Thus, without loss of generality we can replace (B.3) by $\lim_{\lambda \xrightarrow{\tau_0} 0} \gamma(\mathscr{S}_\lambda) = \inf_{k\in\mathbb{N}^*}\{a_k/k^N\}$.

Proof of Lemma B.1. For (a), fix $Q \in cub(\mathbb{R}^N)$. It is easy to see that for every $k \in \mathbb{N}^*$ and $\varepsilon > 0$ small enough, there exist $k_\varepsilon \in \mathbb{N}^*$ and $z_\varepsilon \in \mathbb{Z}^N$ such that

$$(k_\varepsilon - 2)]0,k[^N + k(z_\varepsilon + \hat{e}) \subset \frac{1}{\varepsilon}Q \subset k_\varepsilon]0,k[^N + kz_\varepsilon$$

where $\hat{e} := (1,1,\ldots,1)$. By subadditivity and $\mathbb{Z}^N$-invariance, we have

$$\mathscr{S}_\lambda\left(\frac{1}{\varepsilon}Q\right) \leqslant (k_\varepsilon - 2)^N \mathscr{S}_\lambda(]0,k[^N) + \mathscr{S}_\lambda\left(\frac{1}{\varepsilon}Q \setminus [(k_\varepsilon - 2)[0,k]^N + k(z_\varepsilon + \hat{e})]\right).$$

Since, up to a set of zero Lebesgue measure, the set $(1/\varepsilon)Q\setminus[(k_\varepsilon-2)[0,k]^N+k(z_\varepsilon+\hat{e})]$ may be written as the disjoint union of $k_\varepsilon^N-(k_\varepsilon-2)^N$ integer translations of open sets contained in $]0,k[^N$, we deduce that

$$\mathscr{S}_\lambda\left(\frac{1}{\varepsilon}Q\right)\leqslant(k_\varepsilon-2)^N\mathscr{S}_\lambda(]0,k[^N)+(k_\varepsilon^N-(k_\varepsilon-2)^N)ck^N.$$

We thus obtain the estimate

$$\frac{\mathscr{S}_\lambda((1/\varepsilon)Q)}{\mathscr{L}_N((1/\varepsilon)Q)}\leqslant\frac{\mathscr{S}_\lambda(]0,k[^N)}{k^N}+\frac{k_\varepsilon^N-(k_\varepsilon-2)^N}{(k_\varepsilon-2)^N}c.$$

Since $k_\varepsilon\to\infty$ as $\varepsilon\to 0$, we deduce that $\limsup_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}}\mathscr{S}_\lambda((1/\varepsilon)Q)/\mathscr{L}_N((1/\varepsilon)Q)\leqslant\frac{a_k}{k^N}$, for every $k\in\mathbb{N}^*$ as we claimed.

To prove (b), we fix $Q\in Cub(\mathbb{R}^N)$ and write for suitable $k_\varepsilon\in\mathbb{N}^*$ and $z_\varepsilon\in\mathbb{Z}^N$

$$\leftrightarrow(k_\varepsilon-2)]0,1[^N+z_\varepsilon+\hat{e}\subset\frac{1}{\varepsilon}Q\subset\,]0,k_\varepsilon[^N+z_\varepsilon.$$

We have

$$\mathscr{S}_\lambda(]0,k_\varepsilon[^N)\leqslant\mathscr{S}_\lambda\left(\frac{1}{\varepsilon}Q\right)+\mathscr{S}_\lambda\left((]0,k_\varepsilon[^N+z_\varepsilon)\setminus\frac{1}{\varepsilon}\bar{Q}\right).$$

Similarly to (b), we can deduce that

$$\gamma(\mathscr{S}_\lambda)\leqslant\frac{\mathscr{S}_\lambda(]0,k_\varepsilon[^N)}{k_\varepsilon^N}\leqslant\frac{\mathscr{S}_\lambda((1/\varepsilon)Q)}{\mathscr{L}_N((1/\varepsilon)Q)}+\frac{k_\varepsilon^N-(k_\varepsilon-2)^N}{(k_\varepsilon)^N}c.$$

Hence

$$\liminf_{\lambda\xrightarrow{\tau_0}0}\gamma(\mathscr{S}_\lambda)\leqslant\liminf_{\substack{(\varepsilon,\lambda)\to(0,0)\\ \lambda\in\tau_0}}\frac{\mathscr{S}_\lambda((1/\varepsilon)Q)}{\mathscr{L}_N((1/\varepsilon)Q)},$$

which completes the proof. □

Appendix C

C.1. A technical lemma

We denote by $\hat{Y}$ the unit cell $[0,1[^{N-1}$. An element $y\in Y$ is denoted by $(\hat{y},y_N)$ where $\hat{y}\in\hat{Y}$ and $y_N\in[0,1[$. Given $\alpha,\beta\in\,]0,\frac{1}{2}[$, let us consider an open set $\omega\subset\subset\hat{Y}$ with Lipschitz boundary and a Lipschitz function $\theta:\omega\to\,]\alpha,1-\beta[\,\subset\,]0,1[$. The set

$$\Sigma:=\{(\hat{y},\theta(\hat{y})):\ \hat{y}\in\omega\}$$

is then a Lipschitz manifold of dimension $N-1$. Fix $\eta_0\in\,]0,\min\{\alpha,1-\beta\}]$ and for every $\eta\in\,]0,\eta_0]$, define the set

$$T_\eta:=\left\{(\hat{y},y_N):\ \hat{y}\in\omega,\ y_N\in\left[\theta(\hat{y})-\frac{\eta}{2},\theta(\hat{y})+\frac{\eta}{2}\right]\right\}.$$

Note that in the example of Section 4.3 we have $\theta\equiv\frac{1}{2}$ and $\eta_0=1/2$.

Lemma C.1. *If* $w \in W_\sharp^{1,p}(]0,1[^N;\mathbb{R}^m)$ *satisfies* $\int_{]0,1[^N\setminus T_{\eta_0}} w\,\mathrm{d}y = 0$, *then there exists a constant* $c > 0$ *such that for every* $\eta \in]0,\eta_0]$ *we have that*

$$\text{(i)}\quad \int_{]0,1[^N\setminus T_\eta} |w|^p\,\mathrm{d}y \leqslant c\int_{]0,1[^N\setminus T_\eta} |\nabla w|^p\,\mathrm{d}y$$

and

$$\text{(ii)}\quad \int_{T_\eta} |w|^p\,\mathrm{d}y \leqslant c\int_{]0,1[^N} |\nabla w|^p\,\mathrm{d}y.$$

Proof. First, let us define $\Theta_\eta^+,\Theta_\eta^-: \hat{Y} \to \mathbb{R}$ by

$$\Theta_\eta^+(\hat{y}) := \begin{cases} \theta(\hat{y}) + \frac{\eta}{2} & \text{if } \hat{y}\in\omega, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

and

$$\Theta_\eta^-(\hat{y}) := \begin{cases} \theta(\hat{y}) - \frac{\eta}{2} & \text{if } \hat{y}\in\omega, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Set $Y_\eta^+ := \{(\hat{y},y_N)\in]0,1[^N: y_N > \Theta_\eta^+(\hat{y})\}$ and $Y_\eta^- := \{(\hat{y},y_N)\in]0,1[^N: y_N < \Theta_\eta^-(\hat{y})\}$, so that $]0,1[^N \setminus T_\eta = Y_\eta^+ \cup Y_\eta^- \cup \{(\hat{y},\frac{1}{2})\in]0,1[^N: \hat{y}\in\hat{Y}\setminus\omega\}$. We first prove

$$\int_{]0,1[^{N-1}} |w(\hat{y},\Theta_\eta^+(\hat{y}))|^p\,\mathrm{d}\hat{y} \leqslant c'\int_{]0,1[^N\setminus T_\eta} |\nabla w|^p\,\mathrm{d}y, \tag{C.1}$$

where c' is a constant which depends on $\alpha,\beta,\omega,\theta$ and p. Let $V,V' \subset\subset \mathbb{R}^{N-1}\times]0,1[$ be two open neighborhoods of T such that $V \subset\subset V'$. Consider a cut-off function φ between V and V', i.e., $\varphi\in D(V')$, $0\leqslant\varphi\leqslant 1$ and $\varphi\equiv 1$ on V. Setting $u := \varphi w$, we have that

$$-w(\hat{y},\Theta_\eta^+(\hat{y})) = \int_{\Theta_\eta^+(\hat{y})}^1 \frac{\partial u}{\partial y_N}(\hat{y},t)\,\mathrm{d}t$$

for $\mathscr{L}_{N-1}$-almost every $\hat{y}\in]0,1[^{N-1}$. By the Hölder inequality we deduce that

$$|w(\hat{y},\Theta_\eta^+(\hat{y}))|^p \leqslant \int_{\Theta_\eta^+(\hat{y})}^1 \left|\frac{\partial u}{\partial y_N}(\hat{y},t)\right|^p\,\mathrm{d}t,$$

and the Fubini theorem yields

$$\int_{]0,1[^{N-1}} |w(\hat{y},\Theta_\eta^+(\hat{y}))|^p\,\mathrm{d}\hat{y} \leqslant \int_{]0,1[^{N-1}}\left(\int_{\Theta_\eta^+(\hat{y})}^1 \left|\frac{\partial u}{\partial y_N}(\hat{y},t)\right|^p\,\mathrm{d}t\right)\mathrm{d}\hat{y}$$

$$= \int_{Y_\eta^+}\left|\frac{\partial u}{\partial y_N}\right|^p\,\mathrm{d}y.$$

But

$$\int_{Y_\eta^+}\left|\frac{\partial u}{\partial y_N}\right|^p\,\mathrm{d}y \leqslant \int_{]0,1[^N\setminus T_\eta} |\nabla u|^p\,\mathrm{d}y = \int_{]0,1[^N\setminus T} |\nabla u|^p\,\mathrm{d}y + \int_{T\setminus T_\eta} |\nabla w|^p\,\mathrm{d}y$$

$$\leqslant 2^p\|\nabla\varphi\|_\infty^p \int_{]0,1[^N\setminus T} |w|^p\,\mathrm{d}y + (2^p+1)\int_{]0,1[^N\setminus T_\eta} |\nabla w|^p\,\mathrm{d}y.$$

Hence, by the Poincaré–Wirtinger inequality

$$\int_{]0,1[^{N-1}} |w(\hat{y},\Theta_\eta^+(\hat{y}))|^p \,\mathrm{d}\hat{y} \leqslant [2^p(c\|\nabla\varphi\|_\infty^p+1)+1]\int_{]0,1[^N\setminus T_\eta} |\nabla w|^p \,\mathrm{d}y,$$

where the constant c depends on $\alpha,\beta,\omega,\theta$ and p. By similar arguments we obtain the inequality

$$\int_{]0,1[^{N-1}} |w(\hat{y},\Theta_\eta^-(\hat{y}))|^p \,\mathrm{d}\hat{y} \leqslant c'\int_{]0,1[^N\setminus T_\eta} |\nabla w|^p \,\mathrm{d}y.$$

To prove (i) we write

$$\int_{]0,1[^N\setminus T_\eta} |w|^p \,\mathrm{d}y = \int_{Y_\eta^+} |\nabla w|^p \,\mathrm{d}y + \int_{Y_\eta^-} |\nabla w|^p \,\mathrm{d}y. \tag{C.2}$$

We have

$$w(\hat{y},y_N) = w(\hat{y},\Theta_\eta^+(\hat{y})) + \int_{\Theta_\eta^+(\hat{y})}^{y_N} \frac{\partial w}{\partial y_N}(\hat{y},t)\,\mathrm{d}t$$

for $\mathscr{L}_{N-1}$-almost every $\hat{y}\in]0,1[^{N-1}$ and $\mathscr{L}_1$-almost every $y_N\in]\Theta_\eta^+(\hat{y}),1[$. Using the Hölder inequality we infer that

$$|w(\hat{y},y_N)|^p \leqslant |w(\hat{y},\Theta_\eta^+(\hat{y}))|^p + \int_{\Theta_\eta^+(\hat{y})}^{1} \left|\frac{\partial w}{\partial y_N}(\hat{y},t)\right|^p \,\mathrm{d}t$$

hence that

$$\int_{\Theta_\eta^+(\hat{y})}^{1} |w(\hat{y},t)|^p \,\mathrm{d}t \leqslant |w(\hat{y},\Theta_\eta^+(\hat{y}))|^p + \int_{\Theta_\eta^+(\hat{y})}^{1} \left|\frac{\partial w}{\partial y_N}(\hat{y},t)\right|^p \,\mathrm{d}t$$

for $\mathscr{L}_{N-1}$-almost every $\hat{y}\in]0,1[^{N-1}$. From (C.1) and the Fubini theorem, it follows that

$$\int_{Y_\eta^+} |w|^p \,\mathrm{d}y \leqslant (c'+1)\int_{]0,1[^N\setminus T_\eta} |\nabla w|^p \,\mathrm{d}y.$$

A similar inequality holds for the second term in the right-hand side of (C.2). Thus (i) follows.

To establish (ii), we apply again the same arguments to obtain

$$\begin{aligned}\int_\omega \left(\int_{\Theta_\eta^-(\hat{y})}^{\Theta_\eta^+(\hat{y})} |w(\hat{y},t)|^p \,\mathrm{d}t\right)\mathrm{d}\hat{y} &\leqslant \int_\omega |w(\hat{y},\Theta_\eta^+(\hat{y}))|^p \,\mathrm{d}\hat{y}\\ &\quad + \int_\omega \left(\int_{\Theta_\eta^-(\hat{y})}^{\Theta_\eta^+(\hat{y})} \left|\frac{\partial w}{\partial y_n}(\hat{y},t)\right|^p \,\mathrm{d}t\right)\mathrm{d}\hat{y}\end{aligned}$$

and finally

$$\int_{T_\eta} |w|^p \,\mathrm{d}y \leqslant \max\{c',1\}\int_{]0,1[^N} |\nabla w|^p \,\mathrm{d}y,$$

which finishes the proof. □

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