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Minimization of the expected compliance as an alternative approach to multiload truss optimization*

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Key words Topology optimization, structural mechanics, multiload compliance minimization, stochastic loading, expected compliance.
Abstract We show that a problem of finding the truss of minimimum expected compliance under stochastic loading conditions is equivalent to the dual of a special convex minimax problem, and therefore may be efficiently solved. This equivalence makes it possible to provide classic multiload compliance minimization problems with interpretations in a probabilistic setting. In fact, we prove that minimizing the expected compliance amounts to solving a multiload like problem associated with a particular finite set of loading scenarios, which depend on the mean and the variance of the perturbations.

1 Introduction

Trusses are designed to support some external nodal as well as self-weight loads, taking into account certain mechanical properties of bar material. Following Ben-Tal *et al.* (1993), Achtziger (1997, §4) and Bendsøe and Sigmund (2003), we focus on the case where the goal is to find the nodal positions (geometry) and bar volumes (topology) which minimize the *compliance* of the truss under mechanical equilibrium and total volume constraints; see problem (3) in §2.1. This design problem is very difficult in general due to nonlinearities caused by geometric variables, and can be simplified by using the so-called *ground structure approach* (Dorn *et al.* 1964): nodal positions are fixed in the reference configuration, and the optimal truss structure is found by considering only bar volumes as design variables. The effect of geometric variables is simulated by a mesh full of nodes and bars, which is referred to as the ground structure. Typically, a large number of potential bar volumes vanish at the optimum.

In this framework, the issue is how to design optimal truss structures that are "robust" under nodal load perturbations (Ben-Tal and Nemirovski 1997). Introducing a *multiload* model is one alternative to deal with the instability of optimal solutions. In its standard form, it consists in minimizing a weighted average of compliances associated with a finite set of loading scenarios; see problem (5) in §2.2 and Achtziger *et al.* (1992). The goal of this paper is to provide a new stochastic basis for the multiload model, extending and improving previous results of the authors (Alvarez 1996; Carrasco 2003). In fact, assuming that nodal loads are subjected to possibly continuous random perturbations, we show that minimizing the *expected compliance* yields to a multiload-like problem in which the loading scenarios have a new interpretation.

The paper is organized as follows. First, in §2 we introduce some notations and definitions, we recall the standard single load and multiload problems, and we describe some properties of optimal solutions. In §3 we assume that the nodal load is a random variable, and we introduce the problem of minimizing the expected compliance. Then, we give explicit analytic formulations of this model depending of whether the random variable is discrete or continuous. Indeed, the main result of this paper is Theorem 1 (see page 4), which states that the minimum expected compliance problem associated with a continuous random perturbation on the nodal load is equivalent to a special multiload-like problem. Next, we give some numerical illustration and we finish the section with a mention on a variant of the design problem including the variance of the compliance. In Appendix A we use perturbation theory (Rockafellar 1974; Rockafellar and Wets 1998) to give an alternative proof of a well-known primal-dual formulation of the design problems considered here. As a by-product, we obtain that the perturbed compliance function Ψ defined in §3 is

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measurable so that the minimum expected compliance problem is well defined. Finally, in Appendix B we provide proofs for some auxiliary results used in the proof of Theorem 1.

Of course, multiload models are just one alternative among others to deal with mechanical instability. In *worst case design* the objective is to minimize the maximum compliance under a set of discrete loading scenarios (Achtziger 1997, 1998,§3). In the same direction, the *ellipsoid method* considers a continuum of primary and secondary loads defined by a particular ellipsoid (Ben-Tal and Nemirovski 1997). On the other hand, *multilevel stochastic programming* problems have been proposed in this context (Marti and Stöckl 1999; Evgrafov *et al.* 2003). In the latter approach, approximation and discretization schemes are usually required in order to estimate expected values. Since we obtain explicit multiload-like formulations for the minimum expected compliance problem considered in this paper, approximation techniques are not needed at all for numerical computations, a key difference with other stochastic models.

Finally, let us mention that other volume as well as buckling constraints can also be considered in the classic setting (see for instance Achtziger 1997; Ben-Tal *et al.* 2000). In this paper, the techniques are bounded to structures with linear response, which for example does not allow to consider trusses with cables and/or in frictionless or frictional contact with obstacles. Also, the convexity of the design problem is essential for the primal-dual formulation, which does not allow to apply the method to structural optimization problems with non-convex performance functional and/or non-convex constraints.

2 Standard minimum compliance truss design

2.1 Brief description of the single load model

By a *truss* we mean a mechanical structure consisting of an ensemble of cylindrical slender bars, connecting some pairs of nodal points in $\mathbb{R}^d$ with either $d=2$ or $d=3$. Bars are supposed to be made of a linearly elastic, isotropic and homogeneous material; long bars overlapping small ones are neglected. Let $n = d\cdot N - s$ be the number of degrees of freedom of a ground structure consisting of $N \geq 2$ nodes, where $s \geq 0$ is the number of fixed nodal coordinate directions (i.e., coordinates corresponding to support conditions are removed). Let $m \geq n$ be the number of potential bars in the truss structure (of course, $m \leq N(N-1)/2$), and denote by $t_i \geq 0$ the volume of the i-th bar with $i \in \{1,\dots,m\}$. Assume that external loads apply only at nodal points, and are described in global reduced coordinates by a vector $f \in \mathbb{R}^n$. Under the assumption that each bar is subjected to only axial tension or compression (neglecting thus large deflections and bending effects), the mechanical response of the truss is described by the elastic equilibrium equation $K(t)u = f$, where $u \in \mathbb{R}^n$ is the nodal displacements vector in global reduced coordinates and $K(t)$ is the *stiffness matrix* of the truss, which has the form

$$K(t) = \sum_{i=1}^{m} t_i K_i. \tag{1}$$

Here, $t \geq 0$ is the member volume vector and $K_i \in \mathbb{R}^{n\times n}$ is the specific stiffness matrix of the i-th bar, that is, $K_i = \frac{E_i}{\ell_i^2}\gamma_i\gamma_i^T$, where E_i is the Young modulus, $\ell_i > 0$ is the length and $\gamma_i \in \mathbb{R}^n$ is a cosines/sines vector describing the orientation of bar number i. From now on, we assume that the ground structure satisfies

$$span\{\gamma_1,\dots,\gamma_m\} = \mathbb{R}^n, \tag{2}$$

which precludes the consideration of mechanisms and rigid body motions.

Remark 1 Every K_i is a dyadic matrix formed from vector γ_i so that $K(t)$ is positive semi-definite for every volume vector $t \geq 0$. Under (2), if $t > 0$ then $K(t)$ is positive definite. □

In order to take into account the self-weight of the structure, under the standard assumption that the weight of a bar is carried equally by the joints at its ends, the equilibrium condition becomes $K(t)u = g(t) + f$, where $g(t) = \sum_{i=1}^m t_i g_i$, with $g_i \in \mathbb{R}^n$ being the specific nodal gravitational force vector due to the i-th bar (see for instance Bendsøe and Sigmund (2003,§4.1.3)). The problem of finding the minimum *compliance* truss for a given volume $V > 0$ of material is given by

$$\min_{t\in\mathbb{R}^m} \{ \tfrac{1}{2}(g(t)+f)^T u \mid K(t)u = g(t)+f,\ u\in\mathbb{R}^n, \\ \textstyle\sum_{i=1}^m t_i = V,\ t_i \geq 0,\ i=1,\dots,m\} \tag{3}$$

Let us denote by $\varDelta_m$ the simplex $\{\lambda \in \mathbb{R}^m \mid \lambda \geq 0,\ \sum_{i=1}^m \lambda_i = 1\}$. Equivalently, taking $t = V\lambda$ with $\lambda \in \varDelta_m$, and replacing K_i and g_i with VK_i and Vg_i respectively, (3) can be written as

$$\min_{\lambda\in\varDelta_m} \left\{ \tfrac{1}{2}(g(\lambda)+f)^T u \mid K(\lambda)u = g(\lambda)+f, u\in\mathbb{R}^n \right\} \tag{4}$$

Remark that compliance, i.e., the value of the objective function in (4), does not depend on the choice of the equilibrium displacement vector $u \in \mathbb{R}^n$ such that $K(\lambda)u = g(\lambda)+f$; indeed, for any such a vector u we have $-\frac{1}{2}(g(\lambda)+f)^T u = \frac{1}{2}u^T K(\lambda)u - (g(\lambda)+f)^T u = \min_{x\in\mathbb{R}^n} \left\{ \frac{1}{2}x^T K(\lambda)x - (g(\lambda)+f)^T x \right\}$.

Remark 2 It is well known that a nice application of convex duality theory permits to reformulate (4) as a quadratic minimax problem in displacements only (Achtziger *et al.* 1992; Ben-Tal and Bendsøe 1993; Achtziger 1997); see Appendix A for a new proof of this property.

Efficient algorithms have been proposed for these problems (Jarre *et al.* 1993; Ben-Tal and Zibulevsky 1997). Numerical results using a single load model show that optimal solutions may be unstable, even under small perturbations in the principal load (Achtziger 1997; Ben-Tal and Nemirovski 1997). In fact, there are several examples that show some optimal structures giving infinite compliance under small perturbations. An alternative to deal with this drawback consists in considering the so called *multiload model*, which we recall briefly in the next section.

2.2 Instabilities and the standard multiload model

The design problem (4) may produce unsatisfactory results in respect to mechanical stability.

Example 1 Fig. 1 illustrates through a very simple example that the optimal substructure can be unstable (see Ben-Tal and Nemirovski 1997). Nodes 1 and 4 are fixed, while the main load is distributed on nodes 2, 3, 5, and 6 as we show in the image on the left. The resulting optimal truss collapses under a small vertical perturbation applied on node 6.

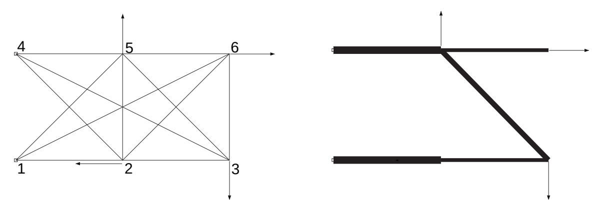

Fig. 1 Toy example: ground structure and unstable optimal solution.

A natural idea to handle this inconvenience is to consider a *multiload model* instead of the single load one, by minimizing a weighted average of the compliances associated with k different loading scenarios (see Ben-Tal and Bendsøe (1993)):

$$\min_{\lambda\in\Delta_m}\{\tfrac{1}{2}\sum_{j=1}^{k}\alpha_j(g(\lambda)+f^j)^T u^j | K(\lambda)u^j = g(\lambda)+f^j,\quad j=1,...,k\} \tag{5}$$

The weight factors $\alpha_j > 0$, $j = 1,\ldots,k$, are supposed to be given. Defining $\hat{K}(\lambda) = \sum_{i=1}^m \lambda_i \hat{K}_i$ with $\hat{K}_i = \mathrm{diag}(\alpha_1 K_i,\ldots,\alpha_k K_i)_{nk\times nk}$, $\hat{f} = (\alpha_1 f^{1T},\ldots,\alpha_k f^{kT})^T$, $\hat{g} = (\alpha_1 g^T,\ldots,\alpha_k g^T)^T$, (5) can be rewritten as

$$\min_{\lambda\in\Delta_m}\left\{\tfrac{1}{2}(\hat{g}(\lambda)+\hat{f})^T\hat{u} | \hat{K}(\lambda)\hat{u} = g(\hat{\lambda})+\hat{f}, \hat{u}\in\mathbb{R}^{kn}\right\} \tag{6}$$

Although this problem is analogous to (4), the increase of the dimension is evident. Remark also the loss of the dyadic structure in $\hat{K}_i$, which precludes the reduction of (6) to a linear programming problem as in Achtziger *et al.* (1992).

The stability of the solution to (5) is partially explained by the following result.

Lemma 1 *Let C be such that $C \geq \frac{1}{2}\max_j f^{jT}u^j$ with $\lambda \in \Delta_m$ and $K(\lambda)u^j = f^j$, $j = 1,\ldots,k$. Then for each $f \in \mathrm{Conv}\{f^1,\ldots,f^k,-f^1,\ldots,-f^k\}$*[1]*, there exists $u \in \mathbb{R}^n$ satisfying $K(\lambda)u = f$ and moreover $C \geq \frac{1}{2}f^T u$.*

Proof of Lemma 1. Suppose $f \in \mathrm{Conv}\{f^1,\ldots,f^k\}$, the general case is similar. Then write $f = \sum_{j=1}^k \mu_j f^j$ where $\mu_j \geq 0$ and $\sum_{j=1}^k \mu_j = 1$. Taking $u = \sum_{j=1}^k \mu_j u^j$, yields $K(\lambda)u = f$. We have $\frac{1}{2}f^T u = f^T u - \frac{1}{2}u^T K(\lambda)u = \sum_{j=1}^k \mu_j(f^{jT}u - \frac{1}{2}u^T K(\lambda)u) \leq \sum_{j=1}^k \mu_j \max_x\{f^{jT}x - \frac{1}{2}x^T K(\lambda)x\} = \sum_{j=1}^k \mu_j \frac{1}{2}f^{jT}u^j \leq C$, which proves the result. □

In virtue of Lemma 1, if $\hat{\lambda}$ solves (5) then the corresponding structure supports not only the k different loading scenarios $\{g(\hat{\lambda})+f^j\}_{j=1}^k$, but the constant $\hat{C} = \frac{1}{2}\max_j(g(\hat{\lambda})+f^j)^T u^j$ prescribes an a priori upper bound on the compliance associated to any load in the convex hull of them. This suggests that in order to obtain robust structures it is not necessary to consider explicitly all loading scenarios but a good representation of them. As we will see in the next section, this property holds true even in a stochastic setting when continuous random perturbations on the loads are considered and the objective is to minimize the expected compliance (see Theorem 1).

3 Random perturbations: minimizing the expected compliance

3.1 General framework

Under the setting of §2.1, let $\xi \in \mathbb{R}^n$ be a perturbation on the nodal load vector $f \in \mathbb{R}^n$ and consider

$$\Psi(\xi,\lambda) = \begin{cases} \frac{1}{2}(g(\lambda)+f+\xi)^T u & \text{if } \lambda\in\Delta_m \text{ and } \\ & \exists u\in\mathbb{R}^n \text{ s.t.} \\ & K(\lambda)u = g(\lambda) \\ & +f+\xi. \\ +\infty & \text{otherwise.}\end{cases} \tag{7}$$

The function $\Psi\colon \mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$ results to be proper (i.e. $\Psi \not\equiv +\infty$), lower semi-continuous and convex (see Lemma 4 in Appendix A). Therefore, for each $\lambda \in \Delta_m$, the function

$$\Psi(\cdot,\lambda)\colon\ \big(\mathbb{R}^n,\mathcal{B}(\mathbb{R}^n)\big)\to\big(\mathbb{R}\cup\{+\infty\},\overline{\mathcal{B}}(\mathbb{R})\big)$$

[1] Here, Conv A stands for the convex hull of the set A. $\mathrm{Conv}\,A = \{x\in\mathbb{R}^n \mid \sum\mu_i x_i,\ x_i\in A,\ \mu_i\geq 0, \sum\mu_i = 1\}$

is l.s.c. and convex, hence it is measurable. Here, $\mathcal{B}(\mathbb{R}^n)$ and $\overline{\mathcal{B}}(\mathbb{R})$ stand for the Borel σ-algebra of $\mathbb{R}^n$ and $\mathbb{R} \cup \{+\infty\}$ respectively.

Next, assume that ξ in $\Psi(\xi, \lambda)$ is a random variable corresponding to an uncertain nodal load perturbation. More precisely, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and consider a measurable function

$$\begin{aligned} \xi\colon\ (\Omega, \mathcal{A}) &\to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \\ \omega &\mapsto \quad \xi(\omega). \end{aligned}$$

Given $A \in \mathcal{B}(\mathbb{R}^n)$, the probability that ξ belongs to A is by definition $\mathbb{P}\{\xi \in A\} := \mathbb{P}\{\omega \in \Omega \mid \xi(\omega) \in A\}$.

As a natural generalization of the standard multiple load model recalled in §2.2, we introduce the *minimum expected compliance design problem* with respect to $\mathbb{P}$ as

$$\min_{\lambda \in \Delta_m} \mathbb{E}_\xi[\Psi(\xi, \lambda)], \tag{$\mathcal{D};\mathbb{P}$}$$

where

$$\mathbb{E}_\xi[\Psi(\xi, \lambda)] = \int_\Omega \Psi(\xi(\omega), \lambda) d\mathbb{P}(\omega).$$

Remark 3 For each $\lambda \in \Delta_m$, we define the vector subspace V_λ of $\mathbb{R}^n$ by $V_\lambda := \operatorname{Im} K(\lambda) = \{K(\lambda)x \mid x \in \mathbb{R}^n\}$. By (2), if $\lambda > 0$ then $K(\lambda)$ is positive definite and therefore $V_\lambda = \mathbb{R}^n$. Otherwise, it may occur that for some $\omega \in \Omega$, the system $K(\lambda)u = g(\lambda) + f + \xi(\omega)$ has no solution. Thus $\mathbb{E}_\xi[\Psi(\xi, \lambda)] = +\infty$ whenever $\mathbb{P}\{g(\lambda) + f + \xi \notin V_\lambda\} > 0$, and the corresponding λ is not a feasible member volume vector for $(\mathcal{D}; \mathbb{P})$. Therefore, this problem may be written as

$$\min_{\lambda \in \Delta_m} \{\mathbb{E}_\xi[\Psi(\xi, \lambda)] \mid g(\lambda) + f + \xi \in \operatorname{Im} K(\lambda) \ \mathbb{P}\text{-a.s.}\}.$$

3.2 Discrete perturbations

In the simple case where the probability is given by $\mathbb{P}_1(\xi \in B) = 1$ if $0 \in B$ and 0 otherwise, we obtain of course $\mathbb{E}_\xi[\Psi(\xi, \lambda)] = \Psi(0, \lambda)$ and $(\mathcal{D}; \mathbb{P}_1)$ recovers the classical single load model (4). More generally, if $\mathbb{P}_2$ is a probability function with finite support given by $\{\xi^1, \ldots, \xi^k\}$, then $(\mathcal{D}; \mathbb{P}_2)$ is the multiload problem (5) with $\alpha_j = \mathbb{P}_2\{\xi = \xi^j\}$ and $f^j = f + \xi^j$, $j = 1, \ldots, k$.

3.3 Continuous perturbations

The next result gives an explicit expression for $(\mathcal{D}; \mathbb{P})$ when the perturbation ξ is a continuous random variable (without atoms). In the sequel, given a square matrix $A = (a_{ij})$, we denote by $\operatorname{tr}(A)$ the trace of A, i.e., $\operatorname{tr}(A) = \sum a_{ii}$.

Theorem 1 *Let $\xi : \Omega \to \mathbb{R}^n$ be a continuous random variable with mean vector $\mathbb{E}(\xi) = 0$ and covariance matrix $\operatorname{Var}(\xi) = PP^T$ with $P \in \mathbb{R}^{n\times k}$ for some $k \geq 1$. Then the corresponding* minimum expected compliance design problem $(\mathcal{D}; \mathbb{P})$ *is given by*

$$\min_{\lambda \in \Delta_m} \{\tfrac{1}{2}\big(g(\lambda) + f\big)^T u + \tfrac{1}{2} \operatorname{tr}(P^T U)\} \tag{8}$$

$$\text{s.t. } K(\lambda)u = g(\lambda) + f, \tag{9}$$

$$K(\lambda)U = P. \tag{10}$$

Here, the value of the objective function is independent of the choice of $u \in \mathbb{R}^n$ and $U \in \mathbb{R}^{n\times k}$ satisfying (9) *and* (10) *respectively.*

Proof. We will use two auxiliary results whose proofs are given in Appendix B.

Lemma 2 *Let V be a nonempty vector subspace of $\mathbb{R}^n$. Under the assumptions of Theorem* 1*, we have:*
(i) $\mathbb{P}\{f + \xi \in V\} = 1$ iff $f \in V$ and $\mathbb{P}\{\xi \in V\} = 1$.
(ii) $\mathbb{P}\{\xi \in V\} = 1$ iff the columns of P are vectors in V.

Lemma 3 *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Under the assumptions of Theorem* 1*, if $\mathbb{P}\{\xi \in \operatorname{Im} A\} = 1$ and $x : \Omega \to \mathbb{R}^n$ is a measurable function satisfying $Ax(\omega) = \xi(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, then $\mathbb{E}(\xi^T x) = \operatorname{tr}(P^T U)$, where $U \in \mathbb{R}^{n\times k}$ is any matrix satisfying $AU = P$.*

Now, we proceed with the proof of Theorem 1. Let $\lambda \in \Delta_m$ be feasible for $(\mathcal{D}; \mathbb{P})$, i.e., $\mathbb{P}\{g(\lambda) + f + \xi \in V_\lambda\} = 1$, where $V_\lambda = \operatorname{Im} K(\lambda)$ (see Remark 3). By Lemma 2, we have $f + g(\lambda) \in V_\lambda$ and $\xi \in V_\lambda$ $\mathbb{P}$-a.s., then there exist $u \in \mathbb{R}^n$ such that $K(\lambda)u = g(\lambda) + f$ and a measurable function $x : \Omega \to \mathbb{R}^n$ such that $K(\lambda)x = \xi$ $\mathbb{P}$-a.s. The existence of such a function is ensured by classical results on measurable selections (see for instance Rockafellar and Wets 1998, Ch. 14). Then $\Psi(\xi(\omega), \lambda) = \frac{1}{2}(g(\lambda) + f + \xi(\omega))^T(u + x(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. By Lemma 3, we get $\mathbb{E}_\xi[\Psi(\xi, \lambda)] = \frac{1}{2}(g(\lambda) + f)^T u + \frac{1}{2}\operatorname{tr}(P^T U)$, where U is such that $K(\lambda)U = P$. This completes the proof. □

Remark 4 Denoting the columns of P by p^j, $j = 1, \ldots, k$, and defining $\hat{f} = (f^T, p^{1T}, \ldots, p^{kT})^T \in \mathbb{R}^{n(k+1)}$, $\hat{g}(\lambda) = (g(\lambda)^T, 0, \ldots, 0)^T \in \mathbb{R}^{n(k+1)}$ and $\hat{K}(\lambda) = \sum_{i=1}^m \lambda_i \hat{K}_i$ with $\hat{K}_i = \operatorname{diag}(K_i, K_i, \ldots, K_i) \in \mathbb{R}^{n(k+1)\times n(k+1)}$, it is easy to see that (8)-(10) may be written as a multiload problem (6). An apparent difference is that self-weight is only associated to the mean load f. However, setting $f := \sum_{j=1}^k \alpha_j f^j$ and $p^j := f - f^j$, $j = 1, \ldots, k$, the multiload model (5) may equivalently be written as (8)-(10). Therefore, Theorem 1 provides a new stochastic approach to the construction of multiload models.

Remark 5 (Random independent perturbations) Let $\xi = \sum_{j=1}^k \varepsilon_j d^j$, where $d^j \in \mathbb{R}^n$ are some given directions and

ε_j are mutually independent real-valued random variables with $\mathbb{E}(\varepsilon_j) = 0$ and $\mathrm{Var}(\varepsilon_j) = \sigma_j^2$. Then the corresponding minimum expected compliance problem is

$$\min_{\lambda \in \Delta_m} \{ \tfrac{1}{2}(g(\lambda) + f)^T u + \tfrac{1}{2} \sum_{j=1}^{k} \sigma_j d^{jT} u^j | K(\lambda)u = g(\lambda) + f, \\ K(\lambda)u^j = \sigma_j d^j, j = 1, ..., k\}$$

3.4 Numerical examples

In this section some classic examples are revisited in the context of the minimum expected compliance model. Since the latter is equivalent to a multiload-like problem, in every case we solve it numerically by considering a well-known minimax dual formulation (see remark 2 and Appendix A). These numerical examples bear only illustrative character and are not intended to demonstrate the efficiency of the numerical method. Consequently, we omit the details concerning implementation as well as output data.

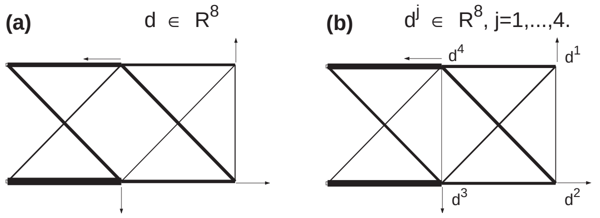

Fig. 2 Toy example (continued): two minimum expected compliance solutions.

Example 1 (continued) We begin with the *toy example* of Fig. 1. In Fig. 2 we give two minimum expected compliance solutions. Image (a) corresponds to an one-dimensional perturbation of the form $\xi = \varepsilon d$ with $d \in \mathbb{R}^8$; this perturbation is orthogonal in $\mathbb{R}^8$ to the main load and models 4 perfectly correlated random loads distributed on nodes 2, 3, 5 and 6. Image (b) illustrates an optimal solution associated with 4 independent and mutually orthogonal perturbations in $\mathbb{R}^8$, each one acting on a different node, so that the actual observed perturbation may be written as $\xi = \varepsilon_1 d^1 + \ldots + \varepsilon_4 d^4$.

Example 2 Consider a *dome* (similar to Achtziger (1997)) with 4 floors as in Fig. 3, where one vertical load is applied just on the top. The optimal solution is shown in the image on the right, which is unstable under small horizontal perturbations. Next, we introduce 2 independent orthogonal perturbations acting on the top along some horizontal directions d^1 and d^2; see Fig. 4 where we illustrate the convex hull of $\{f, d^1, d^2, -f, -d^1, -d^2\}$ (loads for which we have an upper bound for the compliance according to Lemma 1) and the corresponding stable optimal structure. Remark that the cost of adding two extra forces is low when compared with other approaches.

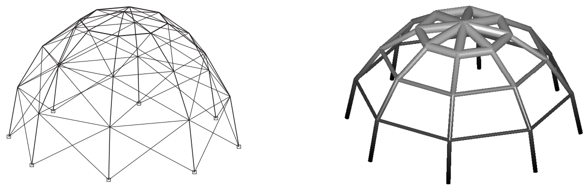

Fig. 3 Dome: ground structure and single-load optimal solution.

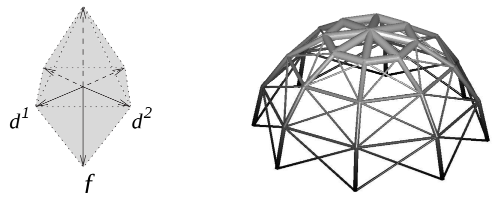

Fig. 4 Dome: 2 orthogonal perturbations at the top and optimal solution.

Example 3 The next structure is an *electricity mast* (Ben-Tal and Bendsøe 1993). The ground structure, main loads and optimal solution obtained by a single-load model are shown in Fig. 5. The central force on the top is in relation 10:1 to the others. Then, we give two minimum expected compliance examples. First we consider an one-dimensional distributed orthogonal perturbation; next, 3 independent perturbations orthogonal to the mean load are considered, two acting on the arms of the structure and one on the top of it.

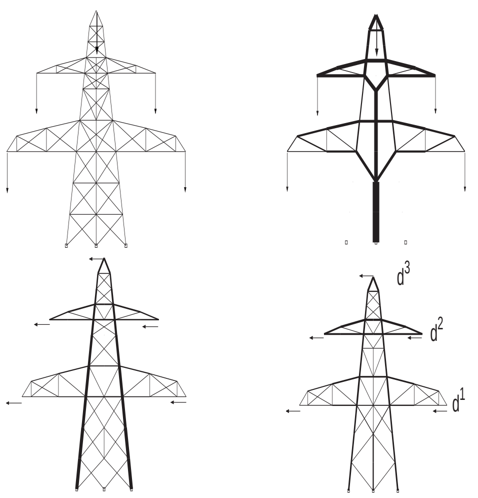

Fig. 5 Electricity mast: ground structure, single-load solution and two minimum expected compliance counterparts.

Example 4 Finally, Fig. 6 shows a ground structure with a set of 3*3*3 nodes in which all nodes are connected by bars (related to Jarre *et al.* (1993)). One load is applied as the first image shows, and the optimal solution is shown in the next image. Then we obtain a stable truss by considering 2 independent orthogonal loads in the plane orthogonal to the main load.

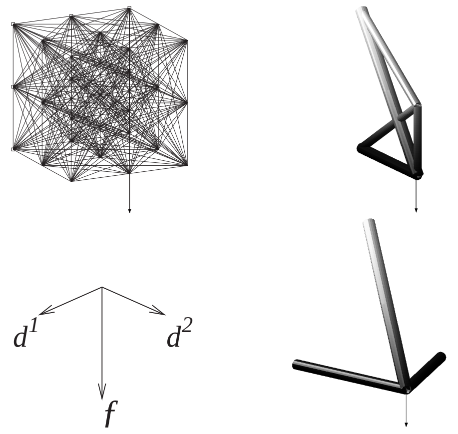

Fig. 6 Ground structure, single-load solution and stable counterpart by considering 2 orthogonal perturbations.

3.5 Stochastic model including variance

The stochastic setting introduced in this paper allows to consider other variants for the optimal design problem. For instance, one may include the variance of the compliance in the objective function and consider

$$\min_{\lambda\in\Delta_m} \{\alpha\mathbb{E}_\xi[\Psi(\xi,\lambda)] + \beta\,\mathrm{Var}_\xi[\Psi(\xi,\lambda)]\}, \qquad (\mathcal{D}_{Var};\mathbb{P})$$

where Ψ is defined by (7) and $\alpha, \beta \geq 0$.

Example 5 Suppose $\xi \sim \mathcal{N}_n(0, PP^T)$, i.e., the distribution of ξ is a n-multivariate normal with mean vector 0 and covariance matrix PP^T. Taking for simplicity $g = 0$, $\alpha = 0$ and $\beta = 1$, we have that $(D_{Var};\mathbb{P})$ is given by

$$\min_{\lambda\in\Delta_m} \{\tfrac{1}{2}\,\mathrm{tr}(P^TU)^2 + f^TUU^Tf \mid K(\lambda)u = f,\ K(\lambda)U = P\}.$$

Indeed, this follows by similar arguments to the proof of Theorem 1 together with the formula $\mathrm{Var}(\xi^TA\xi) = 2\,\mathrm{tr}((A\Gamma)^2) + 4\mu^TA\Gamma A^T\mu$ as $\xi \sim \mathcal{N}_n(\mu,\Gamma)$ (see Seber (1977)). We leave the details to the reader.

It is apparent that this kind of problems is harder to solve than the minimum expected compliance model $(\mathcal{D};\mathbb{P})$. It would be interesting to develop a primal-dual formulation of $(\mathcal{D}_{Var};\mathbb{P})$ in order to implement efficient numerical resolution methods. However, this is beyond the scope of this paper.

Appendix

A Duality via conjugate perturbation functions

We are going to describe a dual formulation of (4) in terms of Fenchel-Rockafellar conjugate perturbation functions. Although this duality is well-known in topology optimization (see Ben-Tal and Bendsøe 1993), the standard approach relies on Lagrangian duality instead of Fenchel-Rockafellar theory. Thus our proof of duality is essentially a method imported from other field and not used before in structural optimization. This has the advantage of providing us with a suitable mathematical framework to handle random load perturbations (see Lemma 4 below).

For each $i \in I := \{1,\ldots,m\}$, let $F_i\colon \mathbb{R}^n \to \mathbb{R}$ be defined by $F_i(x) = \frac{1}{2}x^TK_ix - (g_i+f)^Tx$, where $f \in \mathbb{R}^n$, $g_i \in \mathbb{R}^n$ and $K_i \in \mathbb{R}^{n\times n}$ is a symmetric and positive semi-definite real matrix for every $i \in I$. Consider the following convex minimax problem:

$$\min_{x\in\mathbb{R}^n}\max_{i\in I}\{F_i(x)\}. \qquad (\mathcal{P})$$

In order to dualize $(\mathcal{P})$, we follow Rockafellar (1974) by introducing a convex *perturbation* function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ such that this optimization problem may be written

$$\min_{x\in\mathbb{R}^n} \varphi(x,0). \qquad (11)$$

In fact, for any $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, we set

$$\varphi(x,y) := \max_{i\in I}\{F_i(x) + y_i\}, \qquad (12)$$

which is a continuous convex function. The corresponding dual problem is given by

$$\min_{\lambda\in\mathbb{R}^m} \varphi^*(0,\lambda), \qquad (13)$$

where $\varphi^*\colon \mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$ is the *Legendre-Fenchel transform* of φ, also known as the *conjugate function* of φ, which is defined by

$$\varphi^*(\xi,\lambda) = \sup_{x,y}\{\xi^Tx + \lambda^Ty - \varphi(x,y)\}.$$

Here, $\xi \in \mathbb{R}^n$ is interpreted as a perturbation parameter for the dual problem.

Lemma 4 *Under the previous definitions, $\varphi^* = \Psi$ where Ψ is given by* (7)*. In particular, Ψ is a proper, lower semicontinuous and convex, hence measurable, function.*

Proof. We begin by noticing that $\varphi^*(\xi,\lambda) = +\infty$ whenever $\lambda \notin \Delta_m$. Indeed, assume first that $\lambda_{i_0} < 0$ for some $i_0 \in I$, and take $x = 0$ and $y = -\alpha e_{i_0}$ where $e_{i_0} = (0,\ldots,0,1,0,\ldots,0)$ with α being a positive scalar and e_{i_0} the i_0-th vector of the standard basis of $\mathbb{R}^m$. Then, using that $F_i(0) = 0$, for all $\alpha > 0$ large enough, $\varphi^*(\xi,\lambda) \geq -\alpha\lambda_{i_0}$, and it suffices to let $\alpha \to +\infty$ to obtain $\varphi^*(\xi,\lambda) = +\infty$. Similarly, when $\sum_{i\in I}\lambda_i \neq 1$, taking $x = 0$ and $y = \alpha e$ with $e = (1,\cdots,1)$, we get $\varphi^*(\xi,\lambda) \geq \alpha\left(\sum_{i=1}^m \lambda_i - 1\right)$; letting $\alpha \to \pm\infty$ yields $\varphi^*(\xi,\lambda) = +\infty$. Next, suppose $\lambda \in \Delta_m$. Setting $z_i := F_i(x) + y_i$, we get

$\varphi^*(\xi,\lambda) = \sup_x\{\sum_{i\in I}[\xi_i x_i - \lambda_i F_i(x)] + \sup_z\{\sum_{i\in I}\lambda_i z_i - \max_{i\in I} z_i\}\}$. But $\forall z \in \mathbb{R}^m$, $\sum_{i\in I}\lambda_i z_i \le \max_{i\in I} z_i$ (recall $\lambda \in \Delta_m$) so $\varphi^*(\xi,\lambda) = \sup_x \sum_{i\in I}[\xi_i x_i - \lambda_i F_i(x)] = -\inf_x\left\{\frac{1}{2}x^T K(\lambda)x - (g(\lambda)+f+\xi)^T x\right\}$, which turns out to be a quadratic convex minimization problem (recall that $K(\lambda)$ is a positive semi-definite matrix). In particular, a finite value for this problem is guaranteed iff there exists a global minimum $u \in \mathbb{R}^n$ satisfying the first-order stationary condition: $K(\lambda)u - (g(\lambda)+f+\xi) = 0$. For any such a vector u we have $\frac{1}{2}u^T K(\lambda)u - (g(\lambda)+f+\xi)^T u = -\frac{1}{2}(g(\lambda)+f+\xi)^T u$, which proves the result. □

By Lemma 4, the problem (13) dual to $(\mathcal{P})$ is given by

$$\min_{\lambda\in\Delta_m}\left\{\tfrac{1}{2}(g(\lambda)+f)^T u \mid K(\lambda)u = g(\lambda)+f\right\}. \qquad (\mathcal{D})$$

The duality $(\mathcal{P})$-$(\mathcal{D})$ is summarized in the following theorem, which is a direct consequence of classical results; see Rockafellar and Wets (1998, Ch. 11) and Bonnans and Shapiro (2000, Ch. 2).

Theorem 2 *Suppose that*

$$\exists\lambda_0 \in \Delta_m \textit{ such that } K(\lambda_0) \textit{ is positive definite.} \qquad (14)$$

Then the primal optimal solution set $S(\mathcal{P})$ is nonempty, convex and bounded. Moreover, the dual optimal solution set $S(\mathcal{D})$ is nonempty, convex and bounded; the primal and dual optimal values satisfy $\min(\mathcal{P}) = -\min(\mathcal{D})$; *finally, a pair $(\hat{u},\hat{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$ belongs to $S(\mathcal{P}) \times S(\mathcal{D})$ iff $\hat{\lambda} \in \Delta_m$, $K(\hat{\lambda})\hat{u} = g(\hat{\lambda}) + f$ and $\hat{\lambda}_i = 0$ for all $i \notin I(\hat{u})$ where $I(x) = \{i \in I \mid F_i(x) = \max_{j\in I}\{F_j(x)\}\}$.*

Remark 6 Since $\max_{i\in I}\{F_i(x)\} \ge \frac{1}{2}x^T K(\lambda_0)x - (g(\lambda_0)+f)^T x$, (14) ensures that $(\mathcal{P})$ is *coercive*, hence the first conclusion in Theorem 2 is immediate. □

Remark 7 For this duality result, we only assume that K_i is positive semi-definite; no dyadic structure is needed. □

Remark 8 Under the setting described in §2.1, $(\mathcal{D})$ is precisely the classical normalized truss optimization problem given by (4). If condition (2) holds then (14) is satisfied for any $\lambda_0 > 0$. This duality between $(\mathcal{P})$ and $(\mathcal{D})$ can be efficiently exploited through primal-dual numerical schemes to solve (4); see Ben-Tal and Zibulevsky (1997). □

B
Proofs of some auxiliary results

Proof of Lemma 2. (i) We only prove the "only if" direction, the other one being completely trivial. Since $\mathbb{P}\{f+\xi \in V\} = 1$, we have $V \ni \mathbb{E}(f+\xi) = f$. Hence $f \in V$ and $\mathbb{P}\{\xi \in V\} = 1$.

(ii) Let $V^\perp$ be the orthogonal complement of V. It is clear that $\mathbb{P}\{\xi \in V\} = 1$ iff $\forall h \in V^\perp$, $\mathrm{Var}(h^T\xi) = 0$. But $\mathrm{Var}(h^T\xi) = h^T\Gamma h = \|P^T h\|^2$ (see Seber 1977, Ch. 1), therefore the columns of P belong to $(V^\perp)^\perp = V$. □

Proof of Lemma 3. An easy computation shows that the value of $\xi^T x$ does not depend on x such that $Ax = \xi$. Let us denote by A^- one of the generalized inverses of A, that is, the matrix A^- satisfies $AA^-A = A$. This matrix always exists, but there is no uniqueness in general (see Seber 1977, §3.8.c). As $\xi \in V$ $\mathbb{P}$-a.s., we have that $y := A^-\xi$ satisfies $Ay = \xi$ so that $\xi(\omega)^T x(\omega) = \xi^T(\omega)A^-\xi(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Using a standard formula for the expected value of quadratic form (see Seber 1977, Ch. 1), we obtain $\mathbb{E}(\xi^T x) = \mathbb{E}(\xi^T A^-\xi) = \mathrm{tr}(A^-\Gamma) + \mu^T A^-\mu$, where $\Gamma = \mathrm{Var}(\xi)$ and $\mu = \mathbb{E}(\xi)$. But $\mathbb{E}(\xi) = 0$ and $\mathrm{tr}(A^-\Gamma) = \mathrm{tr}(A^-PP^T) = \mathrm{tr}(P^T A^- P)$ and by Lemma 2(ii) with $V = \mathrm{Im}\, A$ we conclude that $Y := A^-P$ satisfies $AY = P$. Finally, it is clear that the value of $\mathrm{tr}(P^T U)$ is the same for any matrix U satisfying $AU = P$, which proves the result. □

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