On the asymptotic behavior of a system of steepest descent equations coupled by a vanishing mutual repulsion*

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Summary. We investigate the behavior at infinity of a special dissipative system, which consists of two steepest descent equations coupled by a non-autonomous conservative repulsion. The repulsion term is parametrized in time by an asymptotically vanishing factor. We show that under a simple slow parametrization assumption the limit points, if any, must satisfy an optimality condition involving the repulsion potential. Under some additional restrictive conditions, requiring in particular the equilibrium set to be one-dimensional, we obtain an asymptotic convergence result. Finally, some open problems are listed.

Key words: Steepest descent system, optimization, coupled system, slow parametrization, asymptotic selection.
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1 Introduction

Throughout this paper, H is a real Hilbert space with scalar product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $\phi : H \to \mathbb{R}$ be a $\mathcal{C}^1$ function and suppose that the set of critical points of ϕ is nonempty, that is,

$$S := \{x \in H \mid \nabla\phi(x) = 0\} \neq \emptyset.$$

A standard first-order method for finding a point in S consists in following the "Steepest Descent" trajectories:

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$$\text{(SD)} \quad \dot{x} + \nabla\phi(x) = 0, \quad t \geq 0.$$

The evolution equation SD defines a dissipative dynamical system in the sense that every solution $x(t)$ satisfies $\frac{d}{dt}\phi(x(t)) = -\|\nabla\phi(x(t))\|^2$ so that $\phi(x(t))$ decreases as long as $\nabla\phi(x(t)) \neq 0$. Since the stationary solutions of SD are described by S, it is natural to expect the corresponding solution $x(t)$ to approach the set S as $t \to \infty$. Indeed, under additional hypotheses, it is possible to ensure convergence at infinity to a local minimizer of ϕ (we refer the reader to [7, 8] for more details). However, we may be interested in additional information about S when ϕ has multiple critical points. For instance, we would like to compare some of them to select the best ones according to some additional criteria. We could also be interested in some properties of S such as unboundedness directions, symmetries, diameter estimates... A possible strategy may be to "explore" the state space by solving a system of simultaneous SD equations. In order to reinforce the exploration aspect, and motivated by the second-order in time system treated in [11], we propose to introduce a perturbation term which models an asymptotically vanishing repulsion. More precisely, in this paper we study the following non-autonomous coupled system:

$$\text{(SDVR)} \qquad \begin{cases} \dot{x} + \nabla\phi(x) + \varepsilon(t)\nabla V(x-y) = 0, \\ \dot{y} + \nabla\phi(y) - \varepsilon(t)\nabla V(x-y) = 0. \end{cases}$$

Here the function $V : H \to \mathbb{R}$ is of class $\mathcal{C}^1$ and satisfies the repulsion condition

$$\forall x \in H \setminus \{0\},\ \langle \nabla V(x), x \rangle < 0, \tag{1}$$

while the parametrization map $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ tends to zero as $t \to \infty$:

$$\lim_{t\to\infty} \varepsilon(t) = 0. \tag{2}$$

This evolution problem will be referred to as the "Steepest Descent and Vanishing Repulsion" (SDVR) system.

As a simple illustration of the type of behavior that SDVR may exhibit, suppose $H = \mathbb{R}^n$ and consider the case of a quadratic objective function $\phi(x) = \frac{1}{2}\langle Ax, x\rangle$ with $A \in \mathbb{R}^{n\times n}$ being symmetric and positive semi-definite, together with the quadratic repulsion potential $V(x) = -\frac{1}{2}\|x\|^2$. The corresponding SDVR system is

$$\begin{cases} \dot{x} + Ax - \varepsilon(t)(x-y) = 0, \\ \dot{y} + Ay + \varepsilon(t)(x-y) = 0, \end{cases}$$

whose solution is explicitly given by

$$\begin{cases} x(t) = e^{-tA}\left[\frac{x_0+y_0}{2} + \frac{x_0-y_0}{2} e^{2\int_0^t \varepsilon(\tau)d\tau}\right], \\ y(t) = e^{-tA}\left[\frac{x_0+y_0}{2} - \frac{x_0-y_0}{2} e^{2\int_0^t \varepsilon(\tau)d\tau}\right]. \end{cases}$$

As $\varepsilon(t)$ vanishes when $t \to \infty$, if A is positive definite then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$, independently of the improper integral $\int_0^\infty \varepsilon(t)dt$. Suppose now that $\ker A \neq \{0\}$ and take $v \in \ker A \setminus \{0\}$. Remark that v

is a direction of unboundedness for $S = \ker A$. Since $e^{-tA}v = v$, we get $\langle x(t) - y(t), v\rangle = e^{2\int_0^t \varepsilon(\tau)d\tau}\langle x_0 - y_0, v\rangle$. When $\langle x_0 - y_0, v\rangle \neq 0$, the asymptotic behavior along the direction given by v depends strongly on the improper integral $\int_0^\infty \varepsilon(t)dt$. In that case, if $\int_0^\infty \varepsilon(\tau)d\tau = \infty$ then the repulsion forces $x(t)$ and $y(t)$ to diverge towards infinity following opposite directions. Notice that in this example $\inf V = -\infty$ and $\|\nabla V(x)\| \to \infty$ as $\|x\| \to \infty$. However, an analogous divergent behavior can occur for a repulsion potential V that is bounded from below and satisfies $\|\nabla V(x)\| \to 0$ as $\|x\| \to \infty$. For instance, take $H = \mathbb{R}$ and $\phi \equiv 0$ (so that $S = \mathbb{R}$), and suppose that $V \in \mathcal{C}^1(\mathbb{R})$ is such that $V(x) = |x|^{-1}$ for all $|x| \geq 1$. If $y_0 \leq -1$ and $x_0 \geq 1$ then the system we have to solve is given by

$$\begin{cases} \dot{x} - \varepsilon(t)/(x-y)^2 = 0, \\ \dot{y} + \varepsilon(t)/(x-y)^2 = 0. \end{cases}$$

Since $\frac{d}{dt}(x - y) = 2\varepsilon(t)/(x - y)^2$, we have that $x(t) - y(t) = ((x_0 - y_0)^3 + 6\int_0^t \varepsilon(\tau)d\tau)^{1/3}$, which diverges if and only if $\int_0^\infty \varepsilon(t)dt = \infty$.

¿From these examples we infer that the repulsion term $\pm\varepsilon(t)\nabla V(x - y)$ is asymptotically effective as soon as $\varepsilon(t)$ vanishes sufficiently slow as $t \to \infty$, and moreover, it is apparent that the adequate condition is

$$\int_0^\infty \varepsilon(t)\,dt = \infty. \tag{3}$$

Such a "slow parametrization" condition has already been pointed out by many authors in various contexts (*cf.* [3, 4, 9, 11]). Since $\varepsilon(t)$ vanishes when $t \to \infty$, it is quite easy to prove the convergence of the gradients $\nabla\phi(x)$ and $\nabla\phi(y)$ toward 0. The examples above show that under unboundedness of S we may observe divergence to infinity. Divergence can be prevented under coercivity of ϕ and the natural question that arises is the convergence of the trajectory $(x(t), y(t))$ as $t \to \infty$. This is a difficult problem due to the non convexity of the repulsive potential V (see [10] for positive results in a convex framework). In this direction, a one-dimensional convergence result has been obtained in [11] for a second-order in time version of SDVR.

The paper is organized as follows. In section 2, we state some general convergence properties for the SDVR system and we show that the slow parametrization assumption (3) forces the limit points to satisfy an optimality condition involving ∇V and the normal cone of S. This normal condition[3] is new and allows to reformulate some results of [11] in a more elegant way. In section 3 we derive a sharp convergence result when the equilibrium set S is one-dimensional. In the last section, we precise our results when ϕ is the square of a distance function. Due to the first-order (in time) structure of SDVR, our asymptotic selection results are sharper than in [11].

[3] This optimality condition has been found independently by M.-O. Czarnecki (University Montpellier II).

Notations. We use the standard notations of convex analysis. In particular, given a convex set $C \subset H$, we denote by $d_C(x)$ (resp. $P_C(x)$) the distance of the point $x \in H$ to the set C (resp. the best approximation to x from C). For every $x \in C$, the set $N_C(x)$ stands for the normal cone of C at x. Given any set $D \subset H$, the closed convex hull of D is denoted by $\overline{\mathrm{co}}(D)$. Given $a, b \in H$, we define $[a,b] = \{a+\lambda(b-a) \mid \lambda \in [0,1]\}$ and $]a,b[= \{a+\lambda(b-a) \mid \lambda \in]0,1[\}$.

2 General asymptotic results

¿From now on, suppose that the functions $\phi : H \to \mathbb{R}$, $V : H \to \mathbb{R}$ and $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$, which are assumed to be of class $\mathcal{C}^1$, satisfy the following set of hypotheses $(\mathcal{H})$:

$$(\mathcal{H}_1)\ \begin{cases} i - \phi \text{ and } V \text{ are bounded from below on } H, \text{ with } \inf V = 0. \\ ii - \nabla\phi \text{ and } \nabla V \text{ are Lipschitz continuous on the bounded sets of } H. \end{cases}$$

$$(\mathcal{H}_2)\ \begin{cases} i - \text{ The map } \varepsilon \text{ is non-increasing, } i.e.\ \dot\varepsilon(t) \le 0 \quad \forall t \in \mathbb{R}_+. \\ ii - \text{ The map } \varepsilon \text{ is Lipschitz continuous on } \mathbb{R}_+. \\ iii - \lim\limits_{t\to\infty} \varepsilon(t) = 0. \end{cases}$$

Let us begin our study of SDVR by noticing that it can be rewritten as a single vectorial equation in $H^2 = H \times H$. Indeed, let us set $X = (x,y) \in H^2$, $\Phi(X) = \phi(x) + \phi(y)$ and $U(X) = V(x-y)$. With such notations, SDVR is equivalent to

$$\dot X + \nabla\Phi(X) + \varepsilon(t)\nabla U(X) = 0, \tag{4}$$

where Φ and U are differentiable functions on H^2 satisfying the analogue to $(\mathcal{H}_1)$, that is

$$(\mathcal{H}_1^{vec})\ \begin{cases} i - \Phi \text{ and } U \text{ are bounded from below on } H^2, \text{ with } \inf U = 0. \\ ii - \nabla\Phi \text{ and } \nabla U \text{ are Lipschitz continuous on the bounded sets of } H^2. \end{cases}$$

Set

$$E(t) = \Phi(X(t)) + \varepsilon(t)U(X(t)) = \phi(x(t)) + \phi(y(t)) + \varepsilon(t)V(x(t)-y(t)).$$

By differentiating E with respect to time, we obtain

$$\dot E = -\|\dot X\|^2 + \dot\varepsilon\, U(X) = -\|\dot x\|^2 - \|\dot y\|^2 + \dot\varepsilon\, V(x-y) \le 0.$$

Thus E is non-increasing, defining a Lyapounov-like function for (4). This is a useful tool in the study of the asymptotic stability of equilibria. Lyapounov methods and other powerful tools (like the Lasalle invariance principle) have been developed to study such a question. We refer the reader to the abundant literature on this subject; see, for instance, [2, 13, 14]. In this specific case, some standard arguments relying on the non-increasing and bounded from below function $E(t)$ permit to prove the next result, which we state without proof.

Proposition 2.1 *Assume that* $(\mathcal{H}_1^{vec})$ *and* $(\mathcal{H}_2)$ *hold. Then:*

(i) For every $X_0 \in H^2$*, there exists a unique solution* $X : \mathbb{R}_+ \to H^2$ *of* (4)*, which is of class* $\mathcal{C}^1$ *and satisfies* $X(0) = X_0$*. Moreover,* $\dot{X} \in L^2([0,\infty); H^2)$.

(ii) Assuming additionally that $\{X(t)\}_{t\geq 0}$ *is bounded in* H^2 *(which is the case for example if* Φ *is coercive, i.e.* $\lim_{\|X\|\to\infty} \Phi(X) = \infty$*), then* $\lim_{t\to\infty} \dot{X}(t) = 0$ *and* $\lim_{t\to\infty} \nabla\Phi(X(t)) = 0$.

(iii) If Φ *is convex and* $\{X(t)\}_{t\geq 0}$ *is bounded then* $\lim_{t\to\infty} \Phi(X(t)) = \inf \Phi$.

The natural question that arises is the convergence of the trajectory $X(t)$ as $t \to \infty$. When $\varepsilon \equiv 0$, (4) reduces to the steepest descent dynamical system associated with Φ. In that case, there are different conditions ensuring the asymptotic convergence towards an equilibrium. For instance, it is well-known that under convexity of Φ, the trajectories weakly converge to a minimum of Φ (*cf.* Bruck [8]). This last result can be generalized when ε tends to zero fast enough; indeed, we have

Proposition 2.2 *In addition to* $(\mathcal{H}_1^{vec})$ *and* $(\mathcal{H}_2)$*, assume that* Φ *is convex with* $\operatorname{Argmin}\Phi \neq \emptyset$*. If* $\int_0^\infty \varepsilon(t)\,dt < \infty$ *then every solution* $X(t)$ *of* (4) *weakly converges to a minimum of* Φ *as* $t \to \infty$*.*

We omit the proof of this result because it is similar to that given in [1] for second-order in time systems, which has been revisited with slight variants in [4, 5, 9, 11]. Notice that under the conditions of Proposition 2.2, any minimizer of Φ is asymptotically attainable. As the following result shows, that is not the case when the parametrization $\varepsilon(t)$ satisfies (3).

Lemma 2.1 *Assume that* $(\mathcal{H}_1^{vec})$, $(\mathcal{H}_2)$ *and* (3) *hold. Let* $X(t)$ *be a solution to* (4) *and suppose that* $X(t) \to X_\infty$ *strongly as* $t \to \infty$*. Then:*
(i) (Convex case)[4] *If* Φ *is convex then* $X_\infty \in \operatorname{Argmin}\Phi$ *and*

$$-\nabla U(X_\infty) \in N_{\operatorname{Argmin}\Phi}(X_\infty). \tag{5}$$

(ii) (General case) We have $X_\infty \in C := \{X \in H^2 \mid \nabla\Phi(X) = 0\}$ *and*

$$-\nabla U(X_\infty) \in \bigcap_{W\in\mathcal{N}(X_\infty)} \overline{\operatorname{co}}\Big(\mathbb{R}_+\nabla\Phi(W)\Big), \tag{6}$$

where $\mathcal{N}(X_\infty)$ *denotes the set of neighborhoods of* X_∞ *and the set* $\mathbb{R}_+\nabla\Phi(W)$ *is defined by* $\mathbb{R}_+\nabla\Phi(W) := \{\lambda\,\nabla\Phi(x) \mid \lambda \in \mathbb{R}_+,\, x \in W\}$.

[4] This result has been obtained simultaneously by M.-O. Czarnecki (University Montpellier II).

Proof. (i) From Proposition 2.1 (*ii*), $\lim_{t\to\infty} \nabla\Phi(X(t)) = 0$ and hence $X_\infty \in \operatorname{Argmin}\Phi$. Let $w \in \operatorname{Argmin}\Phi$ so that $\nabla\Phi(w) = 0$. By convexity, $\nabla\Phi$ is monotone and we have

$$\forall v \in H^2, \quad \langle \nabla\Phi(v), v - w\rangle \geq 0. \tag{7}$$

Taking the scalar product of (4) by $X(\cdot) - w$ and integrating on $[0, t]$, we obtain

$$\tfrac{1}{2}\|X(t)-w\|^2 - \tfrac{1}{2}\|X(0)-w\|^2 + \int_0^t \langle \nabla\Phi(X(s)) + \varepsilon(s)\nabla U(X(s)), X(s)-w\rangle\, ds = 0.$$

Using (7), we get $\int_0^t \varepsilon(s)\langle \nabla U(X(s)), X(s)-w\rangle\, ds \leq \frac{1}{2}\|X(0)-w\|^2 - \frac{1}{2}\|X(t)-w\|^2$. Recalling that $\int_0^\infty \varepsilon(t)\, dt = \infty$, we deduce that

$$\langle \nabla U(X_\infty), X_\infty - w\rangle = \lim_{t\to\infty} \langle \nabla U(X(t)), X(t) - w\rangle \leq 0,$$

otherwise, we would have $\lim_{t\to\infty} \int_0^t \varepsilon(s)\langle \nabla U(X(s)), X(s)-w\rangle\, ds = \infty$, which is impossible. This being true for any $w \in \operatorname{Argmin}\Phi$, we conclude that (5) holds.

(ii) Again, $X_\infty \in C$ due to Proposition 2.1 (*ii*). Next, let $W \in \mathcal{N}(X_\infty)$ and $v \in H^2$. Suppose that for every $w \in W$, $\langle \nabla\Phi(w), v\rangle \leq 0$. Since $X(t) \to X_\infty$, there exists $t_0 \geq 0$ such that for all $t \geq t_0$, $X(t) \in W$, and consequently

$$\forall t \geq t_0,\ \langle \nabla\Phi(X(t)), v\rangle \leq 0. \tag{8}$$

Integrating (4) on $[t_0, t]$ we obtain $\int_{t_0}^t \varepsilon(s)\langle \nabla U(X(s)), v\rangle\, ds = \langle X(t_0) - X(t), v\rangle - \int_{t_0}^t \langle \nabla\Phi(X(s)), v\rangle\, ds$. From (8), we get $\int_{t_0}^t \varepsilon(s)\langle \nabla U(X(s)), v\rangle ds \geq \langle X(t_0) - X(t), v\rangle$, $\forall t \geq t_0$. By (3), we deduce that

$$\langle \nabla U(X_\infty), v\rangle = \lim_{t\to\infty} \langle \nabla U(X(t)), v\rangle \geq 0.$$

This proves that, for every $v \in H^2$ and $w \in W$, if $\langle \nabla\Phi(w), v\rangle \leq 0$ then $\langle \nabla U(X_\infty), v\rangle \geq 0$, which amounts to

$$\forall v \in (\mathbb{R}_+ \nabla\Phi(W))^o,\ \langle \nabla U(X_\infty), v\rangle \geq 0, \tag{9}$$

where $(\mathbb{R}_+ \nabla\Phi(W))^o$ stands for the polar cone of the conic hull of $\nabla\Phi(W)$. By (9), the vector $-\nabla U(X_\infty)$ belongs to $(\mathbb{R}_+ \nabla\Phi(W))^{oo}$, the polar cone of $(\mathbb{R}_+ \nabla\Phi(W))^o$. Finally the bipolar theorem (*cf.* for example [6]) ensures that $-\nabla U(X_\infty) \in \overline{\operatorname{co}}\,(\mathbb{R}_+ \nabla\Phi(W))$, which completes the proof. ■

Remark 2.1 Condition (5) for the convex case expresses a necessary condition for X_∞ to be a local minimum of the function U on the set $\operatorname{Argmin}\Phi$. In the general case, the set arising in (6) is closely related to the normal cone to C at X_∞. However, Lemma 2.1(i) cannot be viewed as a special case of Lemma 2.1(ii).

3 Convergence for a one-dimensional equilibrium set

When ϕ has non-isolated critical points, the general results of the previous section for the vectorial form (4) of SDVR do not ensure the asymptotic convergence of the solution $(x(t), y(t))$ under the slow parametrization condition (3). If ϕ and V are both convex then it is possible to prove the asymptotic convergence to a pair (x_∞, y_∞) that minimizes $(x, y) \mapsto V(x-y)$ on $\mathrm{Argmin}\,\phi \times \mathrm{Argmin}\,\phi$ (see [10]). Although the repulsion condition (1) is not compatible with the convexity of V, the asymptotic selection principle given by Lemma 2.1 establishes that the "candidates" to be limit points must satisfy an analogous extremality condition depending on $U(x, y) = V(x-y)$. In a one-dimensional scalar setting, a convergence theorem for a second-order in time system involving a repulsion term has been proved in [11]. Next, we show that this type of result is valid for SDVR. To our best knowledge, convergence in the general higher dimensional case is an open problem.
From now on, we assume the following hypotheses on the objective function ϕ: for every bounded sequence $(x_n) \subset H$,

$$\lim_{n\to\infty} \|\nabla\phi(x_n)\| = 0 \Rightarrow \lim_{n\to\infty} d_S(x_n) = 0, \tag{10}$$

$$\text{the map } \phi \text{ is coercive and } S = [a, b] \text{ for some } a, b \in H. \tag{11}$$

If $a \neq b$ then we suppose that for every $x \in H$,

$$\text{if } P_\Delta(x) \in S \text{ then } \nabla\phi(x) \text{ is orthogonal to } \Delta, \tag{12}$$

where Δ is the straight line $\Delta := \{a + \lambda(b-a) \mid \lambda \in \mathbb{R}\}$.

Remark 3.1 Condition (10) holds automatically when $\dim H < \infty$, but (11) and (12) are stringent. Take $\phi := f\, o\, d_{[a,b]}$ where $f \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R})$ and $d_{[a,b]}$ refers to the distance function to the segment $[a, b]$. If the function f is such that $f'(0) = 0$ and $f'(x) > 0$ for every $x > 0$, then the function ϕ satisfies (10), (11) and (12). Note that the function ϕ is a C^1 function due to the assumption $f'(0) = 0$.

On the repulsion potential V, we assume that there exists a scalar function $\gamma : H \to \mathbb{R}_{++}$ such that

$$\forall x \in H,\ \nabla V(x) = -\gamma(x)x. \tag{13}$$

Theorem 3.1 *Under hypotheses* $(\mathcal{H})$*, let* $(x(t), y(t))$ *be a solution to SDVR. If* (10)-(13) *hold then:*

(i) *There exists* $(x_\infty, y_\infty) \in [a, b]^2$ *such that* $\lim_{t\to\infty}(x(t), y(t)) = (x_\infty, y_\infty)$.
(ii) *Suppose that* $a \neq b$ *and let us denote by* Γ_a *(resp.* Γ_b*) the connected component of* $\mathrm{cl}\,(\Delta \setminus S)$ *such that* $a \in \Gamma_a$ *(resp.* $b \in \Gamma_b$*). Assume that* $x_\infty = y_\infty = \ell$ *and* $P_\Delta(x(0)) \neq P_\Delta(y(0))$*. Then* ℓ *equals* a *or* b *and*
- $\ell = a$ *implies* $\Big(P_\Delta(x(t)), P_\Delta(y(t))\Big) \in \Gamma_a^2$ *for every* $t \geq 0$.
- $\ell = b$ *implies* $\Big(P_\Delta(x(t)), P_\Delta(y(t))\Big) \in \Gamma_b^2$ *for every* $t \geq 0$.

(iii)*Suppose that the slow parametrization condition* (3) *holds. If* $P_\Delta(x(0)) \neq P_\Delta(y(0))$ *then* $(x_\infty, y_\infty) \in \{a,b\}^2$. *When in addition* $a \neq b$, *if* $x_\infty = y_\infty = a$ *(resp.* $x_\infty = y_\infty = b$*), then we have* $(P_\Delta(x(t)), P_\Delta(y(t))) \in \Gamma_a^2$ *(resp.* $(P_\Delta(x(t)), P_\Delta(y(t))) \in \Gamma_b^2$*) for every* $t \geq 0$.

Proof. (i) From the coercivity of ϕ, we deduce the boundedness of the map $t \mapsto (x(t), y(t))$ and hence in view of Proposition 2.1 (ii), we have $\lim_{t\to\infty} \nabla\phi(x(t)) = \lim_{t\to\infty} \nabla\phi(y(t)) = 0$. From assumption (10), it ensues that

$$\lim_{t\to\infty} d_S(x(t)) = \lim_{t\to\infty} d_S(y(t)) = 0. \tag{14}$$

If $a = b$ the set S is reduced to the singleton $\{a\}$ and the convergence of $x(t)$ and $y(t)$ toward a is immediate. Now assume that the segment line S is not trivial. Since $S \subset \Delta$, we have for every $x \in H$, $\|x - P_\Delta(x)\| = d_\Delta(x) \leq d_S(x)$. Hence, in view of (14), we obtain

$$\lim_{t\to\infty} \|x(t) - P_\Delta(x(t))\| = \lim_{t\to\infty} \|y(t) - P_\Delta(y(t))\| = 0.$$

As a consequence, the convergence of $x(t)$ (resp. $y(t)$) as $t \to \infty$ is equivalent to the convergence of $P_\Delta(x(t))$ (resp. $P_\Delta(y(t))$, which amounts to the convergence of $\langle x(t), b-a\rangle$ (resp. $\langle y(t), b-a\rangle$) as $t \to \infty$. For every $t \geq 0$, set $\alpha(t) := \langle x(t), b-a\rangle$ and $\beta(t) := \langle y(t), b-a\rangle$. From SDVR, we obtain

$$\dot\alpha(t) + \langle \nabla\phi(x(t)), b-a\rangle - \varepsilon(t)\,\gamma(x(t)-y(t))\,(\alpha(t)-\beta(t)) = 0. \tag{15}$$

$$\dot\beta(t) + \langle \nabla\phi(y(t)), b-a\rangle + \varepsilon(t)\,\gamma(x(t)-y(t))\,(\alpha(t)-\beta(t)) = 0. \tag{16}$$

We have that $\{\langle x, b-a\rangle \mid x \in S\} = [\lambda, \mu]$ for some $\lambda < \mu$. It is immediate to check that, for every $x \in H$, $\langle x, b-a\rangle \in [\lambda,\mu]$ is equivalent to $P_\Delta(x) \in S$, so that we can reformulate assumption (12) as

$$\langle x, b-a\rangle \in [\lambda,\mu] \Rightarrow \langle \nabla\phi(x), b-a\rangle = 0. \tag{17}$$

In particular, for every $t \geq 0$, we have that $\alpha(t) \in [\lambda,\mu]$ (resp. $\beta(t) \in [\lambda,\mu]$) implies $\langle \nabla\phi(x(t)), b-a\rangle = 0$ (resp. $\langle \nabla\phi(y(t)), b-a\rangle = 0$). Since the ω-limit sets of $\{x(t)\}_{t\geq 0}$ and $\{y(t)\}_{t\geq 0}$ are included in S, it is clear that:

$$\left[\liminf_{t\to\infty} \alpha(t), \limsup_{t\to\infty} \alpha(t)\right] \subset [\lambda,\mu] \quad \text{and} \quad \left[\liminf_{t\to\infty} \beta(t), \limsup_{t\to\infty} \beta(t)\right] \subset [\lambda,\mu].$$

We are now going to prove the convergence of $\alpha(t)$ and $\beta(t)$ as $t \to \infty$ by distinguishing three cases:

Case 1: For all $t \geq 0$, we have $\min\{\alpha(t), \beta(t)\} \geq \mu$ or $\max\{\alpha(t), \beta(t)\} \leq \lambda$.

Without loss of generality, we can assume that for every $t \geq 0$, $\alpha(t) \geq \mu$ and $\beta(t) \geq \mu$. We deduce that $\liminf_{t\to\infty} \alpha(t) \geq \mu$ and $\liminf_{t\to\infty} \beta(t) \geq \mu$. Since $\limsup_{t\to\infty} \alpha(t) \leq \mu$ and $\limsup_{t\to\infty} \beta(t) \leq \mu$, we conclude that

$\lim_{t\to\infty}\alpha(t) = \lim_{t\to\infty}\beta(t) = \mu$.

Case 2: There exist $c \in]\lambda, \mu[$ and $t_0 \geq 0$ such that either $\alpha(t_0) < c < \beta(t_0)$ or $\beta(t_0) < c < \alpha(t_0)$.

Suppose $\alpha(t_0) < c < \beta(t_0)$. Let us first prove that

$$\forall t \geq t_0,\ \alpha(t) < c < \beta(t). \tag{18}$$

Let us set $t_\infty := \sup\{t \geq t_0, \quad \forall u \in [t_0, t], \quad \alpha(u) < c < \beta(u)\}$. Let us argue by contradiction and assume that $t_\infty < \infty$. We then have:

$$\forall t \in [t_0, t_\infty[,\ \alpha(t) < c < \beta(t). \tag{19}$$

¿From the continuity of the maps $t \mapsto \alpha(t)$ and $t \mapsto \beta(t)$, we have $\alpha(t_\infty) = c$ or $\beta(t_\infty) = c$. Without any loss of generality, let us assume that $\alpha(t_\infty) = c$. Using again the continuity of the map α, there exists $t_1 \in [t_0, t_\infty]$ such that $\forall t \in [t_1, t_\infty]$, $\alpha(t) \geq \lambda$. Let us now use the differential equation (15) satisfied by α. Since $\alpha(t) \in [\lambda, c]$ for every $t \in [t_1, t_\infty]$, we deduce from (17) that $\langle \nabla\phi(x(t)), b - a\rangle = 0$. On the other hand, the sign of $\alpha - \beta$ is negative on $[t_1, t_\infty]$, so that equation (15) yields $\forall t \in [t_1, t_\infty]$, $\dot\alpha(t) \leq 0$. As a consequence, we have $c = \alpha(t_\infty) \leq \alpha(t_1)$, which contradicts (19). Therefore, we conclude that $t_\infty = +\infty$, which ends the proof of (18).

Case 2.a: First assume that $\alpha(t) \geq \lambda$ for every $t \geq t_0$. From (17) and the fact that $\alpha(t) \in [\lambda, c]$, we deduce that $\langle \nabla\phi(x(t)), b - a\rangle = 0$. This combined with (15) and the negative sign of $\alpha(t) - \beta(t)$ implies that $\dot\alpha(t) \leq 0$ for every $t \geq t_0$. As a consequence, $\lim_{t\to\infty}\alpha(t)$ exists.

Case 2.b: Now assume that there exists $t_1 \geq t_0$ such that $\alpha(t_1) < \lambda$. Let us first prove that

$$\forall t \geq t_1, \quad \alpha(t) \leq \lambda. \tag{20}$$

Let us argue by contradiction and assume that there exists $t_2 \geq t_1$ such that $\alpha(t_2) > \lambda$. Let

$$t_3 := \inf\{t \in [t_1, t_2], \quad \forall u \in [t, t_2], \quad \alpha(u) \geq \lambda\}.$$

From the continuity of α, we have $\alpha(t_3) = \lambda$. The definition of t_3 shows that $\alpha(t) \geq \lambda$ for every $t \in [t_3, t_2]$. In particular, we have $\alpha(t) \in [\lambda, c]$, which in view of (17) implies that $\langle \nabla\phi(x(t)), b - a\rangle = 0$. This combined with (15) and the negative sign of $\alpha(t) - \beta(t)$ yields $\dot\alpha(t) \leq 0$ for every $t \in [t_3, t_2]$. Hence, we infer that $\lambda < \alpha(t_2) \leq \alpha(t_3) = \lambda$, a contradiction which ends the proof of (20). From (20), we deduce that $\limsup_{t\to\infty}\alpha(t) \leq \lambda$. Since on the other hand, $\liminf_{t\to\infty}\alpha(t) \geq \lambda$, we conclude that $\lim_{t\to\infty}\alpha(t) = \lambda$.

The proof of the convergence of $\beta(t)$ follows the same lines and is left to the reader.

Case 3: There exist $c \in]\lambda, \mu[$ and $t_0 \geq 0$ such that $\alpha(t_0) = \beta(t_0) = c$.

It is clear that the constant map $t \in [t_0, \infty[\mapsto (c, c)$ satisfies the differential equations (15) and (16). From the uniqueness of the Cauchy problem at t_0, we deduce that $\alpha(t) = \beta(t) = c$ for every $t \geq t_0$.

We let the reader check that all cases are recovered by the previous three ones.

(ii) If case 2 holds, it is immediate that $\lim_{t\to\infty} \alpha(t) \neq \lim_{t\to\infty} \beta(t)$, thus implying that $\lim_{t\to\infty} x(t) \neq \lim_{t\to\infty} y(t)$. If case 3 occurs, we obtain by reversing the time that $\alpha(0) = \beta(0)$ and hence $P_\Delta(x(0)) = P_\Delta(y(0))$. Therefore, if $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \ell$ and $P_\Delta(x(0)) \neq P_\Delta(y(0))$, case 1 necessary holds which means that

$$\forall t \geq 0,\ (P_\Delta(x(t)), P_\Delta(y(t))) \in \Gamma_a^2 \ \text{ or } \ \forall t \geq 0,\ (P_\Delta(x(t)), P_\Delta(y(t))) \in \Gamma_b^2.$$

In the first eventuality, we have $\ell = a$, while in the second one we obtain $\ell = b$.

(iii) First assume that $x_\infty = y_\infty$. From (ii), we deduce that x_∞ and y_∞ are extremal points of $S = [a, b]$. Now assume that $x_\infty \neq y_\infty$. Let us apply Lemma 2.1 (ii) by taking into account the fact that $\nabla V(x) = -\gamma(x)\, x$ and $\gamma(x) > 0$ for every $x \in H$. Condition (6) yields

$$x_\infty - y_\infty \in \bigcap_{W_1 \in \mathcal{N}(x_\infty)} \overline{\text{co}}(\mathbb{R}_+ \nabla\phi(W_1)) \ \text{ and } \ y_\infty - x_\infty \in \bigcap_{W_2 \in \mathcal{N}(y_\infty)} \overline{\text{co}}(\mathbb{R}_+ \nabla\phi(W_2)).$$

Let us argue by contradiction and assume that $x_\infty \in]a, b[$ (resp. $y_\infty \in]a, b[$). It is then clear that

$$\bigcap_{W_1 \in \mathcal{N}(x_\infty)} \overline{\text{co}}\,(\mathbb{R}_+ \nabla\phi(W_1)) \subset \Delta_0^\perp \quad \text{and} \quad \bigcap_{W_2 \in \mathcal{N}(y_\infty)} \overline{\text{co}}\,(\mathbb{R}_+ \nabla\phi(W_2)) \subset \Delta_0^\perp,$$

where $\Delta_0 := \Delta - \Delta = \mathbb{R}\,(b-a)$. Therefore $x_\infty - y_\infty \in \Delta_0^\perp$. Since $x_\infty - y_\infty \in \Delta_0$, we conclude that $x_\infty = y_\infty$, a contradiction. The rest of the statement is an immediate consequence of (ii). ■

4 Further convergence results

Under the assumption of slow parametrization, Theorem 3.1 shows that, either the solutions x and y of SDVR converge to the opposite extremities of S, or they have the same limit. Since our aim is a global exploration of S, the second case clearly appears as the pathological one. Our purpose in this section is to find sufficient conditions on ϕ and V ensuring the convergence toward the opposite extremities of ϕ. We will restrict the analysis to the functions of the form $\phi := d_S^2$.

Lemma 4.1 *Under the hypotheses of Theorem 3.1, take* $\phi(x) = \frac{\delta}{2}\|x-p\|^2$ *for some* $\delta \in \mathbb{R}_+$ *and* $p \in H$*. Suppose moreover that the map* γ *in* (13) *satifies* $\liminf_{x\to 0}\gamma(x) > 0$*. If* (3) *holds then for every straight line* L *going through the point* p *and satisfying* $P_L(x(0)) \neq P_L(y(0))$*, there exists* $T \geq 0$ *such that* $p \in]P_L(x(t)), P_L(y(t))[$ *for all* $t \geq T$*.*

Proof. Set $x_0 = x(0)$ and $y_0 = y(0)$. Let us denote by u a director vector of L. The assumption $P_L(x_0) \neq P_L(y_0)$ amounts to saying that $\langle x_0, u\rangle \neq \langle y_0, u\rangle$. Without any loss of generality, one can assume that $\langle x_0, u\rangle > \langle y_0, u\rangle$. Taking into account the particular form of ϕ and V and adding (resp. subtracting) the first and second equation of SDVR, we find respectively

$$\dot{x}(t) + \dot{y}(t) + \delta(x(t) + y(t) - 2p) = 0$$

$$\dot{x}(t) - \dot{y}(t) + \delta(x(t) - y(t)) - 2\varepsilon(t)\,\gamma\,(x(t) - y(t))\,(x(t) - y(t)) = 0.$$

Taking the scalar product of these equations by the vector u and setting $\alpha(t) := \langle x(t), u\rangle$ (resp. $\beta(t) := \langle y(t), u\rangle$), we obtain:

$$\dot{\alpha}(t) + \dot{\beta}(t) + \delta(\alpha(t) + \beta(t) - 2\langle p, u\rangle) = 0 \tag{21}$$

$$\dot{\alpha}(t) - \dot{\beta}(t) + \delta(\alpha(t) - \beta(t)) - 2\varepsilon(t)\,\gamma\,(x(t) - y(t))\,(\alpha(t) - \beta(t)) = 0. \tag{22}$$

It is clear in view of equation (22) that if the quantity $\alpha(t) - \beta(t)$ takes the value 0 for some $t \geq 0$ then $\alpha(t) - \beta(t) = 0$ for every $t \geq 0$. Since by assumption $\alpha(0) - \beta(0) > 0$, we deduce that $\alpha(t) - \beta(t) > 0$ for every $t \geq 0$. From the assumption $\liminf_{x\to 0}\gamma(x) > 0$, there exist $\eta > 0$ and $m > 0$ such that, for every $\|x\| \leq \eta$, we have $\gamma(x) \geq m$. Since ϕ admits p as a unique strong minimum, we clearly have $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = p$ and hence $\lim_{t\to\infty} x(t) - y(t) = 0$. We deduce the existence of $t_0 \geq 0$ such that, for every $t \geq t_0$, we have $\gamma(x(t) - y(t)) \geq m$. This last inequality combined with (22) gives

$$\dot{\alpha}(t) - \dot{\beta}(t) + \delta(\alpha(t) - \beta(t)) \geq 2m\,\varepsilon(t)(\alpha(t) - \beta(t)).$$

Multiplying by $e^{\delta t}$, we obtain

$$\frac{d}{dt}\left[e^{\delta t}(\alpha(t) - \beta(t))\right] \geq 2m\,\varepsilon(t)\,e^{\delta t}(\alpha(t) - \beta(t)).$$

By integrating this differential equation between t_0 and t, we find:

$$\alpha(t) - \beta(t) \geq (\alpha(t_0) - \beta(t_0))\,e^{-\delta(t-t_0)}\exp\int_{t_0}^{t} 2m\,\varepsilon(s)\,ds. \tag{23}$$

On the other hand, a simple integration of (21) on $[t_0, t]$ yields

$$\alpha(t)+\beta(t)-2\langle p,u\rangle=(\alpha(t_0)+\beta(t_0)-2\langle p,u\rangle)\,e^{-\delta(t-t_0)}. \qquad (24)$$

Relations (23) and (24) imply that

$$\alpha(t)-\langle p,u\rangle \geq \tfrac{e^{-\delta(t-t_0)}}{2}\Big(\alpha(t_0)+\beta(t_0)-2\langle p,u\rangle+(\alpha(t_0)-\beta(t_0))e^{\int_{t_0}^{t}2m\varepsilon(s)ds}\Big).$$
$$\beta(t)-\langle p,u\rangle \leq \tfrac{e^{-\delta(t-t_0)}}{2}\Big(\alpha(t_0)+\beta(t_0)-2\langle p,u\rangle-(\alpha(t_0)-\beta(t_0))e^{\int_{t_0}^{t}2m\varepsilon(s)ds}\Big).$$

Since $\int_{t_0}^{\infty}\varepsilon(s)\,ds=\infty$, we obtain the existence of $T\geq t_0$ such that $\beta(t)<\langle p,u\rangle<\alpha(t)$ for every $t\geq T$. This means that $p\in]P_L(x(t)),P_L(y(t))[$ for every $t\geq T$. ∎

Remark 4.1 The assumption $\liminf_{x\to 0}\gamma(x)>0$ means that the repulsion term $\nabla V(x)$ is not negligible with respect to x when $x\to 0$. Suppose that the function V is defined by $V(x):=\theta(\|x\|^2)$, where $\theta:\mathbb{R}_+\to\mathbb{R}$ is a decreasing function of class $\mathcal{C}^1$. In this case, the condition $\liminf_{x\to 0}\gamma(x)>0$ is equivalent to $\theta'(0)<0$.

In the next theorem, we assume that the function ϕ equals $\phi:=d^2_{[a,b]}$, for some $a,\,b\in H$. We show that the assumption $\liminf_{x\to 0}\gamma(x)>0$ implies that the trajectories x and y converge to opposite extremities of the segment line $[a,b]$. In this case, the repulsion term is strong enough to push the trajectories x and y away from one another.

Theorem 4.1 *Consider a segment line $[a,b]\subset H$, included in some straight line Δ and let us define the function ϕ by $\phi:=\frac{\delta}{2}\,d^2_{[a,b]}$ for some $\delta>0$. Under the hypotheses of Theorem 3.1, we suppose moreover that the map γ in* (13) *satifies $\liminf_{x\to 0}\gamma(x)>0$, and that the slow parametrization condition* (3) *holds. Let $(x,y):\mathbb{R}_+\to H^2$ be the unique trajectory of SDVR with initial conditions $(x_0,y_0)\in H^2$ satisfying $P_\Delta(x_0)\neq P_\Delta(y_0)$. Then we have*

$$\lim_{t\to\infty}(x(t),y(t))=(a,b) \qquad \text{or} \qquad \lim_{t\to\infty}(x(t),y(t))=(b,a).$$

Proof. When $a=b$, the function ϕ admits the real a as a unique strong minimum and we obviously have $\lim_{t\to\infty}x(t)=\lim_{t\to\infty}y(t)=a$. From now on, let us assume that $a\neq b$. In view of Remark 3.1, the function $\phi:=\frac{\delta}{2}\,d^2_{[a,b]}$ satisfies hypotheses (10)-(12). Hence Theorem 3.1 applies and one of the following cases holds
(i) $\lim_{t\to\infty}(x(t),y(t))=(a,b)$ or $\lim_{t\to\infty}(x(t),y(t))=(b,a)$.
(ii) $\lim_{t\to\infty}(x(t),y(t))=(a,a)$ and $\forall t\geq 0$, $(P_\Delta(x(t)),P_\Delta(y(t)))\in\Gamma_a^2$.
(iii) $\lim_{t\to\infty}(x(t),y(t))=(b,b)$ and $\forall t\geq 0$, $(P_\Delta(x(t)),P_\Delta(y(t)))\in\Gamma_b^2$.
Let us argue by contradiction and assume that case (i) does not hold. Without any loss of generality, we can assume that case (ii) holds. On the half-space E_a defined by $E_a:=\{x\in H,\quad P_\Delta(x)\in\Gamma_a\}$, the function ϕ coincides with the function $x\mapsto\frac{\delta}{2}\|x-a\|^2$. From Lemma 4.1 applied with $p=a$ and the straight line Δ, we obtain the existence of $t_0\geq 0$ such that $a\in]P_\Delta(x(t)),P_\Delta(y(t))[$ for

$t \geq t_0$. This shows that either $P_\Delta(x(t)) \notin \Gamma_a$ or $P_\Delta(y(t)) \notin \Gamma_a$, which gives a contradiction. ■

When the assumption $\liminf_{x\to 0} \gamma(x) > 0$ does not hold, it is possible to choose initial conditions so as to force the corresponding trajectories to converge toward the same limit. The next proposition provides us with a counterexample in the case $H = \mathbb{R}$.

Proposition 4.1 *Take any function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\phi(x) = x^2/2$ for every $x \in \mathbb{R}_+$. Assume that the functions $V : H \to \mathbb{R}$ and $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $(\mathcal{H})$. Suppose that there exist $M > 0$ and $\delta > 1$ such that,*

$$\forall x \in \mathbb{R}, \quad |V'(x)| \leq M\,|x|^\delta.$$

Let $(x, y) : \mathbb{R}_+ \to \mathbb{R}^2$ be the unique trajectory of SDVR with initial conditions (x_0, y_0). Then there exist $r > 0$ and a function $\theta : [0, r[\to \mathbb{R}_+$ such that, for every $x_0 > 0$ and $y_0 > 0$ with $|y_0 - x_0| < r$,

$$\theta(|y_0 - x_0|) \leq x_0 + y_0 \quad \Longrightarrow \quad \forall t \geq 0, \quad x(t) \geq 0 \quad \textit{and} \quad y(t) \geq 0. \tag{25}$$

For such initial conditions, we have $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$.

Proof. Let us consider the function $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined by $\tilde{\phi}(x) = x^2/2$ for every $x \in \mathbb{R}$ and let $(\tilde{x}, \tilde{y})$ be the unique trajectory of SDVR associated with $\tilde{\phi}$. If $(\tilde{x}, \tilde{y})$ is proved to satisfy the property (25), then $(\tilde{x}, \tilde{y})$ is also the solution of SDVR associated with any function ϕ coinciding with $\tilde{\phi}$ on $\mathbb{R}_+$. As a consequence, without loss of generality, we can assume that $\phi = \tilde{\phi}$. The SDVR system then reduces to:

$$\text{(SDVR)} \qquad \begin{cases} \dot{x}(t) + x(t) + \varepsilon(t)V'(x-y)(t) = 0 \\ \dot{y}(t) + y(t) - \varepsilon(t)V'(x-y)(t) = 0. \end{cases}$$

By adding the first and the second equation of SDVR, we obtain $\dot{x}(t) + \dot{y}(t) + x(t) + y(t) = 0$, which immediately yields

$$x(t) + y(t) = (x_0 + y_0)\, e^{-t}. \tag{26}$$

Without any loss of generality, one may assume that $y_0 < x_0$. We then have $y(t) < x(t)$ for every $t \geq 0$. From the assumption on V, we have for every $t \geq 0$, $V'(x-y)(t) \geq -M\,(x(t) - y(t))^\delta$. Let us subtract the first and the second equation of SDVR by taking into account the previous inequality

$$\dot{x}(t) - \dot{y}(t) + x(t) - y(t) - 2M\varepsilon(t)(x(t) - y(t))^\delta \leq 0.$$

We now multiply by e^t and set $u(t) = e^t(x(t) - y(t))$ to obtain

$$\dot{u}(t) \leq 2M\varepsilon(t)e^t(x(t) - y(t))^\delta = 2M\varepsilon(t)e^{-(\delta-1)t}\, u^\delta(t).$$

Let us integrate the previous inequality on $[0, t]$ to find

$$-\frac{1}{\delta-1}\left(\frac{1}{u^{\delta-1}(t)}-\frac{1}{u^{\delta-1}(0)}\right)\leq 2M\int_0^t \varepsilon(s)e^{-(\delta-1)s}ds.$$

Setting

$$C=2M(\delta-1)\int_0^{\infty}\varepsilon(s)e^{-(\delta-1)s}\,ds,$$

we deduce

$$\frac{1}{u^{\delta-1}(t)}\geq\frac{1}{u^{\delta-1}(0)}-C. \tag{27}$$

Setting $r=C^{-\frac{1}{\delta-1}}$, we observe that if $u(0)=x_0-y_0<r$ then the second member of (27) is positive. Inequality (27) is then equivalent to

$$u(t)\leq\left(\frac{1}{u^{\delta-1}(0)}-C\right)^{-\frac{1}{\delta-1}}=(x_0-y_0)\,(1-C\,(x_0-y_0)^{\delta-1})^{-\frac{1}{\delta-1}}.$$

Defining the function $\theta:[0,r[\to\mathbb{R}_+$ by $\forall z\in[0,r[,\ \theta(z)=z\,(1-C\,z^{\delta-1})^{-\frac{1}{\delta-1}}$, the previous inequality can be rewritten as

$$x(t)-y(t)\leq\ \theta(x_0-y_0)\,e^{-t}. \tag{28}$$

Note that the previous inequality remains true when $x_0=y_0$, in which case $x(t)=y(t)$ for every $t\geq 0$. By combining (26) and (28), we finally obtain $y(t)\geq\frac{e^{-t}}{2}(x_0+y_0-\theta(x_0-y_0))$. It is then clear that $\theta(x_0-y_0)\leq x_0+y_0$ implies $y(t)\geq 0$ for every $t\geq 0$. Since $x(t)\geq y(t)$, we also have $x(t)\geq 0$ for every $t\geq 0$. ■

5 Open questions and further remarks

Below are listed some open questions and possible directions for future investigation. Assumptions of Theorem 3.1 are very stringent: the set S of equilibria of ϕ is one-dimensional and the level curves of ϕ are colinear to the direction of S. We conjecture that the result of Theorem 3.1 remains true without assumption (12). More generally, the extension of Theorem 3.1 to the case of multidimensional equilibrium sets is open. The proof technique that we use in the paper cannot be immediately extended to these situations.
¿From Theorem 3.1, the trajectories x and y of SDVR may possibly coincide at the limit when $t\to+\infty$, even if the function V modelizes a repulsive potential. To avoid this eventuality, a natural idea consists in introducing a "singular" potential V defined on $H\setminus\{0\}$ such that $\lim_{x\to 0}V(x)=+\infty$. This type of potential plays a central role in gravitational or electromagnetic theories. For example, when $V(x)=1/\|x\|$ it corresponds to the electric potential between two particles having the same sign. For further details, we refer the reader to [12], where the author studies the dynamics of a pair of oscillators

coupled by a singular potential.
Another extension consists in studying the system of $N \geq 3$ steepest descent equations coupled by a mutual repulsion. For large values of N, such a coupled system could help in finding a global description of the set of minima of ϕ and also estimates of its size.
For numerical purposes, it would be interesting to study a discretized version of SDVR by using a finite differencing scheme. These developments are out of the scope of this paper but certainly indicate a matter for future research.

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