DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **15**, Number **3**, July **2006**

Website: http://aimSciences.org

pp. **921–938**

ASYMPTOTIC SELECTION OF VISCOSITY EQUILIBRIA OF SEMILINEAR EVOLUTION EQUATIONS BY THE INTRODUCTION OF A SLOWLY VANISHING TERM

Felipe Alvarez

Departamento de Ingeniería Matemática
Centro de Modelamiento Matemático, Universidad de Chile
Av. Blanco Encalada 2120, 837-0459 Santiago, Chile

Alexandre Cabot

Département de Mathématiques, Université de Limoges
123, avenue Albert Thomas, Limoges, France

(Communicated by Roger Temam)

Abstract. The behavior at infinity is investigated of global solutions to some nonautonomous semilinear evolution equations with conservative and convex nonlinearities. It is proved that the trajectories converge to viscosity stationary solutions as time goes to infinity, that is, they evolve towards stationary solutions that are minimal with respect to a generalized viscosity criterion. Hierarchical viscosity selections and applications to specific nonlinear PDE are given.

1. **Introduction.** Let V and H be Hilbert spaces such that $V \subset H \subset V'$ with dense and continuous injections (V' stands for the topological dual of V). Given a bilinear continuous form $a : V \times V \to \mathbb{R}$, we define the linear continuous operator $A : V \to V'$ by $\langle Av, w\rangle_{V',V} = a(v, w)$. Consider a global solution $u(t)$ of the semilinear evolution problem:

$$u'(t) + Au(t) + f(u(t)) = 0, \tag{1}$$

where the nonlinearity $f : V \to H$ is supposed to be locally Lipschitz continuous and conservative, that is, $f(v) = F'(v)$ for some $F \in \mathcal{C}^1(V; \mathbb{R})$. Let us introduce the energy functional $E : V \to \mathbb{R}$ defined by $E(v) = \frac{1}{2}a(v, v) + F(v)$ so that $E'(v) = Av + f(v)$, and assume that E is convex. Within this framework, the stationary solutions of (1) are the critical points of E, i.e., the minimizers of E in the convex case. Since this dynamical system is dissipative in the sense that

$$\frac{d}{dt}E(u(t)) = -|u'(t)|^2,$$

2000 *Mathematics Subject Classification.* Primary: 35B40, 34G20; Secondary: 35K90, 37N40.
Key words and phrases. Evolution equation, asymptotic behavior, convergence, slow decay, viscosity solution, selection principle.
This work was partially supported by FONDAP grant in Applied Mathematics. The first author was also supported by FONDECYT grants 1020610 and 1050706, and Programa Iniciativa Científica Milenio. The second author was partially supported by ECOS-CONICYT C04E03.

it is natural to expect $u(t)$ to evolve toward a stationary solution. However, the set $S_0 := \operatorname{Argmin} E$ of the stationary solutions of (1) may consist of non isolated points; in that case, asymptotic convergence may fail without additional conditions. It is well-known that under global monotonicity of E', which amounts to convexity of E, it is possible to overcome such a lack of local uniqueness, ensuring the (possibly weak) convergence of $u(t)$, as $t \to \infty$, to a stationary solution (see for instance [6, 7]).

On the other hand, in many PDE applications a particular stationary solution is more interesting than others due to physical, economic or design considerations. When global convergence of trajectories holds, one could let the trajectory reach a particular target equilibrium by appropriately adjusting the initial conditions. Nevertheless, in many practical situations it is not possible to have an accurate control of the initial state. An alternative approach consists in introducing a term into the system which forces convergence to the desired stationary solution, independently of the initial state. Such a restoring term should vanish at infinity in order to recover, at least asymptotically, an equilibrium point of (1).

The above discussion motivates the following abstract evolution equation of the first-order in time:

$$u'(t) + Au(t) + f(u(t)) + \varepsilon(t)g(u(t)) = 0, \tag{2}$$

where $g \in \mathcal{C}(V; H)$ is monotone and conservative: $g(v) = G'(v)$ for some given convex function $G \in C^1(V; \mathbb{R})$ which, following [2], will be referred to as the "viscosity function". The minimizers of G on S_0 will be called "G-viscosity stationary solutions". The parameterization $\varepsilon(t)$ is supposed to be positive and $\varepsilon(t) \to 0$ as $t \to \infty$. We focus our attention on the parameterizations $t \mapsto \varepsilon(t)$ which satisfy in addition the following *slow decay* property:

$$\int_0^\infty \varepsilon(t)dt = \infty.$$

Such a condition has already been pointed out for the steepest descent method combined with Tikhonov viscosity-regularization in convex minimization [3] as well as for the stabilization of nonlinear oscillators [1, 4, 8]. For strongly convex viscosity functions, in those works it is proved that under appropriate conditions the trajectories tend to minimize the function G over the set S_0, provided that the slow decay property holds. In [3], conditions relying on the behavior of some approximate stationary solutions are required as well. In a rather different direction, the second author has studied in [9] a finite dimensional version of (2) and has proved that this asymptotic viscosity selection principle holds even under a lack of strong convexity. In this paper, we improve and generalize this type of asymptotic selection result, extending it to the setting of abstract semicoercive nonlinear equations. More precisely, we derive a general condition $(\mathcal{C})$ under which the trajectories of (2) are shown to satisfy $\lim_{t\to\infty} d_H(u(t), \operatorname{Argmin}_{S_0} G) = 0$, where $d_H(., \operatorname{Argmin}_{S_0} G)$ denotes the distance to the set $\operatorname{Argmin}_{S_0} G$ with respect to the norm $|\cdot|$ in H. Condition $(\mathcal{C})$ is sufficiently general to cover various situations occurring for example when the injection $V \hookrightarrow H$ is compact or when the operator g is strongly monotone. We also show that, if the parameterization map ε does not tend to zero "too slowly" (in a sense to be made precise), then the orbit $u(.)$ weakly converges in V. The question of the convergence in the case of a very slow control remains open without additional conditions.

When the selection principle associated with the viscosity function G is not sufficient to determine completely the limit points of u, we introduce a "second-order viscosity" as follows:

$$u'(t) + Au(t) + f(u(t)) + \varepsilon(t)g(u(t)) + \varepsilon_2(t)g_2(u(t)) = 0, \tag{3}$$

where the function $g_2 \in \mathcal{C}(V; H)$ is monotone, conservative and derives from the potential $G_2 \in C^1(V; \mathbb{R})$. The control map $\varepsilon_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be negligible with respect to ε. Setting $S_1 := \operatorname{Argmin}_{S_0} G$, we prove that, under adequate conditions, the solutions u of (3) tend to minimize the function G_2 on the set S_1. This hierarchical minimization result indicates that our techniques are flexible and likely to be generalized for $n \geq 2$ viscosity functions.

The paper is organized as follows. In section 2 we recall some preliminaries about nonlinear evolution equations and the associate functional setting. We also show that the trajectories of (2) are asymptotically attracted by the set $S_0 = \operatorname{Argmin} E$. In the sequel of the paper, the control map ε is assumed to be slow and we prove in section 3 that under general conditions, we have $\lim_{t\to\infty} d_H(u(t), \operatorname{Argmin}_{S_0} G) = 0$ (see Theorem 1). Section 4 deals with the problem of convergence of the trajectories themselves; in this direction, Theorem 2 shows that the orbits of (2) are weakly convergent in V provided that the control map ε is not "too slow". Section 5 is concerned with the generalized evolution equation (3) and the associate hierarchical viscosity selection. Finally, section 6 deals with some applications of the abstract results and describes some possible extensions and generalizations of them.

Let us conclude this introduction by mentioning that in order to enlarge the applicability in PDE theory of the methods of this paper, an interesting extension of our results could be the consideration of a nonlinear monotone operator $A : V \to V'$, with V being a Banach space (not necessary a Hilbert space) embedded in a Hilbert space H. Indeed, the essential property in our analysis is the monotonicity of the operator A, while linearity is apparently only a simplifying condition. In a similar direction, it should be possible to apply these techniques to some classes of evolutionary monotone differential inclusions. None of these aspects is developed in this paper.

2. **Preliminaries.**

Throughout this paper, V stands for a real Hilbert space, whose scalar product and norm are respectively denoted by $((\cdot,\cdot))$ and $\|\cdot\|$. Let H be another real Hilbert space with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. Suppose that V is dense in H with continuous injection: $\exists c > 0, \forall v \in V, |v| \leq c\|v\|$. By duality, the topological dual space H' of H is identified with a dense subspace of the topological dual V' of V. The norm in V' is still denoted by $|\cdot|$. Identifying H with H', we obtain $V \subset H \subset V'$, where each space is dense in the next one, each injection being continuous. Let $\langle\cdot,\cdot\rangle_{V',V}$ be the duality pairing between V' and V. With the previous notations and identifications, we have $\forall v \in V$, $\forall w \in H$, $\langle w, v\rangle_{V',V} = (w, v)$.

Let $a : V \times V \to \mathbb{R}$ be a bilinear continuous function which is associated with the linear continuous operator $A : V \to V'$ given by

$$\langle Av, w\rangle_{V',V} = a(v, w).$$

One may also consider $A : D(A) \subseteq H \to H$ as a linear unbounded operator in H with domain $D(A) = \{v \in V \mid Av \in H\}$. Assume that:

$$(\mathrm{h}_1) \qquad \begin{cases} \text{Symmetry: } \forall v, w \in V,\ a(v, w) = a(w, v). \\ \text{Monotonicity: } \forall v \in V,\ a(v, v) \geq 0. \end{cases}$$

In some applications these conditions are supplemented with the following semicoercivity property: there exist $\lambda \geq 0$ and $\mu > 0$ such that

$$\forall v \in V,\ a(v,v) + \lambda |v|^2 \geq \mu \|v\|^2. \tag{4}$$

Let $f : V \to H$ be a locally Lipschitz continuous, monotone and conservative function. In fact, assume:

(h_2) $\forall v, w \in V$, $(f(v) - f(w), v - w) \geq 0$ and $(f(v), w) = \langle F'(v), w\rangle_{V',V}$ for some $F \in \mathcal{C}^1(V;\mathbb{R})$.

Defining $E : V \to \mathbb{R}$ by

$$E(v) := \tfrac{1}{2} a(v,v) + F(v),$$

we obtain a function of class $\mathcal{C}^1$ whose first derivative is given by $\langle E'(v), w\rangle_{V',V} = a(v,w) + (f(v), w)$, or equivalently

$$E'(v) = Av + f(v).$$

Moreover, E is convex, which amounts to

$$\forall v, w \in V,\ \ a(v, w - v) + (f(v), w - v) \leq E(w) - E(v). \tag{5}$$

Consequently, stationary and minimum points of E coincide, i.e.,

$$\operatorname{Argmin} E = (E')^{-1}(\{0\}) = \{v \in V \mid Av + f(v) = 0\}, \tag{6}$$

where $\operatorname{Argmin} E := \{v \in V \mid E(v) = \inf E\}$. This set is convex, closed and is contained in $D(A)$. Suppose moreover that

(h_3) $S_0 := \operatorname{Argmin} E$ is nonempty.

Remark 1. If in addition we have $f(0) = 0$ then $S_0 = \ker A \cap \{v \in V \mid f(v) = 0\}$. Indeed, if $w \in S_0$ then in particular $(Aw, w) + (f(w), w) = 0$, and by monotonicity of f we have $(f(w) - f(0), w) \geq 0$, hence $(Aw, w) = (f(w), w) = 0$ and therefore $Aw = 0$.

Let $g : V \to H$ be a locally Lipschitz continuous function satisfying the following condition: there exists $G \in \mathcal{C}^1(V;\mathbb{R})$ such that

$$(\mathrm{h}_4) \quad \begin{cases} \forall v, w \in V,\ (g(v) - g(w), v - w) \geq 0 \text{ and } (g(v), w) = \langle G'(v), w\rangle_{V',V}. \\ G \text{ is bounded from below.} \\ S_1 := \operatorname{Argmin}_{S_0} G \text{ is nonempty.} \end{cases}$$

Of course, G is convex and satisfies:

$$\forall v, w \in V,\ \ (g(v), w - v) \leq G(w) - G(v). \tag{7}$$

Given $\varepsilon \in W^{1,\infty}_{loc}((0,\infty);\mathbb{R})$, consider the abstract evolution equation (2). A function $u \in \mathcal{C}([0,\infty); H) \cap \mathcal{C}((0,\infty); V) \cap L^2_{loc}((0,\infty); V) \cap W^{1,2}_{loc}((0,\infty); V')$ is said to be a global solution of (2) if for every $v \in V$,

$$\frac{d}{dt}(u(\cdot), v) + a(u(\cdot), v) + (f(u(\cdot)), v) + \varepsilon(\cdot)(g(u(\cdot)), v) = 0, \tag{8}$$

in the scalar distribution sense on $(0,\infty)$. As for every $v \in V$, $\langle u'(\cdot), v\rangle_{V',V} = \frac{d}{dt}(u(\cdot), v)$ in the scalar distribution sense (see [11, 15]), (8) may be written in vectorial form:

$$u'(\cdot) + Au(\cdot) + f(u(\cdot)) + \varepsilon(\cdot) g(u(\cdot)) = 0 \ \text{ in } L^2_{loc}((0,\infty); V'). \tag{9}$$

In particular, (2) holds as an equality in V' for a.e. $t > 0$. From now on, we assume the existence of global solutions of (2) satisfying

$$\dot{L}_u(t) \leq -|u'(t)|^2 + \dot{\varepsilon}(t)[G(u(t)) - \inf G], \text{ a.e. } t \geq 0, \tag{10}$$

where

$$L_u(t) := E(u(t)) - \inf E + \varepsilon(t)[G(u(t)) - \inf G], \tag{11}$$

is supposed to be absolutely continuous on $[0, \infty)$. For general results concerning existence and uniqueness of global solutions, we refer the reader to [6, 11, 12, 13, 14, 17].

We are interested in the asymptotic behavior as $t \to \infty$ of global solutions to (2), under the assumption that the map $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\lim_{t\to\infty} \varepsilon(t) = 0$ and $\dot{\varepsilon}(t) \leq 0$ for a.e. $t \geq 0$. Under this condition, (10) shows that L_u is nonincreasing on $[0, \infty)$. This property is crucial for the asymptotic analysis of (2); in particular, it implies that the trajectories of (2) are minimizing for the energy functional E.

Proposition 1. *Under* (h_1)-(h_4), *if u is a global solution of* (2) *with* $\lim_{t\to\infty} \varepsilon(t) = 0$ *and* $\dot{\varepsilon}(t) \leq 0$ *for a.e.* $t \geq 0$, *then* $u' \in L^2((0,\infty); V')$ *and* $L_u(t) \to 0$ *as* $t \to \infty$, *with* $L_u(t)$ *being defined by* (11). *In particular,* $\lim\limits_{t\to\infty} E(u(t)) = \min E$.

Proof. In view of (10) and since $\dot{\varepsilon} \leq 0$ a.e. on $(0, \infty)$, we have $\dot{L}_u(t) \leq -|u'(t)|^2$ for a.e. $t \geq 0$. Integrating on $[0, \infty)$, we obtain $\int_0^\infty |u'(s)|^2\, ds \leq E(0)$, *i.e.*, $u' \in L^2(0, \infty; V')$. Let us now prove that $\lim_{t\to\infty} L_u(t) = 0$. Since $\dot{L}_u \leq 0$ a.e. on $(0, \infty)$, L_u is nonincreasing and $l := \lim_{t\to\infty} L_u(t)$ exists. We argue by contradiction and assume that $l > 0$. Let $z \in S_0$ and consider the absolutely continuous function ψ defined by $\psi(t) := \frac{1}{2}|u(t) - z|^2$. Differentiating ψ, we find (see [15]) that for a.e. $t > 0$, $\dot{\psi}(t) = \langle u'(t), u(t) - z\rangle_{V',V} = -a(u(t), u(t) - z) - (f(u(t)), u(t) - z) - \varepsilon(t)\,(g(u(t)), u(t) - z)$. From inequalities (5) and (7), we then deduce that

$$\dot{\psi}(t) + (E(u(t)) - \min E) + \varepsilon(t)[G(u(t)) - G(z)] \leq 0, \tag{12}$$

or equivalently $\dot{\psi}(t) + L_u(t) + \varepsilon(t)\,(\inf G - G(z)) \leq 0$. Letting $t \to \infty$, we obtain $\limsup_{t\to\infty} \dot{\psi}(t) \leq -l$. Since $l > 0$, we deduce that $\lim_{t\to\infty} \psi(t) = -\infty$, which is impossible. □

Notice that the function G does not arise directly in the conclusions of Proposition 1. This is due to the fact that the parametrization function ε is allowed to decay quickly toward 0 (for instance, $\varepsilon \equiv 0$). We supplement the hypotheses on ε with a "slow decay" condition; in fact, we assume:

(h_5) $\lim\limits_{t\to\infty} \varepsilon(t) = 0$, $\dot{\varepsilon}(t) \leq 0$ for a.e. $t \geq 0$, and $\displaystyle\int_0^\infty \varepsilon(t)dt = \infty$.

The consequences of this additional assumption are investigated in the next section.

3. Minimization of the viscosity G over S_0 under the slow decay condition.

When the spaces H and V are finite dimensional (in which case $H = V$), the asymptotic behaviour of the solutions of the (ordinary) differential equation (2) has been studied in [9]. Roughly speaking, it is shown that under the assumption $\int_0^\infty \varepsilon(t)dt = \infty$, then the solutions of (2) tend to minimize the function G over the set S_0. This result can also be recovered by a repeated application of a former result due to Baillon-Cominetti [5, Theorem 2.1]. In another direction, Attouch and Czarnecki have studied in [4] a second-order in time version of (2) involving a slow control ε and the identity operator $u \mapsto g(u) = u$. They show in an infinite dimensional setting that the associated trajectories strongly converge to the minimal norm element of S_0. The purpose of this section is to improve and extend these convergence results within the abstract framework described is section 2.

3.1. **The main result.** In the sequel, we will assume that the functions E and G satisfy the following general condition: for every sequence (u_n) in V, we have:

$$(\mathcal{C}) \qquad \left.\begin{array}{l} \lim_{n\to\infty} E(u_n) = \min E \\ \limsup_{n\to\infty} G(u_n) \leq \min_{S_0} G \end{array}\right\} \quad \Longrightarrow \quad \lim_{n\to\infty} d_H(u_n, S_1) = 0,$$

where $S_0 := \operatorname{Argmin} E$ and $d_H(u, S_1) = \inf_{v\in S_1} |u - v|$ denotes the distance in H between u and the set $S_1 := \operatorname{Argmin}_{S_0} G$. It is worth noting that condition $(\mathcal{C})$ applies in various situations, e.g. when the injection $V \hookrightarrow H$ is compact (see Corollary 1) or when the function G is strongly convex (see Corollary 3).

Theorem 1. *Under* (h$_1$)-(h$_5$)*, assume that condition* $(\mathcal{C})$ *holds. Then, for any global solution* u *of* (2)*, we have*

$$\lim_{t\to\infty} d_H(u(t), S_1) = 0.$$

Proof. The proof relies on the study of the function $h(t) = \frac{1}{2} d_H(u(t), S_1)^2$. We claim that h is absolutely continuous and moreover $\dot{h}(t) = \langle u'(t), u(t) - P_{S_1}(u(t))\rangle_{V',V}$ for a.e. $t > 0$ (see [15]), where P_{S_1} is the orthogonal projection on S_1 with respect to $(\cdot,\cdot)$. Therefore, we infer from (2) that $\dot{h}(t) = -a(u(t), u(t) - P_{S_1}(u(t))) - (f(u(t)), u(t) - P_{S_1}(u(t))) - \varepsilon(t)\,(g(u(t)), u(t) - P_{S_1}(u(t)))$, a.e. $t > 0$. From (5) and (7), we then deduce that $\dot{h}(t) + (E(u(t)) - \min E) + \varepsilon(t)[G(u(t)) - \min_{S_0} G] \leq 0$ a.e. in $[0, \infty)$. Hence

$$\dot{h}(t) + \varepsilon(t)[G(u(t)) - \min_{S_0} G] \leq 0, \text{ a.e. } t > 0. \tag{13}$$

The main idea of the proof is now to respectively distinguish the cases where $G(u(t)) > \min_{S_0} G$ and $G(u(t)) \leq \min_{S_0} G$. Precisely, we distinguish the two cases:

(a) $\exists T \geq 0$, $\forall t \geq T$, $G(u(t)) > \min_{S_0} G$.

(b) $\forall T \geq 0$, $\exists t \geq T$, $G(u(t)) \leq \min_{S_0} G$.

Case (a). We assume that there is $T \geq 0$, such that, for every $t \geq T$, $G(u(t)) \geq \min_{S_0} G$. We deduce from inequality (13) that, for every $t \geq T$, $\dot{h}(t) \leq 0$ and hence $\lim_{t\to\infty} h(t)$ exists. Integrating (13) on $[T, \infty[$ we deduce that $\int_T^\infty \varepsilon(t)[G(u(t)) - \min_{S_0} G]dt < \infty$. Since $\int_0^\infty \varepsilon(t)dt = \infty$, we infer that $\liminf_{t\to\infty} G(u(t)) = \min_{S_0} G$. Let $t_n \to \infty$ be such that $\lim_n G(u(t_n)) = \min_{S_0} G$. From Proposition 1, we have $\lim_{n\to\infty} E(u(t_n)) = \lim_{t\to\infty} E(u(t)) = \min E$. Applying condition $(\mathcal{C})$ with the sequence $(u(t_n))$, we obtain that $\lim_{n\to\infty} d_H(u(t_n), S_1) = 0$ and since the map $t \mapsto d_H(u(t), S_1)$ is convergent, we conclude that $\lim_{t\to\infty} d_H(u(t), S_1) = 0$.

Case (b). We now assume that, for every $T \geq 0$, there exists some $t \geq T$ such that $G(u(t)) < \min_{S_0} G$.

For every $t \geq 0$, let us define

$$\tau(t) := \sup\left\{t' \in [0, t] \;\middle|\; G(u(t')) \leq \min_{S_0} G\right\}.$$

First notice that, since case (b) holds, the quantity $\tau(t)$ is well-defined as soon as t is large enough. The following inequality holds for t large enough:

$$d_H(u(t), S_1) \leq d_H(u(\tau(t)), S_1). \tag{14}$$

Indeed, if $G(u(t)) \leq \min_{S_0} G$, then $\tau(t) = t$ and (14) follows immediately. Assume that $G(u(t)) > \min_{S_0} G$. For every $t' \in [\tau(t), t]$, $G(u(t')) \geq \min_{S_0} G$. From (13), we deduce that $\dot{h} \leq 0$ a.e. in $[\tau(t), t]$, which immediately yields (14). Now, if we are able to prove that $\lim_{t\to+\infty} d_H(u(\tau(t)), S_1) = 0$, then inequality (14) immediately implies that $\lim_{t\to+\infty} d_H(u(t), S_1) = 0$. From Proposition 1, we have

$$\lim_{t\to+\infty} E(u(\tau(t))) = \lim_{t\to+\infty} E(u(t)) \min E. \tag{15}$$

On the other hand, the definition of $\tau(t)$ gives $G(u(\tau(t))) \leq \min_{S_0} G$, so that

$$\limsup_{t\to+\infty} G(u(\tau(t))) \leq \min_{S_0} G. \tag{16}$$

From (15), (16) and condition $(\mathcal{C})$, we deduce that $\lim\limits_{t\to+\infty} d_H(u(\tau(t)), S_1) = 0$, which concludes the proof of (b). □

Remark 2. The main idea of the proof of Theorem 1 consists in the study of the function $h(t) = \frac{1}{2} d_H(u(t), S_1)^2$ by the distinction of cases (a) and (b). In the context of control and stabilization of nonlinear oscillators, this type of proof has been initiated by Attouch-Czarnecki [4].

3.2. **Special cases of the main result.** In this section we exhibit some situations where condition $(\mathcal{C})$ arising in Theorem 1 holds. Let us begin with a technical lemma which explores the links between strong convergences in H and V for bounded and minimizing energy sequences under the semicoercivity condition (4).

Lemma 1. *Suppose* (h_1)-(h_3) *together with* (4)*, and let* $(u_n) \subset V$.
(i) *If* $\sup_n |u_n| < \infty$ *and* $\sup_n E(u_n) < \infty$ *then* $\sup_n \|u_n\| < \infty$.
(ii) *If* $\lim_n |u_n - p| = 0$ *for some* $p \in S_0$ *and* $\lim_n E(u_n) = \min E$*, then* $\lim_n \|u_n - p\| = 0$.

Proof. (i) Since F is convex continuous in V, we have $F(v) \geq b\|v\| - c$ for some constants $b, c \in \mathbb{R}$ (possibly with $b < 0$). By semicoercivity, $\frac{\mu}{2}\|v\|^2 - \frac{\lambda}{2}|v|^2 \leq \frac{1}{2}a(v,v) = E(v) - F(v) \leq E(v) - b\|v\| + c$. Hence, $\frac{\mu}{2}\|v\|^2 + b\|v\| \leq E(v) + \frac{\lambda}{2}|v|^2 + c$. From this estimate, we deduce that (u_n) is bounded in V.
(ii) Assume that $E(u_n) \to \min E$ and $u_n \to p$ strongly in H with $p \in S_0$. From (i) it follows that (u_n) is bounded in V, hence $u_n \rightharpoonup p$ for the weak topology of V. It remains to prove that $u_n \to p$ strongly in V. By semicoercivity, we have $\mu\|v - p\|^2 \leq \lambda|v - p|^2 + a(v - p, v - p)$, which we rewrite $\frac{\mu}{2}\|v - p\|^2 \leq \frac{\lambda}{2}|v - p|^2 + E(v) + \frac{1}{2}a(p,p) - a(v,p) - F(v)$. Since $u_n \rightharpoonup p$ weakly in V, we have $\lim_n a(u_n, p) = a(p,p)$. On the other hand, by the weak lower semi-continuity of the continuous convex function $F : V \to \mathbb{R}$, we deduce that $\liminf_n F(u_n) \geq F(p)$. Therefore $\limsup_n [\frac{1}{2}a(p,p) - a(u_n,p) - F(u_n)] \leq -\frac{1}{2}a(p,p) - F(p) = -E(p)$. Recalling that $\lim_n E(u_n) = \min E$ and $E(p) = \min E$ (since $p \in S_0$), we obtain that $\limsup_n \|u_n - p\|^2 \leq \frac{\lambda}{\mu} \lim_n |u_n - p|^2 = 0$, which completes the proof. □

Corollary 1 (Compact injection $V \hookrightarrow H$)**.** *Under* (h_1)-(h_5) *and* (4)*, suppose that the injection of* V *into* H *is compact. Assume moreover that the following condition holds: there exists* $M_0 > \min E$ *and* $M_1 > \min_{S_0} G$ *such that*

$$\textit{the set} \quad [E \leq M_0] \cap [G \leq M_1] \quad \textit{is bounded in } H. \tag{17}$$

Then, any global solution u of (2) *satisfies* $\lim_{t\to+\infty} d_H(u(t), S_1) = 0.$

Proof. It suffices to apply Theorem 1 by showing that condition $(\mathcal{C})$ holds. For that purpose, let us consider a sequence (u_n) in V satisfying $\lim_{n\to\infty} E(u_n) = \min E$ and $\limsup_{n\to\infty} G(u_n) \leq \min_{S_0} G$. First remark that condition (17) implies the boundedness of (u_n) in H and hence the boundedness of the sequence $(d_H(u_n, S_1))_n$ in $\mathbb{R}$. Let $\alpha \geq 0$ be a cluster point of the sequence $d_H(u_n, S_1)$ when $n \to \infty$: there exists a sequence $n_k \to \infty$ such that $\lim_{k\to\infty} d_H(u_{n_k}, S_1) = \alpha$. By Lemma 1(i), (u_n) is bounded in V, hence relatively compact for the weak topology of V. As a consequence, there exists a subsequence of (u_{n_k}), still denoted by (u_{n_k}) which weakly converges to $\overline{u}$ in V. By the lower semicontinuity of E with respect to the weak topology in V, we have $E(\bar{u}) \leq \liminf_{k\to\infty} E(u_{n_k}) = \lim_{n\to+\infty} E(u_n) = \min E$, that is, $\bar{u} \in \operatorname{Argmin} E = S_0$. Similarly, the lower semicontinuity of G with respect to the weak topology in V yields: $G(\bar{u}) \leq \liminf_{k\to\infty} G(u_{n_k}) \leq \limsup_{n\to\infty} G(u_n) \leq \min_{S_0} G$, and hence $\bar{u} \in [G \leq \min_{S_0} G] \cap S_0 = S_1$. Since the injection $V \hookrightarrow H$ is compact, $u_{n_k} \to \bar{u} \in S_1$ strongly in H and hence $\lim_{k\to\infty} d_H(u_{n_k}, S_1) = d_H(\bar{u}, S_1) = 0$. Since 0 is the unique cluster point of the bounded sequence $(d_H(u_n, S_1))_n$, we conclude that $\lim_{n\to+\infty} d_H(u_n, S_1) = 0$ and hence condition $(\mathcal{C})$ is satisfied. □

Remark 3. Corollary 1 holds without the semicoercivity condition (4) by replacing (17) with

$$\textit{the set} \quad [E \leq M_0] \cap [G \leq M_1] \quad \textit{is bounded in } V.$$

On the other hand, when $H = V = \mathbb{R}^n$ we recover the result of [9, Corollary 2.2].

Let us now turn to another class of examples in which condition $(\mathcal{C})$ is fulfilled. We assume that the function G satisfies the following property:

$$\forall v \in S_1, \quad \forall w \in V, \quad G(w) - \min_{S_0} G \geq (g(v), w - v) + \beta(d_H(w, S_1)), \tag{18}$$

where the function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\lim_{t\to\infty} \beta(t)/t = \infty$ and $[\beta(t_n) \to 0 \Rightarrow t_n \to 0]$, for every sequence $(t_n) \subset \mathbb{R}_+$. Condition (18) can be viewed as a kind of (relaxed) strong convexity assumption.

Corollary 2. *Under* (h_1)-(h_5) *and* (4)*, assume that S_1 is bounded in H and $G : V \to \mathbb{R}$ satisfies* (18)*. Then, any global solution u of* (2) *satisfies*

$$\lim_{t\to+\infty} d_H(u(t), S_1) = 0.$$

Proof. Let us prove that condition $(\mathcal{C})$ of Theorem 1 is satisfied. For that purpose, let us consider a sequence (u_n) in V satisfying $\lim_{n\to\infty} E(u_n) = \min E$ and $\limsup_{n\to\infty} G(u_n) \leq \min_{S_0} G$. We first show that the sequence (u_n) is bounded in H. For every $M_1 > \min_{S_0} G$, there exists $n_0 \in \mathbb{N}$ such that

$$\{u_n,\, n \geq n_0\} \subset [G \leq M_1]. \tag{19}$$

Fix some $\bar{v} \in S_1$ in (18) and use the Cauchy-Schwarz inequality to obtain:

$$\forall w \in V, \quad G(w) \geq \min_{S_0} G - |g(\bar{v})|\, |w - \bar{v}| + \beta(d_H(w, S_1)).$$

Denoting by $\delta(S_1)$ the diameter in H of the bounded set S_1, we have $|w - \bar{v}| \leq d_H(w, S_1) + \delta(S_1)$, so that the previous inequality implies

$$\forall w \in V, \quad G(w) \geq \min_{S_0} G - |g(\bar{v})|\, \delta(S_1) - |g(\bar{v})|\, d_H(w, S_1) + \beta(d_H(w, S_1)).$$

Since $\lim_{t\to\infty}\beta(t)/t=\infty$, we deduce that $\lim_{|w|\to\infty}G(w)=\infty$. Therefore the sublevel set $[G\le M_1]$ is bounded in H and from (19), the sequence (u_n) is bounded in H.

Let $\alpha\ge 0$ be a cluster point of the bounded sequence $d_H(u_n,S_1)$ when $n\to\infty$. There exists a sequence $n_k\to\infty$ such that $\lim_{k\to\infty}d_H(u_{n_k},S_1)=\alpha$. By Lemma 1(i), (u_n) is bounded in V, hence relatively compact for the weak topology of V. As a consequence, there exists a subsequence of (u_{n_k}), still denoted by (u_{n_k}) which weakly converges to $\bar u$ in V. By the lower semicontinuity of E with respect to the weak topology in V, we have $E(\bar u)\le\liminf_{k\to\infty}E(u_{n_k})=\lim_{n\to+\infty}E(u_n)=\min E$, that is, $\bar u\in\operatorname{Argmin}E=S_0$. Similarly, the lower semicontinuity of G with respect to the weak topology in V yields: $G(\bar u)\le\liminf_{k\to\infty}G(u_{n_k})\le\limsup_{n\to\infty}G(u_n)\le\min_{S_0}G$, and hence $\bar u\in[G\le\min_{S_0}G]\cap S_0=S_1$. On the other hand, from (18) we deduce that

$$\beta(d_H(u_{n_k},S_1))\le G(u_{n_k})-\min_{S_0}G-\ (g(\bar u),u_{n_k}-\bar u). \tag{20}$$

Since the injection from V into H is continuous, the sequence (u_{n_k}) weakly converges to $\bar u$ in H, thus implying that $\lim_{k\to\infty}(g(\bar u),u_{n_k}-\bar u)=0$. Taking the upper limit in (20) when $k\to\infty$ and using the fact that $\limsup_{k\to\infty}G(u_{n_k})\le\min_{S_0}G$, we obtain $\limsup_{k\to\infty}\beta(d_H(u_{n_k},S_1))\le 0$. Since on the other hand, $\liminf_{k\to\infty}\beta(d_H(u_{n_k},S_1))\ge 0$, we deduce $\lim_{k\to\infty}\beta(d_H(u_{n_k},S_1))=0$, which combined with the assumption on β finally yields $\lim_{k\to\infty}d_H(u_{n_k},S_1)=0$. Since 0 is the unique cluster point of the bounded sequence $(d_H(u_n,S_1))_n$, we conclude that $\lim_{n\to+\infty}d_H(u_n,S_1)=0$ and hence condition $(\mathcal{C})$ is satisfied. □

In the next corollary, we specialize the setting of Corollary 2 by assuming that the function G is strongly convex in V with respect to the norm of H, *i.e.,*

$$\forall v,w\in V,\quad G(w)-G(v)\ge(g(v),w-v)+\beta(|w-v|), \tag{21}$$

where the function $\beta:\mathbb{R}_+\to\mathbb{R}_+$ is such that $\lim_{t\to\infty}\beta(t)/t=\infty$ and $[\beta(t_n)\to 0\Rightarrow t_n\to 0]$, for every sequence $(t_n)\subset\mathbb{R}_+$. In this case, the set S_1 is reduced to a singleton so that we obtain the strong convergence of the trajectories of (2).

Corollary 3 (g strongly monotone)**.** *Under* (h_1)-(h_5) *and* (4)*, suppose that the function $G:V\to\mathbb{R}$ satisfies the strong convexity property* (21)*. Then the following holds:*
(i) There exists $p\in V$ such that $\operatorname{Argmin}_{S_0}G=\{p\}$.
(ii) If u is a solution of (2) *then $\lim_{t\to\infty}u(t)=p$ strongly in V.*

Proof. (i) Let us first remark that the strict convexity of G implies that $\operatorname{Argmin}(G+\delta_{S_0})=\operatorname{Argmin}_{S_0}G$ contains at most one point. Let us now prove that $\operatorname{Argmin}_{S_0}G\neq\emptyset$. From a classical result, it suffices to prove that the function $G+\delta_{S_0}$ is coercive for the strong topology of V, *i.e.,* $\lim_{||w||\to\infty}(G+\delta_{S_0})(w)=\infty$. Let us argue by contradiction and assume that there exist $M\in\mathbb{R}$ and a sequence (w_n) in V such that $\lim_{n\to\infty}||w_n||=\infty$ and $(G+\delta_{S_0})(w_n)\le M$. This means that, for every $n\in\mathbb{N}$, $w_n\in S_0$ and

$$G(w_n)\le M. \tag{22}$$

Since $\sup_n||w_n||=\infty$ and $(w_n)\subset S_0$, we deduce in view of Lemma 1(i) that $\sup_n|w_n|=\infty$. Hence there exists a subsequence (w_{n_k}) such that $\lim_{k\to\infty}|w_{n_k}|=\infty$. On the other hand, taking $v=0$ in inequality (21), we obtain

$$G(w_{n_k})-G(0)\ge(g(0),w_{n_k})+\beta(|w_{n_k}|)\ge-|g(0)|\,|w_{n_k}|+\beta(|w_{n_k}|).$$

From the assumption on β, we have $\lim_{k\to\infty} \beta(|w_{n_k}|)/|w_{n_k}| = \infty$, so that $\lim_{k\to\infty} G(w_{n_k}) = \infty$ and we obtain a contradiction with (22).

(ii) Since $\operatorname{Argmin}_{S_0} G = \{p\}$, it is immediate that inequality (21) implies (18). Hence Corollary 2 applies and $u(t)$ strongly converges to p in H when $t \to \infty$. The strong convergence in V is then an immediate consequence of Lemma 1 (ii). □

When g is the identity operator, the associate function $G = |\,.\,|^2/2$ is trivially strongly convex. In this case, Corollary 3 implies the strong convergence (in V) of the trajectories toward the element of minimal norm of S_0 . This type of stabilization result associated with the Tikhonov regularization [18] has already been pointed out in the context of nonlinear ODE's (see for example [3, 4]).

4. **Convergence toward some $\bar{u} \in \operatorname{Argmin}_{S_0} G$.** The purpose of this section is to determine sufficient conditions ensuring the convergence of the trajectory toward a particular point of $S_1 = \operatorname{Argmin}_{S_0} G$. If S_1 is reduced to a singleton, the trajectory u strongly converges in H (and even in V) in view of Theorem 1. In the general case the convergence can be obtained by strengthening the assumptions on the map ε. Before stating the result, let us introduce the function $\omega : \mathbb{R}_+ \to \mathbb{R}$ defined by:

$$\omega(t) := \inf_{v\in V} [E(v) - \min E \,+\, \varepsilon(t)\,(G(v) - \min_{S_0} G)]. \tag{23}$$

The function ω is minorized by

$$\omega(t) \geq -\varepsilon(t)\,(\min_{S_0} G - \inf G).$$

The following proposition establishes that the negative part $\omega^-(t) = \max\{-\omega(t), 0\}$ of the quantity $\omega(t)$ is negligible with respect to $\varepsilon(t)$ when $t \to \infty$.

Proposition 2. *Assume that the functions E and $G : V \to \mathbb{R}$ satisfy the hypotheses of Theorem 1. Suppose moreover that the S_1 is bounded in H. Then $\omega(t)$ defined by (23) satisfies: $\lim_{t\to\infty} \omega^-(t)/\varepsilon(t) = 0$.*

Proof. Let us argue by contradiction and assume that the function ω satisfies $\liminf_{t\to\infty} \omega(t)/\varepsilon(t) < 0$. Then there exist $\eta > 0$ and a sequence $(\varepsilon(t_n))$ tending toward 0 such that

$$\forall n \in \mathbb{N}, \quad \inf_{v\in V} [E(v) - \min E \,+\, \varepsilon(t_n)\,(G(v) - \min_{S_0} G)] \leq -\eta\, \varepsilon(t_n).$$

Therefore, there exists a sequence (v_n) in V such that

$$\forall n \in \mathbb{N}, \quad E(v_n) - \min E \,+\, \varepsilon(t_n)\,(G(v_n) - \min_{S_0} G) \leq -\frac{\eta}{2}\, \varepsilon(t_n). \tag{24}$$

Noticing that $G(v_n) \geq \inf G$ and taking the upper limit when $n \to +\infty$, we find $\limsup_{n\to+\infty} E(v_n) \leq \min E$ and hence

$$\lim_{n\to+\infty} E(v_n) = \min E. \tag{25}$$

Since on the other hand, $E(v_n) \geq \min E$, we infer from (24) that

$$\limsup_{n\to+\infty} G(v_n) \leq \min_{S_0} G - \frac{\eta}{2}. \tag{26}$$

From (25), (26) and condition $(\mathcal{C})$, we have $\lim_{n\to+\infty} d_H(v_n, S_1) = 0$. Since the set S_1 is bounded in H, we deduce that the sequence (v_n) is bounded in H. From Lemma 1 (i), the sequence (v_n) is also bounded in V. Therefore there exist $\bar{v} \in$

V and a subsequence (v_{n_k}) of (v_n) that weakly converges to $\bar{v}$ in V. From the closedness of E (resp. G) with respect to the weak topology in V and inequality (25) (resp. (26)), we deduce respectively that

$$E(\bar{v}) \leq \liminf_{k\to+\infty} E(v_{n_k}) = \lim_{k\to+\infty} E(v_{n_k}) = \min E,$$

$$G(\bar{v}) \leq \liminf_{k\to+\infty} G(v_{n_k}) \leq \limsup_{k\to+\infty} G(v_{n_k}) \leq \min_{S_0} G - \frac{\eta}{2}.$$

The first inequality implies that $\bar{v} \in S_0$ and the second one gives the contradiction. □

Remark 4. For the previous result it suffices to have $\lim_{t\to\infty} \varepsilon(t) = 0$.

Let us now give examples for which it is possible to compute explicitly the map ω, or at least a lower bound for ω.

Proposition 3. *Assume that the functions E and $G : V \to \mathbb{R}$ satisfy the hypotheses of Theorem* 1.
(a) *If* $\operatorname{Argmin} E \cap \operatorname{Argmin} G \neq \emptyset$ *then* $\omega \geq 0$.
(b) *Assume that there exist $a > 0$, $b > 0$ and $p \geq 1$ such that*
(i) $E - \min E \geq a\, d_H(., S_0)^p$,
(ii) $G - \min_{S_0} G \geq -b\, d_H(., [G \geq \min_{S_0} G])$.
Then there exist $\alpha \geq 0$ and $q > 1$ such that $\omega(t) \geq -\alpha\, \varepsilon(t)^q$ for t large enough (when $p > 1$ the exponent q is the conjugate of p, i.e., $q = 1/(1 - 1/p)$).

Proof. (a) Notice that the assumption $\operatorname{Argmin} E \cap \operatorname{Argmin} G \neq \emptyset$ implies that $\operatorname{Argmin}_{S_0} G = \operatorname{Argmin} E \cap \operatorname{Argmin} G$ and $\min_{S_0} G = \min G$. As a consequence,

$$E - \min E + \varepsilon(t)\,(G - \min_{S_0} G) E - \min E + \varepsilon(t)\,(G - \min G) \geq 0$$

and $\omega \geq 0$.
(b) Since the set S_0 is included in the set $[G \geq \min_{S_0} G]$, we have $d_H(., S_0) \geq d_H(., [G \geq \min_{S_0} G])$ which combined with the assumption on G, implies $G - \min_{S_0} G \geq -b\, d_H(., S_0)$. Taking into account assumption (i), we deduce from the previous inequality that, for every $v \in H$,

$$E(v) - \min E + \varepsilon(t)\,(G(v) - \min_{S_0} G) \geq a\, d_H(v, S_0)^p - b\,\varepsilon(t)\, d_H(v, S_0).$$

First assume that $p = 1$. It is then immediate that we have, for t large enough

$$E(v) - \min E + \varepsilon(t)\,(G(v) - \min_{S_0} G) \geq 0,$$

so that $\omega \geq 0$. Now assume that $p > 1$. An elementary computation then shows that

$$a\, d_H(v, S_0)^p - b\,\varepsilon(t)\, d_H(v, S_0) \geq -b \left(\frac{b}{p\,a}\right)^{\frac{1}{p-1}} \left(\frac{p-1}{p}\right) \varepsilon(t)^{\frac{p}{p-1}},$$

so that the expected inequality holds with $\alpha = b \left(\frac{b}{p\,a}\right)^{\frac{1}{p-1}} \left(\frac{p-1}{p}\right)$ and $q = \frac{p}{p-1}$. □

Let us return to the convergence of the trajectory u associated with (2). Proposition 2 shows the existence of a gap between the functions $t \mapsto \varepsilon(t)$ and $t \mapsto \omega^-(t)$. The next theorem shows how to exploit this property to deduce the convergence of u.

Theorem 2. *Under the hypotheses of Theorem 1, assume moreover that*

$$\int_0^{+\infty} \omega^-(t) < +\infty,$$

where the function $\omega : \mathbb{R}_+ \to \mathbb{R}$ *is defined by* (23). *Then, any global solution* u *of* (2) *weakly converges in* V *to some* $\bar{u} \in S_1$. *Moreover,* $u(t) \to \bar{u}$ *strongly in* V *iff* $u(t) \to \bar{u}$ *strongly in* H.

Proof. Given any $z \in S_1$, let us define the map ψ by $\psi(t) := \frac{1}{2}|u(t) - z|^2$. The same computation as in the proof of Proposition 1 shows that

$$\dot{\psi}(t) + \Big(E(u(t)) - \min E + \varepsilon(t)\,(G(u(t)) - \min_{S_0} G)\Big) \leq 0, \quad \text{a.e. } t > 0, \tag{27}$$

and hence

$$\dot{\psi}(t) - \omega^-(t) \leq \dot{\psi}(t) + \omega(t) \leq 0, \text{ a.e. } t > 0.$$

This implies that $\dot{\psi}^+(t) \leq \omega^-(t)$, a.e. $t > 0$. By using the fact that $\int_0^{+\infty} \omega^-(t) < +\infty$, we obtain in view of the following elementary lemma that $\lim_{t\to+\infty} \psi(t)$ exists and hence

$$\lim_{t\to+\infty} |u(t) - z| \quad \text{exists for any } z \in S_1. \tag{28}$$

Lemma 2. *Let* $\psi : \mathbb{R}_+ \to \mathbb{R}$ *be an absolutely continuous function which is bounded from below and such that* $\dot{\psi}^+ \in L^1(0, +\infty)$. *Then,* $\lim_{t\to+\infty} \psi(t)$ *exists.*

The proof of Lemma 2 is immediate and we leave it to the reader. Let us now return to the proof of Theorem 2. From (28) the trajectory u is bounded in H. It is also bounded in V in view of Lemma 1(i). Let $\bar{u} \in V$ be a cluster point of $\{u(t) \mid t \to \infty\}$ for the weak topology of V. There exist $\bar{u} \in V$ and a sequence $t_n \to +\infty$ such that $\lim_{n\to\infty} u(t_n) = \bar{u}$ weakly in V. Since the injection $V \hookrightarrow H$ is continuous, the sequence $u(t_n)$ weakly converges to $\bar{u}$ in H. By the weak lower semicontinuity of $d_H(., S_1)$, we obtain in view of Theorem 1

$$d_H(\bar{u}, S_1) \leq \liminf_{n\to\infty} d_H(u(t_n), S_1) = \lim_{t\to\infty} d_H(u(t), S_1) = 0,$$

and hence $\bar{u} \in S_1$. To prove the uniqueness of the cluster points of $\{u(t) : t \to \infty\}$, we apply the following argument due to Opial [16]. Let $\bar{u}_1, \bar{u}_2 \in S_1$ be two cluster points of $\{u(t) : t \to \infty\}$ for the weak topology of V. In view of (28), the quantity $l_i := \lim_{t\to\infty} |u(t) - \bar{u}_i|^2$ exists for each $i = 1, 2$. Take a sequence $t_k \to \infty$ such that $u(t_k) \rightharpoonup \bar{u}_1$ weakly in V. Since the injection from V into H is continuous, $u(t_k) \rightharpoonup \bar{u}_1$ weakly in H. From the identity $|u(t) - \bar{u}_1|^2 - |u(t) - \bar{u}_2|^2 = |\bar{u}_1 - \bar{u}_2|^2 + 2(\bar{u}_1 - \bar{u}_2, \bar{u}_2 - u(t))$, we deduce that $l_1 - l_2 = -|\bar{u}_1 - \bar{u}_2|^2$. Similarly, if we take $t_j \to \infty$ such that $u(t_j) \rightharpoonup \bar{u}_2$ then $l_1 - l_2 = |\bar{u}_1 - \bar{u}_2|^2$. Consequently, $|\bar{u}_1 - \bar{u}_2| = 0$. This establishes the uniqueness of the cluster points of $\{u(t) : t \to \infty\}$ for the weak topology of V. Hence $u(t) \rightharpoonup \bar{u}$ weakly in V as $t \to \infty$ for some $\bar{u} \in S_1$. The equivalence between the strong convergence in H and in V is an immediate consequence of Lemma 1 (ii). ☐

Remark 5. The assumptions of Theorem 2 relative to the maps ε and ω can be rewritten as $\varepsilon \notin L^1(0, +\infty)$ and $\omega^- \in L^1(0, +\infty)$. The question of the convergence is open when $\omega^- \notin L^1(0, +\infty)$, which corresponds to the "very slow" case.

Now consider the special case where $S_0 \cap \operatorname{Argmin} G \neq \emptyset$. From Proposition 3 (a), it ensues that $\omega \geq 0$ so that the condition $\omega^- \in L^1(0, +\infty)$ is automatically satisfied. This remark gives rise to the following corollary of Theorem 2.

Corollary 4. *Under the hypotheses of Theorem* 1*, assume moreover that* $S_0 \cap \{v \in V \,|\, g(v) = 0\} \neq \emptyset$*. Then, the trajectory* u *of* (2) *weakly converges in* V *to some* $\bar{u} \in S_0 \cap \{v \in V \,|\, g(v) = 0\}$*. Moreover,* $u(t) \to \bar{u}$ *strongly in* V *iff* $u(t) \to \bar{u}$ *strongly in* H*.*

Proof. Immediate from Proposition 3 (a) and Theorem 2. □

The next proposition shows that Corollary 4 holds true without assuming condition $(\mathcal{C})$ if one strengthens the hypotheses on ε.

Proposition 4. *Under* (h_1)-(h_5)*, suppose that the set* $S_0 \cap \{v \in V \,|\, g(v) = 0\}$ *is non empty. Denoting by* u *a global solution of* (2)*, the following holds:*
(i) $L_u(t) = o(1/t)$ *as* $t \to \infty$*. In particular,* $E(u(t)) = \min E + o(1/t)$ *and* $G(u(t)) = \min G + o(1/(t\,\varepsilon(t)))$ *as* $t \to \infty$*.*
(ii) Assume moreover that there exists $m > 0$ *such that* $\varepsilon(t) \geq m/t$ *for* t *large enough. Then there exists* $\bar{u} \in S_0 \cap \{v \in V \,|\, g(v) = 0\}$ *such that* $u(t) \rightharpoonup \bar{u}$ *weakly in* V *as* $t \to \infty$*.*

Proof. (i) Let $z \in S_0 \cap \{v \in V \,|\, g(v) = 0\}$ and consider the function ψ defined by $\psi(t) = \frac{1}{2}|u(t) - z|^2$. The same computation as in the proof of Theorem 2 shows that inequality (27) holds true. Since $S_0 \cap \{v \in V \,|\, g(v) = 0\} \neq \emptyset$, we have $\min G = \min_{S_0} G$, so that inequality (27) becomes

$$\dot{\psi}(t) + L_u(t) \leq 0, \text{ a.e. } t > 0. \tag{29}$$

This implies in particular that $\dot{\psi}(t) \leq 0$ a.e. on $[0, \infty)$ and hence ψ is non increasing; thus

$$\lim_{t\to\infty} |u(t) - z| \text{ exists for any } z \in S_0 \cap \{v \in V \,|\, g(v) = 0\}. \tag{30}$$

Coming back to inequality (29) and integrating on $[0, t]$, we obtain $\psi(t) - \psi(0) + \int_0^t L_u(s)ds \leq 0$ and therefore $\int_0^\infty L_u(s)\,ds \leq \psi(0) < \infty$, *i.e.* $L_u \in L^1(0, \infty; \mathbb{R}_+)$. Since L_u is nonincreasing, we deduce from a classical result that $L_u(t) = o(1/t)$ as $t \to \infty$.

(ii) From (30), $(u(t))_{t>0}$ is bounded in H. By Lemma 1(i), $(u(t))_{t>0}$ is bounded in V. Let $\bar{u} \in V$ be a cluster point of $\{u(t) \mid t \to \infty\}$ for the weak topology of V. We have $u(t_k) \rightharpoonup \bar{u}$ weakly in V for some sequence $t_k \to \infty$. By the weak lower semicontinuity of E, we obtain $E(\bar{u}) \leq \liminf_{k\to\infty} E(u(t_k)) = \lim_{t\to\infty} E(u(t)) = \min E$, hence $\bar{u} \in S_0$. Since $\varepsilon(t) \geq m/t$ for t large enough, we deduce from (i) that $\lim_{t\to\infty} G(u(t)) = \min G$. The weak lower semi-continuity of G yields $G(\bar{u}) \leq \liminf_{k\to\infty} G(u(t_k)) = \lim_{t\to\infty} G(u(t)) = \min G$, hence $\bar{u} \in \operatorname{Argmin} G = \{v \in V \,|\, g(v) = 0\}$. Finally, we conclude that $\bar{u} \in S_0 \cap \{v \in V \,|\, g(v) = 0\}$. To prove the uniqueness of the cluster points of $\{u(t) \mid t \to \infty\}$, we apply the same arguments as in the proof of Theorem 2. The weak convergence of the trajectory u immediately follows. □

5. **Toward hierarchical viscosity selection.** According to the previous results of this paper, the evolution equation (2) generates trajectories u which approach the set of minimizers of G over the set $S_0 = \operatorname{Argmin} E$. When such a viscosity selection principle is not sufficient to completely characterize the limit points of

u, it seems natural to introduce a second term into the dynamical system. Given $\varepsilon_2 \in W^{1,\infty}_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ and $g_2 \in \mathcal{C}(V;H)$ consider the following abstract evolution equation of the first-order in time:

$$u'(t) + Au(t) + f(u(t)) + \varepsilon(t)g(u(t)) + \varepsilon_2(t)g_2(u(t)) = 0, \ t > 0. \tag{31}$$

Suppose that:
(h_6)

$$\begin{cases} \forall v, w \in V, \ (g_2(v) - g_2(w), v - w) \geq 0 \text{ and } g_2 = {G_2}' \text{ for some } G_2 \in \mathcal{C}^1(V;\mathbb{R}). \\ \\ G_2 \text{ is bounded from below.} \\ \\ S_2 := \operatorname{Argmin}_{S_1} G_2 \text{ is nonempty.} \end{cases}$$

It is then natural to expect the trajectories of (31) to minimize the function G_2 over the set $S_1 = \operatorname{Argmin}_{S_0} G$, provided that the parametrization map ε_2 tends sufficiently slow toward 0. This question of hierarchical minimization has been addressed by many authors (see for example Attouch [2] for a discussion on this subject in an abstract stationary setting and Cominetti-Courdurier [10] for penalty steepest descent trajectories in convex programming).

In the next theorem, we will assume condition $(\mathcal{C})$ (*cf.* Theorem 1) and the analogous one for the set S_2: for every sequence (u_n) in V,

$$(\mathcal{C}_2) \qquad \begin{cases} \lim_{n\to\infty} d_H(u_n, S_1) = 0 \\ \limsup_{n\to\infty} G_2(u_n) \leq \min_{S_1} G_2 \end{cases} \implies \lim_{n\to\infty} d_H(u_n, S_2) = 0.$$

Theorem 3. *Under* (h_1)-(h_6), *suppose that S_1 is bounded in H and that* $\sup_{S_1} G_2 < \infty$. *Assume moreover that conditions* $(\mathcal{C})$ *and* $(\mathcal{C}_2)$ *hold. We are given a map* $\varepsilon_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying:*

$$\lim_{t\to+\infty} \varepsilon_2(t)/\varepsilon(t) = 0, \ \lim_{t\to+\infty} \omega^-(t)/\varepsilon_2(t) = 0, \ \textit{and} \ \int_0^\infty \varepsilon_2(t)\,dt = \infty, \tag{32}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}$ *is defined by* (23). *Then, any global solution u of* (31) *satisfies* $\lim_{t\to+\infty} d_H(u(t), S_2) = 0$.

Proof. Notice that since the set S_1 is bounded in H, we have $\lim_{t\to+\infty} \omega^-(t)/\varepsilon(t) = 0$ in view of Proposition 2. As a consequence, the choice of the map ε_2 satisfying (32) is always possible. The results of Proposition 1 for (2) can be immediately extended to (31) by replacing $L_u(t) = E(u(t)) - \inf E + \varepsilon(t)\,(G(u(t)) - \inf G)$ by

$$L_{u,2}(t)E(u(t)) - \inf E + \varepsilon(t)\,(G(u(t)) - \inf G) + \varepsilon_2(t)\,(G_2(u(t)) - \inf G_2)$$

in the proof of Proposition 1. We then obtain that $\lim_{t\to+\infty} E(u(t)) = \inf E$. We decompose the proof into two steps: first we prove that $\lim_{t\to+\infty} d_H(u(t), S_1) = 0$ and secondly that $\lim_{t\to+\infty} d_H(u(t), S_2) = 0$.
Step 1. Setting $h(t) = \frac{1}{2} d_H(u(t), S_1)^2$, the same computation as in the proof of Theorem 1 shows that

$$\dot{h}(t) + \varepsilon(t)\,(G(u(t)) - \min_{S_0} G) + \varepsilon_2(t)\,(G_2(u(t)) - G_2(P_{S_1}(u(t)))) \leq 0, \ \text{a.e. } t > 0.$$

The function G_2 is minorized by $\inf G_2$ on H and majorized by $\sup_{S_1} G_2 < \infty$ on the set S_1. Setting $M := \sup_{S_1} G_2 - \inf G_2 \geq 0$, we deduce that, for every $t \geq 0$,

$$G_2(u(t)) - G_2(P_{S_1}(u(t))) \geq -M.$$

This inequality combined with the previous one yields

$$\dot{h}(t) + \varepsilon(t)\,(G(u(t)) - \min_{S_0} G - M\,\varepsilon_2(t)/\varepsilon(t)) \leq 0, \text{ a.e. } t > 0.$$

The rest of the proof consists in distinguishing the following cases:
(a$_1$) $\exists T \geq 0, \quad \forall t \geq T, \quad G(u(t)) > \min_{S_0} G + M\,\varepsilon_2(t)/\varepsilon(t).$
(b$_1$) $\forall T \geq 0, \quad \exists t \geq T, \quad G(u(t)) \leq \min_{S_0} G + M\,\varepsilon_2(t)/\varepsilon(t).$
The arguments of the proof of Theorem 1 still apply here in so far as the quantity $\varepsilon_2(t)/\varepsilon(t)$ tends to 0 when $t \to +\infty$. As a consequence, we obtain that $\lim_{t\to+\infty} d_H(u(t), S_1) = 0$. The details are left to the reader.

Step 2. Let us define the map h_2 by

$$h_2(t) = \tfrac{1}{2} d_H(u(t), S_2)^2.$$

The same computation as in the proof of Theorem 1 leads to

$$\dot{h}_2(t) + E(u(t)) - \min E + \varepsilon(t)(G(u(t)) - \min_{S_0} G) + \varepsilon_2(t)(G_2(u(t)) - \min_{S_1} G_2) \leq 0, \quad \text{a.e. } t > 0,$$

and hence, in view of the definition of the map ω

$$\dot{h}_2(t) + \varepsilon_2(t)\left(G_2(u(t)) - \min_{S_1} G_2 - \omega^-(t)/\varepsilon_2(t)\right) \leq 0, \text{ a.e. } t > 0. \tag{33}$$

We are led to distinguish the following cases:
(a_2) $\exists T \geq 0, \quad \forall t \geq T, \quad G_2(u(t)) > \min_{S_1} G_2 + \omega^-(t)/\varepsilon_2(t).$
(b_2) $\forall T \geq 0, \quad \exists t \geq T, \quad G_2(u(t)) \leq \min_{S_1} G_2 + \omega^-(t)/\varepsilon_2(t).$

Case (a_2). We assume that there is $T \geq 0$ such that, for every $t \geq T$, $G_2(u(t)) > \min_{S_1} G_2 + \omega^-(t)/\varepsilon_2(t)$. We deduce from inequality (33) that, for every $t \geq T$, $\dot{h}_2(t) \leq 0$ and hence $\lim_{t\to\infty} h_2(t)$ exists. Integrating (33) on $[T, \infty[$ we deduce that

$$\int_T^\infty \varepsilon_2(t)[G_2(u(t)) - \min_{S_1} G_2 - \omega^-(t)/\varepsilon_2(t)]dt < \infty.$$

Since $\int_0^\infty \varepsilon_2(t)dt = \infty$, we infer that

$$\liminf_{t\to\infty}\,[G_2(u(t)) - \min_{S_1} G_2 - \omega^-(t)/\varepsilon_2(t)] = 0,$$

and hence $\liminf\limits_{t\to\infty} G_2(u(t)) = \min_{S_1} G_2$. Let $t_n \to \infty$ be such that $\lim_{n\to\infty} G_2(u(t_n)) = \min_{S_1} G_2$. From step 1, we have $\lim_{n\to\infty} d_H(u(t_n), S_1) = \lim_{t\to\infty} d_H(u(t), S_1) = 0$. Applying ($\mathcal{C}_2$) with the sequence $(u(t_n))$, we obtain that $\lim_{n\to\infty} d_H(u(t_n), S_2) = 0$ and since the map $t \mapsto d_H(u(t), S_2)$ is convergent, we conclude that $\lim\limits_{t\to\infty} d_H(u(t), S_2) = 0$.

Case (b_2). We now assume that, for every $T \geq 0$, there exists some $t \geq T$ such that $G_2(u(t)) \leq \min_{S_1} G_2 + \omega^-(t)/\varepsilon_2(t)$. For every $t \geq 0$, let us define

$$\tau_2(t) := \sup\left\{t' \in [0, t] \;\middle|\; G_2(u(t')) \leq \min_{S_1} G_2 + \omega^-(t')/\varepsilon_2(t')\right\}.$$

First notice that, since case (b$_2$) holds, the quantity $\tau_2(t)$ is well-defined as soon as t is large enough. The following inequality holds for t large enough:

$$d_H(u(t), S_2) \leq d_H(u(\tau_2(t)), S_2). \tag{34}$$

If $G_2(u(t)) \leq \min_{S_1} G_2 + \omega^-(t)/\varepsilon_2(t)$, then $\tau_2(t) = t$ and (34) follows immediately. Now assume that $G_2(u(t)) > \min_{S_1} G_2 + \omega^-(t)/\varepsilon_2(t)$. For every $t' \in [\tau_2(t), t]$,

$G_2(u(t')) \geq \min_{S_1} G_2 + \omega^-(t')/\varepsilon_2(t')$. From (33), we deduce that $\dot{h}_2(t') \leq 0$, which immediately yields (34).
If we are able to prove that $\lim_{t\to+\infty} d_H(u(\tau_2(t)), S_2) = 0$, then inequality (34) immediately implies that $\lim_{t\to+\infty} d_H(u(t), S_2) = 0$. From step 1, we have

$$\lim_{t\to+\infty} d_H(u(\tau_2(t)), S_1) = \lim_{t\to+\infty} d_H(u(t), S_1) = 0. \tag{35}$$

On the other hand, the definition of $\tau_2(t)$ gives

$$G_2(u(\tau_2(t))) \leq \min_{S_1} G_2 + \omega^-(\tau_2(t))/\varepsilon_2(\tau_2(t)),$$

so that

$$\limsup_{t\to+\infty} G_2(u(\tau_2(t))) \leq \min_{S_1} G_2. \tag{36}$$

From (35), (36) and condition $(\mathcal{C}_2)$, we deduce that $\lim_{t\to+\infty} d_H(u(\tau_2(t)), S_2) = 0$, which concludes the proof of case (b_2). □

As for condition $(\mathcal{C})$, it can be shown that condition $(\mathcal{C}_2)$ is automatically satisfied if the injection $V \hookrightarrow H$ is compact. This remark gives rise to the following corollary of Theorem 3.

Corollary 5. *Under* (h_1)-(h_6) *and* (4)*, assume that the injection of V into H is compact. Suppose that $\sup_{S_1} G_2 < \infty$. Assume moreover that the following condition holds: there exists $M_0 > \min E$ and $M_1 > \min_{S_0} G$ such that*

$$\textit{the set} \quad [E \leq M_0] \cap [G \leq M_1] \quad \textit{is bounded in } H. \tag{37}$$

We are given a map $\varepsilon_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (32)*. Then, any global solution u of* (31) *satisfies $\lim_{t\to+\infty} d_H(u(t), S_2) = 0$.*

Proof. It suffices to check that the assumptions of Theorem 3 are satisfied. First remark that condition (37) immediately yields the boundedness of the set S_1 in H. On the other hand, we have shown in the proof of Corollary 1 that the compactness of the injection $V \hookrightarrow H$ together with (37) implies condition $(\mathcal{C})$. We let the reader check that condition $(\mathcal{C}_2)$ can be deduced from the same compactness arguments. □

When S_2 is not reduced to a singleton the above methods can be iterated in order to introduce new viscosity criteria. We do not go further on this matter here.

6. **A simple illustration in PDE theory.** The convergence results of this paper are intended to stress the role of asymptotically vanishing viscosity terms in the selection of special limit solutions of the associated stationary problems. In order to give a flavor of the possible applications of our abstract results, we next discuss briefly a simple example in PDE theory.

Given $N \geq 1$ and an open set $\Omega \subset \mathbb{R}^N$ with boundary $\partial\Omega$ sufficiently regular, we first take $H = L^2(\Omega)$, $V = H^1(\Omega)$ and $a(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx$ so that $A = -\Delta$, which is monotone and semicoercive in $H^1(\Omega)$. Thus (h_1) is satisfied. Suppose that $f = f(x,s) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function which is nondecreasing with respect to $s \in \mathbb{R}$, and define $F(x,s) := \int_0^s f(x,r)dr$. For simplicity of notation, we write $F(u)$ instead of $\int_\Omega F(x,u(x))dx$. Under standard regularity and growing conditions on f, we have that $F \in C^1(H^1(\Omega);\mathbb{R})$ with $F'(u) = f(\cdot, u(\cdot)) \in L^2(\Omega)$. Moreover, since $f(x,\cdot)$ is nondecreasing, F' satisfies the monotonicity condition (h_2). Let S_0 be the set of minimizers of the energy functional $E(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx + F(u)$ that are characterized as the solutions to the Neumann boundary value problem

$$\begin{array}{rcll} -\Delta u + f(x,u) & = & 0 & \text{in} \quad \Omega, \\ \nabla u \cdot \mathbf{n} & = & 0 & \text{on} \quad \partial\Omega. \end{array} \tag{38}$$

Suppose that S_0 is nonempty, that is, (h_3) holds. The set S_0 may be nontrivial; if, for instance, $f = f(x)$ and $f \in L^2(\Omega)$ with $\int_\Omega f(x)dx = 0$ then $S_0 = \hat{u} + \mathbb{R}$ for any particular solution $\hat{u}$ of (38). Consider next a global solution $u = u(x,t)$ of the equation

$$\begin{array}{rcll} u_t - \Delta u + f(x,u) + \varepsilon(t)g(x,u) & = & 0 & \text{in} \quad \Omega\times]0,\infty[, \\ \nabla u \cdot \mathbf{n} & = & 0 & \text{on} \quad \partial\Omega\times]0,\infty[, \end{array} \tag{39}$$

where $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ corresponds to a generalized viscosity nonlinearity and $\varepsilon(t) \searrow 0$. Similarly, under appropriate conditions on g, we have that $G'(u) = g(\cdot, u(\cdot))$ where $G(u) = \int_\Omega G(x, u(x))dx$ with $G(x,s) := \int_0^s g(x,r)dr$. Assume that ($h_4$) and ($h_5$) are satisfied.

The results of Section 3 provide several criteria ensuring the $L^2(\Omega)$-convergence of the global solutions of (39) towards $S_1 = \operatorname{Argmin}_{S_0} G$, the set of all the G-viscosity solutions of (38), as $t \to \infty$. In fact, if Ω is bounded so that the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, and G is coercive in $L^2(\Omega)$, then $d_{L^2(\Omega)}(u(\cdot,t), S_1) \to 0$ by Corollary 1. If G is strongly convex then $u(\cdot,t) \to p$ strongly in $H^1(\Omega)$ with p being the unique minimizer of G on S_0, even without compactness of the injection by virtue of Corollary 3.

For instance, if $g(x,s) = s$ then

$$\begin{array}{rcll} u_t - \Delta u + f(x,u) + \varepsilon(t)u & = & 0 & \text{in} \quad \Omega\times]0,\infty[, \\ \nabla u \cdot \mathbf{n} & = & 0 & \text{on} \quad \partial\Omega\times]0,\infty[, \end{array}$$

and $G(u) = \frac{1}{2}\int_\Omega u(x)^2 dx$, hence $u(\cdot,t)$ converges strongly in $H^1(\Omega)$ to the minimal L^2-norm solution of (38). On the other hand, the choice $g(s) = [s-b]_+ - [a-s]_+$ for $a < b$, where $[s]_+ = \max\{s, 0\}$, yields

$$\begin{array}{rcll} u_t - \Delta u + f(x,u) + \varepsilon(t)\{[u-b]_+ - [a-u]_+\} & = & 0 & \text{in} \quad \Omega\times]0,\infty[, \\ \nabla u \cdot \mathbf{n} & = & 0 & \text{on} \quad \partial\Omega\times]0,\infty[, \end{array}$$

and $G(u) = \frac{1}{2}\int_\Omega \{[u(x)-b]_+^2 + [a-u(x)]_+^2\}dx$. Appropriate application of the abstract results gives $d_{L^2(\Omega)}(u(\cdot,t), S_1) \to 0$, but it is not evident whether $u(\cdot,t)$ strongly converges or not to a particular function in S_1. However, if there exists $u \in S_0$ such that $a \le u(x) \le b$ for almost every $x \in \Omega$, Corollary 4 ensures the weak convergence in $H^1(\Omega)$ to a function in S_1. If the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact then the convergence holds for the strong topology of $H^1(\Omega)$ by virtue of Corollary 4.

Analogous convergence results are valid for an evolution problem of the type

$$\begin{array}{rcll} u_t - \Delta u - \lambda_1 u + f(x,u) + \varepsilon(t)g(x,u) & = & 0 & \text{in} \quad \Omega\times]0,\infty[, \\ u & = & 0 & \text{on} \quad \partial\Omega\times]0,\infty[, \end{array}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. In this case, we take $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ and $a(u,v) = \int_\Omega [\nabla u(x) \cdot \nabla v(x) - \lambda_1 u(x)v(x)]dx$ so that $A = -\Delta - \lambda_1 I$ is monotone and semicoercive, and the corresponding stationary problem

$$\begin{array}{rcll} -\Delta u - \lambda_1 u + f(x,u) & = & 0 & \text{in} \quad \Omega\times]0,\infty[, \\ u & = & 0 & \text{on} \quad \partial\Omega\times]0,\infty[, \end{array}$$

may have a nontrivial set S_0 of solutions depending on the choice of the conservative term $f = f(x,u)$. Other special cases are obtained when $-\Delta$ in $H^1(\Omega)$ is replaced with the bi-Laplacian operator Δ^2 in $H^2(\Omega)$, or $-\Delta - \lambda_1 I$ in $H_0^1(\Omega)$ is replaced with $\Delta^2 - \mu_1 I$ where μ_1 is the first eigenvalue of Δ^2 in $H_0^2(\Omega)$. Certainly, nonhomogeneous operators, coupled systems of differential equations and other boundary conditions (mixed Neumann-Dirichlet, spatial-periodicity,...) can also be considered.

REFERENCES

[1] F. Alvarez and H. Attouch, *Convergence and asymptotic stabilization for some damped hyperbolic equations with non-isolated equilibria*, ESAIM: Control, Optimisation and Calculus of Variations, **6** (2001), 539–552.

[2] H. Attouch, *Viscosity solutions of minimization problems*, SIAM J. Optimization, **6** (1996), 769–806.

[3] H. Attouch and R. Cominetti, *A dynamical approach to convex minimization coupling approximation with the steepest descent method*, J. Differential Equations, **128** (1996), 519–540.

[4] H. Attouch and M.-O. Czarnecki, *Asymptotic control and stabilization of nonlinear oscillators with non isolated equilibria* , J. Differential Equations, **179** (2002), 278–310.

[5] J.-B. Baillon and R. Cominetti, *A convergence result for non-autonomous subgradient evolution equations and its application to the steepest descent exponential penalty trajectory in linear programming*, J. Funct. Anal., **187** (2001), 263–273.

[6] H. Brézis, Opérateurs Maximaux Monotones dans les Espaces de Hilbert et Équations d'Évolution, Lecture Notes 5, North Holland, 1972.

[7] R.E. Bruck, *Asymptotic convergence of nonlinear contraction semigroups in Hilbert space*, Journal of Functional Analysis, **18** (1975), 15–26.

[8] A. Cabot, *Inertial gradient-like dynamical system controlled by a stabilizing term*, J. Optim. Theory Appl., **120** (2004), 275–303.

[9] A. Cabot, *The steepest descent dynamical system with control. Applications to constrained minimization*, ESAIM: Control, Optimization and Calculus of Variations, **10** (2004), 243–258.

[10] R. Cominetti and M. Courdurier, *Coupling general penalty schemes for convex progamming with the steepest descent and the proximal point algorithm*, SIAM Journal on Optimization, **13** (2002), 745–765.

[11] R. Dautray and J.-L. Lions, Analyse Mathématique et Calcul Numérique, Vol. 8, Evolution: Semi-Groupe, Variationnel, Masson, Paris, 1988.

[12] A. Haraux, Nonlinear Evolution Equations. Global Behavior of Solutions, Lecture Notes in Mathematics 841, Springer-Verlag, Berlin-New York, 1981.

[13] K. Ito and F. Kappel, Evolution Equations and Approximations, Series on Advances in Mathematics for Applied Sciences 61, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[14] G. Lieberman, Second Order Parabolic Differential Equations, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[15] J.-L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications, Die Grundlehren der mathematischen Wissenschaften, Springer-Verlag, New York-Heidelberg, 1972.

[16] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. of the American Math. Society, **73** (1967), 591–597.

[17] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences 68, Springer-Verlag, New York, 1988.

[18] A. N. Tikhonov, V. Y. Arsenine, Méthodes de Résolution de Problèmes Mal Posés, MIR, 1976.

Received April 2005; revised November 2005.

E-mail address: falvarez@dim.uchile.cl
E-mail address: alexandre.cabot@unilim.fr