The heavy ball with friction dynamical system for convex constrained minimization problems

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Résumé. Le système dynamique "boule pesante avec frottement" associé à l'équation $\ddot{u} + \gamma\dot{u} + \nabla\Phi(u) = 0$ est un oscillateur non linéaire avec amortissement ($\gamma > 0$). Dans [2], Alvarez a démontré que si H est un espace de Hilbert réel et si $\Phi : H \to I\!R$ est une fonction convexe lisse dont le minimum est atteint, alors chaque trajectoire $t \to u(t)$ converge faiblement vers un minimiseur de Φ. Nous montrons un résultat similaire dans le cas convexe avec contraintes en considérant un système dynamique du type gradient-projection. Ce résultat étend, par l'utilisation de techniques différentes, des résultats obtenus par Antipin [1].

Abstract. The "heavy ball with friction" equation $\ddot{u} + \gamma\dot{u} + \nabla\Phi(u) = 0$ is a nonlinear oscillator with damping ($\gamma > 0$). In [2], Alvarez proved that when H is a real Hilbert space (possibly infinite dimensional) and $\Phi : H \to I\!R$ is a smooth convex function whose minimal value is achieved, then each trajectory $t \to u(t)$ weakly converges to a minimizer of Φ. We prove a similar result in the convex constrained case by considering the gradient-projection dynamical system

$$\ddot{u} + \gamma\dot{u} + u - \mathrm{proj}_C(u - \mu\nabla\Phi(u)) = 0,$$

where C is a closed convex subset of H. This result extends, by using different technics, previous results by Antipin [1].

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1 Introduction

Throughout the paper, H is a real Hilbert space, $\langle .,. \rangle$ denotes the associated scalar product and $|.|$ stands for the corresponding norm, $|v|^2 = \langle v, v \rangle$ for $v \in H$. Let us consider a function $\Phi : H \to \mathbb{R}$ that is assumed to be convex and continuously differentiable, which is the objective function, and a closed convex nonempty subset C of H, which is the set of constraints.

We are interested in time second-order differential systems whose trajectories asymptotically converge as $t \to +\infty$ towards minimizers of the convex constrained problem

$$(\mathcal{P}) \qquad \min\{\Phi(v) : v \in C\}.$$

When $C = H$, that is in the unconstrained case, and for Φ smooth possibly non convex, the asymptotical behavior of the non-linear oscillator with damping ($\gamma > 0$) system

$$\text{(HBF)} \qquad \ddot{u} + \gamma \dot{u} + \nabla \Phi(u) = 0,$$

also called the "heavy ball with friction" system, has been considered by several authors: Antipin [1], Attouch-Goudou-Redont [4], Alvarez [2], Haraux-Jendoubi [10]. In the convex unconstrained case, a general asymptotical result has been obtained by Alvarez [2] who proved that, when the set ArgminΦ of the minimizers of Φ is nonempty, then every trajectory of the (HBF) system weakly converges to a minimizer of Φ.

The Alvarez result presents striking similarities (the use of the Opial lemma, weak convergence of the trajectories) with the Bruck theorem [8]. This last paper deals, as an important particular case, with the asymptotic behavior (as $t \to +\infty$) of the trajectories of the generalized steepest descent equation

$$\text{(SD)} \qquad \dot{u} + \partial \varphi(u) \ni 0,$$

where φ is a convex function, which may now take the value $+\infty$ and is only assumed to be lower semicontinuous.

A natural extension of the Bruck theorem to the second-order setting would consist in considering

$$\ddot{u} + \gamma \dot{u} + \partial \varphi(u) \ni 0.$$

Indeed, this is a quite interesting system from a mechanical point of view which modelizes shocks, as soon as φ is not continuous. But, in this situation, $\dot{u}$ may be discontinuous and $\ddot{u}$ has to be interpreted as a measure, which makes this system quite involved and not easy to deal with from a numerical point of view.

In this paper, we follow a *different approach*. In order to study the problem $(\mathcal{P})$ we need only to consider functions φ of the following form

$$\varphi = \Phi + \delta_C,$$

where δ_C is the indicator of C, i.e. $\delta_C(v) = 0$ for $v \in C$ and $+\infty$ elsewhere. In that case, it can be shown, see Brezis [6], that the generalized steepest descent equation can be equivalently interpreted as a gradient-projection method

$$\dot{u}(t) = \mathrm{proj}_{T_C(u)}(-\nabla\Phi(u)),$$

where $T_C(u)$ is the tangent cone to C at u. This suggests that looking at various formulations of the *gradient-projection* method may be more tractable from a numerical point of view, and well fitted to the dynamical system approach.

As noticed in Antipin [1], one can reformulate the optimality condition for $(\mathcal{P})$

$$\nabla\Phi(u) + N_C(u) \ni 0,$$

where $N_C(u)$ is the outwards normal cone to C at u, as

$$u - \mathrm{proj}_C(u - \mu\nabla\Phi(u)) = 0,$$

where $\mu > 0$ is a parameter. By taking $\mu > 0$ adequately chosen, one can prove that the operator $Au = u - \mathrm{proj}_C(u - \mu\nabla\Phi(u))$ is a nice maximal monotone operator (indeed it is a cocoercive maximal monotone operator). We thus consider the associated second order differential system

$$\ddot{u} + \gamma\dot{u} + u - \mathrm{proj}_C(u - \mu\nabla\Phi(u)) = 0$$

and prove in Theorem 3.1 that each trajectory of this differential system weakly converges as $t \to +\infty$ to an element of the set $A^{-1}(0)$, which is indeed a global solution of the minimization problem $(\mathcal{P})$.

As basic ingredients for the proof of Theorem 3.1, we use the cocoerciveness of the operator A and the Opial Lemma. The proof readily extends (theorem 3.2) to the case $A = I - T$ where T is a general contraction.

2 A gradient-projection formulation of problem $(\mathcal{P})$

Let us now make the following assumptions on Φ :

(H_ϕ) $\Phi : H \to I\!R$ is a convex, continuously differentiable function whose gradient $\nabla\Phi$ is Lipschitz continuous, namely there exists a constant $L \geq 0$ such that for all v, $w \in H$

$$|\nabla\phi(v) - \nabla\phi(w)| \leq L|v - w|.$$

On the other hand, C is a nonempty, closed convex subset of H.

We consider the convex constrained minimization problem

$$(\mathcal{P}) \qquad \min\{\Phi(v) : v \in C\}$$

and assume that the set $S := \{v \in C : \Phi(v) = \inf_C \Phi\}$ of the optimal solutions (also called minimizers) of $(\mathcal{P})$ is nonempty. Clearly S is a nonempty closed convex subset of H.

We use standard notions of convex analysis and formulate $(\mathcal{P})$ as

$$\min\{\Phi(v) + \delta_C(v) : v \in H\}, \tag{1}$$

where δ_C is the indicator function of C, $\delta_C(v) = 0$ when $v \in C$ and $+\infty$ elsewhere. The optimality condition for (1) reads as follows

$$\nabla\Phi(u) + N_C(u) \ni 0, \tag{2}$$

where $N_C(u)$ is the outwards normal cone to C at u and is equal to the subdifferential of the indicator function of C at u.

Equivalently, given a parameter $\mu > 0$ (which for the moment is arbitrary and whose value will be precised later on), one can reformulate (2) as

$$-\mu\nabla\Phi(u) \in N_C(u),$$

or equivalently,

$$u - \mu\nabla\Phi(u) \in u + N_C(u). \tag{3}$$

As a basic tool, we use the classical relation

$$(I + N_C)^{-1} = \mathrm{proj}_C$$

to obtain as an equivalent optimality condition for $(\mathcal{P})$

$$u = \mathrm{proj}_C(u - \mu\nabla\Phi(u)). \tag{4}$$

Note that because of the convexity assumption on Φ, and the positivity of μ, (4) is a necessary and sufficient condition of optimality for $(\mathcal{P})$. So, we may work either with (4) or with the problem $(\mathcal{P})$. It is worthwhile to introduce the operator $T_\mu : H \to H$ defined by

$$T_\mu(v) := \mathrm{proj}_C(v - \mu\nabla\Phi(v)).$$

Let us summarize in the following statement the properties of T_μ.

Lemma 2.1 *Let us assume that $0 < \mu \leq \frac{2}{L}$. Then T_μ is a contraction from H into H. More precisely,*

$$\forall v, w \in H \quad |T_\mu v - T_\mu w| \leq |v - w|^2 - \frac{\mu}{2}(2 - \mu L)|\nabla\phi(v) - \nabla\phi(w)|^2.$$

Proof of lemma 2.1. Since proj_C is a contraction, we just need to prove that, for $0 < \mu \leq \frac{2}{L}$, the operator $v \mapsto v - \mu\nabla\phi(v)$ is a contraction. Given arbitrary $v, w \in H$

$$\begin{aligned}|(v - \mu\nabla\phi(v)) - (w - \mu\nabla\phi(w))|^2 &= |(v - w) - \mu(\nabla\phi(v) - \nabla\phi(w))|^2 \\ &= |v - w|^2 - 2\mu\langle\nabla\phi(v) - \nabla\phi(w), v - w\rangle + \mu^2|\nabla\phi(v) - \nabla\phi(w)|^2.\end{aligned}$$

In order to prove that $v \mapsto v - \mu\nabla\phi(v)$ is a contraction, we need equivalently to show that

$$-2\mu\langle\nabla\phi(v) - \nabla\phi(w), v - w\rangle + \mu^2|\nabla\phi(v) - \nabla\phi(w)|^2 \leq 0,$$

that is,

$$\langle\nabla\phi(v) - \nabla\phi(w), v - w\rangle \geq \frac{\mu}{2}|\nabla\phi(v) - \nabla\phi(w)|^2.$$

Indeed, this last property, called firmly non expansiveness of $\nabla\phi$ or *cocoerciveness* of $\nabla\phi$, is equivalent to the $\frac{2}{\mu}$-Lipschitz property of $\nabla\Phi$, see lemma 2.2 below. So, for $0 < \mu \leq \frac{2}{L}$, we obtain that T_μ is a contraction.

Lemma 2.2 (Baillon-Haddad [5]) *For an operator $\nabla\Phi$ (the gradient of a convex function), the following properties (i) and (ii) are equivalent*

(i) $\forall v, w \in H$, $|\nabla\phi(v) - \nabla\phi(w)| \leq L|v - w|$

(ii) $\forall v, w \in H$, $\langle\nabla\phi(v) - \nabla\phi(w), v - w\rangle \geq \frac{1}{L}|\nabla\phi(v) - \nabla\phi(w)|^2$

One may notice that (ii) can be viewed as a dual property of (i). Indeed, property (ii) is equivalent to

$$\langle(\nabla\phi)^{-1}(v) - (\nabla\phi)^{-1}(w), v - w\rangle \geq \frac{1}{L}|v - w|^2.$$

Since $(\nabla\phi)^{-1} = \partial\phi^*$, the above lemma expresses the Lipschitz continuity of $\nabla\phi$ in terms of the strong coerciveness of $\partial\phi^*$.

Let us return to the optimality condition (4) which can be expressed as

$$A_\mu u = 0 \tag{5}$$

where $A_\mu v := v - T_\mu v$. The properties of the operator $A_\mu = I - T_\mu$ follow from the general properties of operators of the form $I - T$ where T is a contraction, as stated in the following lemma:

Lemma 2.3 *Let $T : H \to H$ be a contraction. Then the operator $A = I - T$ is a maximal monotone operator which is $\frac{1}{2}$-cocoercive, that is*

$$\forall v, w \in H, \quad \langle Av - Aw, v - w\rangle \geq \frac{1}{2}|Av - Aw|^2.$$

Proof of lemma 2.3. A direct computation yields

$$\begin{aligned}&\langle Av - Aw, v - w\rangle - \tfrac{1}{2}|Av - Aw|^2\\ &= \langle v - Tv - w + Tw, v - w\rangle - \tfrac{1}{2}|v - w|^2 - \tfrac{1}{2}|Tv - Tw|^2 + \langle v - w, Tv - Tw\rangle\\ &= \tfrac{1}{2}(|v - w|^2 - |Tv - Tw|^2) \geq 0,\end{aligned}$$

where this last inequality expresses that T is a contraction. Thus, A is a monotone operator which is everywhere defined, which implies that A is a maximal monotone operator, see [7].

Remark : Following Bruck [8], $A = I - T$ is a demipositive maximal monotone operator. This class of operators is interesting because it is for operators of this class, which contains subdifferentials of closed convex functions and operators $A = I - T$ where T is a contraction, that Bruck [8] proved the weak asymptotical convergence of the trajectories of the associated first order differential system. In [2], Alvarez extended the Bruck result to the second order differential system for $A = \nabla\Phi$. Here, we are doing the parallel extension by considering operators $A = I - T$.

All these considerations lead us to consider the time second order differential system

$$\ddot{u} + \gamma\dot{u} + u - \mathrm{proj}_C(u - \mu\nabla\Phi(u)) = 0$$

with $0 < \mu \leq \frac{2}{L}$ and $\gamma > 0$.

3 The main result

We can now state the following

Theorem 3.1 *Let $\Phi : H \to I\!R$ be a convex, continuously differentiable function, whose gradient $\nabla\Phi$ is Lipschitz continuous with Lipschitz constant L. Let C be a closed convex nonempty subset of H, and let us assume that $S := \mathrm{Argmin}_C\Phi$, the set of minimizers of*

$$(\mathcal{P}) \qquad \min\{\Phi(v) : v \in C\}$$

is nonempty.

Let us consider the second order differential system with Cauchy data u_0, v_0:

$$(E_{\gamma,\mu}; u_0, v_0) \qquad \begin{cases} \ddot{u} + \gamma\dot{u} + u - \mathrm{proj}_C(u - \mu\nabla\Phi(u)) = 0\\ u(0) = u_0, \dot{u}(0) = v_0\end{cases}$$

Let us assume that $\gamma > \sqrt{2}$ and $0 < \mu \leq \frac{2}{L}$. Then for each u_0 and v_0 belonging to H, there exists a unique solution $u \in \mathcal{C}^2([0, +\infty[; H)$ of $(E_{\gamma,\mu}; u_0, v_0)$ which satisfies :

(i) $\dot{u} \in L^2(0, +\infty; H)$, $\ddot{u} \in L^2(0, +\infty; H)$

(ii) $\lim_{t\to+\infty} \dot{u}(t) = \lim_{t\to+\infty} \ddot{u}(t) = 0$

(iii) there exists $\bar{u} \in S = \text{Argmin}_C \Phi$ *such that* $u(t) \rightharpoonup \bar{u}$ *weakly in* H *as* $t \to +\infty$.

Proof of theorem 3.1. The equation $(E_{\gamma,\mu}; u_0, v_0)$ can equivalently be written as

$$\ddot{u} + \gamma \dot{u} + Au = 0 \tag{6}$$

where $A = I - T$ and $Tv = \text{proj}_C(v - \mu\nabla\Phi(v))$.

It follows from lemma 2.1 that, for $0 < \mu \leq \frac{2}{L}$, the operator T is a contraction and from lemma 2.3 that $A = I - T$ is a $\frac{1}{2}$-cocoercive operator, that is, for any $v, w \in H$

$$\langle Av - Aw, v - w\rangle \geq \frac{1}{2}|Av - Aw|^2. \tag{7}$$

Indeed, these are the only ingredients that we need for the proof of theorem 3.1, in addition to the fact that $S = \text{Argmin}_C \Phi = \{v \in H : Av = 0\} \neq \emptyset$.

Clearly, since A is Lipschitz continuous, by the Cauchy-Lipschitz theorem there exists a unique solution to $(E_{\gamma,\mu}; u_0, v_0)$, with $u \in \mathcal{C}^2([0, +\infty[; H)$. Take now $z \in S = A^{-1}(0)$ and define the auxiliary function $h(t) := \frac{1}{2}|u(t) - z|^2$. We have

$$\begin{aligned} \dot{h}(t) &= \langle u(t) - z, \dot{u}(t)\rangle \\ \ddot{h}(t) &= |\dot{u}(t)|^2 + \langle u(t) - z, \ddot{u}(t)\rangle. \end{aligned}$$

It follows that

$$\ddot{h} + \gamma\dot{h} = |\dot{u}|^2 + \langle u - z, \ddot{u} + \gamma\dot{u}\rangle,$$

which, by using (6) yields

$$\ddot{h} + \gamma\dot{h} + \langle Au, u - z\rangle = |\dot{u}|^2.$$

Since $Az = 0$, by using the $\frac{1}{2}$-cocoerciveness of A as stated in (7), we obtain

$$\ddot{h} + \gamma\dot{h} + \frac{1}{2}|Au|^2 \leq |\dot{u}|^2,$$

that is, by using (6) again,

$$\ddot{h} + \gamma\dot{h} + \frac{1}{2}|\ddot{u} + \gamma\dot{u}|^2 \leq |\dot{u}|^2. \tag{8}$$

This is the basic inequality from which we are going to derive various estimations on u. Let us rewrite (8) as

$$\ddot{h} + \gamma\dot{h} + \frac{\gamma}{2}\frac{d}{dt}|\dot{u}(t)|^2 + (\frac{\gamma^2}{2} - 1)|\dot{u}|^2 + \frac{1}{2}|\ddot{u}|^2 \leq 0. \tag{9}$$

Since $\gamma > \sqrt{2}$, we have

$$\ddot{h} + \gamma \dot{h} + \frac{\gamma}{2}\frac{d}{dt}|\dot{u}(t)|^2 \leq 0,$$

which implies that the function $t \to \dot{h} + \gamma h + \frac{\gamma}{2}|\dot{u}(t)|^2$ is *decreasing*. Thus, for any $t \geq 0$

$$\begin{aligned} \dot{h}(t) + \gamma h(t) + \tfrac{\gamma}{2}|\dot{u}(t)|^2 &\leq \dot{h}(0) + \gamma h(0) + \tfrac{\gamma}{2}|v_0|^2 \\ &\leq |u_0 - z||v_0| + \tfrac{\gamma}{2}|u_0 - z|^2 + \tfrac{\gamma}{2}|v_0|^2 := C_z \end{aligned} \tag{10}$$

which implies

$$\dot{h}(t) + \gamma h(t) \leq C_z. \tag{11}$$

After integration of (11) we obtain

$$h(t) \leq \frac{1}{2}|u_0 - z|^2 + \frac{1}{\gamma}C_z,$$

which implies that each trajectory of the differential system (6) is bounded. We need now to obtain estimations on $\dot{u}$ and $\ddot{u}$.

Let us integrate (9) from 0 to t to obtain

$$\dot{h}(t) + \gamma h(t) + \frac{\gamma}{2}|\dot{u}(t)|^2 + (\frac{\gamma^2}{2} - 1)\int_0^t |\dot{u}(s)|^2 ds + \frac{1}{2}\int_0^t |\ddot{u}(s)|^2 ds \leq C_z. \tag{12}$$

From (12) we derive that

$$\dot{h}(t) + \frac{\gamma}{2}|\dot{u}(t)|^2 \leq C_z$$

(notice that all the other terms are nonnegative) which by definition of $h(.)$ yields

$$\langle u(t) - z, \dot{u}(t)\rangle + \frac{\gamma}{2}|\dot{u}(t)|^2 \leq C_z. \tag{13}$$

Since $|u(t)|$ remains bounded on $[0, +\infty[$, the inequality (13) clearly implies that $|\dot{u}(t)|$ also remains bounded on $[0, +\infty[$:

$$\sup_{t \in [0, +\infty[} |\dot{u}(t)| < +\infty. \tag{14}$$

Using that $u(.)$ and $\dot{u}(.)$ remain bounded on $[0, +\infty[$ we deduce that $\dot{h}(t) = \langle u(t) - z, \dot{u}(t)\rangle$ is also bounded on $[0, +\infty[$.

This information and (12) immediately yield that

$$\int_0^{+\infty} |\dot{u}(s)|^2 ds < +\infty \tag{15}$$

and

$$\int_0^{+\infty} |\ddot{u}(s)|^2 ds < +\infty. \tag{16}$$

Let us now return to (6). Since u and $\dot{u}$ are bounded on $[0, +\infty[$ and A is Lipschitz continuous we deduce from (6) that

$$\sup_{t \in [0,+\infty[} |\ddot{u}(t)| < +\infty. \tag{17}$$

We now observe that the function $g(t) = \dot{u}(t)$ satisfies both

$$g \in L^2(0, +\infty; H) \quad \text{and} \quad \dot{g} \in L^\infty(0, +\infty; H).$$

According to a classical result, these two properties imply that

$$\lim_{t \to +\infty} \dot{u}(t) = 0. \tag{18}$$

Let us now prove that $\lim_{t \to +\infty} \ddot{u}(t) = 0$. To that end, let us write the equation (6) at the points t and $t + \varepsilon$, make the difference and divide by ε. We obtain that $u_\varepsilon(t) := \frac{1}{\varepsilon}(\dot{u}(t+\varepsilon) - \dot{u}(t))$ satisfies

$$\dot{u}_\varepsilon(t) + \gamma u_\varepsilon(t) = f_\varepsilon(t) \tag{19}$$

where $f_\varepsilon(t) = -\frac{1}{\varepsilon}(Au(t+\varepsilon) - Au(t))$.

We know that A is a 2-Lipschitz continuous operator on H. Hence,

$$|f_\varepsilon(t)| \leq \frac{2}{\varepsilon}|u(t+\varepsilon) - u(t)] \leq 2 \sup_{s \in [t,+\infty[} |\dot{u}(s)|.$$

Let us write $g(t) := 2 \sup_{s \in [t,+\infty[} |\dot{u}(s)|$.

We notice that, since $\lim_{t \to +\infty} \dot{u}(t) = 0$, we have that $\lim_{t \to +\infty} g(t) = 0$.

Let us integrate (19) to obtain

$$|u_\varepsilon(t)| \leq e^{-\gamma t}|u_\varepsilon(0)| + e^{-\gamma t} \int_0^t e^{\gamma s} g(s) ds,$$

which easily implies

$$\lim_{t \to +\infty} (\sup_{\varepsilon > 0} |u_\varepsilon(t)|) = 0.$$

Since, for all $t \geq 0$ the following inequality holds

$$|\ddot{u}(t)| \leq \sup_{\varepsilon > 0} |u_\varepsilon(t)|$$

we conclude that

$$\lim_{t\to+\infty} \ddot{u}(t) = 0. \tag{20}$$

We are now in position to apply Opial's lemma, which we recall for the convenience of the reader.

Lemma 3.1 (Opial) *Let H be a Hilbert space and $u : [0,+\infty[\to H$ be a function such that there exists a non void set $S \subset H$ which verifies :*

(a) $\forall t_n \to +\infty$ with $u(t_n) \rightharpoonup \bar{u}$ weakly in H, we have $\bar{u} \in S$.

(b) $\forall z \in S$, $\lim_{t\to+\infty} |u(t) - z|$ exists.

Then, $u(t)$ weakly converges as $t \to +\infty$ to an element $\bar{u}$ of S.

End of the proof of theorem 3.1. Let us apply the Opial lemma with $S = A^{-1}(0) = \mathrm{Argmin}_C \Phi$.

(a) By equation (6) we have

$$-\ddot{u}(t) - \gamma\dot{u}(t) = Au(t). \tag{21}$$

We know, by (18) and (20) that the left member of (21) strongly converges to zero as $t \to +\infty$. Let $u(t_n) \rightharpoonup \bar{u}$ weakly in H. Since the maximal monotone operator A is closed in $H \times H$ for the topology $w - H \times s - H$, see [6] for example, we obtain $A\bar{u} = 0$, i.e. $\bar{u} \in A^{-1}(0) = S$.

(b) Let us return to (9) which implies, as we have already noticed, that the function $\psi(t) := \dot{h}(t) + \gamma h(t) + \frac{\gamma}{2}|\dot{u}(t)|^2$ is decreasing. Hence $\lim_{t\to+\infty} \psi(t)$ exists. On the other hand, by (18) we know that $\lim_{t\to+\infty} \dot{u}(t) = 0$. Since $\dot{h}(t) = \langle u(t) - z, \dot{u}(t)\rangle$ and $u(t)$ is bounded, we also have $\lim_{t\to+\infty} \dot{h}(t) = 0$. By combining all these results we obtain

$$\lim_{t\to+\infty} \psi(t) = \gamma \lim_{t\to+\infty} h(t) \quad \text{exists.}$$

Hence, for any $z \in S$, $\lim_{t\to+\infty} |u(t) - z|$ exists.

So, the two assumptions of lemma 3.1 are satisfied, which completes the proof of theorem 3.1.

Remarks : 1) Clearly, if, by an a priori estimate, we know that the trajectory of the differential system $(E_{\gamma,\mu}; u_0, v_0)$ remains in a ball $\mathbb{B}(0, R)$, $0 \le R < \infty$ we need only to assume $\nabla\phi$ to be Lipschitz continuous on bounded sets.

2) All the proof of theorem 3.1 works without any modification for a general operator $A = I - T$ where T is just assumed to be a contraction on H. So, it is worthwhile to state it independently.

Theorem 3.2 *Let $T : H \to H$ be a contraction, i.e.*

$$|Tv - Tw| \leq |v - w| \quad \text{for all } v, w \in H.$$

Let us consider the second order differential system with Cauchy data u_0, v_0:

$$(E_\gamma; u_0, v_0) \qquad \begin{cases} \ddot{u} + \gamma\dot{u} + u - Tu = 0 \\ u(0) = u_0, \dot{u}(0) = v_0 \end{cases}$$

Let us assume that $\gamma > \sqrt{2}$. Then for each u_0 and v_0 belonging to H, there exists a unique solution $u \in \mathcal{C}^2([0, +\infty[; H)$ of $(E_\gamma; u_0, v_0)$ which satisfies :

(i) $\dot{u} \in L^2(0, +\infty; H)$, $\ddot{u} \in L^2(0, +\infty; H)$

(ii) $\lim_{t \to +\infty} \dot{u}(t) = \lim_{t \to +\infty} \ddot{u}(t) = 0$

(iii) there exists $\bar{u} \in \mathrm{Fix}\, T := \{v \in H : Tv = v\}$ such that $u(t) \rightharpoonup \bar{u}$ weakly in H as $t \to +\infty$.

Remark : It is an interesting question, for numerical purposes, to consider discretized versions, explicit or implicit, of theorems 3.1 and 3.2. This is naturally suggested by previous results obtained by Alvarez [2] in the unconstrained case.

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