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A unified approach to the asymptotic almost-equivalence of evolution systems without Lipschitz conditions

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ABSTRACT

We study the asymptotic behavior of almost-orbits of evolution systems in Banach spaces without any continuity assumptions on either the space or the time dependence. We establish, in a unified framework, standard convergence, ergodic convergence and almost-convergence of almost-orbits for both the weak and the strong topologies on the basis of the analogue behavior of orbits.

1. Introduction and preliminaries

Let C be a nonempty Borel subset of a Banach space $(X, \|\cdot\|)$. An *evolution system* on C is a two-parameter family $U = \{U(t,s) \mid t \geq s \geq 0\}$ of possibly nonlinear maps from C into itself satisfying:

(i) $\forall t \geq 0, \forall x \in C, U(t,t)x = x$.
(ii) $\forall t \geq s \geq r \geq 0, \forall x \in C, U(t,s)U(s,r)x = U(t,r)x$.

The evolution system U is *Lipschitz* if there exists a constant $L > 0$ such that $\|U(t,s)x - U(t,s)y\| \leq L\|x - y\|$ for all $t \geq s \geq 0, x, y \in C$. An operator semigroup $T = \{T(t)\}$ defines an *autonomous* evolution system via $U(t,s) = T(t-s)$.

An *orbit* of U is a function $u : [0,\infty) \to C$ such that

$$\forall t \geq 0, \forall h \geq 0, \quad u(t+h) = U(t+h,t)u(t).$$

More generally, a function $u \in L^\infty_{\mathrm{loc}}(0,\infty; C)$ is an *almost-orbit* of U if

$$\lim_{t\to\infty} \sup_{h\geq 0} \|u(t+h) - U(t+h,t)u(t)\| = 0. \tag{1}$$

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The term "almost-orbit" was introduced by Miyadera and Kobayasi in [1] as a perturbed solution to the evolution equation which generates U. Intuitively, the perturbation asymptotically vanishes fast enough as time goes to infinity. In fact, the model example is given by the differential inclusion

$$\dot{u}(t) + Au(t) \ni f(t), \quad t > 0, \tag{2}$$

where A is m-accretive. If $f \in L^1(0, \infty; X)$ then any integral solution u of (2) (see [2,3]) is an almost-orbit of the semigroup generated by $-A$ on $C = \overline{D(A)}$, where $D(A)$ is the domain of A; see [1, Proposition 7.1] for all details.

In the context of nonlinear contractions in Banach spaces, some criteria were given in [1] for ensuring a certain asymptotic behavior of almost-orbits. The same approach was used in [4,5]. A similar analysis was carried out in [6,7] for the so called "uniformly asymptotically almost-nonexpansive curves", a concept that includes almost-orbits of almost-nonexpansive semigroups in Hilbert spaces. Recent attempts to deal with almost-orbits of non-Lipschitz semigroups can actually be found in [8–13], but in the framework of asymptotic nonexpansiveness. We shall not go into the details, but just mention that these previous results hold essentially for semigroups and require strong regularity assumptions with respect to space/time dependence. More results on the asymptotic behavior of almost-orbits of nonexpansive semigroups can be found in [14] and the references therein.

Recently, we developed in [15,16] an *asymptotic almost-equivalence* theory, which is based on the following general principle: for a given evolution system, almost-orbits preserve the asymptotic convergence properties of orbits. In fact, as we show in [16], standard convergence, ergodic convergence and Lorentz's almost-convergence of almost-orbits of a given Lipschitz evolution system, for both the weak and the strong topologies, can be obtained from the corresponding convergence property of orbits. The reader may also find several applications and examples in those articles. We wish to underscore the fact that [15,16] extended significantly some previous results of this kind that can be found in [17,18,1,19], which are valid only for continuous-time solutions and discrete iterative methods associated with autonomous differential inclusions of the type $\dot{u}(t) + Au(t) \ni 0$.

In [16] each type of convergence was treated separately and the Lipschitz property of the evolution systems was essential for our proof techniques. The goal of this paper is to give a new unified approach which permits us to deal with several kinds of convergence (including standard, ergodic and Lorentz's) without further assumptions on the space or time dependence of the evolution systems. In fact, neither Lipschitz continuity nor asymptotic nonexpansiveness will be imposed here.

2. Convergence of means with respect to probability measures

Throughout this paper, almost-orbits are assumed to be measurable and locally bounded, and hence locally integrable on $[0, \infty)$. In [16] we considered three different notions of asymptotic convergence as time goes to infinity: (standard) convergence, ergodic convergence and Lorentz's almost-convergence. They can be applied to either the strong or the weak topology of the underlying Banach space $(X, \|\cdot\|)$.

By standard convergence we mean the following: given $y \in X$, a function $v : [0, \infty] \to X$ is strongly convergent to y if $\lim_{t\to\infty} \|v(t) - y\| = 0$, and weakly convergent to y if $\lim_{t\to\infty} \langle v(t) - y, x^* \rangle = 0$ for every $x^* \in X^*$ where X^* is the dual space of X and $\langle \cdot, \cdot \rangle$ is the duality product.

Ergodic convergence is a weaker notion: a function $v \in L^\infty_{\mathrm{loc}}(0, \infty; X)$ is strongly (resp. weakly) *ergodically convergent* if the current mean value

$$\overline{v}(t) := \frac{1}{t} \int_0^t v(\xi)\, \mathrm{d}\xi$$

has a strong (resp. weak) limit as $t \to \infty$. Next, given $h \geq 0$, we denote by v_h the translation defined by

$$v_h(t) := v(h + t).$$

Notice that

$$\overline{v_h}(t) = \frac{1}{t} \int_0^t v(h + \xi)\mathrm{d}\xi = \left(\frac{t+h}{t}\right) \frac{1}{t+h} \int_0^{t+h} v(\eta)\mathrm{d}\eta - \frac{1}{t} \int_0^h v(\xi)\mathrm{d}\xi.$$

Thus, if v is ergodically convergent to some $y \in X$, so is v_h for each $h \geq 0$. If the latter holds uniformly in h, then we say that v is almost-convergent in the sense of Lorentz [20]. More precisely, v is strongly (resp. weakly) *almost-convergent* to some $y \in X$ if $\overline{v_h}(t)$ converges strongly (resp. weakly) to y as $t \to \infty$ uniformly in $h \geq 0$.

Almost-convergence is an intermediate notion between ergodic convergence and convergence. Of course, almost-convergence implies ergodic convergence. On the other hand, v is convergent if, and only if, it is almost-convergent and *asymptotically regular* in the sense that the difference $v(t + h) - v(t)$ converges to zero as $t \to \infty$ for each $h \geq 0$, for the corresponding topology (see [20]). Thus almost-convergence supplemented with asymptotic regularity provides a criterion for convergence. This approach has been applied to study the asymptotic behavior of semigroups in [21].

In order to unify all these notions we will introduce some general ideas of convergence with respect to a time-indexed family of probability measures. Let μ be a probability measure on $[0, \infty)$. A function $v \in L^\infty_{\mathrm{loc}}(0, \infty; X)$ is *μ-integrable* if the *μ-mean of* v on $[0, \infty)$, $\mu(v) = \int_0^\infty v(\xi)\mathrm{d}\mu(\xi)$, exists.

Definition 2.1. Given a family $\{\mu_t\}_{t\geq 0}$ of probability measures on $[0,\infty)$, a function $v \in L^\infty_{\mathrm{loc}}(0,\infty;X)$ is *$\{\mu_t\}$-integrable* if $\mu_t(v)$ exists for all $t \geq 0$. We say *v converges to y in μ_t-mean* for the topology τ if $y = \tau - \lim_{t\to\infty} \mu_t(v)$.

Example 2.2. Let $v \in L^\infty_{\mathrm{loc}}(0,\infty;X)$. If $\mu_t = \delta_t$ is the Dirac mass at t, then $\mu_t(v) = v(t)$ and convergence in μ_t-mean is standard convergence. If $\mathrm{d}\mu_t(\xi) = \frac{1}{t}\chi_{[0,t]}(\xi)\mathrm{d}\xi$, where χ_A is the characteristic function of the set A, then $\mu_t(v) = \frac{1}{t}\int_0^t v(\xi)\mathrm{d}\xi = \overline{v}(t)$ and convergence in μ_t-mean is ergodic convergence.

Given $v \in L^\infty_{\mathrm{loc}}(0,\infty;X)$ and $h \geq 0$, we set $v_h(t) = v(h+t)$ for $t \geq 0$. If there is $y \in X$ such that

$$y = \tau - \lim_{t\to\infty} \mu_t(v_h) = \tau - \lim_{t\to\infty} \int_0^\infty v(h+\xi)\mathrm{d}\mu_t(\xi)$$

uniformly in $h \geq 0$, for τ the strong (weak) topology, we say v converges strongly (weakly) to *y in μ_t-mean, uniformly with respect to translations.*

Example 2.3. If μ_t is the Dirac mass at t, then $\mu_t(v_h) = v(t+h)$ and convergence in μ_t-mean recovers standard convergence, which is automatically uniform with respect to translations. If $\mathrm{d}\mu_t(\xi) = \frac{1}{t}\chi_{[0,t]}(\xi)\mathrm{d}\xi$, then $\mu_t(v_h) = \frac{1}{t}\int_0^t v(h+\xi)\mathrm{d}\xi$. In this case, convergence in μ_t-mean uniformly with respect to translations is exactly Lorentz's almost-convergence.

3. Asymptotic almost-equivalence results

In this section we show how to deduce the asymptotic behavior of the almost-orbits based on the information concerning the orbits. We begin by stating the hypotheses. Let $\{\mu_t\}_{t\geq 0}$ be a family of probability measures on $[0,\infty)$.

Hypothesis **H**: For each $\varepsilon > 0$, $K > 0$ and $\{\mu_t\}$-integrable function g with $\lim_{t\to\infty}\int_0^\infty g(\xi)\mathrm{d}\mu_t(\xi) = L$ for some $L \in X$, there exists $T > 0$ such that for all $t \geq T$ one has

$$\left\| \int_0^\infty g(\xi)\,\mathrm{d}\mu_t(\xi+K) - L \right\| < \varepsilon.$$

Hypothesis **H** essentially expresses that the family $\{\mu_t\}$ does not accumulate any mass on bounded sets. In particular, one has $\lim_{t\to\infty}\mu_t(B) = 0$ for each bounded set B. However, Hypothesis **H** is slightly stronger than the latter condition:

Example 3.1. Define $n(\xi) = \sum_{k\geq 0}\chi_{[2k,2k+1)}(\xi)$ and $\hat{n}(\xi) = n(\xi+1)$ so that $n^2 \equiv n$ and $n\hat{n} \equiv 0$. Let $\mathrm{d}\mu_t(\xi) = \alpha^{-1}(t)n(\xi)\chi_{[0,t]}(\xi)\mathrm{d}\xi$, where $\alpha(t) = \int_0^t n(\xi)\mathrm{d}\xi$. Then $\mu_t(B) \to 0$ for every bounded set B (this is obvious) but $\{\mu_t\}$ does not fulfill Hypothesis **H**. To see this, simply notice that $\int_0^\infty n(\xi)\mathrm{d}\mu_t(\xi) = 1$ while $\int_0^\infty n(\xi)\mathrm{d}\mu_t(\xi+1) = \alpha^{-1}\int_1^{t-1}\hat{n}(\xi)n(\xi)\mathrm{d}\xi = 0$ for all t.

For the weak topology we consider the following version of Hypothesis **H**:

Hypothesis **w-H**: For each $\varepsilon > 0$, $K > 0$, $x^* \in X^*$ and $\{\mu_t\}$-integrable function g with $w - \lim_{t\to\infty}\int_0^\infty g(\xi)\mathrm{d}\mu_t(\xi) = L$ for some $L \in X$, there exists $T > 0$ such that for all $t \geq T$ one has

$$\left| \left\langle \int_0^\infty g(\xi)\mathrm{d}\mu_t(\xi+K) - L, x^* \right\rangle \right| < \varepsilon.$$

The families described in Example 2.2 do satisfy Hypotheses **H** and **w-H**. This is trivial if μ_t is the Dirac mass at t. If $\mathrm{d}\mu_t(\xi) = \frac{1}{t}\chi_{[0,t]}(\xi)$, then for t large enough, $\int_0^\infty g(\xi)\mathrm{d}\mu_t(\xi+K) = \left(\frac{t-K}{t}\right)\frac{1}{t-K}\int_0^{t-K} g(\xi)\mathrm{d}\xi$, which tends to L as $t \to \infty$ whenever $\frac{1}{t}\int_0^t g(\xi)\mathrm{d}\xi$ does so.

Theorem 3.2. *If $\{\mu_t\}$ satisfies hypothesis* **H** *and each orbit of an evolution system U converges strongly in μ_t-mean, so does every $\{\mu_t\}$-integrable almost-orbit of U. The same holds for the weak topology provided $\{\mu_t\}$ satisfies hypothesis* **w-H** *and X is weakly complete.*[1]

Proof. Suppose u is a $\{\mu_t\}$-integrable almost-orbit of U and let $\varepsilon > 0$. Choose $S > 0$ such that

$$\sup_{h\geq 0}\|u(t+h) - U(t+h,t)u(t)\| < \varepsilon/6$$

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[1] A Banach space is *weakly complete* if every weak Cauchy sequence converges weakly to an element in X. The spaces ℓ^1, L^1 and all reflexive Banach spaces have this property. It is not the case if X contains c_0, though.

for all $t \geq S$. Define

$$\zeta(S) = \lim_{t\to\infty} \int_0^\infty U(S+\xi, S)u(S)\,\mathrm{d}\mu_t(\xi). \tag{3}$$

By hypothesis, there is T_1 such that

$$\left\| \zeta(S) - \int_0^\infty U(S+\xi, S)u(S)\mathrm{d}\mu_t(\xi) \right\| < \varepsilon/6$$

for all $t \geq T_1$. We have

$$\begin{aligned}\|\mu_t(u) - \zeta(S)\| \leq &\int_0^S \|u(\xi)\|\,\mathrm{d}\mu_t(\xi) + \int_S^\infty \|u(\xi) - U(\xi, S)u(S)\|\mathrm{d}\mu_t(\xi)\\ &+ \left\| \zeta(S) - \int_0^\infty U(S+\xi, S)u(S)\mathrm{d}\mu_t(\xi+S) \right\|.\end{aligned}$$

For the first term, since $\lim_{t\to\infty} \mu_t([0, S]) = 0$, we can take T_2 such that

$$\mu_t([0, S]) < \varepsilon/6C$$

for all $t \geq T_2$, where $C = \sup_{0\leq\xi\leq S} \|u(\xi)\|$. The second term is less than $\varepsilon/6$. By Hypothesis **H** there is T_3 such that the last term is less than $\varepsilon/6$ whenever $t \geq T_3$. Hence if $t \geq T = \max\{T_1, T_2, T_3\}$, we have

$$\|\mu_t(u) - \zeta(S)\| < \varepsilon/2$$

for all $h \geq 0$. We have found $T > 0$ such that

$$\|\mu_t(u) - \mu_s(u)\| < \varepsilon$$

for all $t, s \geq T$ and therefore $\mu_t(u)$ converges to some y as $t \to \infty$.

Under Hypothesis **w-H**, an analogue argument shows that if the orbits of U converge weakly in μ_t-mean, then $\lim_{t,s\to\infty} \langle \mu_t(u) - \mu_s(u), x^*\rangle = 0$ for each $x^* \in X^*$. If X is weakly complete then $\{\mu_t(u)\}$ converges weakly as $t \to \infty$. □

The uniformity in $h \geq 0$ requires a slightly stronger assumption on $\{\mu_t\}$ (that still holds for the families mentioned in Example 2.2) for proving the equivalence results:

Hypothesis $\mathbf{H_u}$: For each $\{\mu_t\}$-integrable g with $\lim_{t\to\infty} \int_0^\infty g(\xi)\mathrm{d}\mu_t(\xi) = L$, each $\varepsilon > 0$ and $K > 0$, there exists $T > 0$ such that for all $t \geq T$ and $k \in [0, K]$ one has $\left\| \int_0^\infty g(\xi)\mathrm{d}\mu_t(\xi+k) - L \right\| < \varepsilon$.

And for the weak topology:

Hypothesis $\mathbf{ww\text{-}H_u}$: For each $\{\mu_t\}$-integrable g with $\text{w-}\lim_{t\to\infty} \int_0^\infty g(\xi)\mathrm{d}\mu_t(\xi) = L$, each $\varepsilon > 0$, $K > 0$ and $x^* \in X^*$, there exists $T > 0$ such that for all $t \geq T$ and $k \in [0, K]$ one has $\left| \langle \int_0^\infty g(\xi)\mathrm{d}\mu_t(\xi+k) - L, x^*\rangle \right| < \varepsilon$.

Theorem 3.3. *Let U be an ES. If $\{\mu_t\}$ satisfies hypothesis $\mathbf{H_u}$ and $U(t, s)x$ converges strongly in μ_t-mean, uniformly with respect to translations for all x and s, then so does every $\{\mu_t\}$-integrable almost-orbit. The same holds for the weak topology provided $\{\mu_t\}$ satisfies hypothesis $\mathbf{ww\text{-}H_u}$ and X is weakly complete.*

Proof. We begin like in the proof of Theorem 3.2 by taking a $\{\mu_t\}$-integrable almost-orbit u of U and some $\varepsilon > 0$, and define $\zeta(S)$ by (3). By hypothesis, there is T_1 such that

$$\left\| \zeta(S) - \int_0^\infty U(S+h+\xi, S)u(S)\mathrm{d}\mu_t(\xi) \right\| < \varepsilon/6$$

for all $t \geq T_1$ and $h \geq 0$ (the convergence is uniform in $h \geq 0$). We divide the rest of the proof into two parts:

$0 \leq h \leq S$: As in the proof of Theorem 3.2 we have

$$\begin{aligned}\|\mu_t(u_h) - \zeta(S)\| \leq &\int_0^{S-h} \|u(h+\xi)\|\mathrm{d}\mu_t(\xi) + \int_{S-h}^\infty \|u(h+\xi) - U(h+\xi, S)u(S)\|\mathrm{d}\mu_t(\xi)\\ &+ \left\| \zeta(S) - \int_0^\infty U(S+\xi, S)u(S)\mathrm{d}\mu_t(\xi+(S-h)) \right\|.\end{aligned}$$

For the first term, since $\mu_t([0, S]) \to 0$ as $t \to \infty$, we can take T_2 such that $\mu_t([0, S]) < \varepsilon/6C$ for all $t \geq T_2$, where $C = \sup_{0\leq\xi\leq S} \|u(\xi)\|$. The second term is always less than $\varepsilon/6$. Finally, we use hypothesis $\mathbf{H_u}$ to find T_3 such that the last term is less than $\varepsilon/6$ whenever $t \geq T_3$. Hence if $t \geq T = \max\{T_1, T_2, T_3\}$, we have

$$\|\mu_t(u_h) - \zeta(S)\| < \varepsilon/2$$

for all $h \geq 0$.

$h \geq S$: We have

$$\|\mu_t(u_h) - \zeta(S)\| \leq \int_0^\infty \|u(h+\xi) - U(h+\xi,S)u(S)\| \, \mathrm{d}\mu_t(\xi) + \left\|\zeta(S) - \int_0^\infty U(h+\xi,S)u(S)\mathrm{d}\mu_t(\xi)\right\|,$$

whenever $t \geq T_1$. Each term is less than $\varepsilon/6$, so

$$\|\mu_t(u_h) - \zeta(S)\| < \varepsilon/3 < \varepsilon/2$$

for all $t \geq T_1$ and $h \geq S$.

Finally, $\|\mu_t(u_h) - \zeta(S)\| < \varepsilon/2$ for all $t \geq T$ and $h \geq 0$. Of course, this implies

$$\|\mu_t(u_h) - \mu_s(u_k)\| < \varepsilon$$

for all $t, s \geq T$ and $h, k \geq 0$ and so u is strongly convergent in μ_t-mean, uniformly with respect to translations. □

Remark 3.4. The weak completeness assumption in Theorems 3.2 and 3.3 can be dropped if the evolution system U is Lipschitz. This is proved in [16] when the family $\{\mu_t\}$ is one of those in Example 2.2.

Theorems 3.2 and 3.3, along with Remark 3.4, give:

Corollary 3.5. *Let U be an evolution system. If every orbit of U*

(i) *converges,*
(ii) *converges ergodically, or*
(iii) *almost-converges*

for the strong topology, then so does every almost-orbit. The same holds for the weak topology provided X is weakly complete or U is Lipschitz.

Remark 3.6. Notice that [4,1] contain very particular cases of Corollary 3.5.

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