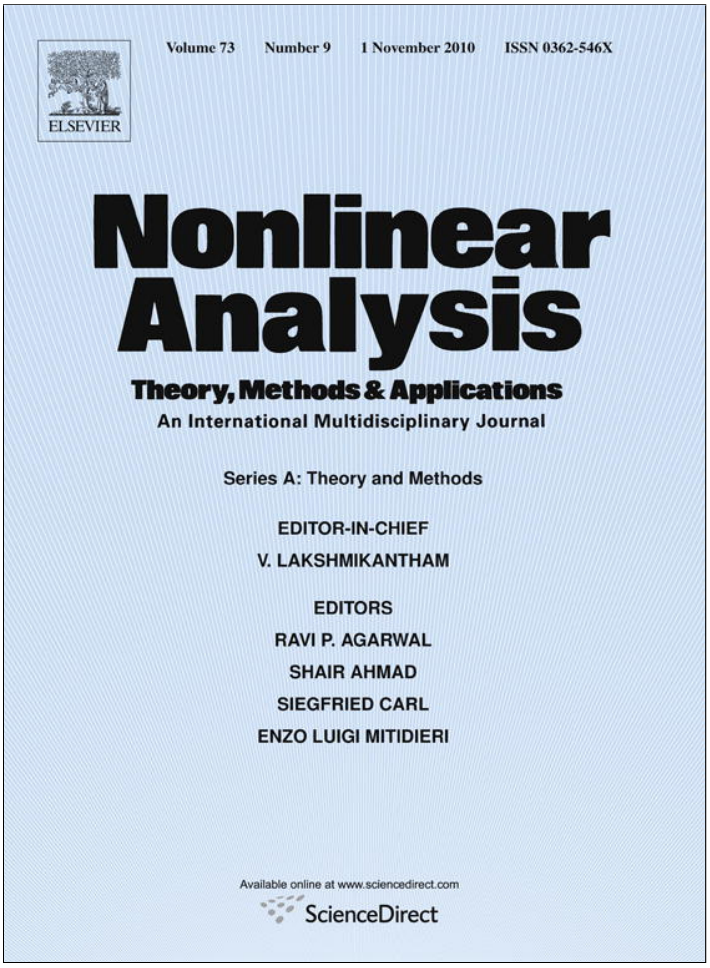
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Asymptotic almost-equivalence of Lipschitz evolution systems in Banach spaces ☆

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ABSTRACT

In this paper we introduce a notion of asymptotic almost-equivalence of two evolution systems and provide simple tests that guarantee that two evolution systems have the same qualitative asymptotic properties. In this way we are able to unify and extend many previously known results and also to understand what is behind equally behaved systems. In particular, we establish convergence, ergodic convergence and almost-convergence of almost-orbits both for the weak and the strong topologies based on the behavior of the orbits.

1. Introduction

Roughly speaking, a dynamical system in discrete (resp. continuous) time is a rule that determines a sequence (resp. trajectory) departing from certain initial data and which evolves in some space. This notion covers, for instance, iterative algorithms and well-posed ordinary differential equations. If any two dynamical systems are close in some sense, it is then natural to expect them to share some properties. This paper deals with some of the asymptotic convergence properties that are common to systems which can be considered equivalent in a sense to be made precise later on.

We shall model dynamical systems by means of an abstract definition, which is similar to the idea of a semiflow (see, for example [1]). More precisely, let C be a nonempty Borel subset of a Banach space $(X, \|\cdot\|)$. An *evolution system* on C is a two-parameter family $U = \{U(t,s) \mid t \geq s \geq 0\}$ of maps from C into itself satisfying:

(i) $\forall t \geq 0, \forall x \in C, U(t,t)x = x$.
(ii) $\forall t \geq s \geq r \geq 0, \forall x \in C, U(t,s)U(s,r)x = U(t,r)x$.

An evolution system U is said to be L-Lipschitz on C if

$$\forall t \geq s \geq 0,\ \forall x, y \in C, \quad \|U(t,s)x - U(t,s)y\| \leq L\|x - y\|, \tag{1.1}$$

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for a given constant $L > 0$. If this holds true for $L = 1$ then U is said to be *nonexpansive*. We say that U is *autonomous* if for all $t, s \geq 0$ we have $U(t+s, s) = U(t, 0)$. For such an evolution system, we set $S(t) := U(t, 0)$ so that $U(t, s) = S(t-s)$ for all $t \geq s \geq 0$. The family $S = \{S(t) \mid t \geq 0\}$ is a *semigroup*, that is $S(0)x = x$ and $S(t)S(s)x = S(t+s)x$ for all $t, s \geq 0$ and $x \in C$. Thus one can identify the class of all autonomous evolution systems on C with the semigroups on C. For instance, consider an initial value problem of the type

$$\begin{cases} \dot{u}(t) = F(t, u(t)), & t > t_0 \\ u(t_0) = x^0 \in C \end{cases} \tag{1.2}$$

for some vector field $F : (0, \infty) \times C \to X$. Assume that (1.2) is well-posed in the sense that for every $t_0 \geq 0$ and $x^0 \in C$ there exists a unique continuous function $u(\cdot; t_0, x^0) \in \mathcal{C}([t_0, \infty); X)$ which satisfies (1.2), possibly in a weak or generalized sense (typically, an integral solution in the sense of Bénilan [2]; see also [3]). In addition, suppose that C is *invariant* under the dynamics induced by (1.2), that is, for all $x^0 \in C$ and $t \geq t_0$ we have $u(t; t_0, x^0) \in C$. In such a case, we get an evolution system on C by setting

$$U(t, t_0)x^0 := u(t; t_0, x^0) \tag{1.3}$$

for all $t \geq t_0 \geq 0$ and $x^0 \in C$. If the vector field in (1.2) does not depend on t, that is, $F(t, \cdot) \equiv F(\cdot)$ for some $F : C \to X$, then the corresponding evolution system is autonomous.

Under appropriate conditions, the first equation in (1.2) may be replaced with a differential inclusion of the form $\dot{u}(t) \in F(t, u(t))$, for a suitable set-valued map $F : (0, \infty) \times C \to \mathcal{P}(X)$. If X is a real Hilbert space and taking $F(t, x) = -Ax$, where $A : X \to \mathcal{P}(X)$ is a maximal monotone operator, then (1.2) induces a nonexpansive semigroup on $C = \overline{D(A)}$ (see [4]). See also [5–7] for (possibly time-dependent) differential equations or inclusions associated with m-accretive operators on Banach spaces. In any case, under uniqueness of solution for (1.2), by definition (1.3) we have that $\forall t \geq s \geq t_0$, $u(t; t_0, x^0) = U(t, s)u(s; t_0, x^0)$. More generally, we say that a measurable and locally bounded function $u \in L^\infty_{\mathrm{loc}}(0, \infty; C)$ is an *orbit* of an arbitrary evolution system U on C if for some $t_0 \geq 0$ we have that

$$\forall t \geq s \geq t_0, \quad u(t) = U(t, s)u(s). \tag{1.4}$$

The abstract notion of an evolution system can also be used to model *discrete-time dynamics* such as those associated with iterative methods in optimization and fixed point theory. Indeed, consider a family $\{F_n\}$ of functions from C into C and an increasing sequence $\{\sigma_n\}$ of positive real numbers such that $\sigma_n \to \infty$. Let us define the piecewise constant function $\nu : [0, \infty) \to \mathbb{N}$ by

$$\nu(t) := \max\{0, \max\{n \in \mathbb{N} \mid \sigma_n \leq t\}\},$$

and set

$$V(t, s) := \prod_{n=\nu(s)+1}^{\nu(t)} F_n$$

for any $t \geq s \geq 0$, the product representing here the composition of maps. Then V is an evolution system on C. If each F_n is L_n-Lipschitz and the product $\prod_{n=1}^{\infty} L_n$ is bounded from above by $L > 0$, then such a V is L-Lipschitz. Notice that if $\{x^n\} \subset C$ is a sequence such that

$$x^n = F_n(x^{n-1}), \quad n = 1, 2, \ldots$$

then we have that

$$\forall m \geq n \geq 0, \quad x^m = V(\sigma_m, \sigma_n)x^n.$$

Natural questions are those of when and to what extent two evolution systems U and V have the same asymptotic behavior as time goes to infinity. In particular, if for one of these systems every orbit converges to a stationary point in some sense, we want to know whether such a property is preserved for the other system. The goal of this paper is to develop tools for answering these questions and to address similar issues for general Lipschitz evolution systems under minimal assumptions. Therefore we focus on the mathematical theory that enables us to compare different evolution systems and not on particular comparisons. We are specially interested in the case where one seeks to compare discrete-time dynamics with continuous-time dynamics.

Our approach is based on the notion of the *almost-orbit* (see Section 2 for the precise definition), which was introduced by Miyadera and Kobayasi in [8]. Roughly speaking, an almost-orbit is a sort of relaxed or perturbed trajectory of a given evolution system such that the perturbation vanishes fast enough as time goes to infinity, in a sense to be made precise later, in order to preserve many asymptotic properties of actual orbits. Therefore, if two evolution systems are close enough that every orbit of one of them is an almost-orbit of the other one, we say that the systems are *asymptotically almost-equivalent*. If this is so, it is natural to expect the two systems to share much of their asymptotic behavior in terms of boundedness, convergence in several senses and other related properties of their orbits.

Several works have been devoted to investigating the asymptotic behavior of almost-orbits. Let us mention, for instance, [8,9] where some criteria are given for ensuring certain asymptotic behavior of almost-orbits of nonexpansive semigroups, and [10] where the author carries out a similar analysis for the so-called uniformly asymptotically almost nonexpansive curves, a concept that includes almost-orbits of (almost) nonexpansive semigroups in Hilbert space. Ergodic theorems for almost-orbits of nonlinear semigroups are given in [11,12]. On the other hand, see [13,14,8,15] and the discussion in Section 2 for previous results on how, under special conditions, the asymptotic behavior of certain discrete iterative processes, such as backward differencing or proximal-resolvent iterations, can be related to some continuous-time evolution systems associated with maximal monotone operators on Hilbert spaces or m-accretive operators on Banach spaces. Finally, it is important to mention that our work is closely related to [16]. In both papers, convergence properties of the orbits of certain evolution systems are proved to be inherited by the almost-orbits. However, there is a rather fundamental difference with relevant consequences. The analysis carried out in [16] is valid only for nonexpansive semigroups, while our setting covers all Lipschitz evolution systems. The advantage of such an extension is that we account for systems that are neither dissipative nor autonomous. In particular, we are able to establish convergence results for the almost-orbits of nonautonomous differential inclusions governed by (families of) operators that need not be accretive. We can also deal with algorithms whose calculus rule is updated at each iteration. Let us mention that the framework of [16] does not cover some of the examples and applications given in this paper (see, for instance, Example 4, Propositions 6.1 and 6.4).

The paper is organized as follows: In Section 2 we introduce the basic convergence notions and discuss the definition of almost-orbits. Section 3 contains the main theoretical results on the preservation of the asymptotic properties. In Section 4 we present the idea of asymptotic almost-equivalence and give some examples, old and new. We provide a glance at the applicability of this theory in Sections 5 and 6. More precisely, in Section 5 we revisit some classical results concerning evolution equations governed by m-accretive operators, the proximal point algorithm and Mann's iterations, emphasizing the relationship between the three systems. In Section 6 we present new stability results for nonautonomous differential inclusions governed by families of monotone operators in Hilbert space along with robustness results for general nonexpansive algorithms. To conclude, we underscore the distinctive features of this approach, mention several areas of application and present guidelines for future research in Section 7.

Some of these results were obtained in the second author's Ph.D. dissertation [17].

2. Preliminaries

In this paper we consider three different notions of asymptotic convergence as time goes to infinity: (standard) convergence, ergodic convergence and almost-convergence. They can be applied to either the strong or the weak topology of the underlying Banach space $(X, \|\cdot\|)$. More precisely, given $y \in X$, a function $v : [0, \infty] \to X$ is strongly convergent to y if $\lim_{t\to\infty} \|v(t) - y\| = 0$, and weakly convergent to y if $\lim_{t\to\infty} \langle v(t) - y, x^*\rangle = 0$ for every $x^* \in X^*$ where X^* is the dual space of X and $\langle\cdot,\cdot\rangle$ is the standard duality product. On the other hand, the weakest notion of convergence is the ergodic one: given a measurable and locally bounded function $v \in L^\infty_{\text{loc}}(0, \infty; X)$, we say that v is strongly (resp. weakly) *ergodically convergent* if the mean $\overline{v}(t)$ has a strong (resp. weak) limit as $t \to \infty$, where

$$\overline{v}(t) := \frac{1}{t}\int_0^t v(\xi)\mathrm{d}\xi.$$

Given $h \geq 0$, we denote by v_h the translation defined by

$$v_h(t) := v(h + t).$$

Notice that

$$\overline{v_h}(t) = \frac{1}{t}\int_0^t v(h+\xi)\mathrm{d}\xi = \left(\frac{t+h}{t}\right)\frac{1}{t+h}\int_0^{t+h} v(\eta)\mathrm{d}\eta - \frac{1}{t}\int_0^h v(\xi)\mathrm{d}\xi.$$

Thus, if v is ergodically convergent to some $y \in X$, so does v_h for each $h \geq 0$. But if the latter holds uniformly in h, then we say that v is almost-convergent in the sense of Lorentz [18]. More precisely, v is strongly (resp. weakly) *almost-convergent* to some $y \in X$ if $\overline{v_h}(t)$ converges strongly (resp. weakly) to y as $t \to \infty$ uniformly in $h \geq 0$.

Remark 2.1. Almost-convergence is a notion intermediate between ergodic convergence and convergence. Of course, almost-convergence implies ergodic convergence. On the other hand, v is convergent if, and only if, it is almost-convergent and *asymptotically regular* in the sense that the difference $v(t+h) - v(t)$ converges to zero as $t \to \infty$ for each $h \geq 0$, for the corresponding topology (see [18]). Thus almost-convergence supplemented with asymptotic regularity provides a criterion for convergence. This approach has been applied to study the asymptotic behavior of semigroups; see, for instance, [19]. □

The previous notions of convergence will be applied to the so-called *almost-orbits* of an arbitrary evolution system U on some nonempty Borel subset C of X. The following terminology was introduced by Miyadera and Kobayasi in [8] for the particular case of semigroups.

Definition 2.2. A function $u \in L^\infty_{loc}(0, \infty; C)$ is an *almost-orbit* of an evolution system U if

$$\lim_{t\to\infty} \sup_{h\geq 0} \|u(t+h) - U(t+h,t)u(t)\| = 0. \tag{2.1}$$

Of course, orbits as defined by (1.4) are also almost-orbits of the same evolution system.

Remark 2.3. It is easy to see that if U is Lipschitz on C and u is an almost-orbit of U, then so is any function $v \in L^\infty_{loc}(0, \infty; C)$ satisfying $\lim_{t\to\infty} \|v(t) - u(t)\| = 0$. □

The original definition in [8] required u to be continuous. We drop out any continuity condition to cover, in particular, the piecewise constant orbits of evolution systems such as (4.9).

Remark 2.4. Let A be an m-accretive operator on X and $f \in L^1_{loc}(0, \infty; X)$. An integral solution of

$$\dot{u}(t) + Au(t) \ni f(t), \quad t > 0 \tag{2.2}$$

is a continuous function u satisfying

$$\|u(t) - x\|^2 - \|u(r) - x\|^2 \leq 2\int_r^t \langle f(\xi) + y, u(\xi) - x\rangle \, d\xi$$

for each $y \in Ax$ and $0 \leq r \leq t$ (see [2]). It is easy to see that if $f \in L^1(0, \infty; X)$ then any integral solution u of (2.2) is an almost-orbit of the semigroup generated by $-A$ on $C = \overline{D(A)}$; see [8, Proposition 7.1]. The asymptotic behavior of integral solutions of (2.2) depend on the operator A and the geometric properties of the space X. In Section 5 we discuss the asymptotic behavior of these semigroups in the Hilbert space setting. For a more complete survey on this topic see [20], which includes relevant extensions to more general Banach spaces. □

Remark 2.5. As we mentioned before, our definition of an almost-orbit is based on the one given in [8]. In recent studies concerning the asymptotic behavior of nonexpansive semigroups, other authors have explored different definitions. Let T be a nonexpansive semigroup.

1. In [11] the almost-orbits are required to verify
$$\lim_{t,h\to\infty} \|u(t+h) - T(h)u(t)\| = 0.$$
2. Later, in [16] the authors define *nearly almost-orbits* to satisfy
$$\inf_{t\geq 0} \sup_{h\geq 0} \|u(t+h) - T(h)u(t)\| = 0.$$

Both definitions are slightly weaker than (2.1). However, the examples given in the cited references all reduce to Examples 1 and 3, which also comply with our definition (2.1). □

Remark 2.6. If (X, d) is a complete metric space, the Lipschitz condition (1.1) on a evolution systems reads $d(U(t,s)x, U(t,s)y) \leq Ld(x,y)$ and the limit (2.1) in the definition of almost-orbit can be rephrased as $\lim_{t\to\infty} \sup_{h\geq 0} d(u(t+h), U(t+h,t)u(t)) = 0$. □

3. Preservation of convergence properties

We begin this section by providing a basic asymptotic property concerning the set of almost-orbits of a given evolution system. It generalizes [8, Lemma 3.1]:

Lemma 3.1. *Let U be L-Lipschitz, and u_1, u_2 be two almost-orbits of U. Then*

$$\limsup_{t\to\infty} \|u_1(t) - u_2(t)\| \leq L \liminf_{t\to\infty} \|u_1(t) - u_2(t)\| < \infty.$$

Consequently:

(i) *If one almost-orbit of U is bounded, then every almost-orbit of U is so.*
(ii) *If 0 is a cluster point of $\|u_1(t) - u_2(t)\|$ then $\lim_{t\to\infty} \|u_1(t) - u_2(t)\| = 0$.*
(iii) *If $L = 1$ then the limit $\lim_{t\to\infty} \|u_1(t) - u_2(t)\|$ always exists.*

Proof. For $i = 1, 2$ let $\psi_i(t) = \sup_{h\geq 0} \|u_i(t+h) - U(t+h,t)u_i(t)\|$. Then

$$\|u_1(t+h) - u_2(t+h)\| \leq \psi_1(t) + \psi_2(t) + L\|u_1(t) - u_2(t)\|$$

for every $h \geq 0$. Hence $\limsup_{h\to\infty} \|u_1(h) - u_2(h)\| \leq \psi_1(t) + \psi_2(t) + L\|u_1(t) - u_2(t)\| < \infty$ and finally $\limsup_{h\to\infty} \|u_1(h) - u_2(h)\| \leq L \liminf_{t\to\infty} \|u_1(t) - u_2(t)\|$. □

Remark 3.2. It is easy to see that the analogue to Lemma 3.1 is still true in the framework of Remark 2.6. □

We now state and prove the main theoretical result of this paper. It establishes that any of the convergence properties introduced in the previous section on almost-orbits can be reduced to the study of orbits.

Theorem 3.3. *Let U be a Lipschitz evolution system.*

(i) *If every orbit of U is strongly (resp. weakly) convergent, so is every almost-orbit.*
(ii) *If every orbit of U is strongly (resp. weakly) ergodically convergent, so is every almost-orbit.*
(iii) *If every bounded orbit of U is strongly (resp. weakly) almost-convergent, so is every bounded almost-orbit.*

Proof. (i) Let τ denote the hypothesized topology. And suppose that the τ-limit of $U(t,s)x$ as $t \to \infty$ exists for all x and s. Let u be an almost-orbit of U. Take $p \geq 0$ and set

$$\zeta(p) = \tau - \lim_{t\to\infty} U(t,p)u(p).$$

We have

$$\zeta(p+h) - \zeta(p) = \tau - \lim_{t\to\infty} \{U(t,p+h)u(p+h) - U(t,p)u(p)\}.$$

Let $h \geq 0$. If $t \geq p+h$ then $U(t,p)u(p) = U(t,p+h)U(p+h,p)u(p)$. By virtue of the Lipschitz property (1.1), for all $t \geq p+h$ we have that

$$\|U(t,p+h)u(p+h) - U(t,p)u(p)\| \leq L\|u(p+h) - U(p+h,p)u(p)\|,$$

for some $L > 0$. By the τ-lower semicontinuity of the norm we get

$$\|\zeta(p+h) - \zeta(p)\| \leq L\|u(p+h) - U(p+h,p)u(p)\|.$$

Since u is an almost-orbit of U, the right-hand side tends to zero as $p \to \infty$ uniformly in $h \geq 0$. Therefore $\{\zeta(p) : p \to \infty\}$ is a Cauchy net that converges strongly to a limit, namely ζ_∞. Finally, we can express $u(p+h) - \zeta_\infty$, for all $p, h \geq 0$, as

$$u(p+h) - \zeta_\infty = [u(p+h) - U(p+h,p)u(p)] + [U(p+h,p)u(p) - \zeta(p)] + [\zeta(p) - \zeta_\infty].$$

Given $\varepsilon > 0$ we can choose p large enough that the first and third terms on the right-hand side are less than ε in norm, uniformly in h for the first term. Next for such a fixed p, we let $h \to \infty$ so that the second term τ-converges to zero. Hence $u(t)$ is τ-convergent to ζ_∞ as $t \to \infty$.

(ii) Let u be an almost-orbit of U. For $p, h \geq 0$ and t sufficiently large, define

$$\sigma_h(t,p) = \frac{1}{t}\int_0^t U(p+h+\xi,p)u(p)\mathrm{d}\xi \tag{3.1}$$

and set $\zeta(p) = \tau - \lim_{t\to\infty} \sigma_0(t,p)$, where τ stands for either the strong or the weak topology according to the hypothesis. Notice that

$$[\sigma_0(t,p+h) - \sigma_0(t+h,p)] - [\sigma_h(t,p) - \sigma_0(t+h,p)] = [\sigma_0(t,p+h) - \sigma_h(t,p)]. \tag{3.2}$$

For each $h \geq 0$ we have that

$$\tau - \lim_{t\to\infty} \sigma_h(t,p) = \tau - \lim_{t\to\infty} \sigma_0(t,p) = \tau - \lim_{t\to\infty} \sigma_0(t+h,p).$$

We let $t \to \infty$ in Eq. (3.2) and use the weak lower semicontinuity of the norm to obtain

$$\|\zeta(p+h) - \zeta(p)\| \leq \liminf_{t\to\infty} \|\sigma_0(t,p+h) - \sigma_h(t,p)\| \leq L\,\|u(p+h) - U(p+h,p)u(p)\|,$$

which in turn tends to zero as $p \to \infty$ uniformly in $h \geq 0$. As a consequence, $\zeta(p)$ converges strongly to some ζ_∞ as $p \to \infty$. Finally, for any $p, h \geq 0$ we write

$$\begin{aligned}\bar{u}(p+h) - \zeta_\infty &= \frac{1}{p+h}\int_0^p u(\xi)\mathrm{d}\xi + \left[\frac{h}{p+h}\sigma(h,p) - \zeta(p)\right] + [\zeta(p) - \zeta_\infty] \\ &\quad + \frac{1}{p+h}\int_0^h [u(p+\xi) - U(p+\xi,p)u(p)]\mathrm{d}\xi.\end{aligned}$$

The second term is bounded by $\sup_{k\geq 0}\|u(p+k) - U(p+k,p)u(p)\|$, which is independent of h and tends to zero as $p \to \infty$. The last term converges strongly to zero as $p \to \infty$. Thus, given any $\varepsilon > 0$, we can choose p_ε large enough that the second and fourth terms are both less than ε. Having fixed p_ε, the first term converges strongly to zero as $h \to \infty$ while the third term τ-converges to zero. As a consequence $\bar{u}(t)$ is τ-convergent to ζ_∞ as $t \to \infty$.

(iii) Let u be a bounded almost-orbit of U, so that $\|u\|_\infty = \sup_t \|u(t)\| < \infty$. Define $\sigma_h(t,p)$ as in (3.1). Since u is an almost-orbit, there exists $p_0 \geq 0$ such that for all $p \geq p_0$, we have

$$\|u(p+h) - U(p+h,p)u(p)\| \leq 1 \quad \text{for all } h \geq 0.$$

Hence, for all $p \geq p_0$ and $h \geq 0$ we get

$$\|U(p+h,p)u(p)\| \leq 1 + \|u\|_\infty.$$

Therefore, by virtue of the hypothesis, for every $p \geq p_0$ there exists $\zeta(p) \in X$ such that for all $h \geq 0$,

$$\zeta(p) = \tau - \lim_{t\to\infty} \sigma_h(t,p),$$

and the convergence is uniform in $h \geq 0$. Next, we prove that $\{\zeta(p) : p \geq 0\}$ is a Cauchy net. For every $p, h \geq 0$ and $t \geq p+h$ we have

$$\|\sigma_0(t,p+h) - \sigma_h(t,p)\| \leq L\|u(p+h) - U(p+h,p)u(p)\|.$$

Let $t \to \infty$ to get

$$\|\zeta(p+h) - \zeta(p)\| \leq L\|u(p+h) - U(p+h,p)u(p)\|,$$

which tends to 0 as $p \to \infty$ uniformly in $h \geq 0$. Hence $\zeta(p) \to \zeta_\infty$ as $p \to \infty$, for some ζ_∞. For any $p, h, k \geq 0$ we write

$$\overline{u_n}(p+h) - \zeta_\infty = \frac{1}{p+h}\int_0^p u(k+\xi)\mathrm{d}\xi + \left[\frac{h}{p+h}\sigma_n(h,p) - \zeta(p)\right]$$
$$+ \frac{1}{p+h}\int_0^h [u(p+k+\xi) - U(p+k+\xi,p)u(p)]\mathrm{d}\xi + [\zeta(p) - \zeta_\infty].$$

The first term on the right-hand side is bounded by $p/(p+h)\|u\|_\infty$, independently of k. The second term is bounded by $\sup_{q\geq 0}\|u(p+q) - U(p+q,p)u(p)\|$, which is independent of h and k, and tends to zero as $p \to \infty$. The last term converges strongly to zero as $p \to \infty$. Thus, given any $\varepsilon > 0$, we can choose p_ε large enough that the second and fourth terms are both less than ε. Then, for such p_ε, the first term converges strongly to zero as $h \to \infty$ while the third term τ-converges to zero, both uniformly in k. As a consequence $\overline{u_n}(t)$ is τ-convergent to ζ_∞ as $t \to \infty$ uniformly in k. □

Remark 3.4. Theorem 3.3(i) is inspired from the analysis developed by Passty in [15] for an m-accretive operator A which is single-valued and Lipschitz continuous. In that context, [15] establishes a rigorous and precise connection between the asymptotic convergence of the semigroup generated by $-A$, with the convergence of *infinite products of resolvents* of the type $x^n = J^A_{\lambda_n}(x^{n-1})$ for $J^A_{\lambda_n} = (I+\lambda_n A)^{-1}$ where $\lambda_n > 0$ (see Example 3). □

Remark 3.5. Theorem 3.3(iii) was proved in [8] under additional assumptions: (i) U is a strongly continuous semigroup of contractions, (ii) the set of stationary points is nonempty, and (iii) for the weak topology, the space X is assumed to be *weakly sequentially complete*, which means that every weak Cauchy sequence converges weakly to an element in X. The spaces ℓ^1, L^1 and all reflexive Banach spaces have this property. This is not the case if X contains c_0, though. □

Remark 3.6. Compared with Theorem 3.3(i)–(ii), the hypotheses and conclusion in (iii) seem weaker. By Lemma 3.1(i), for Theorem 3.3(iii) to be useful, one must prove that the system has at least one bounded almost-orbit. In practice, this step tends to be necessary for proving that the orbits are convergent. In many applications one has to do it anyway. □

Remark 3.7. Part (i) holds in the metric framework of Remark 2.6 (for the strong topology). □

4. Asymptotic almost-equivalence for discrete-time and continuous-time dynamics

Definition 4.1. We say that two evolution systems U and V are *asymptotically almost-equivalent* (AAE for short) if every orbit of one of them is an almost-orbit of the other one.

Remark 4.2. When U is an autonomous nonexpansive evolution system and V is AAE to U, then V is said to be an *asymptotic semigroup*, according to the terminology used in [15]. □

Remark 4.3. Observe that if we assume that for each $r > 0$ there is $G_r : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty}\sup_{h\geq 0} G_r(t+h,t) = 0$ and $\|U(t,s)x - V(t,s)x\| \leq G_r(t,s)$ for all $x \in B(0,r)$ and $t \geq s \geq 0$ then every bounded orbit of U is an almost-orbit of V and vice versa. If $G_r \equiv G$, the boundedness assumption is unnecessary. □

Let U and V be two AAE evolution systems according to Definition 4.1. By Lemma 3.1, if one almost-orbit of U *or* V is bounded, every almost-orbit of U *and* V is so. Furthermore, as a direct consequence of Theorem 3.3, the orbits of U and V have, essentially, the same asymptotic behavior in terms of the three notions of convergence considered in this paper. More precisely, we have the following:

Corollary 4.4. *Let U, V be asymptotically almost-equivalent Lipschitz evolution systems. Then:*

(i) *Every orbit of U is strongly (resp. weakly) convergent if, and only if, every orbit of V is so.*
(ii) *Every orbit of U is strongly (resp. weakly) ergodically convergent if, and only if, every orbit of V is so.*
(iii) *Every bounded orbit of U is strongly (resp. weakly) almost-convergent if, and only if, every orbit of V is so.*

Remark 4.5. Even if all the orbits of an evolution system U are known to converge at a certain rate, nothing can be said about the rate of convergence of its almost-orbits (see Remark 2.3). Therefore, two AAE evolution systems may have different convergence rates. □

Now let us restate some well-known results in terms of the asymptotic almost-equivalence of suitable evolution systems.

Example 1. From now on, A stands for an m-accretive operator on X. By virtue of [21,8] (see Remark 2.4), the dynamics on $C = \overline{D(A)}$ associated with the differential inclusion

$$\dot{u}(t) + Au(t) \ni 0 \tag{4.1}$$

is asymptotically almost-equivalent to

$$\dot{u}(t) + Au(t) \ni f(t), \tag{4.2}$$

whenever $f \in L^1(0, \infty; X)$. More precisely, let S_A be the nonexpansive semigroup generated by $-A$ on $\overline{D(A)}$, and denote by U_A the autonomous evolution system corresponding to S_A, that is, $U_A(t, s) = S_A(t - s)$ for any $t \geq s \geq 0$. In other words, for any $x \in C$, the function $u(t; s, x) := U_A(t, s)x$ is an integral solution of the differential inclusion (4.1) with initial condition $u(s) = x$ at $t = s$. On the other hand, let $U_{A,f}$ be the nonautonomous nonexpansive evolution system defined by the integral solutions of (4.2). If $f \in L^1(0, \infty; X)$ then the evolution systems U_A and $U_{A,f}$ are AAE according to Definition 4.1. □

As we have already noticed, systems which are intrinsically discrete-time ones, such as those associated with iterative algorithms, can be regarded as continuous-time evolution systems by interpolation. We now restate some of the definitions in such a setting. Let $\{\sigma_n\}$ be a strictly increasing unbounded sequence of positive numbers. Given any sequence $\{x^n\}$ in C, the piecewise constant interpolation of the points $(\sigma_n, x^n) \in \mathbb{R} \times X$ defines a continuous-time trajectory $v(t)$ such that $v(t) \equiv x^n$ for $\sigma_n \leq t < \sigma_{k+1}$, that is to say,

$$v(t) = x^{\nu(t)}, \quad t \geq 0, \tag{4.3}$$

where

$$\nu(t) := \max\{0, \max\{n \in \mathbb{N} \mid \sigma_n \leq t\}\}, \tag{4.4}$$

with the convention that $\max \emptyset = -\infty$. Of course, $v(t)$ is not continuous as a function of time unless $\{x^n\}$ is a constant sequence. Observe that $v(t)$ converges to x^* as $t \to \infty$ if, and only if, x^n converges to x^* as $n \to \infty$. In a similar fashion, $v(t)$ is ergodically convergent to x^* as $t \to \infty$ if, and only if, the sequence $\{\bar{x}^n\}$ of means

$$\bar{x}^n = \frac{1}{\sigma_n} \sum_{k=1}^{n} \lambda_k x^k$$

converges to x^* as $n \to \infty$. In this case we say $\{x^n\}$ is ergodically convergent to x^* as $n \to \infty$.

If $\{x^n\}$ is generated by an iterative method of the type

$$x^n = F_n(x^{n-1}), \quad n = 1, 2, \ldots \tag{4.5}$$

for some suitable family of applications $\{F_n\}$ from C into C, then we can define an evolution system on C by setting

$$V(t, s) := \prod_{n=\nu(s)+1}^{\nu(t)} F_n, \quad t \geq s \geq 0, \tag{4.6}$$

where the product operation stands for the composition of maps, under the convention that $\prod_{n=k}^{k} F_n = I$. If each F_n is L_n-Lipschitz and the product $\prod_{n=1}^{\infty} L_n$ is bounded from above by $L > 0$, then such a V is L-Lipschitz. If $F_n \equiv F$ then V is autonomous. Notice that if $\{x^n\}$ is generated by (4.5) and $v(t)$ is the corresponding piecewise constant interpolation given by (4.3)–(4.4), then we get $\forall t \geq s \geq 0,\ v(t) = V(t, s)v(s)$, and so in particular

$$\forall m \geq n \geq 0, \quad x^m = V(\sigma_m, \sigma_n)x^n.$$

In this sense, the pair $\{\sigma_n\}, \{x^n\}$ is a discrete orbit of V.

We say that the discrete-time evolution system given by a pair $\{\sigma_n\}, \{F_n\}$ is AEE to a given evolution system U if the corresponding evolution system V defined by (4.4) and (4.6) is AAE to U.

Remark 4.6. By definition, the discrete-time evolution systems induced by two families $\{F_n^1\}$ and $\{F_n^2\}$ are AAE for a given $\{\sigma_n\}$ if, and only if, the corresponding evolution systems V^1 and V^2 defined by (4.4) and (4.6) are AAE. By construction, the latter is equivalent to

$$\lim_{n\to\infty}\left[\sup_{k\in\mathbb{N}}\left\|x_1^{n+k}-V^2(\sigma_{n+k},\sigma_n)x_1^n\right\|\right]=\lim_{n\to\infty}\left[\sup_{k\in\mathbb{N}}\left\|x_2^{n+k}-V^1(\sigma_{n+k},\sigma_n)x_2^n\right\|\right]=0,$$

where $x_i^n = F_n^i(x_i^{n-1})$ for $i = 1, 2$ and $n \in \mathbb{N}$. See Proposition 6.1 for an example. □

Example 2. Consider the so-called *Mann iteration* for finding a fixed point of a nonexpansive map $T : C \to C$ on a nonempty, closed and convex set C:

$$x^n = (1-\lambda_n)x^{n-1} + \lambda_n T(x^{n-1}), \quad n = 1, 2, \dots \tag{4.7}$$

where $\{\lambda_n\} \subset (0, 1)$ and the starting point x^0 is given in C. Mann's iteration (4.7) may be rewritten as

$$(x^n - x^{n-1})/\lambda_n + x^{n-1} = T(x^{n-1}),$$

which in turn can be viewed as an explicit Euler integration scheme applied to the differential equation

$$\dot{u}(t) + u(t) = T(u(t)). \tag{4.8}$$

The latter has the form $\dot{u}(t) + Au(t) = 0$ for $A = I - T$, which is a monotone operator by virtue of the nonexpansiveness of T.

We claim that the evolution systems corresponding to (4.7) and (4.8) are AEE when $\{\lambda_n\} \in \ell^2 \setminus \ell^1$. More precisely, set

$$V_T(t,s) := \prod_{n=\nu(s)+1}^{\nu(t)} ((1-\lambda_n)I + \lambda_n T), \tag{4.9}$$

for $\nu(t)$ given by (4.4) with

$$\sigma_n := \sum_{j=0}^{k} \lambda_j, \tag{4.10}$$

where $\lambda_0 = 0$. We thus get a nonexpansive evolution system V_T on C. If $\lambda_n \equiv \lambda \in (0, 1)$ then V_T is autonomous, of course. On the other hand, let U_T be the nonexpansive autonomous evolution system generated by $A = I - T$, that is, the nonexpansive semigroup associated with the solutions to (4.1) with $Ax = x - Tx$. We have thus two evolution systems associated with T, namely V_T given by (4.9) which is a piecewise constant interpolation for the iteration (4.7), and U_T which is generated by the continuous-time dynamics (4.8). For both systems we have that $x^* \in C$ is a stationary point, that is $V_T(t, s)x^* = x^* = U_T(t, s)x^*$ for all $t \geq s \geq 0$, if and only if x^* is a fixed point of T.

Proposition 4.7. *If $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ then every bounded orbit of U_T is an almost-orbit of V_T and vice versa.*

Proof. Let $n \geq k$ and $t \geq s$. Corollary 3.12 in [22] gives

$$\|V_T(\sigma_n,\sigma_n)y - U_T(t,s)y\| \leq \|y - T(y)\|\sqrt{[(\sigma_n-\sigma_n)-(t-s)]^2+\tau_n-\tau_n}. \tag{4.11}$$

Let u be a bounded orbit of U_T. Then $C = \sup_{t\geq 0}\|u(t) - T(u(t))\| < \infty$ and hence

$$\begin{aligned}\|V_T(t+h,t)u(t)-u(t+h)\| &\leq C\sqrt{\left[\sigma_{\nu(t+h)}-\sigma_{\nu(t)}-h\right]^2+\tau_{\nu(t+h)}-\tau_{\nu(t)}}\\ &\leq C\sqrt{3\sum_{j\geq\nu(t)}\lambda_j^2},\end{aligned}$$

which tends to 0 as $t \to \infty$ uniformly in $h \geq 0$. The converse is similar. □

Example 3. In the late 1970's there was intensive research activity regarding the asymptotic behavior of the differential inclusion (4.1) and the proximal iterations

$$\frac{x^n - x^{n-1}}{\lambda_n} + Ax^n \ni 0 \tag{4.12}$$

where A is a maximal monotone operator in a Hilbert space and $\{\lambda_n\}$ is a sequence of positive numbers. Observe that (4.12) is an implicit discretization scheme for (4.1) and can be written in resolvent form as

$$x^n = (I + \lambda_n A)^{-1}x^{n-1}.$$

Define the nonexpansive evolution system by

$$W_A(t,s) = \prod_{n=\nu(s)+1}^{\nu(t)} (I+\lambda_n A)^{-1},$$

where, as before, $\nu(t)$ is given by (4.4) with $\{\sigma_n\}$ given by (4.10). We have the following:

Proposition 4.8. *Under the previous definitions and hypotheses, we have that:*

(i) *If $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ then W_A and U_A are AAE.*
(ii) *If X is a Hilbert space and $A = \partial f$, the subdifferential of a closed, proper and convex function $f : X \to \mathbb{R} \cup \{\infty\}$, then $W_{\partial f}$ and $U_{\partial f}$ are AAE under the weaker condition $\{\lambda_n\} \notin \ell^1$.*

For the proof of Proposition 4.8(i) in the general case, see [14]. The fact that the orbits of W_A are almost-orbits of U_A had already been proved in [8]. This was shown earlier in [15] by assuming A to be single-valued and Lipschitz continuous. For Proposition 4.8(ii), the reader is referred to the paper [13] by Güler, where the author proves that Baillon's counterexample in [23] for the strong convergence of the solutions of the differential inclusion (4.1) also provides a counterexample for the strong convergence of the proximal point algorithm (4.12). □

Example 4. In each of the previous examples, one of the evolution systems is indeed a semigroup. However, one can establish asymptotic equivalence between two truly nonautonomous systems.

First recall that the solutions of (4.1) are ergodically convergent but not convergent in general. A classical idea for forcing strong convergence is to consider the Tikhonov-like regularization

$$\dot{u}(t) + Au(t) + \varepsilon(t)u(t) \ni 0, \tag{4.13}$$

where ε is positive and $\lim_{t\to\infty} \varepsilon(t) = 0$. Strong convergence to the least-norm element of $A^{-1}0$ has been proved under additional conditions on ε. A recent result in this sense is given in [24] when ε has bounded variation and $\int_0^\infty \varepsilon(t)\mathrm{d}t = \infty$. In the cited reference, the authors take $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $A(x,y) = (1-y, x-1)$ and exhibit a positive function ε_0 such that $\lim_{t\to\infty} \varepsilon_0(t) = 0$ and $\int_0^\infty \varepsilon_0(t)\mathrm{d}t = \infty$ but strong convergence fails. They also show that the same counterexample works for the corresponding diagonal proximal point algorithm

$$\frac{x^n - x^{n-1}}{\lambda_n} + Ax^n + \varepsilon_0(\sigma_n)x^n \ni 0 \tag{4.14}$$

for a certain choice of the step sizes $\{\lambda_n\}$, where σ_n is given by (4.10). Although stated in a different way, they essentially prove that (4.14) is AAE to (4.13) for $\varepsilon = \varepsilon_0$. □

5. A glance at applications I: some classical results revisited

In this section we revisit some classical results concerning the evolution equation (4.1), the proximal point algorithm (4.12) and Mann's iterations (4.7). The aim is to illustrate the kinds of conclusions that one can draw by comparing different evolution systems.

We begin by noticing that in view of Theorem 3.3 and Proposition 4.8, several known weak and strong convergence results for the proximal point algorithm (4.12) are direct consequences of the corresponding results for the continuous-time differential inclusion (4.1). More precisely, in Hilbert spaces we have the following:

1. Strong convergence when A is odd: Theorem 1 in [25] implies Theorem II.1 in [26].
2. Weak convergence in average for arbitrary A: Theorem 1 in [27] implies Theorem III.1 in [26].
3. Weak convergence when A is *demipositive*: Theorem 2 in [28] implies Theorem 10 in [29].

In the following parts of this section we give a new look at classical conditions ensuring the strong convergence of the evolution systems defined by (4.1), (4.7) and (4.12), respectively.

5.1. Strong monotonicity and minimal assumptions on the space

From now on, let $(X^*, \|\cdot\|_*)$ be the topological dual of the Banach space $(X, \|\cdot\|)$ and denote by $\langle\cdot,\cdot\rangle$ the duality pairing $\langle\cdot,\cdot\rangle_{X,X^*}$. We say that X is *smooth* if its norm is Gâteaux-differentiable. By Corollary 5.4.18 in [30] this is equivalent to the duality mapping being single-valued. More precisely, for each $x \in X$ there exists a unique $j(x) \in X^*$ such that $\|j(x)\|_* = \|x\|$ and $\langle x, j(x)\rangle = \|x\|^2$. In Hilbert space one has $j(x) = x$.

An operator $A : X \to 2^X$ is *strongly monotone* if there exists $\alpha > 0$ such that

$$\langle x^* - y^*, j(x-y)\rangle \geq \alpha\|x-y\|^2 \tag{5.1}$$

for all $[x, x^*]$ and $[y, y^*]$ in the graph of A.

First consider the differential inclusion (4.1). Set $\mathcal{S} = A^{-1}0$. Notice that if A is strongly monotone then $\mathcal{S}$ contains at most one element.

Proposition 5.1. *Let X be a smooth Banach space. Let $A : X \to 2^X$ be strongly monotone with $\mathcal{S} \neq \emptyset$. Every solution of (4.1) converges strongly to the unique $p \in \mathcal{S}$.*

Proof. Let $\mathcal{S} = \{p\}$. Since the norm of X is Gâteaux-differentiable we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u(t)-p\|^2 = \langle \dot{u}(t), j(u(t)-p)\rangle \leq -\alpha\|u(t)-p\|^2$$

almost everywhere and so $u(t)$ converges to p. □

The differential inclusion (4.1) has a unique solution for every initial condition in $\overline{D(A)}$ provided A is m-accretive. These solutions are defined for all $t \geq 0$.

According to Proposition 4.8, a similar conclusion can be derived for proximal iterations.

Corollary 5.2. *Let X be a smooth Banach space. Assume $A : X \to 2^X$ is strongly monotone and $\mathcal{S} \neq \emptyset$. Any sequence x_n verifying (4.12) with $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ converges strongly to the unique $p \in \mathcal{S}$.*

The sequence given by (4.12) is well-defined for every initial condition if A is m-accretive.

We are able to prove strong convergence under the same hypotheses as for Proposition 5.1. Observe that the set C is not assumed to be bounded.

Corollary 5.3. *Let C be a closed convex subset of a smooth Banach space X. Let $T : C \to C$ be nonexpansive with a fixed point. If $A = I - T$ is strongly monotone, every sequence x_n verifying (4.7) with $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ converges strongly to the unique fixed point of T.*

In terms of T, the strong monotonicity of $A = I - T$ is equivalent to T verifying

$$\langle Tx - Ty, j(x-y)\rangle \leq w\|x-y\|^2$$

for some $w < 1$ and all $x, y \in C$. This property is known as *quasi-dissipativity*.

The hypotheses on the space seem to be the weakest available in the literature.

5.2. Pazy's convergence condition

The strong monotonicity assumption is rather restrictive. We shall mention other conditions that ensure strong convergence of the three methods. In this part we assume that X is smooth, reflexive and strictly convex. This implies that the duality mapping is single-valued and the *nearest-point mapping* is well-defined for each closed convex set C. More precisely, for each $x \in X$ there exists a unique point $P_C x$ that minimizes the distance from x to C. Moreover, if $A : X \to 2^X$ is accretive then $\mathcal{S}$ is closed and convex.

An operator $A : X \to 2^X$ satisfies *Pazy's convergence condition* (from [31]; see also [32] or [33]) if $\mathcal{S} \neq \emptyset$ and for every bounded sequence (x_n, y_n) in the graph of A one has

$$\lim_{n\to\infty} \langle y_n, j(x_n - P_{\mathcal{S}} x_n)\rangle = 0 \Longrightarrow \liminf_{n\to\infty} \|x_n - P_{\mathcal{S}} x_n\| = 0. \tag{5.2}$$

Strongly monotone operators have this property as well as those having compact resolvent. It is important to notice that this condition does not require $\mathcal{S}$ to be a singleton. The results listed below are still true if (5.2) holds only for the element of minimal norm: $y_n = P_{Ax_n} 0$. This covers operators which are strongly monotone in the sense of Pazy [34].

The following is [33, Theorem 5.1]:

Proposition 5.4. *Let X be smooth, reflexive and strictly convex and let $A : X \to 2^X$ be m-accretive and satisfy Pazy's convergence condition. Then every almost-orbit of the evolution system U_A defined by (4.1) converges strongly to some $p \in \mathcal{S}$.*

Remark 5.5. Using Theorem 3.3(i), the proof of Proposition 5.4 presented in [33] can be simplified and shortened by proving the result for the orbits only. □

Remark 5.6. In [9], the author had proved Proposition 5.4 assuming X and X^* to be uniformly convex. That result is an immediate consequence Theorem 3.3(i) and [32, Theorem 1]. □

The following new result can be obtained via Propositions 4.8 and 5.4:

Corollary 5.7. *Let X be smooth, reflexive and strictly convex and let $A : X \to 2^X$ be m-accretive and satisfy Pazy's convergence condition. Then every sequence satisfying (4.12) with $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ converges strongly to a zero of A.*

Let $A = I - T$, where $T : C \to C$ is nonexpansive. In terms of T, condition (5.2) states that

$$\lim_{n\to\infty} \langle x_n - Tx_n, j(x_n - P_{\mathcal{S}}x_n)\rangle = 0 \Longrightarrow \liminf_{n\to\infty} \|x_n - P_{\mathcal{S}}x_n\| = 0 \tag{5.3}$$

for each bounded sequence x_n in C. Following the arguments in [32] one can prove that if X and X^* are uniformly convex then Mann's iterations also converge strongly to a fixed point of T provided the sequence $(x_n - x_{n-1})/\lambda_n$ is bounded and a ℓ^2-type summability condition involving the modulus of uniform convexity of X holds. By using Proposition 4.7 one gets rid of the boundedness hypothesis and simplifies the summability condition and weakens the hypotheses on the space. This yields the following new result.

Corollary 5.8. *Let X be smooth, reflexive and strictly convex, let C be closed and convex and let $T : C \to C$ be nonexpansive and satisfy (5.3). Then every sequence satisfying (4.7) with $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ converges strongly to a fixed point of T.*

5.3. A solution set with nonempty interior

Strong convergence also holds if X and X^* are uniformly convex and $\operatorname{int} \mathcal{S} \neq \emptyset$. Recall that uniformly convex spaces are reflexive and strictly convex. Observe also that if X^* is uniformly convex then X is smooth. The following is [32, Theorem 4]:

Proposition 5.9. *Let X and X^* be uniformly convex and let $A : X \to 2^X$ be m-accretive with $\operatorname{int} \mathcal{S} \neq \emptyset$. Then every solution of (4.1) converges strongly to some $p \in \mathcal{S}$.*

The following result [32, Theorem 2] can also be obtained via Propositions 5.9 and 4.8:

Corollary 5.10. *Let X and X^* be uniformly convex and let $A : X \to 2^X$ be m-accretive with $\operatorname{int} \mathcal{S} \neq \emptyset$. Then every sequence satisfying (4.12) with $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ converges strongly to a zero of A.*

Following the arguments in the cited reference one can prove that, under the same hypotheses, Mann's iterations also converge strongly to a fixed point of T under additional boundedness and summability assumptions involving the modulus of uniform convexity of X. By using Proposition 4.7 one obtains, as before, a new result under simpler hypotheses:

Corollary 5.11. *Let X and X^* be uniformly convex, let C be closed and convex and let $T : C \to C$ be a nonexpansive function whose fixed point set has nonempty interior. Then every sequence satisfying (4.7) with $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ converges strongly to a fixed point of T.*

6. A glance at applications II: robustness and stability

In this section we show that small perturbations are negligible for the asymptotic behavior of dissipative systems.

6.1. Robustness of nonexpansive algorithms

Let $\{F_n\}$ be a family of nonexpansive functions on a Banach space X. The family $\{F_n\}$ defines a 1-Lipschitz discrete evolution system $\{U_m^n\}$ by

$$U_m^n = \prod_{n=m+1}^{n} F_n \tag{6.1}$$

for $n \geq m$. The following particular cases are relevant in applications:

1. The proximal point algorithm (4.12), where $F_n = (I + \lambda_n A)^{-1}$. More generally, one can consider a diagonal proximal algorithm with $F_n = (I + \lambda_n A_n)^{-1}$.
2. Mann's iterations (4.7), where $F_n = \lambda_n T + (1 - \lambda_n)I$.
3. Successive approximations: $F_n \equiv T$.

Suppose a sequence $\{\hat{x}^n\}$ is computed approximately so as to satisfy

$$\left\|\hat{x}^n - F_n(\hat{x}^{n-1})\right\| \leq \varepsilon_n. \tag{6.2}$$

The general algorithm described by (6.1) is robust in the following sense:

Proposition 6.1. *Any sequence $\hat{x}^n$ satisfying (6.2) with $\{\varepsilon_n\} \in \ell^1$ enjoys the same asymptotic properties (to the extent of this paper) as those computed exactly following (6.1).*

Proof. By Remark 4.6, it suffices to verify that such a sequence is an almost-orbit of the (discrete-time) evolution system defined by (6.1). This is a consequence of the following fact: For $p \geq 1$ one has

$$\begin{aligned}\|U_n^{n+p}\hat{x}^n - \hat{x}^{n+p}\| &\leq \|U_n^{n+p}\hat{x}^n - U_{n+p-1}^{n+p}\hat{x}^{n+p-1}\| + \|U_{n+p-1}^{n+p}\hat{x}^{n+p-1} - \hat{x}^{n+p}\| \\ &\leq \|U_n^{n+p-1}\hat{x}^n - \hat{x}^{n+p-1}\| + \varepsilon_{n+p}.\end{aligned}$$

Inductively one obtains

$$\|U_n^{n+p}\hat{x}^n - \hat{x}^{n+p}\| \leq \sum_{n=n+1}^{n+p} \varepsilon_n$$

and the result follows immediately. □

Particular instances of this property had already appeared in the literature for the proximal iterations (4.12). In [35] the authors use a form of Proposition 6.1 in order to prove the robustness of an alternating algorithm for constrained variational inequalities (see Propositions 5.3, 5.4 and 5.5 in the cited reference).

Regarding the proximal point algorithm, in [29] the authors prove the following[1]:

Proposition 6.2. *Let A be a maximal monotone operator on a Hilbert space H. Take $\{\lambda_n\} \notin \ell^1$ and let $\{x^n\}$ verify (4.12).*

1. *If $\{\lambda_n\} \notin \ell^2$ then x^n converges weakly.*
2. *If A is demipositive then x^n converges weakly.*
3. *If $A = \partial\phi$ and ϕ is proper, closed, convex and even, then x^n converges strongly.*

Then they prove[2] that the same is true for a sequence satisfying (6.2) provided $\{\varepsilon_n\} \in \ell^1$. This is also a consequence of Propositions 6.1 and 6.2. In a similar fashion, all the corollaries in Section 5 remain true if the sequences are computed approximately following (6.2) with $\{\varepsilon_n\} \in \ell^1$.

By Proposition 6.1 along with Theorems II.1 and III.1 in [26] we have the following robustness result for ergodic convergence of the proximal point algorithm:

Corollary 6.3. *With the notation of Proposition 6.2, let $\{\hat{x}_n\}$ satisfy (6.2). Then $\hat{x}_n$ is weakly ergodically convergent. If A is odd, the convergence is strong.*

See [20] for a survey on the asymptotic behavior of sequences satisfying (4.12), which is the same as for those satisfying (6.2) with $F_n = (I + \lambda_n A)^{-1}$.

6.2. Perturbed monotone differential inclusions

Let $A(t)$ be a family of maximal monotone operators on a Hilbert space H. Consider the differential inclusion

$$\dot{u}(t) + A(t)u(t) \ni 0. \tag{6.3}$$

We are not interested in existence results here, so we shall assume that for each initial condition in H, inclusion (6.3) does have a solution (unique by monotonicity). The interested reader may consult [36–38] or [39]. Denote by U the corresponding evolution system. Assume also that for every $R > 0$ there exists $M > 0$ such that $\|x\| \leq R$ implies $\|U(t,s)x\| \leq M$. This occurs, for instance, if $A(t) \equiv A$ and $A^{-1}(0) \neq \emptyset$.

Now consider the differential inclusion

$$\dot{v}(t) + A(t)v(t) + \varepsilon(t)Bv(t) + \eta(t) \ni 0, \tag{6.4}$$

where ε, η, B are to be specified later. We shall prove that every bounded function v satisfying (6.4) is an almost-orbit of U under the following assumptions:

Let $\varepsilon \in L^1(0,\infty;\mathbb{R})$ and $\eta \in L^1(0,\infty;H)$ and suppose that the function $B : H \to H$ satisfies $\|B\zeta\| \leq b\|\zeta\|$ for some $b > 0$ and all $\zeta \in H$.

This setting includes Tikhonov's regularization with fast parameterization (see [24]) and the quasiautonomous differential inclusion in Example 1 (see [8]).

Proposition 6.4. *With the notation and hypotheses above, every bounded function v satisfying (6.4) is an almost-orbit of U.*

1 Proposition 8 and Theorems 9 and 10 in the cited reference.

2 Remark 14 in the cited reference.

Proof. Let v satisfy (6.4). Define $X_s(t) = U(t,s)v(s)$. We shall prove that $\|X_t(t+h) - v(t+h)\|$ tends to 0 as $t \to \infty$ uniformly in $h \geq 0$. Fix t and define $\psi_t(h) = \frac{1}{2}\|X_t(t+h) - v(t+h)\|^2$. Simple computations and our assumption on B yield

$$\psi_t'(h) \leq K\,|\varepsilon(t+h)|\,\|X_t(t+h)\|\,\|X_t(t+h) - v(t+h)\| + \|\eta(t+h)\|\,\|X_t(t+h) - v(t+h)\|$$

for almost every $h \geq 0$. Since y is bounded, our hypothesis on U implies

$$\|X_t(t+h) - y(t+h)\|^2 \leq C\int_t^{t+h} [|\varepsilon(\tau)| + \|\eta(\tau)\|]\,\mathrm{d}\tau$$

for some constant $C > 0$ and all $h \geq 0$. Since ε and η are integrable the right-hand tends to 0 as $t \to \infty$ uniformly in $h \geq 0$. $\square$

This shows that one can perturb systems which are known to have convergent trajectories and preserve this asymptotic behavior. Observe that this is done in a rather straightforward manner thanks to Proposition 6.4. Several differential inclusions which are well-known in the literature concerning penalization schemes coupled with the steepest descent method fit this framework. Just to mention a few classics, [40, Theorem 3.4] as well as all the results of [41, Section 3] and [42, Section 4] remain true if one replaces a differential inclusion of the type

$$\dot{u}(t) + \partial f(u(t), r(t)) \ni 0$$

by

$$\dot{u}(t) + \partial f(u(t), r(t)) + \varepsilon(t)Bu(t) + \eta(t) \ni 0,$$

provided ε, B and η satisfy the hypotheses above.

It is also important to observe that in [24, Section 4.1] the authors use a form of Proposition 6.4 in order to prove that the asymptotic behavior of an evolution equation with Tikhonov regularization is somewhat independent of the local regularity of the parameter function $\varepsilon(\cdot)$.

6.3. *Nonlinear oscillators with damping*

Let us consider the following second-order system:

$$\ddot{u}(t) + \gamma\dot{u}(t) + \nabla\Phi(u(t)) = 0, \tag{6.5}$$

where $\Phi \in \mathcal{C}^1(H;\mathbb{R})$ is bounded from below and $\nabla\Phi$ is locally Lipschitz.

By introducing the variable $\mathbf{u} = \binom{u}{\dot{u}}$ the system can be reinterpreted in the product space $\mathbf{H} = H \times H$ as

$$\dot{\mathbf{u}} + \mathbf{A}\mathbf{u} = 0, \quad \text{where } \mathbf{A} = \begin{pmatrix} 0 & -I \\ \nabla\Phi & \gamma I \end{pmatrix}.$$

However, the results of Section 6.2 cannot be directly translated because the operator $\mathbf{A}$ need not be monotone, even if Φ is convex. Therefore, this second-order system requires an independent analysis.

The asymptotic behavior of the solutions of (6.5) has been established in [43, Theorem 2.1]:

Proposition 6.5. *If Φ is convex and $u \in \mathcal{C}^2$ satisfies (6.5), then $\dot{u} \in L^2$, $\lim_{t\to\infty} \dot{u}(t) = 0$ and $\lim_{t\to\infty} \Phi(u(t)) = \inf\Phi$. Moreover, if $\operatorname{Argmin}\Phi \neq \emptyset$, then $u(t)$ converges weakly to a minimizer of Φ as $t \to \infty$. Strong convergence occurs if $\nabla\Phi$ is strongly monotone, if Φ is even or if the set $\operatorname{Argmin}\Phi$ has nonempty interior.*

Remark 6.6. The arguments in the proof of the preceding result also imply that for every $R > 0$ there exists $M_R > 0$ such that $\sup_{t\geq 0}\|u(t)\| \leq M_R$ for every solution u of (6.5) such that $\|u(0)\| \leq R$ (cf. Section 6.2). $\square$

Let us consider now the following version of (6.5) adding a Tikhonov regularization:

$$\ddot{u}(t) + \gamma\dot{u}(t) + \nabla\Phi(u(t)) + \varepsilon(t)u(t) = 0, \tag{6.6}$$

The system (6.6) is studied in [44] under certain regularity assumptions on the function $\varepsilon(\cdot)$. In particular they prove the following:

Proposition 6.7. *Let $\varepsilon \in L^1(0,\infty;\mathbb{R}_+)$ and let $v \in \mathcal{C}^2$ be a bounded solution of (6.6). Then $\dot{v} \in L^2$, $\lim_{t\to\infty} \dot{v}(t) = 0$ and $\lim_{t\to\infty} \nabla\Phi(v(t)) = 0$.*

Observe that if Φ is convex one additionally has $\lim_{t\to\infty} \Phi(v(t)) = \inf\Phi$.

Using Proposition 6.11 we shall prove that every bounded solution of (6.6) is an almost-orbit of (6.5).

For $t \geq t_0 \geq 0$ and $u_0, v_0 \in H$ we define $\mathcal{U}(t,t_0)[u_0,v_0] = [u(t),\dot{u}(t)]$ as the solution of (6.5) with initial condition $u(t_0) = u_0$ and $\dot{u}(t_0) = v_0$.

Proposition 6.8. *Let Φ be convex and let $\varepsilon \in L^1(0, \infty; \mathbb{R}_+)$. Every bounded solution of (6.6) is an almost-orbit of $\mathcal{U}$.*

Proof. Let v satisfy (6.6). As in the proof of Proposition 6.4 set $[X_s(t), \dot{X}_s(t)] = \mathcal{U}(t, s)[v(s), \dot{v}(s)]$. We must prove that

$$\alpha(t, h) = \|X_t(t+h) - v(t+h)\| \quad \text{and} \quad \beta(t, h) = \|\dot{X}_t(t+h) - \dot{v}(t+h)\|$$

both tend to zero as $t \to \infty$ uniformly in $h \geq 0$.

Fix t and set $E(h) = \frac{1}{2}\|\dot{X}_t(t+h)\|^2 + \Phi(X_t(t+h))$, so $E'(h) = -\gamma\|\dot{X}_t(t+h)\|^2 \leq 0$. Then

$$\begin{aligned} \frac{1}{2}\beta(t, h)^2 &\leq \|\dot{X}_t(t+h)\|^2 + \|\dot{v}(t+h)\|^2 \\ &\leq \|\dot{v}(t)\|^2 + 2(\Phi(v(t)) - \Phi^*) + \|\dot{v}(t+h)\|^2, \end{aligned} \tag{6.7}$$

whence $\beta(t, h)$ tends to 0 as $t \to \infty$ uniformly in $h \geq 0$.

Now define $\psi(h) = \frac{1}{2}\alpha(t, h)^2$. Observing that $\langle \zeta, \zeta - \xi \rangle \geq -\frac{1}{4}\|\xi\|^2$ for all $\zeta, \xi \in H$ we get

$$\psi''(h) + \gamma\psi'(h) \leq \beta(t, h)^2 + \frac{1}{4}\varepsilon(t+h)\|X_t(t+h)\|^2. \tag{6.8}$$

Integrating (6.8) and using the fact that $|\psi'(h)| \leq \alpha(t, h)\beta(t, h)$ we obtain

$$\frac{\gamma}{2}\alpha(t, h) \leq \alpha(t, h)\beta(t, h) + \int_0^h \beta(t, \eta)^2 \, \mathrm{d}\eta + \frac{M_R}{4}\int_0^h \varepsilon(t+\eta)\mathrm{d}\eta,$$

where $R = \sup_{t \geq 0}\|v(t)\|$ (see Remark 6.6).

Using (6.7) and the fact that $\alpha\beta \leq \frac{\gamma}{4}\alpha^2 + \frac{1}{4\gamma}\beta^2$ for all $\alpha, \beta \in \mathbb{R}$ we obtain

$$\frac{\gamma}{4}\alpha(t, h) \leq \frac{1}{4\gamma}\beta(t, h) + 2\int_t^{t+h} \|\dot{X}_t(\eta)\|^2 \, \mathrm{d}\eta + 2\int_t^{\infty} \|\dot{v}(\eta)\|^2 \, \mathrm{d}\eta + \frac{M_R}{4}\int_t^{\infty} \varepsilon(\eta)\mathrm{d}\eta.$$

It suffices to show that $\int_t^{t+h} \|X_t'(\eta)\|^2 \, \mathrm{d}\eta$ tends to zero as $t \to \infty$ uniformly in $h \geq 0$. To see this, recall first that $E'(h) = -\gamma\|X_t'(t+h)\|^2$. Integrating we get

$$\gamma\int_t^{t+h} \|X_t(\eta)\|^2 \, \mathrm{d}\eta = E(t) - E(t+h) \leq \frac{1}{2}\|y'(t)\|^2 + (\Phi(y(t)) - \Phi^*)$$

and the conclusion is straightforward. □

Corollary 6.9. *Assume $\varepsilon \in L^1(0, \infty; \mathbb{R}_+)$ and Φ is convex with Argmin $\Phi \neq \emptyset$. If $v \in \mathcal{C}^2$ satisfies (6.5), then $v(t)$ converges weakly to a minimizer of Φ as $t \to \infty$. Strong convergence occurs in the following cases:*

1. *$\nabla\Phi$ is strongly monotone;*
2. *Φ is even;*
3. *Argmin Φ has nonempty interior.*

Proof. The boundedness of v is proved in [44]. One concludes combining Theorem 3.3 with Propositions 6.5 and 6.8. □

Remark 6.10. The weak convergence and the first two results on strong convergence were proved in [44]. The third one is new. □

Since the solutions of (6.6) are bounded, this equation can be seen as a particular case of

$$\ddot{u}(t) + \gamma\dot{u}(t) + \nabla\Phi(u(t)) = f(t), \tag{6.9}$$

where Φ is sufficiently regular and $f \in L^1(0, \infty; H)$. This setting is studied in [45], where the authors prove the following[3]:

Proposition 6.11. *Let $f \in L^1(0, \infty; H)$ and let $v \in \mathcal{C}^2$ be a bounded solution of (6.9). Then $\dot{v} \in L^2$, $\lim_{t\to\infty}\dot{v}(t) = 0$ and $\lim_{t\to\infty}\nabla\Phi(v(t)) = 0$.*

The arguments in the proof of Proposition 6.8 can be easily modified to yield:

Proposition 6.12. *Let Φ be convex and let $f \in L^1(0, \infty; H)$. Every bounded solution of (6.9) is an almost-orbit of $\mathcal{U}$.*

As a consequence we obtain:

[3] Actually the cited reference deals with a more general setting with respect to the left-hand side of (6.9).

Corollary 6.13. *Let $v \in \mathcal{C}^2$ satisfy (6.9) with Argmin $\Phi \neq \emptyset$ and $f \in L^1(0, \infty; H)$. Then $v(t)$ converges weakly to a minimizer of Φ as $t \to \infty$. Convergence is strong in the following cases:*

1. *$\nabla\Phi$ is strongly monotone;*
2. *Φ is even;*
3. *Argmin Φ has nonempty interior.*

Proof. Boundedness of v is proved in [45]. Then it suffices to combine Theorem 3.3 with Propositions 6.5 and 6.12. □

Remark 6.14. The weak convergence was proved in [45]. The results on strong convergence are new. □

In connection with the system (6.5) it is worth mentioning that a proximal-type discrete version, namely

$$\frac{u^{k+1} - 2u^k + u^{k-1}}{\lambda_k^2} + \gamma \frac{u^{k+1} - u^k}{\lambda_k} + \nabla\Phi(u^{k+1}) = 0, \tag{6.10}$$

is studied in [43]. It turns out that the sequence $\{u^k\}$ also converges weakly as $k \to \infty$ to a minimizer of Φ. An interesting (and challenging!) open problem is determining whether the systems (6.5) and (6.10) are AAE. We conjecture that the answer is affirmative. If this is the case, then the strong convergence properties given by Proposition 6.5 concerning the evolution system (6.5) can be translated to the system (6.10).

7. Concluding remarks

This work has the following distinctive features:

1. Our results apply to a very broad class of evolution systems. Unlike previous works, ours does not assume the evolution systems to be autonomous and nonexpansive. This considerably enlarges the field of applications and allows us to develop asymptotic equivalence results, which have a symmetric character.
2. It is possible and straightforward to develop this theory in the framework of Lipschitz representations of semitopological semigroups (see, for instance, [16]). Since essentially all relevant applications concern evolution systems in continuous and discrete time, we provide a unified but simple (meaning not unnecessarily sophisticated) presentation of the results for these kinds of systems.
3. The notion of asymptotic almost-equivalence helps us understand the relationship between systems whose orbits have the same asymptotic behavior. We revisit several classical results in the light of these ideas.
4. We present a list of simple but illustrative examples and significant applications to give a flavor of the scope and versatility of the results and the underlying notions.

The tools developed here are potentially useful in different scenarios, as follows. In general *asymptotic analysis*, information on the asymptotic behavior of a system can be derived from the study of one that is AAE (as in [15,13,21]). In *numerical analysis*, they can be used to determine whether a discretization has the same asymptotic properties as the continuous-time model. For instance, it would be possible to find out *a priori* whether one must take averages in order to approximate the solution of a problem. In *perturbation theory*, they can be used to establish how much a system can be perturbed without changing its asymptotic behavior. This could help with predicting or controlling the effect of errors and noises. For *ill-posed problems*, they can be used to get an idea of what kinds of perturbations can force a system to converge when it does not. For example, in some optimization problems it is known that a viscosity term can force a nonconverging system to converge (see [41,44] or [24]). Finally, since the AAE results hold in general Banach spaces, it would be interesting to explore applications to *PDE's* outside the classical Hilbert setting—in particular, for scalar conservation laws (degenerate) parabolic equations in L^1 and Hamilton–Jacobi equations in $\mathcal{C}^0$ (see, for instance, [46,47]).

An interesting question is the extension of the framework developed here to evolution systems that are not globally Lipschitz. Of course, not all the results in the previous sections are true without the Lipschitz assumption. For instance, in general having a bounded almost-orbit does not imply that all the almost-orbits are bounded (take $U(t, s)x = e^{(t-s)}x$). The first work that contains equivalence-like results for general evolution systems seems to be [11], where they study strongly continuous semigroups which are "asymptotically nonexpansive in the intermediate sense", by requiring additional (and strong) regularity conditions with respect both to time and to space. In [48], the authors present asymptotic equivalence results without any Lipschitz assumption. It is somewhat surprising that a great number of global and asymptotic properties of the orbits of any evolution system are inherited by all the almost-orbits without Lipschitz conditions. The only price to pay is a minor topological assumption on the space X for those results concerning weak convergence (X is required to be weakly complete; see Remark 3.5).

Another important but difficult question is the comparison between the attractors of two given evolution systems. We believe that the notion of asymptotic almost-equivalence is not the most appropriate and some new ideas are required to address such an issue.

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