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ROBUST PORTFOLIO OPTIMIZATION IN A COINTINUOUS TIME FINANCIAL MARKET: NON COMPACT OR LINEAR UNCERTAINTY CASE

THESIS TOWARDS OBTAINING THE DEGREE OF MATHEMATICAL CIVIL ENGINEER

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The classical problem of portfolio optimization, on which the agent chooses her strategies so as to maximize her expected utility has been extensively studied for a long time. However only recently the interest for taking into account the fact that a particular model selection is in itself risky, has thriven. The problem of utility maximization on which an agent considers model uncertainty or ambiguity is referred to as robust portfolio optimization.

The main theme of this thesis is to study the above problem when the set of uncertain models is non compact or when it arises from linear constraints. Neither if these scenarios for the robust problem have been studied in generality in the literature. When the set of models stems from linear constraints, the problem in this piece of work is solved initially for the complete and incomplete market cases by assuming the usual weak compactness assumption of the models set usual in the literature. This is done by means of entropy minimization techniques, as developed amongst others by C. Léonard. Next, the robust problem in a complete market is solved only under a certain weakly closedness assumption in a relevant Orlicz space, plus some requirements on the economical ingredients of the problem. At this point, the results derived under the compactness assumptions of A.Schied and H. Föllmer, are yet again obtained. With all this, the aforementioned methods of C. Léonard come in handy again to solve the linear constraints case but now without compactness. Among other things, this allows for a new result of the type of primal-dual equality for the conjugate function of the robust utility and a characterization for the least favorable measure (or model). A simple example is also presented, which lies out of the grasp of the theory previously developed in the literature, but that is tackled by means of the results oobtained in this thesis.

Finally the relationship between robust optimization and the concept of weak information as introduced by F. Baudoin, as well as an application to flows of informations is illustrated. Regarding weak information, it is shown how utility maximization under it (insider trading) plus the calculus of variations allow to solve certain robust problems. Pertaining flows of information, a way to use the results of C. Léonard regarding minimization of entropy under flows of marginals constraints is proposed and discussed, as well as how to use them in order to solve the robust problem associated to these flows.

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TO MY FAMILIES AND FRIENDS ... AND ESPECIALLY TO MY MOTHER

Índice general

Capítulo 1

Introduction

After the 2008 financial crisis, whose consequences are still very concrete, the "model risk" concept acquired renewed importance. In a few words, this risk corresponds to the damage or harm that a model of a real phenomenon can generate, for example by underestimating a certain quantity or merely by being overly simplified. In finance, price models (amongst others) have always been subjected to this type of risk yet now more than ever emphasis has been put on this topic in order to understand it, model it and finally hedge it.
The portfolio optimization in a financial market problem has been extensively studied for decades. Regardless of whether the market is complete or incomplete, the tools of convex analysis and duality theory have been successfully used in this area. However, and pertaining the previous paragraph, the need for understanding the portfolio optimization problem under the paradigm that the selection of the probabilistic model for the prices is risky in itself arises. The approach through which this problem has been tackled for the last 20 years is based on the notion of "robust utility".

Given the uncertainty or ambiguity on the models (which reflects the fact that in practice a model does not perfectly represent reality), the robust utility associated to a position on the market is understood as the worst expected utility of the agent or investor, taken under every feasible model. In section 2.1 this interpretation is discussed in more detail. Moreover, a link with the axiomatic theory of preferences that allows to model agent's decisions in a world where the probability distributution of her positions are uncertain is reviewed. Such preferences can be represented numerically and with this the form of the robust utility (as a worst case expected utility) is justified. At the end of this section, the link between "insider trading" models and robustness in a financial market is protrayed.

Once the idea of a robust utility is justified, the problem of robust portfolio optimization can be defined. In this problem, the agent or investor wants to choose her strategies in an optimal way so as to maximize her robust utility (the form of the uncertainty or ambiguity being given). Such problems have been studied in depth during the last decade, where the tools of convex analysis and duality remain as important as in the non-robust setting. In section 2.2 some of the general results for the robust optimization problem are outlined.
In a certain way, the robust optimization problem can be conceived as a projection pro-

blem, because essentially the aim is to find the “least favorable” model amongst those deemed plausible. This approach has already been developed in the literature, with fairly similar results as those presented in 2.2. The problem of projection by means of convex functional (in integral form) has been studied for several years and the literature on the topic abounds. Moreover, the case where the set on which the projection is being made is determined by linear (convex) constraints, the solution and dual interpretation of the projection problem is quite satisfactory, hence in section 2.3 these results are presented. The idea is that in the case of linear uncertainty or ambiguity (where the models set is determined by linear/convex constraints) the projection approach (known as well as mimimization of entropy approach) can yield stronger general results than teh usual ones. Notice that moment type constraints (which include means, covariances and probabilities) as well as constraints on the lwas of certain random variables, correspond to linear functionals with respect to the probability measures under which they are computed.

Sadly, when the models (feasible measures) set comes from a linear uncertainty on the models, it can occur that the set of the densities of the feasible measures calculated with respect to a reference measure may not be weakly compact in the space of integrable functions. This is as of today, to the best of the author’s knowledge, a common requirement to all the approaches for the robust problem. Keeping this in mind, in sections 3.2 y 3.3 the robust optimization problems in complete and incomplete markets (respectively) are explicitly solved, when the model uncertainty is linear and under the assumprion that the set of the models’densities is weakly compact in L^1. Here, the difficulty stems form the required mixture of the results already known on robust optimization with the techniques of minimization of entropy problems, ehich ultimately is reduced to the inclusion of the robust problem into a certain relevant Orlicz space. After this, in section 3.4 it is shown how to solve the robust problem even without the usual weak compacity property. In order to attain this, reflexivity of the relevant Orlicz space is required , which translates into a easily verifiable condition over the ingredients of the robust problem. Roughly, most of the arguments and results in section 2.2 remain valid. Section 3.5 deals with the case of linear uncertainty in this non-compact setting, and finally an example is given in order to stress the ideas presented. Chapter 3 is closed with a discussion on the connexion between “insider trading” under weak information (both in etatic and flow form) and robustness issues.

Next, on chapter 4 the colcusions of the work are given, as well as a series of open problems and directions that where left without being explored or answered in the present work. Briefly, the minimization of entropy techniques allow to solve in detail and generality the robust optimization problems when the uncertainty is of the linear type. Moreover, the robust problem was successfully takled by means of Orlicz space related techniques, even without any compacity assumption on the problem’s ingredients. It remains to be found more complex examples as well as solving in much greater generality the problems associated with weak information flows.

Capítulo 2

Review of the Literature

In the present section several elements present in the literature, which will allow to understand the robust optimization problem as well as the version where the feasible models arise from linear uncertainty, are to be presented. The results listed here are to be repeatedly used in the following parts.

2.1. Context

In this part the concept of robust portfolio optimization is introduced, a motivation for this concept relying on the axiomatic theory of robust preferences is presented, the link between risk measures theory is given and its interpretation under the "insider trading" paradigm is discussed. The very enlightening discussion developed in [FSW09], except from the link with "insider trading", is followed.

2.1.1. Introduction to Robustness in Portfolio Optimization

Financial markets offer a varied gamma of instruments and position for the investor. This in turn, chooses those instruments and positions based on her preferences and wealth. In daily life, however, the outcome of such a decision taking is uncertain because the value of any financial object is uncertain, which is why they are regarded as random variables.

In the classical portfolio optimization problem, preferences (of von Neumann-Morgenstern type) are represented by an expected utility function of the form $\mathbb{E}^{\mathbb{Q}}[U(X)]$, where U is a concave utility function and $\mathbb{Q}$ is a probability measure on the set of possible outcomes, which models investor's preferences. Typically a position X is regarded as attainable if its value (which is computed by means of the equivalent martingale measures, also known as risk neutral measures) does not exceed the initial wealth of the investor. In the last decade however, much attention has been put to the problem of model uncertainty, which has pushed forward the interest in a robust formulation of the optimization problem.

It is hence necessary to define a numeric expression for this new utility, which acknowledges model uncertainty. In the literature, the following general form for such a utility has been adopted:

Definition 2.1.1.
Let U be a concave utility function, $\mathcal{Q}$ a set of probability measures on the set of possible outcomes, and γ a function that penalizes such measures. The **robust utility function** associated with this is:

$$\mathcal{U}(X) = \operatorname*{ínf}_{\mathbb{Q}\in\mathcal{Q}} \left[\mathbb{E}^{\mathbb{Q}}[U(X)] + \gamma(\mathbb{Q})\right]. \tag{2.1.1}$$

The idea behind such an expression is the following: the agent considers a whole class ($\mathcal{Q}$) of probabilistic models on the set of possible outcomes (which correspond to the elements $\mathbb{Q}$ in $\mathcal{Q}$), but these models are regarded in greater or lesser importance through the penalizing term $\gamma(\mathbb{Q})$. Thus, to hedge herself against model uncertainty the investor chooses a "worst case" approach, taking the infimum over the penalized values of each of the models under consideration. It is important to keep in mind that in the case of a financial market model where prices are semimartingales (free of arbitrage), the calculations of the positions adopted by the agent are related to the computation of the expected values of such position under the martingale measures (in the complete case, only one of these exists). Hence besides $\mathcal{Q}$, the set of risk neutral measures has to be taken into account as well.

To clarify this concepts, a couple of examples are introduced:

Example 2.1.1. A real life investor wants to employ the tools of the usual portfolio optimization to assist her decision making. For this, she starts by adjusting a semimartingale model of the prices or the yields to thehistorical series of prices or yields. Thus, a certain component of instant drift is obtained. Evidently, the adjusting process in itself poses a source of uncertainty. Moreover, the a priori constraints that might be imposed to the drift (for instance, that it be piecewise constant, deterministic, reandom, etc.), or the mere selection of say the time window for tha calibration, inflicts yet another error source. Thus, the stochastic price or yield model is uncertain. It seems sensible for instance to establish an interval around the adjusted drift where the "real one" may exist. This can be modelled as a change of probability measure. Therefore, defining $\mathcal{Q}$ as the set of those measures such that the corresponding drift abides in the selected intervals models the abovementioned uncertainty. What is more, any intuition about which of these meausures is more or less desirable or likely can be incorporated into γ.

Example 2.1.2. Let the same situation as in the previous example. In a certain way it does not come out naturally to give an interval or stripe where the "real" drift should live in. After all, in reality the investor only has access to the prices or yields (she cannot observe the drift). For sure a good adjusting procedure should yield a confidence interval for the calibrated parameters, but this says little about they future evolution. Hence, perhaps limiting the values of the model parameters directly might be unrealistic. Instead, it can be conceivable that instead of hedging herself against parameters errors, the investor hedges against her own expectations of what the future prices or yield might wind out. For instance, her best studies, intuitions or insights might give her an idea of the range where the prices will abide or perhaps how certain statistic of them might evolve in the future. Hence it is desirable to consider all the probabilistic models that allow that such prices or its statistics belong to certain ranges, penalized according to a certain criterion through γ.

2.1.2. Motivation from the axiomatic theory of robust preferences

In the theory developed by von Neumann-Morgenstern, an agent may choose between several monetary lotteries (bets), with their odds known. Hence, each lottery is modeled as a real Borel probability measure. The agent has an order or relation preference $\succ$. It is von Neumann-Morgenstern's breakthrough that allowed to give necessary and sufficient conditions for the existence of a numeric representation of these preferences, in the form of:

$$\mu \succ \nu \Leftrightarrow \mathcal{U}(\mu) > \mathcal{U}(\nu),$$

where $\mathcal{U} : \mathcal{M}_{1,c}(S) \to (-\infty, \infty)$ is expressed as $\mathcal{U}(\mu) = \int U(x)\mu(dx)$, for a certain function $U : (-\infty, \infty) \to (-\infty, \infty)$, which is labeled as utility function when increasing and strictly concave (this is connected to the risk aversion of the agent and the economical interpretation of $\succ$).

The limitation of this theory is that the agent is assumed to know perfectly the lotteries, this is, the probability distribution of the monetary bets. In practice, an agent can only guess an approximation of these, which can be modeled as a noise in her knowledge of such measures. To this end, on the probability space $(\Omega, \mathcal{F})$ under consideration, a class $\tilde{\mathcal{X}}$ of Markovian Kernels $\tilde{X}(\omega, dy)$ is defined. Namely, the weights of the lottery are now random.

The question is whether a numeric representation on $\mathcal{M}_{1,c}(S)$ can translate into something similar on $\tilde{\mathcal{X}}$. Leaving out the details, under some axioms on $\succ$ and $\succeq$ plus some topological or regularity conditions on them, the following result is obtained:

Theorem 2.1.1.
There exists a unique extension of $\mathcal{U}$ (utility representation in $\mathcal{M}_{1,c}(S)$) to a numeric representation $\tilde{\mathcal{U}} : \tilde{\mathcal{X}} \to (-\infty, \infty)$. This has the form:

$$\tilde{\mathcal{U}}\left(\tilde{X}\right) = \phi\left(\mathcal{U}\left(\tilde{X}\right)\right) = \phi\left(\int U(x)\tilde{X}(\cdot, dx)\right),$$

where ϕ is a "concave monetary utility functional" definied on $L^\infty(\Omega)$.

The notion of concave monetary utility functional will be introduced shortly. For further references and a greater discussion of this result, check [FSW09]

Functional ϕ admits the following representation:

$$\phi(X) = \inf_{\mathbb{Q} \in \mathcal{Q}} \left[\mathbb{E}^{\mathbb{Q}}[X] - \gamma(\mathbb{Q})\right].$$

Where $\mathcal{Q}$ is a set of probability measures on $(\Omega, \mathcal{F})$, and $\gamma : \mathcal{Q} \to (-\infty, \infty]$ is a penalizing functional. Notice that every $X \in L^\infty(\Omega)$ can be interpreted as an element in $\tilde{\mathcal{X}}$ by means of the identification $X \to \delta_X$, hence it can be written that:

$$\tilde{\mathcal{U}}\left(\delta_X\right) = \phi(U(X)) = \inf_{\mathbb{Q} \in \mathcal{Q}} \left[\mathbb{E}^{\mathbb{Q}}[U(X)] - \gamma(\mathbb{Q})\right]. \quad (2.1.2)$$

The evident similarity between 2.1.1 and 2.1.2 explains the usefullness of this section. Effectively, 2.1.2 justifies the adoption of 2.1.1 as a reasonable definition for a sobust utility function, starting from a rigurous (axiomatic) treatment of a sensible model of preferences under uncertainty. In any case, the sole representation 2.1.1 regarded as a worst case approach towars decission making is valuable in itself.

2.1.3. Connexion with risk measures theory

Given a position in the market, where $(\Omega, \mathcal{F}, \mathbb{P})$ is the set (probability space) of scenarios, it is interesting to consider measuring the risk associated to the (discounted) value $X(\omega)$ of the position. The notion of risk measure $\rho(X)$, interpreted as the minimum quantity of capital to be invested in a riskless fashion along with the position in order to make up an acceptable portfolio, has been considered in an axiomatic way during the las decades and its importance in theoretical an practical finance grows. The following properties, which need not be satisfied by every risk measure, are broadly accepter among the mathematics and finance community:

Definition 2.1.2. Convex and Coherent Risk Measures
A functional $\rho : L^\infty \to (-\infty, \infty)$ is termed a convex risk measure, if it satisfies the following properties $\forall X, Y \in L^\infty$:

- Monotony: if $X \leq Y$, then, $\rho(X) \geq \rho(Y)$.
- Invariance: if $m \in (-\infty, \infty)$, then, $\rho(X + m) = \rho(X) - m$.
- Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$.

If additionally $\forall \lambda \geq 0$, $\rho(\lambda X) = \lambda\rho(X)$ holds (property known as positive homogeneity), ρ is termed coherent.

In order to better understand these axioms, it is enlightening to think of X as the loss associated to a position and of $\rho(X)$ as the amount of capital that invested in a riskless fashion along with the risky position renders an acceptable portfolio for the investor. Now, if $\{X, \rho(X)\}$ quantifies losses and risks, then $\{-X, -\rho(-X)\}$ quantifies profits and utilities. This motivates the necessity of studying $-\rho$ likewise:

Definition 2.1.3. Concave Monetary Utility Functional and its Minimal Penalizing Function
A functional $\phi : L^\infty \to (-\infty, \infty)$ is called a concave monetary utility functional, if $\rho = -\phi$ is a convex risk measure. Associated to this ϕ, is its minimal penalizing function $\gamma : \{\mathbb{Q} : \mathbb{Q} \ll \mathbb{P}\} \to (-\infty, \infty]$, defined by:

$$\gamma(\mathbb{Q}) = \sup_{X \in L^\infty} \left[\phi(X) - \mathbb{E}^{\mathbb{Q}}(X)\right].$$

As it was presented in the previous part, the usefulness of this object (regarding robustness issues) lies on the fact that under some assumptions every concave monetary utility functional ϕ can be understood as the expected value, in the worst case, of the position taken in the market after penalization. More concretely, the almost surely continuity from above of ϕ is equivalent to the representation:

$$\phi(X) = \inf_{\mathbb{Q} \ll \mathbb{P}} [\mathbb{E}^{\mathbb{Q}}(X) + \gamma(\mathbb{Q})]. \quad (2.1.3)$$

At this point, notice that ϕ is coherent if and only if γ takes only 0 and/or $+\infty$ as its values. In this way, $\phi(X) = \inf_{\mathbb{Q}\in\mathcal{Q}}[\mathbb{E}^{\mathbb{Q}}(X)]$, where $\mathcal{Q} = \{\mathbb{Q} \ll \mathbb{P} : \gamma(\mathbb{Q}) < \infty\}$ (known as the maximal representing set).

In spite of the fact that, as illustrated, the portfolio optimization problem is linked to the risk measures context as far as its value function, the financial interpretation has more to do with the duality in financial markets approach. What is more, the techniques needed to solve particular instances of these problems are as varied as stochastic optimal control and BSDE, variational calculus, large deviations theory and inverse statistical problems (this is projection problems, or minimization of entropy problems). In this work this latter approach will be followed when possible, plus the duality techniques, to solve with as much accuracy as plausible the robust problem.

2.1.4. Connexion with Insider Trading models

In insider trading models, a financial agent possesses some kind of knowledge which is not public and that allows her to improve her position in the market as a speculator. In other terms, the optimization problem this agent carries out has “something more” than that of the rest of the people, which allows her to attain better profits. That “extra” thing is information. Whether it be because of some contact within a business sector or company, or because she herself is aware of some still not made public tidings, or because legitimately her studies and expert knowledge of the market thus entail it, this agent (known as “insider”) has within her reach better tools for the decision making process that the rest of the world. In practice, this information has to do with unpublished news (of merges or of internal health of companies for instance), which is quite difficult to quantify, in the mathematical theory this information translates in some kind of knowledge of the future prices.

Amongst the types of insider trading activity that has been modeled, the “initial information”, “progressive information” and “weak information” types are the most important. The initial case consists of the perfect knowledge (this is, under any random outcome) of a future variable, known to the insider from the very beginning of the trading period. The progressive approach is similar to the previous one, but here there is a time-dependent noise that affects the future signal, which vanishes little by little. Both approaches are modeled and studied through the theories of “Enlargement of Filtration” or anticipative calculus techniques (see [Pon05]), mainly. On the other hand, weak information refers to the knowledge of the distribution (law) of some future random variable (see [Bau02]). This future variable could be for instance, some prices in the future, some average prices into the future or ratios, as well as variables indicating whether any of these quantities violates some threshold, or belong to an interval, etc. Also, in any case, once the information is modelled, the way the insider sees the prices and takes her optimal decisions in the marked can be assessed.

In practical terms, it does not seem altogether reasonable that an agent is able to anticipate the value or even the distribution of some future price. On the other hand, arguably an insider might have some idea of mean tendencies or averages of some variables. For instance, she could anticipate in the mean case the final value of some stock, or whether it will belong to some interval, or even that a segment of the market might align itself because of some news and hence that the correlation of the stock's values will increase closer to 1. The common feature of this example is that the insider anticipates a statistic of the prices. Mathematically, these examples can be taken as instances of the following setting, where the marked is comprised of d risky assets and the time horizon is T:

$$\left\{\mathbb{E}\left[F^i\left(S(t_1^i),\ldots,S(t_{n_i}^i)\right)\right] \in C^i : n_i \in \mathbb{N}, i \in \{1,\ldots,N,\}, 0 < t_j^i \leq T\right\}, \tag{2.1.4}$$

where C^i are intervals to simplify and the functions F^i Borel measurable.

In this thesis a rigorous approach to this insider trading problem will be presented. In fact, a much more general context will be regarded (actually, the weak information setting being a subset of it). In order to enforce the constraints 2.1.4 a minimal (leats favorable or worst case, in a sense) model will be seeked. At this point teh resemblance with robust portfolio optimization arises, due to teh fact that an insider with information 2.1.4 would want to maximize over her feasible strategies the following expression:

$$\operatorname*{\acute{i}nf}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[U(X_T)],$$

where X_T is the wealth associated to such strategy and $\mathcal{Q} = \{\mathbb{Q} \ll \mathbb{P} : 2.1.4 \text{ is satisfied under } \mathbb{Q}\}$.

2.2. Duality for the Robust Problem

In this part the work in [SW05], where the robust problem is solved under a compacity condition in the set of feasible models, will be principally followed.

Let there be d stocks and a bond, with its value taken to be constant. Let $S = (S^i)_{1\leq i\leq d}$ be the price process, and $T < \infty$ a finite investment horizon. On the filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F})_{t\leq T}, \mathbb{R})$, the process S is assumed to be a semimartingale. In the following $\mathbb{R}$ will always stand for the reference measure and unless otherwise stated, $\mathbb{E}$ (without superscript) will stand for the mean taken under this measure.

A (self-financing) portfolio π is defined as the couple (x, H), where the constant x is its initial value and $H = (H^i)_{i=1}^d$, which denotes the number of shares under possession, is predictable and S-integrable.

The wealth associated to a portfolio π is defined as the process $X = (X_t)_{t\leq T}$ given by:

$$X_t = X_0 + \int_0^t H_u dS_u. \tag{2.2.1}$$

The set of attainable wealths from x, is defined as:

$$\mathcal{X}(x) = \{X \geq 0 : X \text{ as in } 2{,}2{,}1, \text{ st. } X_0 \leq x\}. \tag{2.2.2}$$

Also, the set of equivalent local martingale measures (or risk neutral) associated to S is defined as:

$$\mathcal{M}^e(S) = \{\mathbb{P} \sim \mathbb{R} : \text{ every } X \in \mathcal{X}(1) \text{ is a } \mathbb{P}\text{-local martingale}\}. \tag{2.2.3}$$

If S winds out to be locally bounded, then the previous set can be more succinctly described as:

$$\mathcal{M}^e(S) = \{\mathbb{P} \sim \mathbb{R} : S \text{ is a } \mathbb{P}\text{-local martingale}\}.$$

A market is coined **completo** if this set is reduces to a singleton, i.e., $\mathcal{M}^e(S) = \{\mathbb{P}\}$.

Definition 2.2.1.
A function $U : (0, \infty) \to (-\infty, +\infty)$ is called a *utility function on* $[0, +\infty)$, if it is strictly increasing, strictly concave and continuously differentiable. Naturally it will be assumed that such a function is extended as $-\infty$ on $(-\infty, 0]$.
It will be classified as satisfying INADA if $U'(0+) = \infty$ and $U'(+\infty) = 0$, and its asymptotic elasticity is defined as $AE(U) = \text{lím sup}_{x\to\infty} \frac{xU'(x)}{U(x)}$.

The following conjugate function of U (which is the Fenchel conjugate of $-U(-\cdot)$) will be of great importance in the following:

$$V(y) = \sup_{x>0}[U(x) - xy], \ \forall y > 0. \tag{2.2.4}$$

Note that for a utility function on $[0, +\infty)$ satisfying INADA, it holds that $V(y) = U \circ I(y) - yI(y)$, where $I = [U']^{-1}$.

In order to represent the ambiguity or uncertainty on the model, the set $\mathcal{Q}$ of feasible probability measures that define the robust problem is introduced. The following assumptions on it will be assumed:

Assumption 2.2.1.

1. $\mathcal{Q}$ is convex.
2. $\mathbb{R}(A) = 0$ if and only if $[\mathbb{Q}(A) = 0, \forall \mathbb{Q} \in \mathcal{Q}]$.
3. The set $\mathcal{Z} := \left\{ \frac{d\mathbb{Q}}{d\mathbb{R}} | \mathbb{Q} \in \mathcal{Q} \right\}$ is closed in $L^0(\mathbb{R})$.

Here, $L^0(\mathbb{R})$ denotes the space of measurable functions equipped with the convergence in probability topology. Notice that as it is shown in *Lemma 3.2* of [SW05], given assumption (1) and (2), number (3) is equivalent to the fact that $\mathcal{Z}$ be a $\sigma(L^1, L^\infty)$-compact set. Moreover, also under assumption 2.2.1, Halmos-Savage Theorem (see *Theorem 1.1* [KS96]) yields the existence of at least one $\mathbb{Q} \in \mathcal{Q}$ equivalent to the reference measure.

In the case of incomplete markets, the following set will come in handy. It generalizes the set of those processes associated with the densities with respect to $\mathbb{Q}$, of the risk neutral measures equivalent to $\mathbb{Q}$:

$$\mathcal{Y}_{\mathbb{Q}}(y) := \{ Y \geq 0 | Y_0 = y \ , \ XY \text{ is } \mathbb{Q} - \text{supermartingale } \forall X \in \mathcal{X}(1) \} . \tag{2.2.5}$$

For the following Theorem, the following definitions will prove useful:

$$u(x) = \sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(U \left(X_T \right) \right) \tag{2.2.6}$$

$$u_{\mathbb{Q}}(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}^{\mathbb{Q}} \left(U \left(X_T \right) \right) \tag{2.2.7}$$

$$v_{\mathbb{Q}}(y) = \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \mathbb{E}^{\mathbb{Q}} \left(V \left(Y_T \right) \right) \tag{2.2.8}$$

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}_e} v_{\mathbb{Q}}(y), \tag{2.2.9}$$

where $\mathcal{Q}_e := \{ \mathbb{Q} \in \mathcal{Q} | \mathbb{Q} \sim \mathbb{R} \}$.
Hence, u is the investor's robust utility and $u_{\mathbb{Q}}$ her "subjective" utility under model $\mathbb{Q}$. On the other hand v and $v_{\mathbb{Q}}$ correspond respectively to the candidate conjugate functions of them.

Following the classical setting (where $\mathcal{Q}$ corresponds to a singleton), in [SW05] the next results are proved:

Theorem 2.2.1 (*Theorem 2.2*, [SW05])**.**
Suppose assumptions 2.2.1, as well as:

$$\exists x > 0, \mathbb{Q}_0 \in \mathcal{Q}_e \ st. \ u_{\mathbb{Q}_0}(x) < \infty. \tag{2.2.10}$$

Then the function u is concave, finite, and satisfies:

$$u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(U \left(X_T \right) \right) = \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{X \in \mathcal{X}(x)} \mathbb{E}^{\mathbb{Q}} \left(U \left(X_T \right) \right) . \tag{2.2.11}$$

Moreover:

$$u(x) = \inf_{y>0} \left(v(y) + xy\right) \ , \text{ and } , \ v(y) = \sup_{x>0} \left(u(x) - xy\right). \tag{2.2.12}$$

In particular, v is convex. Also, their derivatives satisfy:

$$u'(0+) = \infty \ , \text{ and } , \ v'(\infty-) = 0. \tag{2.2.13}$$

Theorem 2.2.2 (*Theorem 2.6*, [SW05])**.**
Assume 2.2.1 and:

$$v_{\mathbb{Q}}(y) < \infty, \forall y > 0, \forall \mathbb{Q} \in \mathcal{Q}_e, \tag{2.2.14}$$

then the derivatives of the value functions satisfy:

$$v'(0+) = -\infty \ , \text{ and } , \ u'(\infty-) = 0, \tag{2.2.15}$$

and $\forall x > 0$, $\exists \hat{X} \in \mathcal{X}(x)$ and a measure $\hat{\mathbb{Q}} \in \mathcal{Q}$ such that:

$$u(x) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left[U\left(\hat{X}_T\right)\right] = \mathbb{E}^{\hat{\mathbb{Q}}} \left[U\left(\hat{X}_T\right)\right] = u_{\hat{\mathbb{Q}}}(x), \tag{2.2.16}$$

this is, the suprema and infima in 2.2.11 are attained. Moreover, there exists $\hat{y}$ in the superdifferencial of u in x, and some $Y \in \mathcal{Y}_{\mathbb{R}}(\hat{y})$ such that:

$$v(\hat{y}) = \mathbb{E}\left[\hat{Z} V\left(\frac{Y_T}{\hat{Z}}\right)\right] \ , \text{ and } , \ \hat{X}_T = I\left(\frac{Y_T}{\hat{Z}}\right) \ (\hat{\mathbb{Q}} - ae.) \tag{2.2.17}$$

where $\hat{Z} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{R}}$ and $I = -V' = (U')^{-1}$. What is more, $\hat{X}Y$ is a $\mathbb{R}$-martingale and v satisfies:

$$v(y) = \inf_{\mathbb{P}\in\mathcal{M}^e(S)} \inf_{\mathbb{Q}\in\mathcal{Q}_e} \mathbb{E}^{\mathbb{Q}}\left[V\left(y\frac{d\mathbb{P}}{d\mathbb{Q}}\right)\right]. \tag{2.2.18}$$

If additionally $AE(U) < 1$, then u is strictly concave, v is continuously differentiable, and:

$$\left\{\hat{X}_T Y_T > 0\right\} = \left\{\hat{Z} > 0\right\} \ (\mathbb{R}\text{-ae.}) \tag{2.2.19}$$

As commented in [SW05], condition 2.2.14 is satisfied as soon as $u_{\mathbb{Q}}$ is finite $\forall \mathbb{Q} \in \mathcal{Q}_e$ and $AE(U) < 1$.

Remark 1.

- The assumption that $\mathcal{Z}$ is closed in L^0 (equivalently, under (1) and (2) in 2.2.1, to being weakly compact in L^1) is crucial on several parts of the arguments used in these Theorems. For example, 2.2.11 is a consequence of it, as well as the expression for $v(\hat{y})$ in 2.2.17
- In [FG06], the authors study this problem as well (plus the case of utility function on the whole of the real line). The approach there is, roughly, that of "projecting" (over $\mathcal{Q}$ and $\mathcal{M}$) by means of a certain functional. In this case, instead of working with processes in $\mathcal{Y}$, the authors work with measures on an extended probability space. For this, the assuemption on $\mathcal{Z}$ are equally necessary, and moreover some structure (topology) on the original probability space is required.

2.3. Minimization of Entropy Functionals

In the literature on Minimization of Entropy (Energy or Divergence) Functionals, the object of study are functional (typically in integral form) acting on a set of measures (or they densities with respect to some reference). The interest is to maximize or minimize such functional, under certain constraints. In this section the article [Léo08], which studeis the problem under convex constraints in great generality, will be followed.

Let $(\Omega, \mathcal{F}, \mathbb{R})$ be a (complete) probability space, and let $\gamma^* : \Omega \times (-\infty, \infty) \to [0, \infty]$ be a measurable function, such that $\gamma^*_\omega(\cdot) := \gamma^*(\omega, \cdot)$ is convex and l.s.c., $\forall \omega$. Let also M_Ω be the space of signed measures on Ω. The **entropy functional**, which will be the objective function of the problem, is defined as:

$$I(\mathbb{Q}) \quad = \quad \begin{cases} \int_\Omega \gamma^*_\omega \left(\frac{d\mathbb{Q}}{d\mathbb{R}}(\omega)\right) \mathbb{R}(d\omega) & \text{if } \mathbb{Q} \ll \mathbb{R} \\ +\infty & \text{otherwise,} \end{cases} \tag{2.3.1}$$

with $\mathbb{Q} \in M_\Omega$.
To simplify (as compared to [Léo08]), the following assumption is considered:

$$\gamma^*_\omega(m) = 0 \iff m = 0. \tag{2.3.2}$$

Notation 2.3.1. Notice that being γ^*_ω convex proper lsc, it is then (for each ω) the conjugate of a certain convex proper lsc function, which will be called γ_ω. Also, define γ_0 as the following even version of γ:

$$\gamma_0(\omega, s) = \text{máx}\{\gamma(\omega, s), \gamma(\omega, -s)\}. \tag{2.3.3}$$

Before proceeding, some notions of Young functions and Orlicz spaces are reminded.

Definition 2.3.1.
A function $\rho : \Omega \times (-\infty, \infty)$ is called a Young function if for $\mathbb{R}-$almost every ω, $\rho(\omega, \cdot)$ is convex, even, $[0, \infty]-$valued, such that $\rho(\omega, 0) = 0$ and $0 < \rho(\omega, s(\omega)) < \infty$, for a certain measurable s.
For a Young function, its **Orlicz** space is defined as:

$$L_\rho(\Omega, \mathbb{R}) := \left\{ u : \Omega \to (-\infty, \infty) | \exists \alpha_0 > 0, \int \rho(\omega, \alpha_0 |u(\omega)|) \mathbb{R}(d\omega) < \infty \right\}, \tag{2.3.4}$$

and its subspace of interest:

$$E_\rho(\Omega, \mathbb{R}) := \left\{ u : \Omega \to (-\infty, \infty) | \forall \alpha > 0, \int \rho(\omega, \alpha |u(\omega)|) \mathbb{R}(d\omega) < \infty \right\}. \tag{2.3.5}$$

Equipped with the norm $||u||_\rho = \text{ínf}\left\{\beta > 0 : \int \rho(\omega, |u(\omega)|/\beta) \leq 1\right\}$ the space L_ρ (of equivalence classes, which will be likewise denoted) is a Banach space, which is embedded continuously in L^1 if the space is of finite measure. Moreover, when a Young function ρ is finite, the topological dual of E_ρ is isomorphic to L_{ρ^*}. See [RR91] for an in-depth discussion over these concepts and properties. On the other hand, L_ρ will stand as an abbreviation of $L_\rho(\Omega, \mathbb{R})$, and when L_{ρ^*} is regarded as a subspace of a dual or the whole dual of something, it will be written as well as $L_{\rho^*}\mathbb{R}$ through the identification of $f \in L_{\rho^*}$ with the measure $f\mathbb{R}$ (or $fd\mathbb{R}$) that has it as its density.

Notation 2.3.2. In order to define the constraints of the problem, the linear spaces $\mathcal{X}_0$ (where the constraints are valued) and $\mathcal{G}_0$ are introduced, where $\mathcal{X}_0 = (\mathcal{G}_0)^*$ (algebraic duality).
The operators $\theta : \Omega \to \mathcal{X}_0$ is defined, as well as $T_0^* : \mathcal{G}_0 \mapsto T_0^*\mathcal{G}_0$ with $T_0^*g(\omega) = \langle g, \theta(\omega)\rangle_{\mathcal{G}_0,\mathcal{X}_0}$.
In the case that $T_0^*\mathcal{G}_0 \subset L_{\gamma_0}$, the constraints operator $T_0 : L_{\gamma_0^*}\mathbb{R} \mapsto \mathcal{X}_0$ with $T_0 f := \int_\Omega \theta df$, for $f \in L_{\gamma_0^*}\mathbb{R} := \left\{\mathbb{Q} \ll \mathbb{R} | \frac{d\mathbb{Q}}{d\mathbb{R}} \in L_{\gamma_0^*}\right\}$, can be constructed, by:

$$\left\langle g, \int \theta dl \right\rangle_{\mathcal{G}_0,\mathcal{X}_0} = \int_\Omega \langle g, \theta(\omega)\rangle_{\mathcal{G}_0,\mathcal{X}_0} l(d\omega).$$

The definition and existence of such a T_0 stems from Hölder's inequality on Orlicz spaces (see A.0.2), becuase then $\int_\Omega \langle g, \theta(\omega)\rangle_{\mathcal{G}_0,\mathcal{X}_0} l(d\omega) \leq 2|| \langle g, \theta\rangle_{\mathcal{G}_0,\mathcal{X}_0} ||_{L_{\gamma_0}} ||dl/d\mathbb{R}||_{L_{\gamma_0^*}}$, from which for every fixed g a linear continuous operator is induced.

With all these ingredients, the following minimization of entropy problem can be presented:

$$\textit{Minimize } I(\mathbb{Q}) \text{ subject to } \int_\Omega \theta d\mathbb{Q} \in C \text{ , } \mathbb{Q} \in L_{\gamma_0^*}\mathbb{R}, \tag{PC}$$

where $C \subset \mathcal{X}_0$ is a convex set. Notice that the constraint is $T_0(d\mathbb{Q}/d\mathbb{R}) \in C$.
Next, the main assumption on the problem's "ingredients", which will allow to get a satisfactory solution of it, are presented:

Assumption 2.3.1.

- $T_0^*\mathcal{G}_0 \subset E_{\gamma_0}$, or equivalently, $\forall g \in \mathcal{G}_0, \int \gamma\left(\langle g, \theta\rangle\right) d\mathbb{R} < \infty$.
- $\gamma^*(\cdot, s)$ is measurable for all s. For $\mathbb{R}-$almost every ω, $\gamma^*(\omega, \cdot)$ is lsc, convex (strictly in its domain) and $[0,\infty]-$valued, such that $[\forall z : \gamma^*(z, m) = 0 \iff m = 0]$.
- $\forall g \in \mathcal{G}_0$, the function $\omega \in \Omega \mapsto \langle g, \theta(\omega)\rangle$ is measurable.
- $\forall g \in \mathcal{G}_0$, $[\langle g, \theta(\cdot)\rangle = 0, \mathbb{R}\text{-cs.} \Rightarrow g = 0]$.

In order to introduce the dual problem to (PC) it is necessary to present some further definitions and notations.

Notation 2.3.3.
Let $\mathcal{G}$ be the completion of $\mathcal{G}_0$ with the norm $|g|_\Gamma := ||\langle g, \theta\rangle||_{\gamma_0}$ (which is isomorphic to the closure of $\{\langle g, \theta\rangle | g \in \mathcal{G}_0\}$ in L_{γ_0}, under the above assumptions), and $\mathcal{X} = \mathcal{G}'$ its topological dual.
It will be convenient to consider the conjugate of the function $\Gamma(g) := \int \gamma(\langle g, \theta\rangle) d\mathbb{R}$, for $g \in \mathcal{G}_0$, namely: $\Gamma^*(x) = \sup_{g \in \mathcal{G}_0} \{\langle g, x\rangle - \Gamma(g)\}$, with $x \in \mathcal{X}_0$.

With this, the Dual problem is:

$$Maximize \inf_{x \in C \cap \mathcal{X}} \langle g, x\rangle - \int \gamma(\langle g, \theta\rangle) d\mathbb{R} \ , \ g \in \mathcal{G}. \tag{DC}$$

Moreover, the following extension of the Dual problem will prove relevant:

Let $K_\gamma := \left\{u \text{ measurable} | \exists a > 0, \int \gamma_\omega(au)\mathbb{R}(d\omega) < \infty\right\}$, and $\tilde{\mathcal{G}} \subset \mathcal{X}^*$ the convex cone which is isomorphic to the $\sigma\left(K_\gamma, L_\pm\right)$ clausure in K_γ of $T_0^*\mathcal{G}_0$. An element u is said to be in the $\sigma\left(K_\gamma, L_\pm\right)$ clausure of a set A, if u_+ and u_- belong respectively to the $\sigma(L_{\gamma_<}, L_{\gamma^*_<})$ and $\sigma(L_{\gamma_>}, L_{\gamma^*_>})$ clausure of the sets A_+ and A_- respectively, where the $\pm$ subscripts denotes the positive and negative parts of an element or set of elements. Also, $\gamma_<(s) := \gamma(|s|)$ and $\gamma_>(s) {=} \gamma(-|s|)$ (see section 3 in [Léo08] for the details). With this, the extended dual is defined as:

$$Maximize \inf_{x \in C \cap \mathcal{X}} \langle \tilde{g}, x\rangle - \int \gamma(\langle \tilde{g}, \theta\rangle) d\mathbb{R} \ , \ \tilde{g} \in \tilde{\mathcal{G}}. \tag{$\tilde{DC}$}$$

For the following key Theorem, the concept of domain, affine hull and "intrinsic core" need to be introduced. Given a real valued function f defined on a topological linear space B, its domain is $dom(f) = \{b \in B | f(b) < \infty\}$. Also, the affine hull aff(A) of $A \subset B$ is the smallest affine subspace containing it, and its "intrinsic core" is $icor(A) = \{a \in A | \forall x \in \text{aff}(A), \exists t > 0 \text{ st. } a + t(x - a) \in A\}$.
Finally, the set C will be required to satisfy a certain closedness condition:

Assumption 2.3.2.
The convex set $C \subset \mathcal{X}_0$ is such that $T_0^{-1}C \cap L_{\gamma_0^*}\mathbb{R}$ is a $\sigma(L_{\gamma_0^*}\mathbb{R}, E_{\gamma_0})$-closed subset of $L_{\gamma_0^*}\mathbb{R}$. In other words:

$$T_0^{-1}C \cap L_{\gamma_0^*}\mathbb{R} = \bigcap_{y \in A} \left\{ f\mathbb{R} \in L_{\gamma_0^*}\mathbb{R} | \int \langle y, \theta\rangle f d\mathbb{R} \geq a_y \right\},$$

for a certain subset A of $\mathcal{X}_0^*$ such that $\langle y, \theta\rangle \in E_{\gamma_0}, \forall y \in A$, and a certain real function $y \in A \mapsto a_y$.

With this, a simplified version of *Theorem 3.2* in [Léo08] is presented:

Theorem 2.3.1.
Suppose assumptions 2.3.2 and 2.3.1 hold, plus the following:

$$For\ \mathbb{R} - ae.\ \omega : \lim_{t \to \pm\infty} \frac{\gamma_\omega^*(t)}{t} = +\infty. \tag{2.3.6}$$

Then:

- *There is dual equality for (PC):* $\inf(PC) = \sup(DC) \in [0, \infty]$.
- *If $C \cap T_0 dom(I) \neq \emptyset$, then (PC) admits a unique solution $\hat{\mathbb{Q}} \in L_{\gamma_0^*}\mathbb{R}$.*

Furthermore, suppose $C \cap icor\left(T_0 dom(I)\right) \neq \emptyset$:

- *Define* $\hat{x} = \int \theta d\hat{\mathbb{Q}}$. *Then there exists* $\tilde{g} \in \tilde{\mathcal{G}}$ *such that:*

$$\begin{cases} (a) & \hat{x} \in C \cap dom\left(\Gamma^*\right), \\ (b) & \langle \tilde{g}, \hat{x} \rangle_{\mathcal{X}_0^*, \mathcal{X}_0} \leq \langle \tilde{g}, x \rangle_{\mathcal{X}_0^*, \mathcal{X}_0}, \forall x \in C \cap dom\left(\Gamma^*\right), \\ (c) & \hat{\mathbb{Q}}(d\omega) = \gamma'_\omega\left(\langle \tilde{g}, \theta(\omega) \rangle\right) \mathbb{R}(d\omega). \end{cases} \tag{2.3.7}$$

  What is more, $\hat{\mathbb{Q}} \in L_{\gamma_0^*}\mathbb{R}$ *and* $\tilde{g} \in \tilde{\mathcal{G}}$ *satisfy 2.3.7 (a,b,c) if and only if* $\hat{\mathbb{Q}}$ *solves* (PC) *and* $\tilde{g}$ *solves* $(\tilde{DC})$.

- *It holds that* $\hat{x} = \int \theta \gamma'\left(\langle \tilde{g}, \theta \rangle\right) d\mathbb{R}$. *Moreover:*

  1. $\hat{x}$ *minimizes* Γ^* *on* C,
  2. $I\left(\hat{\mathbb{Q}}\right) = \Gamma^*\left(\hat{x}\right) = \int \gamma^* \circ \gamma'\left(\langle \tilde{g}, \theta \rangle\right) d\mathbb{R} < \infty$,
  3. $I\left(\hat{\mathbb{Q}}\right) + \int \gamma\left(\langle \tilde{g}, \theta \rangle\right) d\mathbb{R} = \int \langle \tilde{g}, \theta \rangle \, d\hat{\mathbb{Q}}$.

Capítulo 3

The Robust Optimization Problem under Linear Model Uncertainty

In this part the robust optimization problem will be studied, in the case when the uncertainty on the models is "linear" (in the form $T\mathbb{Q} \in C$). In the first sections the compactness assumption on the models set is maintained. Afterwards, this assumption is left aside and it is shown how to tackle the problem under some reflexivity assumption in a certain Orlicz space. With this, in the linear uncertainty case, a general result stronger than those in the literature is devised. Finally, the connexion between the robust problem and the "insider trading" with static and flow weak information setting is obtained.

3.1. Relevant Functions

Let $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\leq T}, \mathbb{R})$ be a filtered complete probability space, where a process S is assumed to be a locally bounded semimartingale. There is an agent (investor) willing to maximize her robust utility in this market. All the notation in section 2 (particulatly, U and V) are inherited to this chapter. From now on U is a utility function on $(0,\infty)$ satisfying INADA.

Now, some definitions that will provide the link between sections 2.2 and 2.3 are provided:

Definition 3.1.1.
For V as in 2.2.4 and $l \geq 0$, the following real function is defined:

$$\gamma_l^*(z) = \begin{cases} \infty & \text{if } z < 0 \\ zV\left(\frac{l}{z}\right) & \text{if } z \geq 0 \end{cases} \tag{3.1.1}$$

Also, it is defined $\eta^*(l,z) := \gamma_l^*(z)$. Notice that γ_l^* is convex lower semi continuous proper (it comes from V, which is convex lsc as it is the Fenchel conjugate of $-U(-\cdot)$, defining $U(x) = -\infty$ if $x < 0$). Hence, $\forall l, \exists \gamma_l$ such that its conjugate is effectively γ_l^*. Notice that γ_l is likewise convex lsc.

Before continuing, some properties of γ_l are to be presented:

Lemma 3.1.1.
Let $\gamma_l^*(\cdot)$ be defined as in 3.1.1. Then:

- *$\gamma_l^*(\cdot)$ is lsc, convex, finite on the positive half line, and is the conjugate function of γ_l, which in turn is convex, lsc and proper. Moreover, $\gamma_l(\cdot) = l\gamma_1(\cdot), \forall l > 0$.*

Further assume that $\gamma_l^*(\cdot)$ is $[0,\infty]-$valued and is such that $[\gamma_l^*(m) = 0 \iff m = 0]$. Then:

- *$\forall l > 0$, $\gamma_l(\cdot)$ is non-negative, not identically null and $\gamma_l(x) = 0$ if $x \leq 0$.*

Finally, if $\gamma_l^*(\cdot)$ is strictly convex in its domain and $\text{lím}_{t\to+\infty} \frac{\gamma_l^*(t)}{t} = +\infty$, then:

- *$\forall l > 0$, $\gamma_l(\cdot)$ is finite and everywhere differentiable.*

Proof. The first point is due to the discussion above, where $\gamma_l := [\gamma_l^*]^*$. Now γ_l^* is finite over the positive reals because $V(y) = U \circ I(y) - yI(y)$ is finite, where I is the inverse function of U'. Also,

$$\gamma_l(x) = \sup_{y>0}[xy - yV(l/y)] = l \sup_{z>0}[xz - zV(1/z)] = l\gamma_1(x).$$

As for the second point, since $\gamma_l(x) = \sup_y[xy - \gamma_l^*(y)]$, then $\gamma_l(x) \geq [-\gamma_l^*(0)] = 0$, ie. it's non-negative. Moreover, $\gamma_l(0) = \sup_y[-\gamma_l^*(y)] \leq 0$, from where $\gamma_l(0) = 0$. If $x < 0$, $\gamma_l(x) = \sup_{y\geq 0}[xy - \gamma_l^*(y)] \leq 0$. Finally γ_l can't be null, because it it were so, γ_l^* would have some infinite value, which is a contradiction.
For the third point, note that $\text{lím}_{t\to+\infty} \frac{\gamma_l^*(t)}{t} = +\infty$ implies that the recession function of γ_l^* is identically infinite, this is, γ_l^* is co-finite in the sense of *Corolary 13.3.1*, from [Roc70], which according to this same result is equivalent to γ_l being finite. Moreover, from *Theorem 26.3* of [Roc70], convexity of $\gamma_l^*(\cdot)$ plus strict convexity in its domain (which implies that this function be essentially strictly convex), imply through this result that $\gamma_l(\cdot)$ be essentially smooth. As this last one is finite, this entails it is everywhere differentiable.

◇

Now, some properties of γ, V and U are stated, as well as connexions amongst them.

Lemma 3.1.2.
If U is a utility function on $(0,\infty)$, such that $U(0+) = 0$ and that satisfies INADA, then V is finite and differentiable (on $(0,\infty)$), strictly decreasing, strictly positive, and satisfies:

$$\text{lím}_{x\to\infty} \frac{V(x)}{x} = 0, \tag{3.1.2}$$
$$V(0) = \text{lím}_{x\to\infty} U(x). \tag{3.1.3}$$

Moreover, function $\eta^*(l,z) := \gamma_l^*(z)$ is convex and continuously differentiable on $(0,+\infty)^2$. If U satisfies as well that $AE(U) < 1$, then $\forall\lambda > 0, \exists : a(\lambda) > 0, b(\lambda) > 0$ such that:

$$V(\lambda y) \leq a(\lambda)V(y) + b(\lambda)(y+1)\ ,\ \forall y. \tag{3.1.4}$$

Proof. Results 3.1.2, 3.1.3 and 3.1.4, plus differentiability of η^* appear in *Lemma 2.1.6* of [Gun06] (noting that under their notation, $x_u = 0$). V is finite because $V(y) = U \circ I(y) - yI(y)$, where I is the inverse of U', and is differentiable because U is strictly concave. Moreover, $V' = -I$, and noting thet $I(\cdot) > 0$, V has to be strictly decreasing. By definition $V(y) \geq U(0+) = 0$, and this plus its strict decreasing feature imply its strict positivity. Last, joint convexity of η^* stems from the observation in (21) in [SW05].

$\diamond$

Assumption 3.1.1. *U is a utility function on $(0, \infty)$, not bounded from above, such that $U(0+) = 0$, and satisfies INADA.*

Remark 2. Under the previous assumptions, and thanks to Lemma 3.1.2, all the properties and assumptions for γ and γ^* in Lemma 3.1.1 are satisfied. Moreover, if more generally $U(0+) > -\infty$ only, after a translation argument it can still be assumed that wlg $U(0+) = 0$.

In defining the relevant Orlicz spaces, it will come in handy to work with function $\gamma_{l,0}(\cdot) := \text{máx}\{\gamma_l(\cdot), \gamma_l(-\cdot)\} = \gamma_l(|\cdot|)$, because γ_l has 0 value over the negative reals. Next a relationship between the conjugates of γ_l and $\gamma_{l,0}$ is established:

Lemma 3.1.3.
Under assumption 3.1.1, it holds that $(\gamma_{l,0})^*(\cdot) = \gamma_l^*(|\cdot|) \leq \gamma_l^*(\cdot), \forall l > 0$

Proof. Since $\gamma_l(|\cdot|) \geq \gamma_l(\cdot)$, then $(\gamma_{l,0})^*(\cdot) \leq \gamma_l^*(\cdot)$
Also $(\gamma_{l,0})^*(y) = \sup_x\{xy - \gamma_l(|x|)\} = \sup_{x>0}\{x|y| - \gamma_l(x)\}$. Hence if $y > 0$, $\sup_{x>0}\{xy - \gamma_l(x)\} = (\gamma_{l,0})^*(y) \leq \gamma_l^*(y) = \text{máx}\{\sup_{x>0}\{xy - \gamma_l(x)\}, \sup_{x\leq 0}\{xy - \gamma_l(x)\}\}$. Now, it will be proved that $\sup_{x>0}\{xy - \gamma_l(x)\} \geq \sup_{x\leq 0}\{xy - \gamma_l(x)\} = \sup_{x\leq 0}\{xy\}$. For this, given $c \leq 0$ it will be proved that $\exists z > 0$ st. $cy \leq zy - \gamma_l(z)$, ie., that $\gamma_l(z) \leq (z - c)y$. Notice $\gamma_l(\cdot)$ under the assumption is continuous. Hence $\exists a_0 > 0$ st. $\forall 0 < a \leq a_0$, $\gamma_l(a) \leq y$. Fix now $0 < a \leq \text{mín}\{a_0, 1\}$, y, $0 < x \leq \text{mín}\{1, \frac{c}{a-1}\}$. By convexity, follows that $\gamma_l(ax) = \gamma_l(ax + 0(1-x)) \leq x\gamma_l(a) \leq xy$. Yet since $x \leq \frac{c}{a-1}$, then $x \leq ax - c$, from where $xy \leq (ax - c)y$, and thus $\gamma_l(ax) \leq (ax - c)y$. Hence taking $z = ax > 0$, it's been shown that $y > 0$, $(\gamma_{l,0})^*(y) = \gamma_l^*(y)$. But $(\gamma_{l,0})^*$ is even, as follows from the begining of the demonstration. This finishes the proof.

$\diamond$

In the following, γ will stand as an abbreviation of γ_1 (thus, $\gamma_l = l\gamma$), and γ_0 as a shorthand for $\gamma_{1,0}$.

3.2. Robust Optimization, Complete case and $\frac{d\mathcal{Q}}{d\mathbb{R}} \subset L^1$ weakly compact

The robust optimization problem introduced in section 2.2 is again considered, yet specialized in the linear ambiguity or uncertainty case. The context and notation of the previous sections is continued, as well as the definitions in 2.2. Moreover, assumption 3.1.1 stands.

Let $\mathcal{X}_0$ and $\mathcal{G}_0$ be linear spaces, such that $\mathcal{X}_0 = (\mathcal{G}_0)^*$ (algebraic duality), and an operator $\theta : \Omega \to \mathcal{X}_0$ and $C \subset \mathcal{X}_0$ a convex. Define $T_0^* g(\omega) = \langle g, \theta(\omega)\rangle_{\mathcal{G}_0,\mathcal{X}_0}$ and $T_0 f := \int_\Omega \theta df$, for $f \in L_{\gamma_0^*}\mathbb{R}$, as in section 2.3 (it is assumed that $T_0^*\mathcal{G}_0 \subset L_{\gamma_0}\mathbb{R}$). Remember the notation on Orlicz spaces introduced there.

In this part the market will be assumed to be complete, with $\mathcal{M}_e = \{\mathbb{P}\}$.

Definition 3.2.1.
The set of feasible models (measures) is defined as:

$$\mathcal{Q} := \left\{ \mathbb{Q} = Z\mathbb{R} \in L_{\eta_0^*}\mathbb{R} | \mathbb{Q} \text{ is a probability measure }, \int \theta Z d\mathbb{R} \in C \right\}, \tag{3.2.1}$$

where $\eta_0(\omega, \cdot) := \eta\left(\frac{d\mathbb{P}}{d\mathbb{R}}(\omega), |\cdot|\right)$, with $\eta(l,z) = \gamma_l(z)$ (γ_l^* defined as in 3.1.1).
As in section 2.2, assumption 2.2.1 is assumed (notice that convexity of $\mathcal{Q}$ holds always). Recall that under this assumptions $\frac{d\mathcal{Q}}{d\mathbb{R}} := \{d\mathbb{Q}/d\mathbb{R} : \mathbb{Q} \in \mathcal{Q}\} \subset L^1$ is weakly compact in L^1 (hence the name of this section).

Assuming all the hypotheses of Theorem 2.2.2, the existance, for every $x > 0$, of a $\hat{y}$in the superdifferential of $u(x)$, a $\hat{Y} \in \mathcal{Y}_\mathbb{R}(\hat{y})$and a $\hat{\mathbb{Q}} \in \mathcal{Q}$ such that the following hold, is assured:

$$v(\hat{y}) = \mathbb{E}^\mathbb{R}\left[\hat{Z}V\left(\frac{\hat{Y}_T}{\hat{Z}}\right)\right] \text{ , con } \hat{Z} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{R}}$$

By *Lemma 4.3* of [KS99], $Y \in \mathcal{Y}_\mathbb{R}(y) \Rightarrow Y_T \leq y\frac{d\mathbb{P}}{d\mathbb{R}}$, and since V is decreasing, necessarily:

$$v(\hat{y}) = \mathbb{E}^\mathbb{R}\left[\hat{Z}V\left(\hat{y}\frac{d\mathbb{P}/d\mathbb{R}}{\hat{Z}}\right)\right] = \mathbb{E}^{\hat{\mathbb{Q}}}\left[V\left(\hat{y}\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}\right)\right]. \tag{3.2.2}$$

Defining $\mathcal{Z} := \left\{\frac{d\mathbb{Q}}{d\mathbb{R}} | \mathbb{Q} \in \mathcal{Q}\right\}$ and $\mathcal{Z}_e = \{Z \in \mathcal{Z} | Zd\mathbb{R} \in \mathcal{Q}_e\}$, the latter can be written equivalently as:

$$v(\hat{y}) = \inf_{Z\in\mathcal{Z}_e} \mathbb{E}^\mathbb{R}\left[ZV\left(\frac{\hat{y}d\mathbb{P}/d\mathbb{R}}{Z}\right)\right] \tag{3.2.3}$$

$$= \inf_{Z\in\mathcal{Z}} \mathbb{E}^\mathbb{R}\left[ZV\left(\frac{\hat{y}d\mathbb{P}/d\mathbb{R}}{Z}\right)\right] \tag{3.2.4}$$

$$= \inf_{Z\in\mathcal{Z}} \int \gamma_\omega^*(Z(\omega))\mathbb{R}(d\omega), \tag{3.2.5}$$

where $\gamma^*_\omega(\cdot) := \gamma^*_{\hat{y}\frac{d\mathbb{P}}{d\mathbb{R}}(\omega)}(\cdot) = \eta^*\left(\hat{y}\frac{d\mathbb{P}}{d\mathbb{R}}(\omega), \cdot\right)$. Equality in 3.2.4 stems from the following Lemma:

Lemma 3.2.1.
If $v(y) < \infty$, then:

$$v(y) = \inf_{Z\in\mathcal{Z}} \mathbb{E}^{\mathbb{R}}\left[ZV\left(\frac{yd\mathbb{P}/d\mathbb{R}}{Z}\right)\right]. \tag{3.2.6}$$

Proof. By *Lemma 4.3* in [KS99], it holds that

$$v(y) := \inf_{\mathbb{Q}\in\mathcal{Q}_e} v_{\mathbb{Q}}(y) = \inf_{Z\in\mathcal{Z}_e} \mathbb{E}^{\mathbb{R}}\left[ZV\left(\frac{yd\mathbb{P}/d\mathbb{R}}{Z}\right)\right],$$

from where $v(y) \geq \inf_{Z\in\mathcal{Z}} \mathbb{E}^{\mathbb{R}}\left[ZV\left(\frac{yd\mathbb{P}/d\mathbb{R}}{Z}\right)\right]$.

Now, replicating the argument in *Lemma 3.5* of [SW05], call $Z_1 \in \mathcal{Z}/\mathcal{Z}_e$ such that $\mathbb{E}^{\mathbb{R}}\left[Z_1V\left(\frac{yd\mathbb{P}/d\mathbb{R}}{Z_1}\right)\right] < \infty$. Since $v(y) < \infty$, $\exists Z_0 \in \mathcal{Z}_e$ such that $\mathbb{E}^{\mathbb{R}}\left[Z_0V\left(\frac{yd\mathbb{P}/d\mathbb{R}}{Z_0}\right)\right] < \infty$. Now, defining function $t \in [0, 1) \mapsto \mathbb{E}^{\mathbb{R}}\left[Z_tV\left(\frac{yd\mathbb{P}/d\mathbb{R}}{Z_t}\right)\right]$, with $Z_t = tZ_1 + (1-t)Z_0$, this winds out being usc (for it is convex and finite). Hence, $\mathbb{E}^{\mathbb{R}}\left[Z_1V\left(\frac{yd\mathbb{P}/d\mathbb{R}}{Z_1}\right)\right] \geq \limsup_{t\to 1} \mathbb{E}^{\mathbb{R}}\left[Z_tV\left(\frac{yd\mathbb{P}/d\mathbb{R}}{Z_t}\right)\right]$, which finishes the proof.

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It is at this point that, thanks to 3.2.5, it becomes apparent how the minimization of entropy methodology enables to reach a solution to this problem. For this, it is defined:

$$I(Z) = I(Zd\mathbb{R}) := \int \gamma^*_\omega(Z(\omega))\mathbb{R}(d\omega). \tag{3.2.7}$$

It is necessary to recall that the minimization of entropy methodology problem as introduced, took place in the space of signed measures with integrable densities in $L_{\eta_0^*}$. Hence, define $\bar{T}_0\mathbb{Q} = \left(\int d\mathbb{Q}, T_0\mathbb{Q}\right)$ and call $\bar{C} = \{1\} \times C$. From this, $[\bar{T}_0\mathbb{Q} \in \bar{C} \iff \mathbb{Q}$ integrates $1, \int \theta d\mathbb{Q} \in C]$.
Notice that if $Z \in L_{\eta_0^*}$ is such that Z_- is not ae null, from the definition of γ, it holds that $I(Z) = +\infty$. Also, $\mathrm{dom}(I) \subset L_{\eta_0^*}d\mathbb{R}$ clearly, which justifies that $\mathcal{Q}$ be submerged a priori in this space. From all this it follows:

$$\mathbb{Q} \in \mathrm{dom}(I) \Rightarrow [\bar{T}_0\mathbb{Q} \in \bar{C} \iff \mathbb{Q} \text{ is prob. meas. }, \int \theta d\mathbb{Q} \in C]. \tag{3.2.8}$$

Notice that likewise $\bar{T}_0\mathbb{Q} = \int(1, \theta)d\mathbb{Q}$ (vector-wise). Henceforth, for convenience, θ will denote this extended $(1, \theta)$. Likewise it will be understood $(-\infty, \infty) \times \mathcal{X}_0$ instead of $\mathcal{X}_0$ (but will be equally written), and $\mathcal{G}_0$ should be modified accordingly.
Keeping all this in mind, defining $\bar{\mathcal{Q}} = \left\{\mathbb{Q} \in L_{\eta_0^*}d\mathbb{R} | \bar{T}_0\mathbb{Q} \in \bar{C}\right\}$, it follows that $v(\hat{y}) = \inf_{\mathbb{Q}\in\bar{\mathcal{Q}}} I(\mathbb{Q})$. This shows that the initial problem (PC) is equivalent to finding v, at the point $\hat{y}$ that comes from 2.2.2.

Henceforth the bar over T, $\mathcal{Q}$ and C will be dropped, as the context should make the situation clear.
With this, it becomes possible to combine Theorems 2.2.2 and 2.3.1 in order to obtain a stronger result. To keep exposition clear, the mimimal relevant assumptions are stated again:

Assumption 3.2.1. On the Constraints

- The convex set $C \subset \mathcal{X}_0$ is such that $T_0^{-1}C \cap L_{\eta_0^*}\mathbb{R}$ is a $\sigma(L_{\eta_0^*}\mathbb{R}, E_{\eta_0})$-closed subset of $L_{\eta_0^*}\mathbb{R}$. This is:

$$T_0^{-1}C \cap L_{\eta_0^*}\mathbb{R} = \bigcap_{y\in A} \left\{ f\mathbb{R} \in L_{\eta_0^*}\mathbb{R} | \int \langle y, \theta \rangle f d\mathbb{R} \geq a_y \right\},$$

  for a certain set A of $\mathcal{X}_0^*$ such that $\langle y, \theta \rangle \in E_{\eta_0}, \forall y \in A$, and a certain real function $y \in A \mapsto a_y$.

- $T_0^*\mathcal{G}_0 \subset E_{\eta_0}$, or equivalently, $\forall g \in \mathcal{G}_0 : \int \gamma\left(\langle g, \theta \rangle\right) d\mathbb{P} < \infty$.
- $\forall g \in \mathcal{G}_0$, the function $\omega \in \Omega \mapsto \langle g, \theta(\omega) \rangle$ is measurable.
- $\forall g \in \mathcal{G}_0$, $[\langle g, \theta(\cdot) \rangle = 0, \mathbb{R}\text{-ae.} \Rightarrow g = 0]$.

Assumption 3.2.2. On the Models Set

- $\mathbb{R}(A) = 0$ if and only if $[\mathbb{Q}(A) = 0, \forall \mathbb{Q} \in \mathcal{Q}]$.
- The set $\mathcal{Z} := \left\{ \frac{d\mathbb{Q}}{d\mathbb{R}} | \mathbb{Q} \in \mathcal{Q} \right\}$ is closed in $L^0(\mathbb{R})$.

Recall that the robust optimization problem is $u(x) = \sup_{X\in\mathcal{X}(x)} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left(U\left(X_T\right)\right)$, where $\mathcal{Q} = \left\{ \mathbb{Q} = Zd\mathbb{R} \in L_{\eta_0^*}\mathbb{R} | \mathbb{Q} \text{ is a prob. meas. }, \int \theta Z d\mathbb{R} \in C \right\}$ and $\eta_0(\omega, \cdot) := \eta\left(\frac{d\mathbb{P}}{d\mathbb{R}}(\omega), |\cdot|\right)$, with $\eta(l, z) = \gamma_l(z)$ and γ_l^* defines as in 3.1.1.

Proposition 3.2.1.
Consider the robust optimization problem in a complete market (with $\mathcal{M}_e = \{\mathbb{P}\}$) as described above.
Let assumptions 3.1.1, 3.2.1 and 3.2.2 hold, as well as:

$$v_{\mathbb{Q}}(y) < \infty, \forall y > 0, \forall \mathbb{Q} \in \mathcal{Q}_e. \tag{3.2.9}$$

Then all the results in Theorem 2.2.2 apply. In particular, for all $x > 0$:

$$u(x) = v(\hat{y}) + x\hat{y} = u_{\hat{\mathbb{Q}}}(x) = \mathbb{E}_{\hat{\mathbb{Q}}}\left[U\left(\hat{X}_T\right)\right], \tag{3.2.10}$$

where $\hat{y}$ is in the super differential of $u(x)$, $\hat{\mathbb{Q}} \in \mathcal{Q}$, $\hat{X} \in \mathcal{X}(x)$ such that $\hat{\mathbb{Q}}-$ae: $\hat{X}_T = [U']^{-1}\left(\hat{y}\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}\right)$.
Moreover:

$$v(\hat{y}) = \mathbb{E}^{\mathbb{R}}\left[\frac{d\hat{\mathbb{Q}}}{d\mathbb{R}} V\left(\hat{y}\frac{d\mathbb{P}/d\mathbb{R}}{d\hat{\mathbb{Q}}/d\mathbb{R}}\right)\right] = \mathbb{E}^{\hat{\mathbb{Q}}}\left[V\left(\hat{y}\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}\right)\right] = \inf_{\mathbb{Q}\in\mathcal{Q}} I(\mathbb{Q}), \tag{3.2.11}$$

with $\mathcal{Q} = \left\{\mathbb{Q} \in L_{\eta_0^*} d\mathbb{R} | T_0\mathbb{Q} \in C\right\}$*, and* I *as in 3.2.7. To this problem (of entropy minimization) Theorem 2.3.1 can be applied. In particular:*

$$v(\hat{y}) = \sup_{G\in\mathcal{G}} \left\{ \inf_{W\in C\cap\mathcal{X}} \langle G, W\rangle - \hat{y}\int \gamma(\langle G,\theta\rangle)d\mathbb{P} \right\}. \tag{3.2.12}$$

Moreover the minimization problem in 3.2.11 possesses always a unique solution.
If additionally $C \cap icor(T_0 dom(I)) \neq \emptyset$*, hence defining* $\hat{W} = \int \theta d\hat{\mathbb{Q}}$*, there exists* $\tilde{G} \in \tilde{\mathcal{G}}$ *such that:*

$$\begin{cases} (a) & \hat{W} \in C \cap dom\,(\Gamma^*)\,, \\ (b) & \left\langle \tilde{G}, \hat{W}\right\rangle_{\mathcal{X}_0^*,\mathcal{X}_0} \leq \left\langle \tilde{G}, W\right\rangle_{\mathcal{X}_0^*,\mathcal{X}_0}, \forall W \in C\cap dom\,(\Gamma^*)\,, \\ (c) & \hat{\mathbb{Q}}(d\omega) = \hat{y}\gamma'\left(\left\langle \tilde{G}, \theta(\omega)\right\rangle\right)\mathbb{P}(d\omega). \end{cases} \tag{3.2.13}$$

What is more, $\hat{\mathbb{Q}} \in L_{\eta_0^*}\mathbb{R}$ *and* $\tilde{G} \in \tilde{\mathcal{G}}$ *satisfy 3.2.13 (a,b,c) if and only if* $\hat{\mathbb{Q}}$ *solves 3.2.11 and* $\tilde{G}$ *solves the following:*

$$Maximize \inf_{W\in C\cap\mathcal{X}} \langle \tilde{G}, W\rangle - \hat{y}\int \gamma(\langle \tilde{G},\theta\rangle)d\mathbb{P}\ ,\ \tilde{G} \in \tilde{\mathcal{G}}. \tag{3.2.14}$$

Remark 3.
The definitions of Γ, $\mathcal{X}$, $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are analogous to those in section 2.3, except from the fact that the role γ_0 plays there, it is now assumed by η_0 (which is the Young function of interest here), ans that that of γ is now assumed by $\eta\left(\frac{d\mathbb{P}}{d\mathbb{R}}(\omega),\cdot\right)$.

Proof. It is a straightforward result from Theorems 2.2.2 and 2.3.1, plus the discussion above.
From assumptions 3.1.1 and 3.2.1, assumptions 2.3.2 and 2.3.1 hold, as well as $\mathbb{R}-$ae: $\text{lím}_{t\to\pm\infty} \frac{\gamma_\omega^*(t)}{t} = +\infty$ thanks to which Theorem 2.3.1 applies.
Thanks to assumptions 3.1.1, 3.2.2 and 3.2.9 the conditions of Theorem 2.2.2 are met.
Note that condition 3.2.9 implies that $C \cap T_0\text{dom}(I) \neq \emptyset$, which explains why 3.2.11 has always a unique solution (see Theorem 2.3.1). Precisely, $v(\hat{y}) < \infty$ due to 3.2.9, and then by 3.2.11, $I(\hat{\mathbb{Q}}) < \infty$ holds, from where $T_0\hat{\mathbb{Q}} \in C \cap T_0\text{dom}(I)$.
Finally, since $\gamma_\omega^*(\cdot) := \gamma^*_{\hat{y}\frac{d\mathbb{P}}{d\mathbb{R}}(\omega)}(\cdot)$, this implies $\gamma_\omega(\cdot) = \hat{y}\frac{d\mathbb{P}}{d\mathbb{R}}(\omega)\gamma(\cdot)$, which explains the presence of $\mathbb{P}$ instead of $\mathbb{R}$ in 3.2.12, 3.2.13, 3.2.14 and in the second point of assumption 3.2.1.

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3.3. Robust Optimization, Incomplete case and $\frac{d\mathcal{Q}}{d\mathbb{R}} \subset L^1$ weakly compact

The tools necessary to solve the incomplete market case are essentially the same as those used in the previous part. Hence, as in section 2.2, for each admissible model $\mathbb{Q} \in \mathcal{Q}$, the following set has to be introduced:

$$\mathcal{Y}_{\mathbb{Q}}(y) := \{Y \geq 0 | Y_0 = y \ , \ XY \text{ is } \mathbb{Q} - \text{supermartingale } \forall X \in \mathcal{X}(1)\} . \tag{3.3.1}$$

This set, that intervenes in the definition of $v_{\mathbb{Q}}$ (see 2.2.8), is irrelevant in the complete case, since in this latter case y times the density with respect to $\mathbb{Q}$ of the unique risk neutral measure, dominates each $Y \in \mathcal{Y}_{\mathbb{Q}}(y)$. Moreover, from the definition of I in 3.2.7 in the complete case ($\mathcal{M}_e(S) = \{\mathbb{P}\}$), the corresponding Orlicz space of interest is $L_{\eta_0^*}$, con $\eta_0^* = \eta(d\mathbb{P}/d\mathbb{R}, \cdot)$, and hence the conditions that enable defining T_0 and θ are met (see the second point in assumption 3.2.1). In the incomplete setting, by contrast, it is no longer a priori clear with wich $Y \in \mathcal{Y}_{\mathbb{Q}}(y)$ to define the corresponding $\eta_0^* = \eta(Y_T, \cdot)$. Due to this observation, a stronger integrability condition on θ has to be enforced in order for the mimization of entropy problem to be well defined.
Except from this, the remaining details are not much different from before.

Under the assumptions of Theorem 2.2.2, the existence, for every $x > 0$, of a $\hat{y}$ in the superdifferential of $u(x)$, a $\hat{Y} \in \mathcal{Y}_{\mathbb{R}}(\hat{y})$ and a $\hat{\mathbb{Q}} \in \mathcal{Q}$ holds, so that:

$$v(\hat{y}) = \mathbb{E}^{\mathbb{R}} \left[\hat{Z} V \left(\frac{\hat{Y}_T}{\hat{Z}} \right) \right] \text{ , con } \hat{Z} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{R}}.$$

Also:

$$v(\hat{y}) = \inf_{Z \in \mathcal{Z}_e} \mathbb{E}^{\mathbb{R}} \left[Z V \left(\frac{\hat{Y}_T}{Z} \right) \right] \tag{3.3.2}$$

$$= \inf_{Z \in \mathcal{Z}} \mathbb{E}^{\mathbb{R}} \left[Z V \left(\frac{\hat{Y}_T}{Z} \right) \right] \tag{3.3.3}$$

$$= \inf_{Z \in \mathcal{Z}} \int \gamma_\omega^*(Z(\omega))\mathbb{R}(d\omega), \tag{3.3.4}$$

where now $\gamma_\omega^*(\cdot) := \gamma_{\hat{Y}_T(\omega)}^*(\cdot) = \eta^* \left(\hat{Y}_T(\omega), \cdot \right)$, and recall that $\eta(l, z) = \gamma_l(z)$. Also let $\eta_0(\omega, \cdot) := \eta \left(\hat{Y}_T^0(\omega), | \cdot | \right)$, where $\hat{Y}^0 := \frac{\hat{Y}}{\hat{y}} \in \mathcal{Y}_{\mathbb{R}}(1)$. Equality in 3.3.3 follows from similar arguments than those of Lemma 3.2.1.

Similarly, the entropy functional I is defined as in 3.2.7 and the discussion around equation 3.2.8 can be repeated in this context. The main difference lies on the second point of the restrictions. This ensures that the constraint operator (and hence set $\mathcal{Q}$) are well defined no matter the optimal $Y \in \mathcal{Y}_{\mathbb{R}}(y)$ be (and no matter the optimal y winds up):

Assumption 3.3.1. On the Constraints

- The convex set $C \subset \mathcal{X}_0$ is such that $T_0^{-1}C \cap L_{\eta_0^*}\mathbb{R}$ is a $\sigma(L_{\eta_0^*}\mathbb{R}, E_{\eta_0})$-closed subset of $L_{\eta_0^*}\mathbb{R}$. This is:

$$T_0^{-1}C \cap L_{\eta_0^*}\mathbb{R} = \bigcap_{y\in A} \left\{ f\mathbb{R} \in L_{\eta_0^*}\mathbb{R} | \int \langle y, \theta\rangle f d\mathbb{R} \geq a_y \right\},$$

for a certain set A of $\mathcal{X}_0^*$ such that $\langle y, \theta\rangle \in E_{\eta_0}, \forall y \in A$, and a certain real function $y \in A \mapsto a_y$.

- $\forall g \in \mathcal{G}_0, \forall Y \in \mathcal{Y}_{\mathbb{R}}(1) : \int Y_T\gamma(\langle g, \theta\rangle)\, d\mathbb{R} < \infty$.
- $\forall g \in \mathcal{G}_0$, the function $\omega \in \Omega \mapsto \langle g, \theta(\omega)\rangle$ is measurable.
- $\forall g \in \mathcal{G}_0$, $[\langle g, \theta(\cdot)\rangle = 0, \mathbb{R}\text{-ae.} \Rightarrow g = 0]$.

With this, Proposition 3.2.1 can be restated:

Proposition 3.3.1.
Consider the robust optimization problem in an incomplete market, as introduced. Let assumptions 3.1.1, 3.3.1 and 3.2.2 hold, as well as:

$$v_{\mathbb{Q}}(y) < \infty, \forall y > 0, \forall \mathbb{Q} \in \mathcal{Q}_e. \tag{3.3.5}$$

Then all the results in Theorem 2.2.2 apply. In particular, for every $x > 0$:

$$u(x) = v(\hat{y}) + x\hat{y} = u_{\hat{\mathbb{Q}}}(x) = \mathbb{E}_{\hat{\mathbb{Q}}}\left[U\left(\hat{X}_T\right)\right], \tag{3.3.6}$$

where $\hat{y}$ is in the super differential of $u(x)$, $\hat{\mathbb{Q}} \in \mathcal{Q}$, $\hat{Y} \in \mathcal{Y}_{\mathbb{R}}(\hat{y})$, $\hat{X} \in \mathcal{X}(x)$ y $\hat{\mathbb{Q}}-cs$: $\hat{X}_T = [U']^{-1}\left(\frac{\hat{Y}_T}{\hat{Z}}\right)$, where $\hat{Z} = d\hat{\mathbb{Q}}/d\mathbb{R}$. Also:

$$v(\hat{y}) = \mathbb{E}^{\mathbb{R}}\left[\hat{Z}V\left(\frac{\hat{Y}_T}{\hat{Z}}\right)\right] = \inf_{\mathbb{Q}\in\mathcal{Q}} I(\mathbb{Q}), \tag{3.3.7}$$

with $\mathcal{Q} = \left\{\mathbb{Q} \in L_{\eta_0^*}d\mathbb{R} | T_0\mathbb{Q} \in C\right\}$, and I as in 3.2.7. To this problem (of entropy minimization) Theorem 2.3.1 can be applied. In particular:

$$v(\hat{y}) = \sup_{G\in\mathcal{G}}\left\{\inf_{W\in C\cap\mathcal{X}}\langle G, W\rangle - \int \hat{Y}_T\gamma(\langle G, \theta\rangle)d\mathbb{R}\right\}. \tag{3.3.8}$$

Moreover the minimization problem in 3.2.11 possesses always a unique solution.
If additionally $C \cap icor(T_0 dom(I)) \neq \emptyset$, then defining $\hat{W} = \int \theta d\hat{\mathbb{Q}}$, there exists $\tilde{G} \in \tilde{\mathcal{G}}$ such that:

$$\begin{cases} (a) & \hat{W} \in C \cap dom\,(\Gamma^*)\,, \\ (b) & \left\langle \tilde{G}, \hat{W}\right\rangle_{\mathcal{X}_0^*,\mathcal{X}_0} \leq \left\langle \tilde{G}, W\right\rangle_{\mathcal{X}_0^*,\mathcal{X}_0}, \forall W \in C \cap dom\,(\Gamma^*)\,, \\ (c) & \hat{\mathbb{Q}}(d\omega) = \hat{Y}_T\gamma'\left(\left\langle \tilde{G}, \theta(\omega)\right\rangle\right)\mathbb{R}(d\omega). \end{cases} \tag{3.3.9}$$

Moreover, $\hat{\mathbb{Q}} \in L_{\eta_0^*}\mathbb{R}$ and $\tilde{G} \in \tilde{\mathcal{G}}$ satisfy 3.3.9 (a,b,c) if and only if $\hat{\mathbb{Q}}$ solves 3.3.7 and $\tilde{G}$ solves the following:

$$\textit{Maximize} \inf_{W\in C\cap\mathcal{X}}\langle \tilde{G}, W\rangle - \int \hat{Y}_T\gamma(\langle \tilde{G}, \theta\rangle)d\mathbb{R}\ ,\ \tilde{G} \in \tilde{\mathcal{G}}. \tag{3.3.10}$$

3.4. Robust Optimization, Complete case without Compactness assumption

In section 3.2 the robust optimozation problem on a completo market was solverd, under the assumption that the set of the densities of the models, $\frac{d\mathcal{Q}}{d\mathbb{R}}$, be wakly compact in L^1 (or equivalently, closed under convergence in probability). Nevertheless, in the linear models constraints case $T_0\mathbb{Q} := \int \theta d\mathbb{Q} \in C$, this assumption can be often violated, as the following example suggests:

Example 3.4.1. In section **??**, an example where the set $\frac{d\mathcal{Q}}{d\mathbb{R}}$ is not closed in L^0, and in fact not even bounded in L^2 (see observation 9 for the importance of this point), is solved. The market consists of a single risky asset, whose price evolves as a deometric brownian motion (with constant volatility y and drift components) and the set of feasible models correspond to those probability measures such that the final price of the asset has a mean greater or equal to an a priori constant A.

It should be noted that, for constraints of the form $\int \theta d\mathbb{Q} = \int \theta \frac{d\mathbb{Q}}{d\mathbb{R}} d\mathbb{R} \leq \alpha$ (thinking of $(-\infty, \infty)^d$, some d, and component-wise inequalities and integration), the corresponding set $\mathcal{Q}$ can sometimes wind out to be closed in L^0 (for instance due to Fatou's Lemma), for θ sufficiently nice. One can device some other contexts in which this will still hold, but clearly these cases do not take advantage of all the generality the theory of entropy minimization entails.

As seen in section 3.2, due to expression 3.2.2, in a complete case settin it should occur that:

$$v(y) = \mathbb{E}^{\hat{\mathbb{Q}}}\left[V\left(y\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}\right)\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}V\left(y\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}\right)\right] = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}V\left(y\frac{d\mathbb{P}}{d\mathbb{Q}}\right)\right] = \inf_{\mathbb{Q}\in\mathcal{Q}} \int \gamma_y^*\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) d\mathbb{P}. \tag{3.4.1}$$

where $\mathbb{P}$ is the unique risk neutral measure and γ_y comes from 3.1.1. The convention $\gamma = \gamma_1$ should be recalled as well as definition $\eta_0(\cdot) := \gamma(|\cdot|)$. Hence note that, instead of the steps followed in section 3.2, $\mathbb{P}$ is regarded now as the natural probability measure (that is why η is defined in terms of γ_1 and not of $\gamma_{\frac{d\mathbb{P}}{d\mathbb{R}}}$ as before).

Since η_0 and η_0^* are, as a result of assumption 3.1.1 and the observations afretwards, Young functions, the spaces E_{η_0} and $L_{\eta_0^*}$ are introduced, as well as the associated, according to definition 2.3.1.
The following set of densities will be of great usefulness in the following:

Definition 3.4.1.
$\mathcal{Z}_{\mathbb{P}} := \frac{d\mathcal{Q}}{d\mathbb{P}} := \left\{\frac{d\mathbb{Q}}{d\mathbb{P}} | \mathbb{Q} \in \mathcal{Q}\right\}$.

The following Lemma enables to gain an understanding of the topology of this latter set, when bounded:

Lemma 3.4.1.
If $\mathcal{Z}_{\mathbb{P}}$ is bounded in $L_{\eta_0^*}$ (ie. $\sup_{Z\in\mathcal{Z}_{\mathbb{P}}} ||Z||_{\eta_0^*} < \infty$), then $\mathcal{Z}_{\mathbb{P}}$ is weakly relatively compact in L^1.

Proof. Taking $k^{-1} = \sup_{Z\in\mathcal{Z}_\mathbb{P}} ||Z||_{\eta_0^*}$ and $G(\cdot) = \eta_0^*(k\cdot)$ (which is likewise a Young function), it holds that $\forall Z \in \mathcal{Z}_\mathbb{P}$, $\mathbb{E}(G(Z)) = \mathbb{E}\left(\eta_0^*\left(k||Z||\frac{Z}{||Z||}\right)\right) \leq k||Z||\mathbb{E}\left(\eta_0^*\left(\frac{Z}{||Z||}\right)\right) \leq k||Z|| \leq 1$, by definition of k and of the norm $||\cdot||_{\eta_0^*}$, and because η_0^* is convex and is worth 0 in 0. Hence, Theorems *de la Vallée Poussin* and *Dunford-Pettis* (see A.0.2 and A.0.3 in the appendix A) allow to conclude.

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Remark 4.
In view of this Lemma, if $\mathcal{Z}_\mathbb{P}$ is bounded in $L_{\eta_0^*}$ and close in L^1 (strongly or weakly, equivalently), then $\mathcal{Q}$ satisfies the conditions in assumption 2.2.1 and hence the resulys in section 3.2 apply. This, in particular, is the case when $L_{\eta_0^*}$ is reflexive and $T_0^{-1}C \cap L_{\eta_0^*}d\mathbb{R}$ is *-weak compact in $L_{\eta_0^*}$ (the latter being a standard assumption when the minimization of entropy methodology it to be used: see the previous sections).

Because of the this, the case when $\mathcal{Z}_\mathbb{P}$ is **not** bounded is motivated. In order to ease notation, the following definitions and results concerning Orlicz spaces need to be introduced:

Definition 3.4.2.
A Young function Φ:

1. satisfies condition Δ_2 (globally), which is denoted $\Phi \in \Delta_2$ ($\Phi \in \Delta_2$ globally), whenever for some constant $k > 0$ and a $x_0 \geq 0$ ($x_0 = 0$ in the global case):

$$\Phi(2x) \leq k\Phi(x)\ ,\ \forall x \geq x_0,$$

2. satisfies condition ∇_2 (globally), which is denoted $\Phi \in \nabla_2$ ($\Phi \in \nabla_2$ globally), whenever for some constant $l > 1$ and a $x_0 \geq 0$ ($x_0 = 0$ in the global case):

$$\Phi(x) \leq \frac{1}{2l}\Phi(lx)\ ,\ \forall x \geq x_0,$$

3. is said to be an **N-function** if: is continuous, $[\Phi(x) = 0 \iff x = 0]$, $\Phi(\cdot) \in [0, \infty)$, $\text{lím}_{x\to 0}\frac{\Phi(x)}{x} = 0$, and $\text{lím}_{x\to\infty}\frac{\Phi(x)}{x} = +\infty$.

Proposition 3.4.1 (*Corollary 12*, [RR91], p. 113)**.**
If the measure space is finite and Φ is an N-function, then [L_Φ is reflexive $\iff$ $\Phi \in \Delta_2 \cap \nabla_2$].

Remark 5.
On the Orlicz space $L_\Phi(\mathbb{P})$ (Φ a Young function), the norm

$$||u||_\Phi = \text{ínf}\left\{\beta > 0 : \int \Phi(u(\omega)/\beta)\mathbb{P}(d\omega) \leq 1\right\}$$

had been defined. Besides, an equivalent norm can be defined as follows:

$$|u|_\Phi = \sup\left\{\int |ug|d\mathbb{P} : \int \Phi^*(|g|)d\mathbb{P} \leq 1\right\}.$$

Thanks to *Proposition 4* in [RR91] p. 61, it holds that $||u||_\Phi \leq |u|_\Phi \leq 2||u||_\Phi, \forall u \in L_\Phi$ Moreover, if Φ is an N-function, due to *Theorem 13* in [RR91] p. 69, it further holds that $\forall u \in L_\Phi(\mathbb{P})$:

$$|u|_\Phi = \inf_{k>0} \left\{ \frac{1}{k} \left(1 + \int \Phi(ku) d\mathbb{P} \right) \right\}, \tag{3.4.2}$$

and what is more, the infimum is attained by a certain $k^* = k^*(u) > 0$.

With all these elements, it becomes apparent to ask oneself about the conditions on the robust optimization problem ingredients that ensure the corresponding aforementioned properties on the Young functions and associated Orlicz spaces:

Lemma 3.4.2.
Suppose assumption holds 3.1.1. Then, for the Young function $\eta_0^*(\cdot) = \gamma^*(|\cdot|) := |\cdot| V(1/|\cdot|)$ (see Lemma 3.1.3) holds:

1. *η_0^* is an N-function.*
2. *If $AE(U) := \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$, then $\eta_0^* \in \Delta_2$.*
3. *If $\exists \alpha \in (0,1), \epsilon > 0, y_0 > 0$ such that $\forall y \geq y_0 : \alpha U'(y) \leq U'\left(\frac{2+\epsilon}{\alpha} y\right)$, then $\eta_0^* \in \nabla_2$.*

Proof. For the first point, from observation 2 (ie. Lemmata 3.1.1 and 3.1.2), it follows necessarily that η_0^* is finite, worth 0 exclusively at the origen, $\lim_{x\to\infty} \eta_0^*(x)/x = V(0) = U(\infty) = \infty$ and $\lim_{x\to 0} \eta_0^*(x)/x = V(+\infty)$. Now, since $V(y) = U(I(y)) - yI(y)$, where $I := (U')^{-1} \geq 0$, it follows $V(U'(x)) \leq U(x)$ from where $U(0+) = 0 \geq V(+\infty)$, due to INADA and assumption 3.1.1. Hence, since $V \geq 0$ (bt Lemma 3.1.2)), it holds $V(+\infty) = 0$ and thus η_0^* is an N-function.
As for the second point, from 3.1.4, follows that for $z > 0$:

$$\eta_0^*(2z) = 2zV(\frac{1}{2z}) \leq a\eta_0^*(z) + b(1+z).$$

Now, given any $c > 0$, for $z \geq \frac{1}{c}$ holds that $1 + \frac{1}{z} \leq 1 + c$. On the other hand, since $\lim_{x\to\infty} \eta_0^*(x)/x = \infty$, then $\exists z_0 > 0$ such that $z \geq z_0 \Rightarrow \eta_0^*(z)/z \geq 1 + c$, from where $\forall z \geq z_c := \max\{1/c, z_0\} : 1 + \frac{1}{z} \leq \frac{\eta_0^*(z)}{z}$, and thus $1 + z \leq \eta_0^*(z)$ for such z. Putting together, $\eta_0^*(2z) \leq k\eta_0^*(z)$ for a certain $k > 0$ and all $z \geq z_c$.
For the third statement, after the change of variables $z := U'(y)$, notice that $y \geq y_0 \iff z \leq z_0 := U'(y_0)$. Hence, it holds $\alpha z \leq U'\left(\frac{2+\epsilon}{\alpha} I(z)\right)$. Note that I is strictly decreasing since U' is so. Thus $\alpha\frac{I(\alpha z)}{I(z)} \geq 2 + \epsilon$, and since $V' = -I$ it follows that $\alpha\frac{V'(\alpha z)}{V'(z)} \geq 2 + \epsilon$ $(\forall z \leq z_0)$. Now, since V is differentiable and $V(0+) = +\infty$, it holds that $\lim_{z\to 0} \frac{V(\alpha z)}{V(z)} = \lim_{z\to 0} \frac{V(\alpha z)'}{V(z)'} = \lim_{z\to 0} \frac{\alpha V'(\alpha z)}{V'(z)} \geq 2 + \epsilon$, by L'Hôpital's rule . Hence, $\exists \bar{z} > 0$ such that $\forall z \leq \bar{z} : \frac{V(\alpha z)}{V(z)} \geq 2$ from where after the change of variables $z = \frac{1}{x}$ and defining $l := \frac{1}{\alpha} > 1$ it holds that $xV\left(\frac{1}{x}\right) \leq \frac{1}{2l} lxV\left(\frac{1}{lx}\right)$, this is, $\eta_0^*(x) \leq \frac{1}{2l}\eta_0^*(lx)$, $\forall x \geq \frac{1}{\bar{z}}$. This completes the proof.

◇

Going back to the robust problem, notice the following: in case $\mathcal{Q}$ be closed in L^0, its convexity implies as well closedness with respect to infinite convex combinations (ie. $\forall \lambda_n \geq 0, \sum_{n\in\mathbb{N}} \lambda_n = 1, \forall \mathbb{Q}_n \in \mathcal{Q} : \sum_{n\in\mathbb{N}} \lambda_n \mathbb{Q}_n \in \mathcal{Q}$). This plus condition $[\mathbb{R}(A) = 0 \iff \forall \mathbb{Q} \in \mathcal{Q}, \mathbb{Q}(A) = 0]$ imply thanks to Halmos-Savage Theorem (see *Theorem 1.1* [KS96]) the existence of at least one $\mathbb{Q} \in \mathcal{Q}_e := \{\mathbb{Q} \in \mathcal{Q} | \mathbb{Q} \sim \mathbb{R}\}$. Nevertheless, in the context of this section it is no longer clear that the set $\mathcal{Q}$ be closed for infinite convex combinations. Therefore, without giving any apriori explicit form as a set of models with linear ambiguity (ie. without introducing T_0, C, etc.), $\mathcal{Q}$ is required to comply with the following hypotheses:

Assumption 3.4.1.

1. $\mathcal{Q}$ is closed for convex combinations of infinitely many elements.

2. $\mathcal{Z}_{\mathbb{P}} = d\mathcal{Q}/d\mathbb{P}$ is a subset $\sigma(L_{\eta_0^*}(\mathbb{P}), E_{\eta_0}(\mathbb{P}))$-closed of $L_{\eta_0^*}(\mathbb{P})$.

3. $[\mathbb{P}(A) = 0 \iff \forall \mathbb{Q} \in \mathcal{Q}, \mathbb{Q}(A) = 0]$.

The following Proposition allows to prove *minimax* equality for the robust utility:

Proposition 3.4.2.
Suppose assumption 3.1.1 holds, and that $\eta_0^* \in \Delta_2$ globally.
Also, suppose that $\exists x > 0, \exists \mathbb{Q}_0 \in \mathcal{Q}_e$ such that $u_{\mathbb{Q}_0}(x) < \infty$, where $\mathcal{Q}$ satisfues assumption 3.4.1.
Then:

$$\forall x > 0, \exists C = C(x) \textit{ tal que } u_Z(x) \geq C||Z||_{\eta_0^*}.$$

Therefore, if $\mathcal{Z}_{\mathbb{P}}$ is not bounded, then:

$$\forall x > 0, u_Z(x) := \sup_{X_T \in \mathcal{X}(x)} \mathbb{E}^{\mathbb{P}}(ZU(X_T)) \longrightarrow +\infty \quad \textit{when } ||Z||_{\eta_0^*} \to \infty, Z \in \mathcal{Z}_{\mathbb{P}}. \tag{3.4.3}$$

Proof. Assume $Z \in \mathcal{Z}_{\mathbb{P}}$ is such that $Zd\mathbb{P} \in \mathcal{Q}_e$. Then, it follows $u_Z(x) = \inf_{y>0} \left[\mathbb{E}^{\mathbb{P}}\left(ZV\left(\frac{y}{Z}\right)\right) + xy\right]$. Since η_0^* is an N-function, it follows that $||Z||_{\eta_0^*} \leq y + \int y\eta_0^*\left(\frac{Z}{y}\right) d\mathbb{P}$, due to observation 5. From this, calling $A_Z(y) = \mathbb{E}^{\mathbb{P}}\left(ZV\left(\frac{y}{Z}\right)\right) + xy$, then $A_Z(y) \geq ||Z||_{\eta_0^*} + (x-1)y$. Thus taking infimum over $\{y > 0\}$, if $x > 1$, then $u_Z(x) \geq ||Z||_{\eta_0^*}$.
In greater generality, from condition Δ_2-global follows $\eta_0^*(2^n x) \leq K^n \eta_0^*(x), \forall x > 0$. Moreover, since η_0^* is increasing on the positive reals (because $(\cdot)V(1/(\cdot))$ is so), necessarily this K ought yo be strictly greater than 1. As above, $\int y\eta_0^*\left(\frac{Z}{y}\right) d\mathbb{P} \geq \frac{1}{K^n}\int y\eta_0^*\left(\frac{Z2^n}{y}\right) d\mathbb{P} \geq \frac{1}{K^n}\left[2^n||Z||_{\eta_0^*} - y\right]$, from where $A_Z(y) \geq \left(\frac{2}{K}\right)^n ||Z||_{\eta_0^*} + y\left(x - \frac{1}{K^n}\right)$. Therefore, given $x > 0$ and choosing n such that $x - \frac{1}{K^n} > 0$, it holds that $u_Z(x) \geq \left(\frac{2}{K}\right)^n ||Z||_{\eta_0^*}$.
With all this, $\forall x > 0, \exists C = C(x) > 0, \forall Zd\mathbb{P} \in \mathcal{Q}_e : u_Z(x) \geq C(x)||Z||_{\eta_0^*}$.
If now $Zd\mathbb{P} \notin \mathcal{Q}_e$ is such that $u_Z(x) = \infty$, the result trivially holds. If in turn $u_Z(x) < \infty$, resorting to *Lemma 3.3* in [SW05], which proves that the function $t \in [0,1] \to u_{t\mathbb{Q}_1+(1-t)\mathbb{Q}_2}(x)$ is continuous $\forall x > 0$, if $\mathbb{Q}_i \in \mathcal{Q}$ are such that $u_{\mathbb{Q}_i} < \infty$, with $i = 1, 2$. Thus, taking $\mathbb{Q}_0$ as in the statement of the present proposition, $t \in (0,1]$ and defining $Z_t = td\mathbb{Q}_0/d\mathbb{P} + (1-t)Z$, follows that $Z_t d\mathbb{P} \in \mathcal{Q}_e$. What is more, $\forall \epsilon > 0, \exists \delta$ such that $t \in (0, \delta) \Rightarrow u_Z(x) \geq u_{Z_t}(x) - \epsilon \geq C(x)||Z_t||_{\eta_0^*} - \epsilon$, where the last equality comes

from the previous paragraph. Hence, taking $\liminf$ when $t \to 0+$, it winds out that $u_Z(x) \geq C(x)||Z||_{\eta_0^*} - \epsilon, \forall \epsilon > 0$ (by lower semicontinuity of the norm). Thus, it follows that $\forall Z \in \mathcal{Z}_{\mathbb{P}}, u_Z(x) \geq C||Z||_{\eta_0^*}$, which completes the Proposition.

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Remark 6.

1. Note that the latter Proposition hold true for $x > 1$, without requiring $\eta_0^* \in \Delta_2$.
2. Notice that necessarily the constant K on the definition of $\eta_0^* \in \Delta_2$ must be such that $K \geq 2$ for the results to hold true, since on the contrary the same calculations lead to $\forall x, Z : U_Z(x) = +\infty$, which contradicts the hypotheses.
3. Although in the context of [SW05] the set $\mathcal{Q}$ is supposed to be closed in L^0, in the proof *Lemma 3.3* there, that hypothesis is not required.

With these ingredients, the following minimax Theorem can be proved:

Theorem 3.4.1.
Consider assumptions 3.1.1 and 3.4.1, and assume $[\exists x > 0, \exists \mathbb{Q}_0 \in \mathcal{Q}_e$ *tal que* $u_{\mathbb{Q}_0}(x) < \infty]$.
Also, suppose that $\eta_0^* \in \Delta_2$ *globally, that the space* $L_{\eta_0^*}$ *is reflexive (ie.* $\eta_0^* \in \nabla_2$ *additionally) and that* $\mathcal{Z}_{\mathbb{P}}$ *is NOT bounded in* $L_{\eta_0^*}$.
Then:

$$u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left(U\left(X_T\right)\right) = \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{X \in \mathcal{X}(x)} \mathbb{E}^{\mathbb{Q}}\left(U\left(X_T\right)\right) = \min_{\mathbb{Q} \in \mathcal{Q}} \sup_{X \in \mathcal{X}(x)} \mathbb{E}^{\mathbb{Q}}\left(U\left(X_T\right)\right) = \inf_{\mathbb{Q} \in \mathcal{Q}_e} \sup_{X \in \mathcal{X}(x)} \mathbb{E}^{\mathbb{Q}}\left(U\left(X_T\right)\right). \tag{3.4.4}$$

Proof. Fix $x > 0$. Let $C = \mathcal{Z}_{\mathbb{P}} \subset L_{\eta_0^*}$, $D := \{g \in L^0 | 0 \leq g \leq X_T, \text{ some } \mathcal{X}(x)\} \subset L^0$ and $F : C \times D \to (-\infty, \infty)$ defined by $F(Z, X) := \mathbb{E}^{\mathbb{P}}\left[ZU\left(X_T\right)\right]$. Since $F(\cdot, X)$ is linear and U bounded from bellow (it is non-negative), follows that $F(\cdot, X)$ is (quasi)convex on the convex C and strongly usc in $L_{\eta_0^*}$ (using Fatou's Lemma). On the other hand, $F(Z, \cdot)$ is (quasi)concave on the convex D and hence, being bounded from bellow, it turns out to be usc on every line segment (as a real concave function with complete domain), and as a matter of fact continuous. Now, both C and D are closed subsets in their respective spaces with their respective topologies: C thanks to assumption 3.4.1 and D thanks to *Proposition 3.1* (part (i)) of [KS99]. Hence, noting that replacing $\mathcal{X}(x)$ by D does not change the value of the robust utility (nor that of u_Z), since U is incresing, it follows that from Proposition 3.4.2, $\sup_{X \in D} F(Z, X) \to +\infty$ as $||Z||_{L_{\eta_0^*}} \to \infty$, $Z \in C$. With all this plus the reflexivity assumption, MiniMax Theorem A.0.4 in the appendix A apply. Finally, for the last equality, the fact that $\mathcal{Q}_e$ is supposed not empty, plus *Lemma 3.3* in [SW05] allow to conclude that, thanks to what has already been proven, $u(x) = \inf_{\mathbb{Q} \in \mathcal{Q}_e} u_{\mathbb{Q}}(x)$, from where conclusion follows readily.

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From Theorem 3.4.4, and in view that for each $\mathbb{Q} \in \mathcal{Q}_e$ the corresponding market inherits the completeness of the market under $\mathbb{R}$ (the reference measure), the usual results of the portfolio optimization on complete markets can be employed: for instance, see

Theorem 2.2.3 in [Gun06], or *Theorem 2.0* in [KS99]. Hence, in particular, given the hypotheses of the previous Theorem, it follows that as in the previous sections:

$$\begin{aligned} u(x) = \inf_{\mathbb{Q}\in\mathcal{Q}_e} \sup_{X\in\mathcal{X}(x)} \mathbb{E}^{\mathbb{Q}}\left(U\left(X_T\right)\right) &= \inf_{\mathbb{Q}\in\mathcal{Q}_e} \inf_{y>0} \left\{ \mathbb{E}^{\mathbb{Q}}\left[V\left(y\frac{d\mathbb{P}}{d\mathbb{Q}}\right)\right] + xy \right\} \\ &= \inf_{y>0} \left\{ \inf_{\mathbb{Q}\in\mathcal{Q}_e} \mathbb{E}^{\mathbb{Q}}\left[V\left(y\frac{d\mathbb{P}}{d\mathbb{Q}}\right)\right] + xy \right\}. \end{aligned}$$

from where the following Corollary stems:

Corollary 3.4.1.
Under the assumptions of the previous Theorem, it holds that:

$$u(x) = \inf_{y>0} \left\{ \inf_{\mathbb{Q}\in\mathcal{Q}_e} \int \left[\frac{d\mathbb{Q}}{d\mathbb{P}} V\left(y\frac{d\mathbb{P}}{d\mathbb{Q}}\right) d\mathbb{P}\right] + xy \right\} = \inf_{y>0} \left\{ \inf_{\mathbb{Q}\in\mathcal{Q}_e} \int \left[\gamma_y^*\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) d\mathbb{P}\right] + xy \right\}. \tag{3.4.5}$$

We will write $I_y(\mathbb{Q}) = \mathbb{E}^{\mathbb{Q}}\left[V\left(y\frac{d\mathbb{P}}{d\mathbb{Q}}\right)\right] = \int\left[\gamma_y^*\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)d\mathbb{P}\right]$. Notice that, as in Lemma 3.2.1, if $y > 0$ is such that $v(y) < \infty$, then:

$$\inf_{\mathbb{Q}\in\mathcal{Q}_e} I_y(\mathbb{Q}) = \inf_{\mathbb{Q}\in\mathcal{Q}} I_y(\mathbb{Q}) = v(y). \tag{3.4.6}$$

Remark 7.
Since the idea is to imitate the results already known in the literature ([KS99] in the non-robust case and [SW05], [Gun06] in the robust case), it is worthwhile pointing out the following comments:

1. The main difference with the context of [SW05] (and [Gun06]) is that for them the weak compactness in L^1 of $\mathcal{Q}$ is an assumption (which, due to its structure, is equivalent to its closedness in L^0). In this section this is not typically the case, and moreover, $\mathcal{Z}_{\mathbb{P}} = d\mathcal{Q}/d\mathbb{P}$ is assumed to be unbounded in $L_{\eta_0^*}$ (see comment 4).

2. The usefulness of the hypothesis that $d\mathcal{Q}/d\mathbb{P}$ be closed in L^0, stems from the following fact (*lemma 3.1* of [KS99]): if f_n is a sequence of non-negative random variables, then there exists another sequence $g_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$ which is convergent in probability (and almos surely) to a random element $g \in [0, +\infty]$. Thanks to this, and to the convexity of $Z \to u_Z(x)$, a sequence $\{Z_n\} \subset \mathcal{Z}_{\mathbb{P}}$ can be found, such that $Z_n \to Z$ in probability, with $Z \in \mathcal{Z}_{\mathbb{P}}$, abd such that $u_{Z_n}(x) \to u(x)$ (the key pont here is that $Z \in \mathcal{Z}_{\mathbb{P}}$ due to the closedness of this set in L^0). Also, from the definition of I_y and convexity of $Z \to ZV\left(\frac{y}{Z}\right)$, a similar argument shows the existence of $Z_n \to Z$ (all of them belonging to $\mathcal{Z}_{\mathbb{P}}$, and convergence in the probability sense) such that $I_y\left(Z_n\mathbb{P}\right) \to \inf_{\mathbb{Q}\in\mathcal{Q}} I_y\left(\mathbb{Q}\right) = v(y)$.

Despite the previous comment, most of the results in [SW05], [Gun06] can be still directly obtained, thanks to the following Lemma (notice, however, that their context is that of a complete market, which complicates the analysis a bit further):

Lemma 3.4.3.
Under the same assumptions of Theorem 3.4.1:
$\forall x > 0$, there exists a $\hat{Z} \in \mathcal{Z}_{\mathbb{P}}$ ($\hat{\mathbb{Q}} = \hat{Z} d\mathbb{P}$) such that:

$$u(x) = u_{\hat{Z}}(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}^{\hat{Q}} \left(U \left(X_T \right) \right). \tag{3.4.7}$$

Moreover, $\hat{Z}$ can be chosen to be the limit in probability and in $L_{\eta_0^*}-$weak of a sequence $\{B_n\}_n \subset \frac{d\mathcal{Q}_e}{d\mathbb{P}}$ such that $u(x) = \lim u_{B_n}(x)$.
Also, $\forall y > 0$ such that $v(y) < \infty$, there is a sequence $\{Z_n\}_n \subset \mathcal{Z}_{\mathbb{P}}$ such that $Z_n \to Z \in \mathcal{Z}_{\mathbb{P}}$, where convergence is in the probability sense and in $L_{\eta_0^*}-$weak, so that:

$$I_y \left(Z_n \mathbb{P} \right) \to v(y). \tag{3.4.8}$$

Proof. For the second statement, since $v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}} I_y(\mathbb{Q})$, let $W_n \in \mathcal{Z}_{\mathbb{P}}$ be such that $I_y(W_n\mathbb{P}) \searrow v(y)$. Due to *lemma 3.1* in [KS99], a sequence $Z_n = \sum_{m \geq n} \lambda_m^n W_m \in \operatorname{conv}(W_n, W_{n+1}, \dots)$ can be picked, so that it is convergent in probability (to a certain Z). Since $\mathcal{Q}$ is closed under convex infinite combinations, it holds that $\forall n : Z_n \in \mathcal{Z}_{\mathbb{P}}$. On the other hand, $v(y) \leq \liminf I_y(Z_n\mathbb{P}) \leq \limsup I_y(Z_n\mathbb{P}) \leq \limsup I_y(W_n\mathbb{P}) = v(y)$, since $I_y(Z_n\mathbb{P}) \leq \sum_{m \geq n} \lambda_m^n I_y(W_m\mathbb{P}) \leq I_y(W_n\mathbb{P})$ (convexity of I, plus the choice of W as a decreasing limit), trom which $v(y) = \lim I_y(Z_n\mathbb{P})$. From this, $\sup_n \mathbb{E}^{\mathbb{P}} \left[\gamma_y^* \left(Z_n \right) \right] =: \kappa < \infty$, and since γ_y^* satisfies the conditions in de la Vallée Poussin's Theorem (see Theorem A.0.2), it follows that this sequence is uniformly integrable. Thus, from Vitali's convergence Theorem, $Z_n \to Z$ in $L^1(\mathbb{P})$. On the other hand, from 3.4.2 (taking $k = y^{-1}$), follows that $||Z_n||_{\eta_0^*} \leq y + I_y(Z_n\mathbb{P}) \leq y + \kappa$. Thus, since $L_{\eta_0^*}$ is reflexive, there exists a subsequence $\left\{ Z_{\sigma(n)} \right\}$ convergent $L_{\eta_0^*}-$weakly to a $\tilde{Z} \in L_{\eta_0^*} \cap \mathcal{Z}_{\mathbb{P}}$ (due to closedness under this topology of this set). Now, since $L_{\eta_0^*}$ is embedded continuously in L^1 (in particular with their weak topologies), necessarily $\tilde{Z} = Z$. This subsequence satisfies the statement.
As for the first statement, existence of $\hat{Z} \in \mathcal{Z}_{\mathbb{P}}$ is due to the "minimum" in 3.4.4. From there follows as well that, a sequence $\{A_n\}_n \subset \frac{d\mathcal{Q}_e}{d\mathbb{P}}$ such that $u_{A_n}(x) \searrow u(x)$ can be chosen. As in the previous paragraph, and out of the convexity of $Z \to u_Z(x)$, a sequence $\{B_n\}_n \subset \frac{d\mathcal{Q}_e}{d\mathbb{P}}$ can be chosen, such that $u(x) = \lim u_{B_n}(x)$, and convergent to a certain B. From the boundedness of $u_{B_n}(x)$, follows thet B_n is bounded in $L_{\eta_0^*}$ (see the estimate in Proposition 3.4.2). Hence, by taking further subsequences if needed, $B_n \to A \in \mathcal{Z}_{\mathbb{P}}$ weakly. Likewise, repeating the proof of Lemma 3.4.1, this se sequence is UI and hence convergent in L^1 to B, and as before it follows that $A = B$. Hence this subsequence and $\hat{Z} = B$ comply with the statement.

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The following Proposition (which summarized the main results in [SW05]), comes out of the previous Lemma and out of the very idea of how starting from a minimizing sequence in $\mathcal{Z}_{\mathbb{P}}$, a similar one can be found, but now convergent in probability and in $L_{\eta_0^*}-$weak to a limit also in $\mathcal{Z}_{\mathbb{P}}$.

Proposition 3.4.3.
Suppose assumptions 3.1.1 and 3.4.1 hold. Moreover, suppose that $\eta_0^* \in \Delta_2$ globally, that the space $L_{\eta_0^*}$ is reflexive (ie. $\eta_0^* \in \nabla_2$ additionally) and that $\mathcal{Z}_{\mathbb{P}}$ is NOT bounded in $L_{\eta_0^*}$.

If $[\exists x > 0, \exists \mathbb{Q}_0 \in \mathcal{Q}_e$ *such that* $u_{\mathbb{Q}_0}(x) < \infty]$, *then* u *is concave, finite, and satisfies the equalities in 3.4.4. What is more,* v *is convex lsc, and conjugate to* u*:*

$$u(x) = \inf_{y>0} (v(y) + xy) \ \ , \ v(y) = \sup_{x>0} (u(x) - xy) \, .$$

If further, suppose $[v_{\mathbb{Q}}(y) < \infty, \forall y > 0, \forall \mathbb{Q} \in \mathcal{Q}_e]$, *then* $\forall x > 0$, *there exists a measure* $\hat{\mathbb{Q}} \in \mathcal{Q}$ *and a* $\mathbb{P}-$*martingale* $\hat{X} \in \mathcal{X}(x)$ *such that:*

$$u(x) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left[U\left(\hat{X}_T\right)\right] = \mathbb{E}^{\hat{\mathbb{Q}}}\left[U\left(\hat{X}_T\right)\right] = u_{\hat{\mathbb{Q}}}(x) = v(\hat{y}) + x\hat{y}, \tag{3.4.9}$$

where $\hat{y}$ *belongs to the super differential of* u *at* x, *and* $\hat{\mathbb{Q}}-$*as:* $\hat{X}_T = [U']^{-1}\left(\hat{y}\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}\right)$, *so that:*

$$v(\hat{y}) = \mathbb{E}^{\hat{\mathbb{Q}}}\left[V\left(\hat{y}\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}\right)\right].$$

If additionally $AE(U) < 1$, *then* u *is strictly concave,* v *is continuously differentiable, and* $\mathbb{P}-$*as:* $\hat{X}_T = [U']^{-1}\left(\hat{y}\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}\right)$.

Proof. Equalities in 3.4.4 are Theorem 3.4.1. Note that Lemmata *3.6* and *3.7* of [SW05] remain valid in this context, since $V^- = 0$ (because $U(0+) = 0$ is assumed) and thanks to the comment before this Proposition plus Lemma 3.4.3, that allows to adapt these proofs to the present setting. With this, the proof of *Theorem 2.2* of [SW05] is the same, which proves the finiteness and concavity of u, the convexity and lower semicontinuity of v, and the fact that they must be conjugate functions.
For the rest of the results, is suffices to notice that *lemma 4.1* of [SW05] follows from the aforementioned comment and Lemma 3.4.3. It should only be noticed, that at the end of the proof of this result in [SW05], where $y_n \to \hat{y}$, it holds that $v_{Z'_n}(y_n) = \mathbb{E}[Z'_n V(y_n/Z'_n)] \to v(\hat{y})$, with Z'_n convergent in probability (and almost surely). Noting that $||Z'_n||_{\eta_0^*} \leq y_n + I_{y_n}(Z'_n d\mathbb{P}) = y_n + v_{Z'_n}(y_n) \leq \kappa$, one can conclude as usual. The next results in [SW05] follow direclty (except for an adaptation to the complete case), since they do not employ the topolpgy of $\mathcal{Q}$. Therefore, *Theorem 2.6* there can be applied. The fact that X is a $\mathbb{P}-$martingale follows from the fact that $X_t\mathbb{E}^{\mathbb{R}}[d\mathbb{P}/d\mathbb{R}|\mathcal{F}_t]$ is an $\mathbb{R}-$martingale, plus Bayes'Theorem for conditional expectations. The fact that if $AE(U) < 1$, then thanks to Theorem 2.2.2, $X_T = 0 \iff \frac{d\hat{\mathbb{Q}}}{d\mathbb{R}} = 0$ ($\mathbb{R}-$as), shows that the expression for X_T is valid $\mathbb{P}-$as, since $I(\infty) := [U']^{-1}(\infty) = 0 = X_T$ in $\frac{d\hat{\mathbb{Q}}}{d\mathbb{R}} = 0$, $\mathbb{P}-$as.

◇

The previous Proposition "solves" the robust optimization problem on a complete market, essentially under the hypotheses that $\eta_0^* \in \nabla_2 \cap [\Delta_2$ global$]$ and that $d\mathcal{Q}/d\mathbb{P}$ be closed under convex infinite combinations, weakly closed and unbounded (in $L_{\eta_0^*}$), for certain reasonable utility functions (assumption 3.1.1). A key point, in equality 3.4.9, is that $u(x) = v(\hat{y}) + x\hat{y}$. This implies, thanks to equation 3.4.5 and the comments around it, that:

$$u(x) = x\hat{y} + \inf_{\mathbb{Q}\in\mathcal{Q}} I_{\hat{y}}(\mathbb{Q}) \tag{3.4.10}$$

with $\hat{y}$ as in Proposition 3.4.3. Moreover, the argument of the infimum 3.4.10 coincides with the $\hat{\mathbb{Q}}$ of this same Proposition. Thus, as in the previous sections, minimization of entropy techniques can be applied on the functional $I_{\hat{y}}$, when $\mathcal{Q}$ stems from linear uncertainty, as it will be presented in short.

3.5. Robust Optimization, Complete Linear case without Compactness

In this part, the minimization of entropy methodology will be applied to the robust optimization problem in a complete market, when the set of feasible models comes out of linear constraints and lacks of a compactness condition (see previous section).
As usual, $\mathbb{P}$ will denote the unique risk neutral measure. For the details of the notation and the precise definitions of the spaces and elements involved, see section 2.3. Here one will go faster.

Let $\mathcal{G}_0$, $\mathcal{X}_0 = (\mathcal{G}_0)^*$, $\theta : \Omega \to \mathcal{X}_0$ and $C \subset \mathcal{X}_0$ a convex. Define (whenever they make sense) $T_0^* g(\omega) = \langle g, \theta(\omega)\rangle_{\mathcal{G}_0,\mathcal{X}_0}$ and $T_0 f := \int_\Omega \theta df$, for $f \in L_{\gamma_0^*}\mathbb{P}$. With this, define the set:

Definition 3.5.1.

$$\mathcal{Q} := \left\{ \mathbb{Q} = Zd\mathbb{P} \in L_{\eta_0^*}\mathbb{P} | \mathbb{Q} \text{ is a probability meas. }, \int \theta Z d\mathbb{P} \in C \right\}. \qquad (3.5.1)$$

As before, one re-defines $T_0\mathbb{Q} := \left(\int d\mathbb{Q}, T_0\mathbb{Q}\right) =: \int (1,\theta) d\mathbb{Q}$ (vector-wise) and $C = \{1\} \times C$, and notices that:

$$\mathbb{Q} \in L_{\eta_0^*} d\mathbb{P} \cap \operatorname{dom}(I_{\hat{y}}) \Rightarrow [T_0\mathbb{Q} \in C \iff \mathbb{Q} \text{ is a probability meas.}, \int \theta d\mathbb{Q} \in C]. \quad (3.5.2)$$

As before, notation will be slightly abused by calling indifferently θ and $\mathcal{X}_0$ to $(1,\theta)$ and $(-\infty,\infty) \times \mathcal{X}_0$ (same thing with $\mathcal{G}_0$, $\mathcal{X}$ and $\mathcal{G}$); context should be self-evident. Also, re-define $\mathcal{Q} = \left\{ \mathbb{Q} \in L_{\eta_0^*} d\mathbb{P} | T_0\mathbb{Q} \in C \right\}$. Thus, from 3.5.2, $v(\hat{y}) = \text{ínf}_{\mathbb{Q}\in\mathbb{Q}} I_{\hat{y}}(\mathbb{Q})$.

Assumption 3.5.1. On the Constraints

- The convex set $C \subset \mathcal{X}_0$ is such that $T_0^{-1}C \cap L_{\eta_0^*}\mathbb{P}$ is a $\sigma(L_{\eta_0^*}\mathbb{P}, E_{\eta_0})$-closed subset of $L_{\eta_0^*}\mathbb{P}$. This is:

$$T_0^{-1}C \cap L_{\eta_0^*}\mathbb{P} = \bigcap_{y\in A} \left\{ f\mathbb{P} \in L_{\eta_0^*}\mathbb{P} | \int \langle y, \theta\rangle f d\mathbb{P} \geq a_y \right\},$$

  for a certain set A in $\mathcal{X}_0^*$ such that $\langle y, \theta\rangle \in E_{\eta_0}, \forall y \in A$, and a certain real function $y \in A \mapsto a_y$.

- $T_0^*\mathcal{G}_0 \subset E_{\eta_0}$, or equivalently, $\forall g \in \mathcal{G}_0 : \int \gamma(\langle g, \theta\rangle) d\mathbb{P} < \infty$.
- $\forall g \in \mathcal{G}_0$, the function $\omega \in \Omega \mapsto \langle g, \theta(\omega)\rangle$ is measurable.
- $\forall g \in \mathcal{G}_0$, $[\langle g, \theta(\cdot)\rangle = 0, \mathbb{P}\text{-as.} \Rightarrow g = 0]$.

With all this, one has the following characterization of $\hat{Q}$:

Proposition 3.5.1.

Suppose assumptions 3.1.1, 3.4.1 and 3.5.1 hold, where the set $\mathcal{Q}$ is as in definition 3.5.1. Also, suppose that $\eta_0^* \in \Delta_2$ globally, that the space $L_{\eta_0^*}$ is reflexive (ie. $\eta_0^* \in \nabla_2$ additionally) and that $\mathcal{Z}_{\mathbb{P}}$ is NOT bounded in $L_{\eta_0^*}$. Suppose further that $[v_{\mathbb{Q}}(y) < \infty, \forall y > 0, \forall \mathbb{Q} \in \mathcal{Q}_e]$.

Then all the results in Proposition 3.4.3 hold (except for those assuming $AE(U) < 1$). Moreover, $\forall x > 0$ ($\hat{y}$ as before):

$$v(\hat{y}) = \mathbb{E}^{\mathbb{R}}\left[\frac{d\hat{\mathbb{Q}}}{d\mathbb{R}} V\left(\hat{y}\frac{d\mathbb{P}/d\mathbb{R}}{d\hat{\mathbb{Q}}/d\mathbb{R}}\right)\right] = \mathbb{E}^{\hat{\mathbb{Q}}}\left[V\left(\hat{y}\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}\right)\right] = \inf_{\mathbb{Q}\in\mathcal{Q}} I(\mathbb{Q}). \tag{3.5.3}$$

To this last problem (of entropy minimization), Theorem 2.3.1 can be applied. In particular:

$$v(\hat{y}) = \sup_{G\in\mathcal{G}}\left\{\inf_{W\in C\cap\mathcal{X}}\langle G, W\rangle - \hat{y}\int \gamma(\langle G,\theta\rangle)d\mathbb{P}\right\}. \tag{3.5.4}$$

Moreover the minimization problem in 3.5.3 possess always a unique solution, $\hat{Q}$ (see Proposition 3.4.3).
If additionally $C \cap icor(T_0 dom(I)) \neq \emptyset$, then defining $\hat{W} = \int \theta d\hat{\mathbb{Q}}$, there exists $\tilde{G} \in \tilde{\mathcal{G}}$ such that:

$$\begin{cases} (a) & \hat{W} \in C \cap dom\,(\Gamma^*)\,, \\ (b) & \left\langle \tilde{G}, \hat{W}\right\rangle_{\mathcal{X}_0^*,\mathcal{X}_0} \leq \left\langle \tilde{G}, W\right\rangle_{\mathcal{X}_0^*,\mathcal{X}_0}, \forall W \in C \cap dom\,(\Gamma^*)\,, \\ (c) & \hat{\mathbb{Q}}(d\omega) = \hat{y}\gamma'\left(\left\langle \tilde{G}, \theta(\omega)\right\rangle\right)\mathbb{P}(d\omega). \end{cases} \tag{3.5.5}$$

What is more, $\hat{\mathbb{Q}} \in L_{\eta_0^*}\mathbb{R}$ and $\tilde{G} \in \tilde{\mathcal{G}}$ satisfy 3.5.5 (a,b,c) if and only if $\hat{\mathbb{Q}}$ solves 3.5.3 and $\tilde{G}$ solves the following problem:

$$Maximize \inf_{W\in C\cap\mathcal{X}}\langle \tilde{G}, W\rangle - \hat{y}\int \gamma(\langle \tilde{G},\theta\rangle)d\mathbb{P}\ ,\ \tilde{G} \in \tilde{\mathcal{G}}. \tag{3.5.6}$$

Proof. Same proof as in Proposition 3.2.1, thanks to Proposition 3.4.3.

◇

Remark 8.

1. For the definitions of $\tilde{\mathcal{G}}$, Γ^* and others, see chapters 2.3 and 3.2.
2. Note that the first point in assumption 3.5.1 is equivalent to the second point in assumption 3.4.1 (weak closedness). Moreover, by definition 3.5.1, this set is immediately convex (yet not necessarily for infinite combinations).

Next, this result together with the last of the previous section will be applied on a simple example of a complete market.

3.5.1. Example

Let $U(x) = \frac{x^\alpha}{\alpha}$, with $\alpha \in (0,1)$ and $x \in (0,\infty)$. Note that this functions is strictly increasing, strictly concave, continuously differentiable on $(0,\infty)$, and satisfies that $U'(0+) = \infty$, $U'(\infty) = 0$. Moreover U is unbounded and $U(0+) = 0$. Thus , it satisfies assumption 3.1.1.

Let $V(y) = \left[\frac{1-\alpha}{\alpha}\right] y^{\frac{\alpha}{\alpha-1}}$. Then $DV(y/D) = \left[\frac{1-\alpha}{\alpha}\right] y^{\frac{\alpha}{\alpha-1}} D^{\frac{1}{1-\alpha}}$

Let

$$\tilde{f}_y(D) = \begin{cases} \left[\frac{1-\alpha}{\alpha}\right] y^{\frac{\alpha}{\alpha-1}} D^{\frac{1}{1-\alpha}} & , \quad D > 0 \\ 0 & , \quad D = 0 \\ \infty & , \quad D < 0 \end{cases} \tag{3.5.7}$$

and let

$$\gamma(D) = \begin{cases} \alpha D^{\frac{1}{\alpha}} & , \quad D > 0 \\ 0 & , \quad D \leq 0 \end{cases} \tag{3.5.8}$$

Hence,

$$\gamma^*(z) = \begin{cases} \infty & , \quad z < 0 \\ 0 & , \quad z = 0 \\ (1-\alpha) z^{\frac{1}{1-\alpha}} & , \quad z > 0 \end{cases} \tag{3.5.9}$$

Now, for $y > 0$ let

$$\gamma_y(D) = \begin{cases} 0 & , \quad D \leq 0 \\ \alpha^{\frac{1}{\alpha}} y D^{\frac{1}{\alpha}} & , \quad D > 0 \end{cases} \tag{3.5.10}$$

Hence $\gamma_y(D) = \gamma(\alpha^{1-\alpha} y^\alpha D)$, and therefore

$$\gamma_y^*(z) = \left[\gamma\left(\alpha^{1-\alpha} y^\alpha D\right)\right]^* = \gamma^*\left(\frac{z}{y^\alpha \alpha^{1-\alpha}}\right) = \begin{cases} \left[\frac{1-\alpha}{\alpha}\right] y^{\frac{\alpha}{\alpha-1}} z^{\frac{1}{1-\alpha}} & , \quad z > 0 \\ 0 & , \quad z = 0 \\ \infty & , \quad z < 0 \end{cases} = \tilde{f}_y(z) \tag{3.5.11}$$

Thus, this γ_y is to be used to solve the problem.

Defining $\eta_0^*(z) = \gamma_1^*(|z|) = \left(\frac{1-\alpha}{\alpha}\right) |z|^{\frac{1}{1-\alpha}}$, one verifies that $\eta_0^*(2z) = k\eta_0^*(z)$ and that $\frac{1}{2l}\eta_0^*(lz) = \frac{1}{2} l^{\frac{\alpha}{1-\alpha}} \eta_0^*(z)$. Thus, taking $l \geq 2^{\frac{1-\alpha}{\alpha}}$, one shows that $\eta_0^* \in \Delta_2 \cap \nabla_2$ globally.

Now the market. Consider on $\left(\Omega, \mathbb{F}, \{\mathcal{F}_t\}_{t=0}^T, \mathbb{R}\right)$, and for $t \leq T$, the diffusion

$$\begin{aligned} dS_t &= S_t\{bdt + \sigma dW_t\}, \\ S_0 &= 1, \end{aligned} \tag{3.5.12}$$

where W is a one-dimentional Brownian motion and the parameters b and σ are constant (this is the most elemental version of a Black-Scholes type model). The explicit solution of the latter is $S_t = \exp\left\{\left(b - \frac{\sigma^2}{2}\right) t + \sigma W_t\right\}$. This model is a complete one, where the unique equivalent martingale measure is defined by $d\mathbb{P} = \exp\left\{-\frac{b}{\sigma}W_t - \frac{b^2}{2\sigma^2}T\right\} d\mathbb{R}$. Under this measure , S can be written as $S_t = \exp\left\{-\frac{\sigma^2}{2}t + \sigma\tilde{W}_t\right\}$, where $\tilde{W}$ is a Brownian motion with respect to this probability measure. Thus, S_T has a lognormal distribution with parameters $B = -\frac{\sigma^2}{2}T$ and $K^2 = \sigma^2 T$ (denoted $S_T \sim \textit{lognormal}(B, K^2)$).

Now consider the following robust optimization problem, where the set of feasible models, $\mathcal{Q}$, stems from the constraint $\int S_T \geq A$, this is, $\theta = S_T$, $T_0\mathbb{Q} = \int S_T d\mathbb{Q}$, $C = [A, +\infty)$ and $\mathcal{Q} := \left\{\mathbb{Q} = Zd\mathbb{P} \in L_{\eta_0^*}\mathbb{P}|\mathbb{Q} \text{ prob. meas. }, \int S_T Z d\mathbb{P} \geq A\right\}$. Note that this set is closed under infinite convex combinations. Moreover, $d\mathcal{Q}/d\mathbb{P}$ is closed in the $\sigma\left(L_{\eta_0^*}, E_{\eta_0}\right)$ topology, since both the functionals identically 1 and S_T belong to E_{η_0} (this latter case, since $\mathbb{E}^{\mathbb{P}}[\eta_0(\lambda S_T)] = c(\lambda, \alpha)\mathbb{E}^{\mathbb{P}}[S_T^{1/\alpha}] < \infty, \forall \lambda > 0$, since lognormals possess finite moments), from where $\mathcal{Q}$ is the inverse image of closed sets through evaluation funtionals, which are always *-weak continuous. Finally, if $\exp\{K^2\} \geq A \geq 1$, then it follows that $\mathbb{Q}_L(d\omega) := \frac{e^{K^2} - L + S_T(L-1)}{e^{K^2}-1}\mathbb{P}(d\omega) \in \mathcal{Q}_e, \forall \exp\{K^2\} \geq L \geq A$, and thus $\mathcal{Q}_e \neq \emptyset$. With all this, the assumpltions 3.4.1 hold. What is more, assumption 3.5.1 is satisfied trivially.

In the following it will be assumed that $\exp\{K^2\} \geq A \geq 1$ and that $\alpha = \frac{1}{2}$. In this case, $\eta_0^*(z) = z^2$, which implies $L_{\eta_0^*} = L^2$ and $||\cdot||_{\eta_0^*} = ||\cdot||_2$. Also:

$$\gamma_y(D) = \begin{cases} 0 & , \quad D \leq 0 \\ \frac{y}{4}D^2 & , \quad D > 0. \end{cases}$$

As it will be presented on comment 9, the set $\mathcal{Q}$ turns out to be unboounded in L^2. With all these elements, the results on this section apply.

From the discussion on the previous section (3.4.6 and 3.5.4, which remain valid for every y, not only for the optimal one), follows:

$$\inf_{\mathbb{Q}\in\mathcal{Q}} I_y(\mathbb{Q}) = \sup_{(z_1,z_2)\in(-\infty,\infty)^2}\left[\inf_{c\geq A} z_1 + cz_2 - \mathbb{E}^{\mathbb{P}}\left(\gamma_y(z_1 + S_T z_2)\right)\right] = \sup_{z_1\in(-\infty,\infty),z_2>0}\left[z_1 + Az_2 - \mathbb{E}^{\mathbb{P}}\left(\gamma_y(z_1 + S_T z_2)\right)\right]. \tag{3.5.13}$$

Hence, 3.5.13 is equal to $\sup_{z_1\in(-\infty,\infty),z_2>0}\left[z_1 + Az_2 - \frac{y}{4}\mathbb{E}^{\mathbb{P}}\left((z_1 + S_T z_2)^2 \mathbb{1}_{z_1+S_T z_2>0}\right)\right]$. Define $\Delta(z_1, z_2) = \mathbb{E}^{\mathbb{P}}\left((z_1 + S_T z_2)^2 \mathbb{1}_{z_1+S_T z_2>0}\right)$.

If $z_1 > 0, z_2 > 0$, then (after some computations):

$$\Delta = \int_0^{\infty} \frac{(z_1 + z_2 x)^2}{xK\sqrt{2\pi}} e^{-\frac{(\log x - B)^2}{2K^2}} = \cdots = z_1^2 + 2z_1 z_2 + e^{K^2} z_2^2.$$

Call now $F(z_1, z_2)$ the function being maximized in 3.5.13. It is not hard to show that function Δ is convex on the whole plane $(-\infty, \infty)^2$ (this is inherited out of the convexity of the function inside the expectation). Therefore F is concave on $(-\infty, \infty)^2$ and thus admits a global maximum.

On $\{z_1 > 0, z_2 > 0\}$, one has that $F(z_1, z_2) = z_1 + Az_2 - \frac{y}{4}\left(z_1^2 + 2z_1 z_2 + e^{K^2} z_2^2\right)$, and thus F is continuosly differentiable twice on such part of the plane. Taking $(a, b) = \left(\frac{2(A-2)}{y\left(e^{K^2}-1\right)}, \frac{2(e^{K^2}-A)}{y\left(e^{K^2}-1\right)}\right)$, it is verified under the assumptions $\exp\{K^2\} \geq A \geq 1$, that $(a, b) \in$

$(0,\infty)^2$ and that $\nabla F(a,b) = 0$. Thus (a,b) is a local maximum of F and hence also a global one. This shows, after some computations, that 3.5.13 is equal to $\frac{1}{y}\left[1+\frac{(A-1)^2}{e^{K^2}-1}\right]$. With all this, finally:

$$u(x) = \inf_{y>0}\left\{xy + \frac{1}{y}\left[1+\frac{(A-1)^2}{e^{K^2}-1}\right]\right\} = 2\sqrt{x\left(1+\frac{(A-1)^2}{e^{K^2}-1}\right)},$$

this is:

$$u(x) = 2\sqrt{x\left(1+\frac{(A-1)^2}{e^{\sigma^2 T}-1}\right)}. \tag{3.5.14}$$

Now, note that the feasibility condition $C \cap \operatorname{icor}(T_0 \operatorname{dom} I_y) \neq \emptyset$, where $C = \{1\}\times[A,\infty)$ and $T_0\mathbb{Q} = \left(\int 1 d\mathbb{Q}, \int S_T d\mathbb{Q}\right)$, holds. Precisely, $\operatorname{dom} I_y = L^2_+$, from where $T_0 \operatorname{dom} I_y \supset B := \{(x,y) : x \geq 0, \exp\{K^2\}x \geq y \geq x\}$ (since $T_0(\alpha\mathbb{Q}_L) = (\alpha, \alpha L), \forall \alpha \geq 0, \exp\{K^2\} \geq L \geq 1$), and thus $\operatorname{icor}(T_0 \operatorname{dom} I_y) \supset \mathring{B}$. Thus, from 3.5.6 and 3.5.5, one concludes that from the tuple (a,b) on the previous paragraph, the optimal measure can be derived (the "least favorable" one), by means of $\hat{\mathbb{Q}} = \hat{y}\gamma'(\langle (a,b),(1,S_T)\rangle)\mathbb{P}(d\omega)$ (where $\gamma = \gamma_1$). From this follows that $\hat{\mathbb{Q}} = \mathbb{Q}_A$, this is:

$$\hat{\mathbb{Q}}(d\omega) := \frac{e^{K^2} - A + S_T(A-1)}{e^{K^2}-1}\mathbb{P}(d\omega). \tag{3.5.15}$$

Remark 9.
In this example one verifies that the set $d\mathcal{Q}/d\mathbb{P}$ is unbounded in $L^2 = L_{\eta_0^*}$ (case $\alpha = 1/2$ and $\exp\{K^2\} \geq A \geq 1$) and also one can check that this set cannot be closed for the convergence in probability topology. Precisely, from the explicit expressions for S_T under $\mathbb{P}$ (where $\tilde{W}$ is a $\mathbb{P}-$brownian motion) it holds that for $\forall Z > T$ the random variables $A_Z := \sqrt{\frac{\pi}{2}}\frac{|\tilde{W}_Z - \tilde{W}_T|}{\sqrt{Z-T}}$ are independent from S_T. A few computations show that $\mathbb{E}^{\mathbb{P}}[A_Z] = 1$. Thus, defining $d\mathbb{Q}^Z = S_T A_Z d\mathbb{P}$, one has that $\forall Z > T : \mathbb{Q}^Z \in \mathcal{Q}$. This is, $\mathbb{Q}^Z$ is a probability measure since $\mathbb{E}^{\mathbb{Q}^Z}(1) = \mathbb{E}^{\mathbb{P}}(S_T A_Z) = \mathbb{E}^{\mathbb{P}}(S_T)\mathbb{E}^{\mathbb{P}}(A_Z) = 1$ and $\mathbb{E}^{\mathbb{Q}^Z}(S_T) = \mathbb{E}^{\mathbb{P}}(S_T^2)\mathbb{E}^{\mathbb{P}}(A_Z) = \exp\{K^2\} \geq A$. Also, the following almost surely limit holds, which follows from the law of the iterated logarithm (since $|\tilde{W}_Z - \tilde{W}_T|$ is of the order of $\sqrt{2(Z-T)\log\log(Z-T)}$ asymptotically): $\lim_{Z\to\infty} A_Z = +\infty$. Thus, the corresponding density $A_Z S_T$ satisfies this last limit in turn. Also $\liminf_{Z\to\infty} ||S_T A_Z||^2_{L^2} \geq \mathbb{E}^{\mathbb{P}}\left[\liminf_{Z\to\infty} S_T^2 A_Z^2\right] = +\infty$, from where $\mathcal{Q}$ is unbounded in L^2. On the other hand, calling $B_Z = \exp\left[\tilde{W}_Z - \tilde{W}_T - \frac{1}{2}(Z-T)\right]$, one notices that the same properties as those for A_Z ($\forall Z > T$) hold, except that in this case $\lim_{Z\to\infty} B_Z = 0$, following now the fact that if $t \to L_t$ is a brownian motion, then $t \to tL_{\frac{1}{t}}$ is likewise a bm., and hence $\lim_{Z\to\infty}\frac{\tilde{W}_Z - \tilde{W}_T}{Z-T} = 0$. Therefore, considering the densities $B_Z S_T$ it follows as before that all of them belong to $\mathcal{Q}$, but their almost sure limit (and in probability) equals to 0, which obviously is not in this set.

3.6. Portfolio Optimization under Weak Information

In this part the relationship between the robust optimization problem and that of "insider trading" is explored. The ensuing discussion is informal and illustrative.

3.6.1. Relationship with the Calculus of Variations

In *section 4* of [Bau02], the concept of "weak information" is developed, within the context of models for insider trading. This consists in the a priori knowledge of the distribution of a $\mathcal{F}_T-$measurable functional, that is used by the agent (the "insider", who possesses this insider information) when making her decision about the optimal strategy for her portfolio. In concrete terms, let Y be a $\mathcal{F}_T-$measurable random variable and ν the "guessed" distribution of Y by the investor (one says that (Y, ν) is the weak information). Suppose that $\mathbb{P}$ i the unique risk neutral measure on this market, and that μ is the distribution of Y under $\mathbb{P}$. Finally, define the subjective probability $d\mathbb{Q}^\nu := \frac{d\nu}{d\mu}(Y)d\mathbb{P}$ (note that under the latter the distribution of Y is exactly ν).

Hence, on *Theorem 19* the author establishes that the utility function of the "insider", given an initial wealth $x > 0$ and weak information ν, is:

$$u(x, \nu) = \int \left[U \circ (U')^{-1}\right] \left(\frac{\Lambda(x)}{\xi(y)}\right) \nu(dy), \tag{3.6.1}$$

where $\xi(y) = \frac{d\nu}{d\mu}(y)$, and where $\Lambda(x)$ is obtained from the following implicit expression:

$$\int (U')^{-1} \left(\frac{\Lambda(x)}{\xi(y)}\right) \mu(dy) = x.$$

It is worth mentioning that this problem can be understood as a robust optimization one, where the set of feasible models corresponds to the absolutely continuous probability measures such that the distribution of Y is ν.

Typically, the variable Y is the final price S_T or an average price, to name a few. Now, in the context of section 3.5, where the model ambiguity is manifested by means of a linear constraint (see definition 3.5.1 and the notation around it), the particular case of moment type constraints cand be considered (thus, θ is a finite vector of $\mathcal{F}_T-$measurable random variables). Then the following interpretetion follows: one could consider all the possible distributions for this θ (call them ν) such that the moment type constraints are satisfied (now the moments are computed under ν), and for every ν fixed, maximize the utility of the investor over all the measures such that the distribution of θ is exactly ν. In other words, the set $\mathcal{Q}$ is partitioned according to the corresponding distributions of θ, this problem is solved for a fix distribution ν (which is analogous to the weak information problem with $Y = \theta$ and ν), and finally one maximizes over all the possible distribution ν that alllow for the moment type constraints to hold (as explained on the previous paragraph, this method yields the same value as the robust utility). This last problem can be tackled by means of the calculus of variations under constraints.

to clarify this point, let $\theta = (f_1(S_T), \ldots, f_n(S_T))$ and $C = (c_1, \ldots, c_n)$. Let μ be the distribution of S_T under $\mathbb{P}$, which is supposed absolutely continuous, and further call $f_0 = 1$ and $c_0 = 1$. Hence, the robust utility (u) associated to the models set $\mathcal{Q} = \left\{\mathbb{Q} \ll \mathbb{P} \text{ probability measure } : \int \theta d\mathbb{Q} = C\right\}$, necessarily needs to satisfy:

$$u(x) = \inf_{\left\{\nu \geq 0 : \int f_i d\nu = c_i, \forall i\right\}} u(x,\nu) = \inf_{\left\{\nu \geq 0 : \int f_i d\nu = c_i, \forall i\right\}} \int \left[U \circ (U')^{-1}\right] \left(\frac{\Lambda(x,\nu)\mu(y)}{\nu(y)}\right) \nu(y) dy, \tag{3.6.2}$$

where Λ is defined by $\int (U')^{-1} \left(\frac{\Lambda(x,\nu)}{\xi(y)}\right) \mu(dy) = x$. The latter is clearly a problem of calculus of variations under constraints, on ν. In this thesis a theoretical study on this particular problem is not undertaken, not to say a generalization to more complicated instances, for example when there are more random variables of interests besides S_T. Nevertheless, in order to show that this method might be useful, the example in **??** is studied yet again. In this case $U(y) = y^\alpha/\alpha$, and as is commented in *example 7* of [Bau02] (plus some arrangement), one obtains that $u(x,\nu) = \frac{x^\alpha}{\alpha}\left[\int \nu(y)^{\frac{1}{1-\alpha}} \mu(y)^{\frac{\alpha}{\alpha-1}}\right]^{1-\alpha}$. Hence, consider the problem:

$$\inf_{\left\{\nu \geq 0 : \int \nu(y)dy = 1, \int y\nu(y)dy \geq A\right\}} \int \nu(y)^{\frac{1}{1-\alpha}} \mu(y)^{\frac{\alpha}{\alpha-1}} dy.$$

Hence the lagrangian for this problems with constraints is introduces (where non-negative is not enforces, but a posteriori verified) $L = \nu^{\frac{1}{1-\alpha}} \mu(y)^{\frac{\alpha}{\alpha-1}} - a\nu - by\nu$, and the Euler-Lagrange equations are writen down (which reduces to $\frac{\partial L}{\partial \nu} = 0$), from where it follows $\nu = \mu \left\{(1-\alpha)(a+by)\right\}^{\frac{1-\alpha}{\alpha}}$, where the constants a, b are to be adjusted such that $\int \nu = 1$ and $\int y\nu(y)dy = A$ (this last equality coming from the fact that $u(x,\nu)$ winds out being increasing in b).
For the case $\alpha = \frac{1}{2}$, the latter can be easily solved by means of the expressions for the moments of the lognormal distribution (remember that μ corresponds to a lognormal with parameter $B = -\frac{\sigma^2}{2}T$ and $K^2 = \sigma^2 T$). Thus, follows:

$$\nu(y) = \frac{\mu(y)}{2}(a + by),$$

where $a = 2\frac{\left(e^{K^2} - A\right)}{\left(e^{K^2} - 1\right)}$ and $b = 2\frac{(A-1)}{\left(e^{K^2} - 1\right)}$. It is not hard to check out that with this ν, it holds that:

$$u(x,\nu) = 2\sqrt{x\left(1 + \frac{(A-1)^2}{e^{\sigma^2 T} - 1}\right)} = u(x),$$

which is equation 3.5.14 exactly. Moreover, from [Bau02] one knows that the least favorable measure, given ν, is $\frac{d\mathbb{Q}^\nu}{d\mathbb{P}} = \frac{d\nu}{d\mu}(S_T)$. From the previous expressions follows that:

$$\frac{\mathbb{Q}^\nu}{d\mathbb{P}} = \frac{e^{K^2} - A + S_T(A-1)}{e^{K^2} - 1},$$

which is equation 3.5.15 for $\hat{\mathbb{Q}}$.

Moreover, with this optimal ν, it follows that the robust optimization problem (the central piece of this thesis) is equivalent to the problem of "insider trading" with weak information (ν, S_T). With this all the results in *section 4* of [Bau02], as well as in [BNN04], where the financial market under $\mathbb{Q}^\nu$ is interpreted, the decomposition of S is found and as well as its drift (by means of a Burgers PDE).

3.6.2. Connexion with the problems of Flow of Weak Information

As commented in the previous section 3.6.1, there is a connexion between the robust optimization problems studied here, and the problem of utility maximization under weak information. As a matter of fact, this last problem can be regarded in turn as a robust optimization one, namely: if Y is $\mathcal{F}_T-$measurable and ν is a distribution, consider $\mathcal{Q}^\nu := \{\mathbb{Q} \ll \mathbb{P} : \mathbb{Q} \circ Y^{-1} = \nu\}$, where $\mathbb{P}$ is the unique rosk neutral measure. Then the problem of utility maximization by means of a weak information (Y, ν), for an initial wealth x, is by definition (see *definition 4* in [BNN04]):

$$\text{Minimize} \sup_{\Theta \in A(S)} \mathbb{E}^{\mathbb{Q}} \left[U \left(x + \int_0^T \Theta_u dS_u \right) \right] \text{ for } \mathbb{Q} \in \mathcal{Q}^\nu. \tag{3.6.3}$$

In this context, the value function $u(x, \nu)$ of this problem is referred to as "the financial value of the weak information" and is interpreted as a worst case utility. Here the connexion with robust optimization becomes apparent (except for the order of min and max). Concretely, in *example 3.12* of [Sch05] it is showed how this problem can be tackled in the robust context, as well as how the conditions on $\mathcal{Q}^\nu$ are met (a reasonable topology over Ω is requires). Baudoin finds by means of ad-hoc techniques the measure Q^ν that solves 3.6.3, namely, $\frac{dQ^\nu}{d\mathbb{P}} = \frac{d\nu}{d\mathbb{P} \circ Y^{-1}}(Y)$, and in fact Schied shows that this is the "least favorable measure" (when $(\Omega, \mathcal{F}_T)$ is a Borel space) associated to $\mathcal{Q}^\nu$; see [Sch05].

A related problem arises when instead of supposedly knowing the distribution of a single random variable, it is the knowledge of a flow of this type of information that the agent has. More precisely, let $\Omega = C\left([0, T], (-\infty, \infty)^d\right)$ and $(\mathcal{F}_t)_{t<T}$ the filtration associated to the coordinate process on Ω. Consider the non-homogeneous generator:

$$Af(t, z) = \sum_i b_i(t, z) \partial_{z_i} f(t, z) + \frac{1}{2} \sum_{i,j} a_{i,j}(t, z) \partial_{z_i} \partial_{z_j} f(t, z), \tag{3.6.4}$$

where the following assumptions are introduced:

Assumption 3.6.1.
$\exists \sigma : [0, T] \times (-\infty, \infty)^d \to (-\infty, \infty)^d$ such that $a = \sigma\sigma^*$, a is bounded and $\sigma, b \in C_b^{1,2,\alpha}\left([0, T], (-\infty, \infty)^d\right)$ (differentiable once in time and twice in space, with bounded Hölder continuous derivatives globally). Hölder continuous derivatives globally).

Under this assumption, the martingale problem with generator A on the domain $C_0^\infty\left((0, T), (-\infty, \infty)^d\right)$ possesses a unique solution $\mathbb{P}$. In the following, X will denote the coordinate process, $M_1\left((-\infty, \infty)^d\right)$ the probability measures on $(-\infty, \infty)^d$ and

$C_{M_1} := C\left([0,T], M_1\left((-\infty,\infty)^d\right)\right)$ (marginal laws's flow).

In [CL95], the problem of finding a probability measure of minimal entropy such that the marginals of X equal a given flow of marginal laws in C_{M_1} is studied. Concretely, the following result is derived there (which is a summarized version of *Theorem 3.6* there):

Theorem 3.6.1.

Under assumptions 3.6.1, let $\nu \in C_{M_1}$ be a given flow of marginal laws, such that $\nu_0 = \mathbb{P} \circ X_0^{-1}$. Also, let $B : [0,T] \times (-\infty,\infty)^d \to (-\infty,\infty)^d$ such that ν is a solution of the following weak Fokker-Planck equation, associated to the generator $A + aB \cdot \nabla$:

$$\int_{[0,T]\times(-\infty,\infty)^d} \left(\partial_t + A + aB(t,z)\cdot\nabla\right) f(t,z)\nu_t(dz)dt = 0 \ ,\ \forall f \in C_0^\infty\left((0,T),(-\infty,\infty)^d\right),$$

and such that the following condition of finite energy is satisfied:

$$\int_{[0,T]\times(-\infty,\infty)^d} B\cdot aB(t,z)\nu_t(dz)dt < \infty.$$

Define:

$$\mathbb{Q}^B(dx) := \mathbb{1}_{\left\{\int_0^T \frac{1}{2} B\cdot aB(t,\cdot)dt<\infty\right\}}(x)\exp\left\{\int_0^T B(t,x_t)\cdot(dx_t - b(t,x_t)dt) - \int_0^T \frac{1}{2} B\cdot aB(t,x_t)dt\right\}\mathbb{P}(dx), \tag{3.6.5}$$

with $x \in \Omega$. Then $\mathbb{Q}^B$ is a probability measure solution to the martingale problem associated with $\partial_t + A + aB\cdot\nabla$ on the domain $C_0^\infty\left((0,T),(-\infty,\infty)^d\right)$ and $\mathbb{Q}^B \circ X_t^{-1} = \nu_t$, for every $0 \le t \le T$.
What is more, if $B \in H_{-1}(\nu)$, then:

$$\begin{aligned} H(\mathbb{Q}^B|\mathbb{P}) &= \text{ínf}\left\{H(\mathbb{Q}|\mathbb{P}) : \mathbb{Q} \text{ is a probability measure on } \Omega, \mathbb{Q}\circ X_t^{-1} = \nu_t, \forall t \le T\right\} \\ &= \int_{[0,T]\times(-\infty,\infty)^d} B\cdot aB(t,z)\nu_t(dz)dt, \end{aligned} \tag{3.6.6}$$

where $H(\cdot|\mathbb{P})$ is the relative entropy:

$$H(\mathbb{Q}|\mathbb{P}) = \begin{cases} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ +\infty & \text{otherwise,} \end{cases}$$

and where $H_{-1}(\nu)$ is the closedness in $L^2(\nu) := \left\{B | \int_0^T B\cdot aB(t,z)\nu_t(dz)dt < \infty\right\}$ of the set $\left\{B : [0,T]\times(-\infty,\infty)^d \to (-\infty,\infty)^d | \text{medibles}, B = \nabla f, f \in C_0^\infty((0,T)\times(-\infty,\infty)^d)\right\}$.

Let $\mathcal{Q}^\nu := \left\{\mathbb{Q} \ll \mathbb{P} : \mathbb{Q} \text{ is a probability measure on } \Omega, \mathbb{Q}\circ X_t^{-1} = \nu_t, \forall t \le T\right\}$. As in the case of weak information, one can be interested in the problem of portfolio optimization with insider information $(X_t,\nu_t)_t$. Note that this problem cannot be tackled a priori by means of Baudoin's approach, since in that case the distribution of the whole process X would need to be known, not just the marginals. As before, this problem can be interpreted as a robust one, where the set of feasible models is now $\mathcal{Q}^{\nu}$. If the utility of the "insider" is defined as in 3.6.3, and is interpreted as a utility function on $(0,\infty)$, then:

$$U_H(x) \quad := \quad \begin{cases} 1+log(x) & \text{if } x>0 \\ -\infty & \text{otherwise,} \end{cases} \tag{3.6.7}$$

then the following holds:

Proposition 3.6.1.
If all the assumptions of Theorem 3.6.1 hold, and $\exists \mathbb{Q}\sim\mathbb{P}, \mathbb{Q}\in\mathcal{Q}^{\nu}$ such that the supremum in 3.6.3 is finite, then for each initial wealth x:

$$\begin{aligned} u_H(x) &:= \inf_{\mathbb{Q}\in\mathcal{Q}^{\nu}} \sup_{\Theta\in A(S)} \mathbb{E}^{\mathbb{Q}}\left[U_H\left(x+\int_0^T \Theta_u dS_u\right)\right] && (3.6.8)\\ &= H\left(\mathbb{Q}^{\nu}|\mathbb{P}\right)+U_H(x) && (3.6.9)\\ &= \int_{[0,T]\times(-\infty,\infty)^d} B\cdot aB(t,z)\nu_t(dz)dt+U_H(x), \end{aligned}$$

where $\mathbb{Q}^B$ and B are defined as in 3.6.6 and $A(S)$ corresponds to the set of predictable processes integrable with respect to the prices S.
The infimum and the supremum are attained.

Proof. Similar arguments to those of Schied-Wu can be used to reduce 3.6.8 to those $\mathbb{Q}\sim\mathbb{P}$. With this, $u_H(x)=\inf_{y>0}\left\{xy+\inf_{\mathbb{Q}\in\mathcal{Q}^{\nu}}\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\tilde{U}_H\left(y\frac{d\mathbb{P}}{d\mathbb{Q}}\right)\right]\right\}$ by means of the classical duality arguments of Schachermayer, where $\tilde{U}_H(z)=-\log(z)\mathbb{1}_{(0,\infty)}$. Now, thanks to 3.6.6, from this follows that $u_H(x)=\inf_{y>0}\{xy-\log(y)+H\left(\mathbb{Q}^{\nu}|\mathbb{P}\right)\}$, and hence conludes.

◇

Evidently it is very reasonable to ask oneself about the existence of optimal strategies. Moreover, instead of beginning with the problem of the "insider" (where the utility has a worst case definition), one could begin from the robust problem where the infimum and supremum in the definition of u_H are exchanged. In the works of C. Léonard this topic has been studied (minimization of entropy) by means of the techniques exposed in section 2.3, but it seems that a more ad-hoc approach (relying on large deviations theory) is more appropriate or directly applicable. For this reason, this problem was not analyzed by means of the results on the previous sections.

If instead of the logarithmic utility U_H a different one would be considered, the problem becomes much harder. Briefly, this is due to the fact that heuristically (by Itô's rule), if $\mathbb{Q}\ll\mathbb{P}$ and Z is its density process, then: $\mathbb{E}^{\mathbb{P}}\left[W(Z_T)\right]=\mathbb{E}^{\mathbb{P}}\left[W(Z_0)\right]+\frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T W''(Z_s)Z_s\beta_s a(X_s)\beta_s\right]$, where X is the coordinate process which is supposed to satisfy along with $\mathbb{P}$ the martingale problem associated to 3.6.4 and β is the stochastic logarith with respect to $M=X-X_0-\int_0 b(X_s,s)ds$. Hence, in the portfolio optimization problem of the previous proposition, $W(z)=z\log(z)$, from where the term $W''(Z_s)Z_s=1$. This highly simplifies the minimization of $\mathbb{E}^{\mathbb{P}}\left[W(Z_T)\right]$ when taking Z over all the $\mathbb{Q}\ll\mathbb{P}$

such that the coordinate process has the definded marginals. Still, a semi satisfactory answer can be derived in the case of power utility functions:

Let $\alpha \in (0,1)$ and define:

$$U_\alpha(x) \;:=\; \begin{cases} \frac{x^\alpha}{\alpha} & \text{if } x > 0 \\ -\infty & \text{otherwise.} \end{cases} \tag{3.6.10}$$

Some tedious computations show that, as in the previous Proposition, the corresponding utility of the "insider" associated to the flow of weak information (or the robust one with $\mathcal{Q}^\nu$) is:

$$u_\alpha(x) := \inf_{\mathbb{Q}\in\mathcal{Q}^\nu} \sup_{\Theta\in A(S)} \mathbb{E}^{\mathbb{Q}}\left[U_\alpha\left(x + \int_0^T \Theta_u dS_u\right)\right] = \frac{x^\alpha}{\alpha}\left[1 + \frac{\alpha R}{2(1-\alpha)^2}\right]^{1-\alpha},$$

where $R := \inf_{\mathbb{Q}\in\mathcal{Q}^\nu} \mathbb{E}^{\mathbb{P}}\left[\int_0^T Z_s^{\frac{1}{1-\alpha}} \beta_s \cdot a(X_s,s)\beta_s\right]$. In the case when $\alpha = 1/2$, one obtains $R = \inf_{\mathbb{Q}\in\mathcal{Q}^\nu} \frac{1}{2} Var\,(Z_T^2)$ and hence the solution to the problem is the minimum variance measure within $\mathcal{Q}^\nu$.

Capítulo 4

Conclusions and Open Questions

The achievements and breakthroughs of this work can be loosely classified around two main ideas: to apply minimization of entropy techniques to the robust optimization problem under linear constraints, and to solve the robust optimization problem without a compactness assumption on the set of models. For both undertakings, it was useful and necessary to establish a relationship between the ingredients of the robust problem and those of Orlicz space theory.

In order to get rid of the weak compactness condition on the set of densities of the feasible models, it was necessary to assume reflexivity of a certain relevant Orlicz space associated to some Young function. Fortunately this condition of reflexivity can still be verified in a relatively simple fashion from the original ingredients of the problem. After this, the main difficulty was to try to follow the arguments in [SW05], adapting them to the present context. Thus, everything is summarized in proposition 3.4.3 essentialy under the assumptions that the utility function does not reach $-\infty$ in a continuous manner, that the Young function belongs to $\nabla_2 \cap [\Delta_2 \text{ global}]$ and that the set of densities of models be convex for infinite combinations, weakly closed and unbounded in the given Orlicz space. With all this, the results in [Léo08] regarding minimization of entropy in propositions 3.2.1, 3.3.1 and 3.5.1 (complete and incomplete cases with compactness and complete case without compactness, respectively), when uncertainty on the models comes in a linear form.

It was not possible to get rid of convexity for infinite combinations of the models set, and is not clear the necessity of this assumption in the linear case. Moreover, for this latter case it was not possible to understand a certain feasibility condition (related to the "intrinsic core" of the image of the domain of a projection functional) in terms of the original ingredients of the problem. On the other hand, in section **??** the developed theory is illustrated in a rather simple example. It would be interesting to find more of these examples, hopefully with greater complexity, and to find out the practical usefulness of the obtained results. As a broader goal, it would have been nice to find a more probabilistic interpretations of the general results derived. For instance, it seem intuitive that the solution of the dual problem should be related with the "drift" of the price under the optimal measure, or at least this measure should solve a certain martingale problem associated to some operator that depended on the solution to the dual problem. This would allow for a trajectory-wise understanding of the market. Also, pertaining to the duality techniques employed, the problem of non-linear constraints (say some family of these) remains open.

In section 3.6 it was illustrated how to use the results of "insider trading" under weak information, in order to solve the robust problem when the uncertainty on the models is of momen-type. Then, an example was carried out and the usefulness of the calculus of variations as a means for solving the robust problem was exemplified. Also, with some limitations, the robust (or "insider trading") problem associated to a flow of weak information (where the marginals of a process are known for each time) was solved. Both the discussion and the arguments given where less sound this time, thus a deep formalization is needed.

Apéndice A

Appendix

The following two classical theorems illustrate the relationship among the concepts of Orlicz spaces, uniform integrability and weak compactness in L^1.

Theorem A.0.2. *de la Vallée Poussin's Theorem*
Let $\mathcal{K}$ be a subset of $L^1(\Omega, \mathbb{P})$. Then the following are equivalent:

1. *$\mathcal{K}$ is uniformly integrable.*
2. *There exists a positive function $G(\cdot)$ defined on $\mathbb{R}_+$, such that $\lim_{t\to\infty} \frac{G(t)}{t} = +\infty$ and:*
$$\sup_{f\in\mathcal{K}} \mathbb{E}(G(f)) < \infty.$$

Proof. See *Theorem* 22, page 24 of chapter II, in [DM78].

◇

On the previous Theorem, G can be chosen to be convex and even a Young function. Even more, it can be chosen an N-function.

Theorem A.0.3. *Dunford-Pettis'compactness criterion*
Let $\mathcal{K}$ be a subset of $L^1(\Omega, \mathbb{P})$. Then the following are equivalent:

1. *$\mathcal{K}$ is uniformly integrable.*
2. *$\mathcal{K}$ is relatively compact in L^1 with its weak topology $\sigma(L^1, L^\infty)$.*
3. *Every sequence in $\mathcal{K}$ possesses a convergent subsequence in the weak topology $\sigma(L^1, L^\infty)$.*

Proof. See *Theorem* 25, page 27 of chapter II, in [DM78].

◇

Proposition A.0.2. *Proposition 1, [RR91]*
If ϕ, γ are Young conjugate functions, then if $f \in L_\phi(\mu)$ and $g \in L_\gamma(\mu)$, the following Hölder inequality holds:
$$\int |fg|d\mu \leq 2||f||_{L_\phi}||g||_{L_\gamma}.$$

Proof. See *Proposition 1*, page 58 of [RR91], plus the comment ("remark") following it.

◇

When the models set is not bounded in a certain reflexive Orlicz space, the following minimax Theorem comes in handy in order to solve the robust optimization problem:

Theorem A.0.4. *Theorem 4*, [Tuy04]*
Let C, D be two closed and convex subsets of two topological vector spaces X e Y respectively. Let there be a function $F : C \times D \to \mathbb{R}$ quasiconvex and lsc in the first variable, and quasiconcave and usc in the second variable on every line segment.
When X is a Banach reflexive space, and $\sup_{y \in D} F(x, y) \to +\infty$ as $||x|| \to \infty$, $x \in C$, it holds that:

$$\sup_{y \in D} \inf_{x \in C} F(x, y) = \inf_{x \in C} \sup_{y \in D} F(x, y) = \min_{x \in \hat{C}} \sup_{y \in D} F(x, y),$$

where $\hat{C} \subset C$ is a compact set.

Proof. This is part $(\tilde{Q})$ of *Theorem 4**, in [Tuy04].

◇

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