Global Carleman estimates for Reissner-Mindlin and Kirchhoff-Love plate models and applications to potential recovery

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Abstract. In this paper we consider two linear plate models, namely, the Reissner-Mindlin system (RM) and the Kirchhoff-Love equation (KL) , which come from the linear elasticity. We prove global Carleman inequalities for both problems with boundary observations and under suitable hypothesis on the parameters. As an application, we use these estimates to study the inverse problem of recovering an spatially dependent potential from knowledge of Neumann boundary data. We obtain L^2-Lipschitz stability for KL and the H^1-Lipschitz stability for RM in both cases under the assumption that the potentials are equal at the boundary.

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1. Introduction

1.1. Reissner-Mindlin plate equation

Let Ω be a bounded domain of $\mathbb{R}^2$ with regular boundary Γ, let Γ_0 be a non-empty open subset of Γ and $T > 0$. Also, let us consider $f \in L^2(0, T; H^1(\Omega))$, $g \in H^1(0, T; L^2(\Omega))$, $(\theta_0^1, \theta_0^2, w_0) \in (H_0^1(\Omega))^3$, $(\theta_1^1, \theta_1^2, w_1) \in (L^2(\Omega))^3$ and for $M > 0$ we define the set of potentials

$$\mathcal{U}_M = \{u \in W^{2,\infty}(\Omega) \mid \|u\|_{W^{2,\infty}} \le M\}.$$

Moreover, we define the set of known trajectories as

$$\mathcal{R} := \{u \in H^1(0, T; H^2(\Omega) \cap W^{1,\infty}(\Omega)) \mid \\ \partial_t^2 u \in L^2(0, T; L^\infty(\Omega) \cap H^1(\Omega)),\ \partial_t^3 u \in L^2(0, T; L^2(\Omega))\},$$

where ∂_t or u_t will denote the time derivative, and we define the norm

$$\|u\|_{\mathcal{R}}^2 := \|u\|^2_{L^2(0,T;W^{1,\infty}(\Omega))} + \|\partial_t u\|^2_{L^2(0,T;W^{1,\infty}(\Omega))} + \|\partial_t^2 u\|^2_{L^2(0,T;L^\infty(\Omega))}.$$

The first problem we consider is the Reissner-Mindlin system with potential $q \in \mathcal{U}_M$:

$$\begin{cases} \theta_{tt} - \mathrm{div}(\sigma(\theta)) - \mu^*\, h_0^{-2}(\nabla w - \theta) = f & \text{in } \Omega \times (0,T) \\ w_{tt} - \mathrm{div}(\mu(\nabla w - \theta)) + qw = g & \text{in } \Omega \times (0,T) \\ \theta = (0,0)^{\mathrm{T}},\ w = 0 & \text{on } \Gamma \times (0,T) \\ \theta(0) = (\theta_0^1, \theta_0^2)^T,\ w(0) = w_0 & \text{in } \Omega \\ \theta_t(0) = (\theta_1^1, \theta_1^2)^T,\ w_t(0) = w_1 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\sigma(\theta) = 2\mu\varepsilon(\theta) + \lambda^*\mathrm{div}(\theta)I$ and $\varepsilon(\theta)$ is the symmetrized gradient of $\theta = (\theta_1, \theta_2)$. The coefficients $\mu^*(x)$ and $\lambda^*(x)$ are defined by

$$\mu^*(x) = 12\mu(x), \quad \lambda^*(x) = \frac{2\mu(x)\lambda(x)}{(\lambda(x) + 2\mu(x))}$$

with μ, λ the Lamé coefficients and $h_0 > 0$ denotes the thickness of the plate which we assume constant and also we suppose $\mu, \lambda \in C^2(\bar{\Omega})$. It is possible to show that the system (1) admits a unique weak solution $(\theta, w) \in C\left([0,T]; (H_0^1(\Omega))^3\right) \cap C^1\left([0,T]; (L^2(\Omega))^3\right)$ (see the section about preliminary results later and [15] for the case where the Lamé parameters are constants). We are using the notation $\nabla, \nabla\cdot, \nabla\wedge$ for the gradient, divergence and curl respectively in the space variables.

We built a global Carleman estimate for the system (1) by using an analogous inequality for the wave equation. The main tool in this construction is the fact that it is possible to split system (1) into six wave equations where each of those corresponde to one of the following functions: $\theta = (\theta^1, \theta^2), \nabla\cdot\theta, \nabla\wedge\theta$, w and w_t. Then, applying the known Carleman inequality for the wave operator over these six equations, we get the estimate by joining them together.

Let us define the operators

$$\mathcal{L}^f(\theta, w) := \theta_{tt} - \mathrm{div}(\sigma(\theta)) - \mu^*\, h_0^{-2}(\nabla w - \theta)$$

$$\mathcal{L}_0^g(\theta, w) = w_{tt} - \mathrm{div}(\mu(\nabla w - \theta))$$

$$\mathcal{L}^g(\theta, w) = w_{tt} - \mathrm{div}(\mu(\nabla w - \theta)) + qw.$$

and let us consider functions

$$(\theta, w) \in \left(H^2(-T,T; H^3(\Omega))\right)^2 \times H^3(-T,T; H^2(\Omega)) \tag{2}$$

or

$$\partial_t^k(\theta, w) \in L^2(-T,T; (H^{3-k}(\Omega))^3) \quad k = 0,1,2,3. \tag{3}$$

and the weight functions

$$\varphi(x,t) = e^{r\psi(x,t)} \tag{4}$$

given r some positive parameter and

$$\psi(x,t) = |x - x_0|^2 - \beta t^2 + M_0, \tag{5}$$

where $0<\beta<1$ and $M_0>0$ is chosen such that $\forall (t,x)\in\Omega\times(-T,T),\ \psi(x,t)\geq 1$.

In order to get the desired estimate, the domain and the parameters must satisfy the following hipothesis

H1.- $\exists x_0\in\mathbb{R}^2\backslash\bar{\Omega}$ with $\Gamma_0\supseteq\Gamma_{x_0}=\{x\in\Gamma,\ (x-x_0)\cdot n(x)>0\}$ (n: unit outward normal).

H2.- $\mu,\ \lambda\in C^2(\bar{\Omega})$ where $\mu(x)\geq\mu_0>0$, $\mu+\lambda\geq\tau_0>0$ and $\exists\theta_0>0$ such that $\forall\ x\in\Omega$:

$$-1+\theta_0<\frac{\nabla\mu(x)\cdot(x-x_0)}{2\mu(x)}<1-\theta_0,$$
$$-1+\theta_0<\frac{\nabla(\lambda^*(x)+2\mu(x))\cdot(x-x_0)}{2(\lambda^*(x)+2\mu(x))}<1-\theta_0.$$

Then, if the above is true we get the next:

Theorem 1 *Under hypothesis* **H1** *and* **H2***, for every $M>0$ there exist $r_0>0$, $s_0>0$ and a positive constant $C=C(\mu,\lambda,h_0,r_0,s_0,\Omega,\beta,x_0,M)$ such that for every potential $q\in\mathcal{U}_M$, $r\geq r_0$, $s\geq s_0$ and for every (θ,w) regular enough (as in (2) or (3)), with $\theta=\vec{0},w=0$ on $\Gamma\times(-T,T)$ and $\theta(\pm T)=\vec{0},\theta_t(\pm T)=\vec{0},w(\pm T)=0,w_t(\pm T)=0,w_{tt}(\pm T)=0$, then*

$$\begin{aligned}
\mathcal{I}_{RM}(\theta,w) &:= sr\int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi\left(\left|\frac{\partial(\nabla\cdot\theta)}{\partial t}\right|^2+\left|\frac{\partial(\nabla\wedge\theta)}{\partial t}\right|^2+\left|\frac{\partial\theta}{\partial t}\right|^2+\left|\frac{\partial w_t}{\partial t}\right|^2\right)\\
&+sr\int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi\left(|\nabla(\nabla\cdot\theta)|^2+|\nabla(\nabla\wedge\theta)|^2+|\nabla\theta|^2+|\nabla w_t|^2+|\nabla w|^2\right)\\
&+\ s^3r^3\int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi^3\left(|\nabla\cdot\theta|^2+|\nabla\wedge\theta|^2+|\theta|^2+|w_t|^2+|w|^2\right)\\
&+\ \sum_{i=1}^{6}\int_{-T}^{T}\int_\Omega\left(\left|P_i^1\zeta_i\right|^2+\left|P_i^2\zeta_i\right|^2\right)\\
&\leq\ C\int_{-T}^{T}\int_\Omega e^{2s\varphi}\left(\left|\mathcal{L}^f\right|^2+\left|\nabla\cdot\mathcal{L}^f\right|^2+\left|\nabla\wedge\mathcal{L}^f\right|^2+|\mathcal{L}^g|^2+\left|\frac{\partial\mathcal{L}^g}{\partial t}\right|^2\right)+\mathcal{B}_{RM}(\theta,w)
\end{aligned}\tag{6}$$

where the boundary terms are given by

$$\begin{aligned}
\mathcal{B}_{RM}(\theta,w) := C\cdot sr\int_{-T}^{T}\int_{\Gamma_0} & e^{2s\varphi}\varphi\left(|(\nabla\theta)n|^2+\left|\frac{\partial w_t}{\partial n}\right|^2+\left|\frac{\partial w}{\partial n}\right|^2\right)(x-x_0)\cdot n\\
&+\ C\cdot sr\int_{-T}^{T}\int_\Gamma e^{2s\varphi}\varphi\left(\left|\frac{\partial(\nabla\cdot\theta)}{\partial t}\right|^2+\left|\frac{\partial(\nabla\cdot\theta)}{\partial n}\right|^2+\left|\frac{\partial(\nabla\cdot\theta)}{\partial\tau}\right|^2\right)\\
&+C\cdot sr\int_{-T}^{T}\int_\Gamma e^{2s\varphi}\varphi\left(\left|\frac{\partial(\nabla\wedge\theta)}{\partial t}\right|^2+\left|\frac{\partial(\nabla\wedge\theta)}{\partial n}\right|^2+\left|\frac{\partial(\nabla\wedge\theta)}{\partial\tau}\right|^2\right)\\
&+\ C\cdot s^3r^3\int_{-T}^{T}\int_\Gamma e^{2s\varphi}\varphi^3\left(|\nabla\cdot\theta|^2+|\nabla\wedge\theta|^2\right),
\end{aligned}$$

and the operators P_i^j are defined in (18) while the functions ζ_i are defined in (19) and (26).

As an application of the previous estimate, we consider the inverse problem of potential recovery. More precisely, we applied the Bukhgeim-Klibanov approach in order to get, for a time T large enough, Lipschitz stability for potentials in $W^{2,\infty}(\Omega)$ with Neumann boundary measurements under **H1**, **H2** and other additional hypothesis.

Let $(\theta(p), w(p))$ be a known solution of (1) with potential $p \in \mathcal{U}_M$, then we have the following

Theorem 2 *Let us assume:*

H1 .- $\exists x_0 \in \mathbb{R}^2 \backslash \bar{\Omega}$ *with* $\Gamma_0 \supseteq \Gamma_{x_0} = \{x \in \Gamma,\ (x - x_0) \cdot n > 0\}$.

H2*.- $\mu,\ \lambda \in C^2(\bar{\Omega})$ *where* $\mu(x) \geq \mu_0$, $\lambda(x) \geq -\lambda_0$, $\frac{2\mu_0}{3} > \lambda_0 > 0$ *and* $\exists \theta_0 > 0$ *such that* $\forall\, x \in \Omega$,

$$0 < \frac{\beta}{\mu_0} < \theta_0, \quad -1 + \theta_0 < \frac{\nabla \mu(x) \cdot (x - x_0)}{2\mu(x)} < 1 - \theta_0,$$
$$-1 + \theta_0 < \frac{\nabla(\lambda^*(x) + 2\mu(x)) \cdot (x - x_0)}{2(\lambda^*(x) + 2\mu(x))} < 1 - \theta_0,$$

H3 .- *The time* T *satisfies* $T\sqrt{\beta} > \sup_{x \in \bar{\Omega}} |x - x_0| =: \rho$.

H4 .- $R(x,t) := w(p) \in \mathcal{R}$, $|R(x,0)| \geq a_0 > 0, \forall x \in \Omega$.

H5 .- $p, q \in \mathcal{U}_M$.

H6 .- $q(x) = p(x)\ \forall x \in \Gamma$.

If $(\theta(q), w(q))$ *is the solution of (1) with potential* q*, then there exists a positive constant* $C = C(h_0, \mu, \lambda, T, M, R)$ *such that*

$$C^{-1} \cdot \|p - q\|_{H^1(\Omega)} \leq \left\| \frac{\partial w(p)}{\partial n} - \frac{\partial w(q)}{\partial n} \right\|_{H^2(0,T\,;\,L^2(\Gamma_0))} + \|\nabla \theta(p) n - \nabla \theta(q) n\|_{H^2(0,T\,;\,L^2(\Gamma_0))} \quad (7)$$
$$+ \|\nabla \cdot \theta(p) - \nabla \cdot \theta(q)\|_{H^2(0,T\,;\,H^1(\Gamma))} + \|\nabla \wedge \theta(p) - \nabla \wedge \theta(q)\|_{H^2(0,T\,;\,H^1(\Gamma))} \cdot$$

Remark 1 *The hypothesis* **H2*** *is a bit more restrictive than* **H2** *because the first one imposes a uniform bound* λ_0 *over* $\lambda(x)$ *which is also related with the uniform bound for* μ *under the inequality* $\frac{2\mu_0}{3} > \lambda_0 > 0$*. The reason why we condider* **H2*** *is to guarantee enough regularity for the solutions of (1) as we will see in the next section.*

1.2. Kirchhoff-Love plate equation

For $N \geq 2$, let Ω be an open bounded subset of $\mathbb{R}^N$ with smooth boundary Γ. Let Γ_0 be as before an open subset of Γ and $T > 0$. For a constant $\kappa_0 > 0$, $g \in L^2(0,T; L^2(\Omega))$, $(k_0, k_1) \in H^3(\Omega) \times H^2(\Omega)$ and a potential $q \in L^\infty(\Omega)$, we define the Kirchhoff-Love differential operators as:

$$L \ := \partial_t^2 - \kappa_0 \Delta \partial_t^2 + \Delta^2,$$
$$L_q := L + q,$$

and we consider the initial-boundary value problem

$$\begin{cases} L_q u = g & \text{en } \Omega \times (0,T) \\ u = 0,\ \Delta u = 0 & \text{en } \Gamma \times (0,T) \\ u(0) = k_0,\ u_t(0) = k_1 & \text{en } \Omega, \end{cases} \tag{8}$$

Similarly to the first plate model, we have the next two theorems where we use the same weight functions as above.

Theorem 3 *For $u \in H^1(-T,T;H^3(\Omega))$ and $v := -\kappa_0 \Delta u + u$ such that $v_{tt} - \kappa_0^{-1}\Delta v - \kappa_0^{-2} v \in L^2(-T,T;L^2(\Omega))$, $u = 0, v = 0$ on $\Gamma \times (-T,T)$ and $v(\pm T) = v_t(\pm T) = 0$ in Ω, $x_0 \in \mathbb{R}^N \backslash \bar{\Omega}$ such that $\Gamma_0 \supset \Gamma_{x_0}$ and $M > 0$, there exist $r_0 > 0$, $s_0 > 0$ and a constant $C = C(r_0, s_0, \kappa_0, \Omega, \beta, x_0, M) > 0$ such that for every $r \geq r_0$, $s \geq s_0$ and $q \in L^\infty(\Omega)$ with $\|q\|_{L^\infty(\Omega)} \leq M$, the following inequality holds*

$$\begin{aligned} \mathcal{I}_{KL}(u) &:= sr \int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi|\Delta u_t|^2 + s^2r^2 \int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi|\nabla u_t|^2 + s^4r^4 \int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi^3|u_t|^2 \\ &+ sr \int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi(|\nabla \Delta u|^2 + s^3r^3 \int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi^3|\Delta u|^2 + s^4r^4 \int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi|\nabla u|^2 \\ &+ s^6r^6 \int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi|u|^2 + \int_{-T}^{T}\int_\Omega |P_1\zeta|^2 + \int_{-T}^{T}\int_\Omega |P_2\zeta|^2 \\ &\leq C \int_{-T}^{T}\int_\Omega e^{2s\varphi}|L_q u|^2 + Cs^4r^4 \int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi}\left(\left|\frac{\partial u_t}{\partial n}\right|^2 + \left|\frac{\partial \Delta u}{\partial n}\right|^2 + \left|\frac{\partial u}{\partial n}\right|^2\right). \end{aligned} \tag{9}$$

where $\zeta = e^{s\varphi} v$ and

$$P_1\zeta := \frac{\partial^2 \zeta}{\partial t^2} - \kappa_0 \Delta\zeta + s^2r^2\varphi^2\left(\left|\frac{\partial\psi}{\partial t}\right|^2 - \kappa_0|\nabla\psi|^2\right)\zeta$$

$$\begin{aligned} P_2\zeta := (M_1 - 1)sr\varphi\left(\frac{\partial\psi}{\partial t} - \kappa_0\Delta\psi\right)\zeta - sr^2\varphi\left(\left|\frac{\partial\psi}{\partial t}\right|^2 - \kappa_0|\nabla\psi|^2\right)\zeta \\ - 2sr\varphi\left(\frac{\partial\psi}{\partial t}\frac{\partial\zeta}{\partial t} - \kappa_0\nabla\psi\cdot\nabla\zeta\right), \end{aligned}$$

with M_1 satisfying

$$\frac{2\beta}{\beta+N} < M_1 < \frac{2}{\beta+N}.$$

Let us assume we know $u(p)$, a solution of (8) with potential $p \in L^\infty(\Omega)$, and $g(x,t)$, k_0 and k_1 are regular enough such that $u(p) \in H^2(0,T;H^3(\Omega))$, then

Theorem 4 *Assuming*

- $\exists x_0 \in \mathbb{R}^2 \backslash \bar{\Omega}$ *with* $\Gamma_0 \supseteq \Gamma_{x_0} = \{x \in \Gamma,\ (x - x_0)\cdot n > 0\}$.
- $T\sqrt{\beta} > \sup_{x\in\bar{\Omega}} |x - x_0| =: \rho$.
- $R(x,t) := u(p) \in H^1(0,T;L^\infty(\Omega))$, *con* $|R(x,0)| \geq a_0 > 0$, $\forall x \in \Omega$.
- $p, q \in L^\infty(\Omega)$ *tal que* $\|p\|_{L^\infty(\Omega)}, \|q\|_{L^\infty(\Omega)} \leq M$.

If $u(q) \in H^2(0,T;H^3(\Omega))$ *is a solution of (8), then there exists a constant*

$$C = C(\kappa_0, T, M, \|R\|_{H^1(0,T;L^\infty(\Omega))}) > 0$$

such that

$$C^{-1}\|q-p\|_{L^2(\Omega)} \leq \left\|\frac{\partial u}{\partial n}(q) - \frac{\partial u}{\partial n}(p)\right\|_{H^2(0,T;L^2(\Gamma_0))} + \left\|\frac{\partial \Delta u}{\partial n}(q) - \frac{\partial \Delta u}{\partial n}(p)\right\|_{H^1(0,T;L^2(\Gamma_0))}. \quad (10)$$

1.3. Related works and scope of the present study

One-measurement inverse stability problems with internal and/or boundary data has been widely studied the last few years by using the known approach introduced by A.L. Bukhgeim and M.V. Klibanov in [2].

Regarding hyperbolic equations, Carleman inequalities for this type of equations are deduced in [6] and [7], while the approach mentioned above is used in [8] in order to obtain a Lipschitz stability estimate for potentials on the wave equation. Another kind of stability is stablished in [9] where the authors obtain a Hölder stability estimate for the principal coefficient of the acoustic wave operator by using Carleman inequalities with right hand side in H^{-1}. In this article, we use a similar Carleman inequality for the wave operator but without null Dirichlet boundary conditions which is less classical.

Inverse problems involving plate equations and more generally, involving the linear elasticity equation, can be divided into two branches of research. The first one studies the one-measurement non-stationary problem for these type of equations where the Bukhgeim-Klibanov approach is widely used and, therefore, the main difficulty is reduced to obtain new global Carleman estimates. On the other hand, the second branch is related with inverse problems in elasticity which are linked with the Calderón problem associated with the bi-laplacian and consequently it requires infinitely many observations.

In order to better describe the contribution of the present study among these works, let us schematize the connection between linear elasticity and different plate models by the following figure, where the rows means the directions in which each model is deduced (see [13]):

3*d* *Linear Elasticity*
↓
2*d* *Reissner-Mindlin plate model* (*or Mindlin-Timoshenko*)
↓
2*d* *Kirchhoff-Love plate model* (*or Kirchhoff*)
↓
2*d* *Euler-Bernoulli plate model*

Regarding the non-stationary problem, the Kirchhoff-Love equation with Navier boundary conditions (i.e. $u|_\Gamma$, $\Delta u|_\Gamma$) is studied from the point of view of the PDE-control theory in [16] and [17], where they also apply Carleman estimates to deduce observability inequalities. Moreover, in [5], the authors obtain an analogous estimate for the Euler-Bernoulli plate equation. In this work we explore the second (Reissner-Mindlin or Mindlin-Timoshenko) and third (Kirchhoff-Love) plate models from an inverse problem perspective by establishing new global Carleman inequalities. Based on the literature review, there was still a gap to fill between global Carleman inequalities for three-dimensional linear elasticity and Euler-Bernoulli plate model, and this was our main motivation.

On the other hand, in the important work [10], O.Y. Imanuvilov and M. Yamamoto proved stability and uniqueness of the density and the Lamé coefficients for the theee-dimensional elastodynamic equation under internal observations using a global Carleman inequality they established for three-dimensional linear elasticity. This paper is similar to that, applying the new Carleman inequalities for the Reissner-Mindlin system and the Kirchhoff-Love plate equations for proving stability inequalities for the zero-order potential inverse problem provided boundary data. Also, we prove existence, uniqueness and regularity of its solutions for these systems when the Lamé coefficients depends on the space variable, generalizing the result obtained in [15], where the author gets similar results but assuming constant and positive coefficients.

Notice that a related important open problem in geophysical lithospheric flexure applications (see [3] and the references therein) is to recover the thickness parameter $h = h(x)$, $\kappa = \kappa(x)$ appearing in these plate models. The present study can be considered a first step in this direction and we consider $h = h_0$ and $\kappa = \kappa_0$ constant and known.

Let us mention some results related to Dirichlet-to-Neumann bi-laplacian inverse problem related to stationary elasticity models, which are different but closely related to the one-measurement time-dependent inverse problems considered here. In [11], uniqueness on the recovery of first-order perturbations $A(x)\cdot D+q(x)$ of the bi-laplacian with Navier boundary conditions and partial data is shown; in [12] it is proved that it is possible to uniquely determine a zero-order perturbation of the poly-harmonic operator (i.e. $(\Delta u)^m + q(x)$, $m \geq 2$) from spectral observations; finally, in [1], the stationary linear isotropic elasticity equation is studied as an application of a more general setting, and the authors obtain uniqueness of the Lamé coefficients from boundary data.

The article is structured as follows: in Section 2 we will show some preliminary results of well-posedness, regularity and continuity (energy estimates) for the Reissner-Mindlin (RM) and Kirchhoff-Love (KL) models, that we will need them in order to demonstrate the main theorems. After that, in Section 3, we will prove the main

Carleman inequalities and stability results related to RM and KL models.

2. Preliminaries: well-posedness, regularity and energy estimates

Before proving the main theorems we will point out some preliminary results which are the existence, uniqeness and regularity for solutions of (1), and some energy estimates that will be useful later in the proof of Theorems 2 and 4.

2.1. Existence, uniqueness and regularity for solutions of the RM-system

The direct problem related to (1) is well posed as we will see in the followings theorems. It is necessary to assume some conditions on μ and λ, namely

$$\mu \geq \mu_0, \quad \lambda \geq -\lambda_0, \quad \frac{2\mu_0}{3} > \lambda_0 > 0.$$

In this section, we will not give the detailed proof of each theorem but we will provide an sketch of the main ideas for proving them. We refer to [15] to get an idea of the proofs where the author states similar results in the case of constant Lamé coefficients.

For $F(x,t) = (f,g) \in L^2(0,T;(L^2(\Omega))^3)$, $k_0 = (\theta_0^1, \theta_0^2, w_0) \in (H_0^1(\Omega))^3$, $k_1 = (\theta_1^1, \theta_1^2, w_1) \in (L^2(\Omega))^3$ and $q \in L^\infty(\Omega)$, weak solutions of (1) are defined as follows.

Definition 1 *We say $u = (\theta, w) \in L^2(0,T;(H_0^1(\Omega))^3)$, with $\theta = (\theta_1, \theta_2)$ and such that $u' \in L^2(0,T;(L^2(\Omega))^3), u'' = L^2(0,T;(H^{-1}(\Omega))^3)$, is a weak solution of (1) if it satisfies*

(i) $\langle u'', v\rangle + B[u,v;t] = (F,v)$, $\forall\, v = (\varphi, z) \in (H_0^1(\Omega))^3$, *a.e. in* $[0,T]$.

(ii) $u(x,0) = k_0(x)$, $u'(x,0) = k_1(x)$,

$$B[u,v;t] := a\int [2\mu\varepsilon(\theta) : \varepsilon(\varphi) + \lambda^* div(\theta) div(\varphi)] - \int b(\nabla w - \theta)\cdot\varphi$$
$$+ \int c\nabla w \cdot \nabla z + \int div(c\theta) z + \int qwz.$$

Theorem 5 (Existence and uniqueness) *There exists a unique weak solution of (1) which belongs to $C(0,T;(H_0^1(\Omega))^3) \cap C^1(0,T;(L^2(\Omega))^3)$ and also it satisfies the estimate:*

$$\|u(t)\|_{C(0,T;(H_0^1(\Omega))^3)} + \|u'(t)\|_{C(0,T;(L^2(\Omega))^3)}$$
$$\leq C\left(\|k_1\|_{(L^2(\Omega))^3} + \|k_0\|_{(H_0^1(\Omega))^3} + \|F\|_{L^2(0,T;(L^2(\Omega))^3)}\right).$$

Theorem 6 (Regularity) *Let m be a positive integer and suppose that*

$$\begin{cases} \mu, \lambda \in C^m(\bar\Omega),\ q \in W^{m-1,\infty}(\Omega) \\ k_0 \in (H^{m+1})^3(\Omega),\ k_1 \in (H^m(\Omega))^3, \\ \frac{d^l}{dt^l} F \in L^2(0,T;(H^{m-l}(\Omega))^3)\ \ (l = 0, ..., m). \end{cases}$$

Moreover, assume the following compatibility conditions (of order m):

$$\begin{cases} k_{0,0} := k_0 \in (H_0^1(\Omega))^3, \ k_{1,1} := k_1 \in (H_0^1(\Omega))^3, \\ k_{0,2l} := \frac{d^{2l-2}}{dt^{2l-2}} F(\cdot,0) - Lk_{0,2l-2} \in (H_0^1(\Omega))^3 \ \ (if\ m = 2l), \\ k_{1,2l+1} := \frac{d^{2l-1}}{dt^{2l-1}} F(\cdot,0) - Lk_{1,2l-1} \in (H_0^1(\Omega))^3 \ \ (if\ m = 2l+1). \end{cases}$$

Then

$$\frac{d^l u}{dt^l} \in L^\infty(0,T;(H^{m+1-l}(\Omega))^3) \quad for\ l = 0, ..., m+1,$$

and u satisifies the estimate:

$$\operatorname*{ess\,sup}_{0\le t\le T} \sum_{l=0}^{m+1} \left\| \frac{d^l u}{dt^l} \right\|_{(H^{m+1-l}(\Omega))^3} \le C \left(\sum_{l=0}^{m} \left\| \frac{d^l F}{dt^l} \right\|_{L^2(0,T;(H^{m-l}(\Omega))^3)} + \|k_0\|_{(H^{m+1}(\Omega))^3} + \|k_1\|_{(H^m(\Omega))^3} \right).$$

Sketch of the Proof: The proof of the existence theorem is classical and based on the Galerkin's method which consists in building a sequence of regular approximate solutions on finite dimensional spaces such that, provided a uniform energy estimate, one obtain the desired weak solution when the sequence tends to the limit. In order to prove the above regularity in $C(0,T;(H_0^1(\Omega))^3) \cap C^1(0,T;(L^2(\Omega))^3)$ it is easy to see that these regular solutions form a Cauchy sequence on this space, therefore it converges to the same weak solution. On the other hand, the uniqueness and regularity theorems can be proved by using Gronwall's inequality as in [4] and [15]. □

2.2. First energy estimate for RM

Definition 2 *We define the energy at time t for a solution of the system (1) by*

$$E_{\theta,w}(t) := \frac{1}{2}\left[\int_\Omega (|\theta_t|^2 + 2\mu|\varepsilon(\theta)|^2 + \lambda^*|div(\theta)|^2) + \int_\Omega (|w_t|^2 + \mu|\nabla w|^2) \right]. \tag{11}$$

Due to the Gronwall's Lemma and a classical procedure involving limits of regular solutions it is easy to show the next lemma.

Lemma 1 *There exists a constant K which depends on the Lamé parameter μ, the thickness h_0 and the L^∞-norm of the potential q, such that*

$$\sqrt{E_{\theta,w}(t)} \le K \left(\sqrt{E_{\theta,w}(0)} + \int_0^T (\|f(x,t)\|_{L^2} + \|g(x,t)\|_{L^2})\, dt \right). \tag{12}$$

Sketch of the Proof: If the solution is not regular enough, one has to apply the following procedure on regular solutions which belong to finite dimensional spaces, and their limit is the unique weak solution of the problem, so at the limit one gets the energy estimate (12). However, let us suppose the weak solution is regular, thus let us multiply the first and the second equation of (1) by θ_t and w_t respectively and integrate over the entire domain. Due to integration by parts, the boundary conditions and the fact that $\sigma(\theta) : \nabla\theta_t = \sigma(\theta) : \varepsilon(\theta_t)$, we obtain

$$\frac{d}{dt}\left(E_{\theta,w}(t)\right) = \int_\Omega \mu^* h_0^2(\nabla w - \theta)\theta_t + \int_\Omega f(x,t)\theta_t + \int_\Omega g(x,t)w_t - \int_\Omega q w w_t - \int_\Omega \operatorname{div}(\mu\theta)w_t.$$

Then, taking absolute value at both sides on the above expression and by Poincare's and Korn's inequality, the time derivative of the energy can be bounded as follows

$$\left|\frac{d}{dt}E_{\xi',y'}(t)\right| \le CE_{\xi',y'}(t) + \sqrt{2}\left(\|f(x,t)\|_{L^2} + \|g(x,t)\|_{L^2}\right)\sqrt{E_{\xi',y'}(t)}, \quad \forall t \in [0,T]$$

where C depends on μ_0, h_0, $\|q\|_{L^\infty(\Omega)}$ and $\|\mu\|_{C^1(\bar{\Omega})}$. Finally, dividing by $2\sqrt{E_{\xi',y'}(t)}$ and applying the Gronwall's Lemma, we obtain the desired estimate. □

2.3. Second energy estimate for RM

Let θ and w be the same functions as in the previous section and let us define the functions $\alpha(x,t), \beta(x,t), \gamma(x,t)$ as follows

$$\alpha(x,t) := \nabla\cdot\theta(x,t), \quad \beta(x,t) := \nabla\wedge\theta(x,t), \quad \gamma(x,t) := \frac{\partial w}{\partial t}. \tag{13}$$

Definition 3 *We define the energy of (α,β,γ) at time t by*

$$E_{\alpha,\beta,\gamma}(t) := \frac{1}{2}\left[\int_\Omega \left(|\alpha_t|^2 + (\lambda^* + 2\mu)|\nabla\alpha|^2 + |\beta_t|^2 + \mu|\nabla\beta|^2 + |\gamma_t|^2 + \mu|\nabla\gamma|^2\right)\right]. \tag{14}$$

As in the previous lemma, by the Gronwall's inequality we have that:

Lemma 2 *There exists a constant $K > 0$ which depends on T, $\|\mu\|_{C^2(\bar{\Omega})}$, $\|\lambda\|_{C^2(\bar{\Omega})}$, the thickness h_0 and $\|q\|_{L^\infty(\Omega)}$, such that*

$$E_{\alpha,\beta,\gamma}(t) \le K\Bigg(E_{\alpha,\beta,\gamma}(0) + E_{\theta,w}(0) + \int_0^T \mathcal{T}(t) + \int_0^T\left[\|g(x,t)\|^2_{L^2(\Omega)} + \|g_t\|^2_{L^2(\Omega)}\right] + \int_0^T\left[\|f\|^2_{(L^2(\Omega))^2} + \|\nabla\cdot f\|^2_{L^2(\Omega)} + \|\nabla\wedge f\|^2_{L^2(\Omega)}\right]\Bigg)$$

where

$$\mathcal{T}(t) = \|\nabla\cdot\xi'\|^2_{L^2(\Gamma)} + \|\nabla\cdot\xi'_t\|^2_{L^2(\Gamma)} + \left\|\frac{\partial(\nabla\cdot\xi')}{\partial n}\right\|^2_{L^2(\Gamma)} + \|\nabla\wedge\xi'\|^2_{L^2(\Gamma)} + \|\nabla\wedge\xi'_t\|^2_{L^2(\Gamma)} + \left\|\frac{\partial(\nabla\wedge\xi')}{\partial n}\right\|^2_{L^2(\Gamma)}$$

and $E_{\theta,w}(t)$ is the energy of (θ,w) defined in (11).

Sketch of the Proof: Here we apply the same idea as in the Lemma 1. In order to get the energy estimate we use Gronwall's lemma, therefore we start by taking divergence on the first equation of (1) and multiplying by α_t, then taking curl on the same equation and multiplying by β_t, and finally differentiating on time the second equation of (1) and multiplying by γ_t. Adding up the three equalities we get the time derivative of the energy (14) at the left hand side and we can estimate the right hand side as follow

$$\begin{aligned}\left|\frac{d}{dt}\left(E_{\alpha,\beta,\gamma}(t)\right)\right| &\leq K_1 E_{\alpha,\beta,\gamma}(t) + K_2 E_{\theta,w}(t)\\ &+ K_3(\|\nabla\cdot f\|^2_{L^2(\Omega)} + \|\nabla\wedge f\|^2_{L^2(\Omega)} + \|g(x,t)\|^2_{L^2(\Omega)} + \|g'(x,t)\|^2_{L^2(\Omega)})\\ &+ K_4\left(\|\nabla\cdot\theta\|^2_{L^2(\Gamma)} + \|\nabla\cdot\theta_t\|^2_{L^2(\Gamma)} + \left\|\frac{\partial(\nabla\cdot\theta)}{\partial n}\right\|^2_{L^2(\Gamma)}\right.\\ &\left. + \|\nabla\wedge\theta\|^2_{L^2(\Gamma)} + \|\nabla\wedge\theta_t\|^2_{L^2(\Gamma)} + \left\|\frac{\partial(\nabla\wedge\theta)}{\partial n}\right\|^2_{L^2(\Gamma)}\right),\end{aligned}$$

where we employed Poincare's inequalities, trace inequalities, L^2-interpolation inequalities over the H^1-seminorm and the L^2-norm of the trace, and the first energy estimate (12). To conclude, we apply Gronwall's lemma. □

2.4. Energy estimate for KL

Let us call $E(t)$ the energy of solutions to (8) at time t which is defined by

Definition 4

$$E(t) := \frac{1}{2}\int_\Omega \left(|(-\kappa_0\Delta w + w)_t|^2 + |-\kappa_0\Delta w + w|^2 + |\Delta w|^2 + |\nabla w|^2 + |w|^2\right).$$

Lemma 3 *For every solution w of (8) there exists a constant $K > 0$ depending on κ_0, T and $\|q\|_{L^\infty(\Omega)}$, such that its energy satisfies*

$$E(t) \leq K(E(0) + \|g\|^2_{L^2(0,T;L^2(\Omega))}).$$

Sketch of the Proof: Again, we assume regular solutions since it is possible to approximate every solution of (8) by smooth functions which satisfy a similar problem in a finite dimensional space.

Noting that we can write (8) as a system of two second-order equations,

$$\begin{cases} -\kappa_0\Delta w + w = \eta & \text{on } \Omega,\ \forall\, 0\leq t\leq T,\\ \eta_{tt} - \kappa_0^{-1}\Delta\eta = g - \kappa_0^{-2}w + \kappa_0^{-2}\eta & \text{on } \Omega\times(-T,T),\end{cases}$$

after multiplying both by η_t, integrating over Ω and using integration by parts we get

$$\frac{d}{dt}E(t) \leq C(E(t) + \|g(x,t)\|_{L^2(\Omega)}\sqrt{E(t)}),$$

where we estimate the right hand side by applying Poincare's inequality. Finally, the Gronwall's lemma allow us to conclude. □

2.5. Carleman estimate for waves with non zero Dirichlet boundary values

Lemma 4 *Let us supose that $\gamma \in C^2(\bar{\Omega})$ and it satisfies*

$$\gamma(x) \geq \gamma_0 > 0, \quad \forall\, x \in \bar{\Omega},$$

$$\exists \theta_0 \in (0,1) \text{ such that } \quad -1+\theta_0 \leq \frac{\nabla\gamma\cdot(x-x_0)}{2\gamma} < 1-\theta_0,\ \forall x \in \bar{\Omega}.$$

Choosing $\beta \in (0,1)$ such that

$$0 < \beta < \theta_0\gamma_0,$$

then for all $M > 0$ there exist $r_0 > 0$, $s_0 > 0$ and a positive constant $C = C(r_0, s_0, \Omega, \beta, x_0, M)$ such that, for every $p \in L^\infty(\Omega)$ with $\|p\|_{L^\infty(\Omega)} \leq M$, for every $r \geq r_0$, $s \geq s_0$ and $v \in L^2(-T,T;H^2(\Omega))$, $v_t \in L^2(-T,T;H^1(\Omega))$, $L_0 v \in L^2(-T,T;L^2(\Omega))$, $v(\pm T) = v_t(\pm T) = 0$,

$$\begin{aligned} & sr\int_{-T}^{T}\int_\Omega \varphi e^{2s\varphi}|v_t|^2 + sr\int_{-T}^{T}\int_\Omega \varphi e^{2s\varphi}|\nabla v|^2 + s^3r^3\int_{-T}^{T}\int_\Omega \varphi^3 e^{2s\varphi}|v|^2 \\ & + \int_{-T}^{T}\int_\Omega |P_1 w|^2 + \int_{-T}^{T}\int_\Omega |P_2 w|^2 \quad \leq \quad C\int_{-T}^{T}\int_\Omega e^{2s\varphi}|v_{tt} - \nabla\cdot(\gamma\nabla v) + pv|^2 \\ & \qquad + Cs^3r^3\int_{-T}^{T}\int_\Gamma \varphi^3 e^{2s\varphi}|v|^2 + Csr\int_{-T}^{T}\int_\Gamma \varphi e^{2s\varphi}|v_t|^2 \\ & \qquad + Csr\int_{-T}^{T}\int_\Gamma \varphi e^{2s\varphi}\left(\left|\frac{\partial v}{\partial n}\right|^2 + \left|\frac{\partial v}{\partial \tau}\right|^2\right). \end{aligned} \tag{15}$$

where the weights functions φ and ψ were defined in (4) and (5), $w(x,t) := e^{s\varphi}v$ and

$$\begin{aligned} P_1(w) &:= w_{tt} - \gamma\Delta w + s^2r^2\varphi^2 w(|\psi_t|^2 - \gamma|\nabla\psi|^2) - 2\nabla\gamma\cdot\nabla w \\ P_2(w) &:= (M_1-1)sr\varphi w(\psi_{tt} - \gamma\Delta\psi) - sr^2\varphi w(|\psi_t|^2 - \gamma|\nabla\psi|^2) \\ &\quad - 2sr\varphi(\psi_t w_t - \gamma\nabla\psi\cdot\nabla w) + sr\varphi w\nabla\gamma\cdot\nabla\psi. \end{aligned}$$

Remark 2 *If in addition we assume that $v = 0$ on Γ then the boundary terms reduce to one integral involving just the normal derivative and we get the usual Carleman estimate for the wave equation with boundary observations. With the previous assumption we just need $v \in L^2(-T,T;L^2(\Omega))$ thanks to hidden regularity.*

3. Proof of the main theorems

3.1. Carleman estimate for RM

In this section, we prove the global Carleman estimate (7) of Theorem 1. As we pointed out in the Introduction, this proof is based on the Carleman inequality for the acoustic wave operator, since the RM-system can be rewritten as a weakly coupled system of six wave equations.

The first step will be to split $\mathcal{L}^f$ into four wave operators so we would be able to apply known Carleman estimates.

1.- Let us remind that $\sigma(\theta) = \mu(\nabla\theta + \nabla\theta^{\mathrm{T}}) + \lambda^*(\nabla\cdot\theta)I$, so if we take divergence we have

$$\nabla\cdot\sigma(\theta) = \mu\Delta\theta + (\mu+\lambda^*)\nabla(\nabla\cdot\theta) + (\nabla\cdot\theta)\nabla\lambda^* + (\nabla\theta + \nabla\theta^{\mathrm{T}})\nabla\mu$$

and replacing the above expression in (1) we directly obtain a wave equation with two components, for θ_1 and θ_2, and also two more equations for $\nabla\cdot\theta$ and $\nabla\wedge\theta$ by taking divergence and curl on it. In sum, the above equations can be written as follows

$$\partial_t^2\nu_i - \gamma_i\Delta\nu_i = F_i + A_i, \quad \text{for } i = 1,2,3,4$$

where

$$\begin{array}{llll} \nu_1 = \theta^1, & \nu_2 = \theta^2, & \nu_3 = \nabla\cdot\theta, & \nu_4 = \nabla\wedge\theta \\ \gamma_1 = \mu(x), & \gamma_2 = \mu(x), & \gamma_3 = \lambda^*(x) + 2\mu(x), & \gamma_4 = \mu(x) \\ F_1 = (\mathcal{L}^f)^1(x,t), & F_2 = (\mathcal{L}^f)^2(x,t), & F_3 = \nabla\cdot\mathcal{L}^f(x,t), & F_4 = \nabla\wedge\mathcal{L}^f(x,t) \end{array}$$

and

$$\begin{array}{rcl} A_1 &=& \mu^*h^{-2}((\nabla w)^1 - \theta^1) + (\mu+\lambda^*)\nabla(\nabla\cdot\theta)^1 + (\nabla\cdot\theta)(\nabla\lambda^*)^1 + [(\nabla\theta + \nabla\theta^{\mathrm{T}})\nabla\mu]^1 \\ A_2 &=& \mu^*h^{-2}((\nabla w)^2 - \theta^2) + (\mu+\lambda^*)\nabla(\nabla\cdot\theta)^2 + (\nabla\cdot\theta)(\nabla\lambda^*)^2 + [(\nabla\theta + \nabla\theta^{\mathrm{T}})\nabla\mu]^2 \\ A_3 &=& \nabla\cdot(\mu^*h^{-2}(\nabla w - \theta)) + 2\nabla(\lambda^* + 2\mu)\nabla(\nabla\cdot\theta) - 2\nabla\mu\cdot(\nabla\wedge(\nabla\wedge\theta)) \\ && +\Delta\lambda^*(\nabla\cdot\theta) + (\nabla\theta + \nabla\theta^{\mathrm{T}}) : \nabla^2\mu \\ A_4 &=& \nabla\wedge(\mu^*h^{-2}(\nabla w - \theta)) + 2\nabla\mu\wedge\nabla(\nabla\cdot\theta) - \nabla\mu\wedge(\nabla\wedge(\nabla\wedge\theta)) \\ && +\nabla\mu\cdot\nabla(\nabla\wedge\theta) + (\nabla\theta + \nabla\theta^{\mathrm{T}})\wedge\nabla^2\mu. \end{array}$$

Assuming enough regularity on $\{\nu_i\}_{i=1}^4$ (as in (2) or (3)) and since every $\nu_i(\pm T) = 0$ in Ω and every p_i satisfies

$$\gamma_i(x) > \vartheta, \quad -1 + \theta_0 < \frac{\nabla\gamma_i(x)\times(x - x_0)}{2\gamma_i(x)} < 1 - \theta_0, \quad \forall x \in \Omega,$$

for some positive constants ϑ ans θ_0, thus the hypothesis of the wave Carleman estimate are satisfied. However, only $\nu_1 = \theta^1$ and $\nu_2 = \theta^2$ vanish at Γ, this implies that ν_1, ν_2 satisfy the usual Carleman estimate for the wave equation with homogeneous Dirichlet boundary conditions (see [14]), but ν_3 and ν_4 satisfy the inequality of the Lemma 4 because they are not null at the boundary. The inequalities are the followings: there exist positive constants C_1, C_2, C_3, C_4 such that, for $i = 1, 2$,

$$sr\int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi\left(\left|\frac{\partial\nu_i}{\partial t}\right|^2 + |\nabla\nu_i|^2\right) + s^3r^3\int_{-T}^{T}\int_\Omega e^{2s\varphi}|\nu_i|^2\varphi^3 + \int_{-T}^{T}\int_\Omega(|P_i^1\zeta_i|^2 + |P_i^2\zeta_i|^2) \tag{16}$$
$$\leq C_i\int_{-T}^{T}\int_\Omega e^{2s\varphi}(|F_i|^2 + |A_i|^2) + C_i sr\int_{-T}^{T}\int_\Gamma e^{2s\varphi}\varphi\left|\frac{\partial\nu_i}{\partial n}\right|^2(x - x_0)\cdot n$$

and for $i = 3, 4$,

$$sr\int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi\left(\left|\frac{\partial\nu_i}{\partial t}\right|^2 + |\nabla\nu_i|^2\right) + s^3r^3\int_{-T}^{T}\int_\Omega e^{2s\varphi}|\nu_i|^2\varphi^3 + \int_{-T}^{T}\int_\Omega(|P_i^1\zeta_i|^2 + |P_i^2\zeta_i|^2)$$

$$\leq C_i \left(sr \int_{-T}^{T}\int_{\Gamma} e^{2s\varphi}\varphi \left(|\partial_t \nu_i|^2 + \left|\frac{\partial \nu_i}{\partial n}\right|^2 + \left|\frac{\partial \nu_i}{\partial \tau}\right|^2 \right) + s^3 r^3 \int_{-T}^{T}\int_{\Gamma} e^{2s\varphi}\varphi^3 |\nu_i|^2 \right) \quad (17)$$
$$+ C_i \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}(|F_i|^2 + |A_i|^2).$$

The terms P_i^j with $j \in \{1,2\}$, $i \in \{1,2,3,4\}$, come from the decomposition of the weighted hyperbolic operators corresponding to the four wave equations already mentioned, $e^{s\varphi}(\partial_t^2 - \nabla \cdot \gamma_i \nabla)\mathrm{e}^{-s\varphi}\nu_i$, and they are defined as follows

$$\left.\begin{aligned} P_i^1\zeta &= \partial_t^2\zeta - \gamma_i\Delta\zeta + s^2r^2\varphi^2\zeta\left(\left|\frac{\partial\psi}{\partial t}\right|^2 - \gamma_i|\nabla\psi|^2\right) - \nabla\gamma_i\nabla\zeta \\ P_i^2\zeta &= (M_i - 1)sr\varphi\zeta\left(\frac{\partial^2\psi}{\partial t^2} - \gamma_i\Delta\psi\right) - sr^2\varphi\zeta\left(\left|\frac{\partial\psi}{\partial t}\right|^2 - \gamma_i|\nabla\psi|^2\right) \\ &\quad -2sr\varphi\left(\frac{\partial\psi}{\partial t}\frac{\partial\zeta}{\partial t} - \gamma_i\nabla\psi\cdot\nabla\zeta\right) + sr\varphi\zeta\nabla\gamma_i\nabla\psi, \end{aligned}\right\} \quad (18)$$

with appropriate constants $M_i > 0$, and

$$\zeta_i = e^{s\varphi}\nu_i, \quad \text{for all } i = 1,2,3,4. \quad (19)$$

The sum of the inequalities (16) and (17) gives us

$$\begin{aligned} & sr\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi\left(\left|\frac{\partial\theta}{\partial t}\right|^2 + |\nabla\theta|^2 + \left|\frac{\partial(\nabla\cdot\theta)}{\partial t}\right|^2 + |\nabla(\nabla\cdot\theta)|^2 + \left|\frac{\partial(\nabla\wedge\theta)}{\partial t}\right|^2 + |\nabla(\nabla\wedge\theta)|^2\right) \\ & + s^3r^3\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi^3(|\theta|^2 + |\nabla\cdot\theta|^2 + |\nabla\wedge\theta|^2) + \sum_{i=1}^{4}\int_{-T}^{T}\int_{\Omega}(|P_i^1\zeta_i|^2 + |P_i^2\zeta_i|^2) \\ & \quad \leq C_\theta\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}(|\mathcal{L}^f|^2 + |\nabla\cdot\mathcal{L}^f|^2 + |\nabla\wedge\mathcal{L}^f|^2) + C_\theta\sum_{i=1}^{4}\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|A_i|^2 \quad (20) \\ & \quad + C_\theta sr\int_{-T}^{T}\int_{\Gamma} e^{2s\varphi}\varphi\left(\left|\frac{\partial\theta^1}{\partial n}\right|^2 + \left|\frac{\partial\theta^2}{\partial n}\right|^2\right)(x - x_0)\cdot n \\ & \quad + C_\theta sr\int_{-T}^{T}\int_{\Gamma} e^{2s\varphi}\varphi\left(|\partial_t(\nabla\cdot\theta)|^2 + \left|\frac{\partial(\nabla\cdot\theta)}{\partial n}\right|^2 + \left|\frac{\partial(\nabla\cdot\theta)}{\partial\tau}\right|^2\right. \\ & \quad \left. + |\partial_t(\nabla\wedge\theta)|^2 + \left|\frac{\partial(\nabla\wedge\theta)}{\partial n}\right|^2 + \left|\frac{\partial(\nabla\wedge\theta)}{\partial\tau}\right|^2\right) \\ & \quad + C_\theta s^3r^3\int_{-T}^{T}\int_{\Gamma} e^{2s\varphi}\varphi^3(|\nabla\cdot\theta|^2 + |\nabla\wedge\theta|^2) \end{aligned}$$

for some constant $C_\theta > 0$. It is easy to note that the terms with θ in the summation on the right hand side of the above inequality can be absorbed by the left hand side by choosing s_0 large enough. In addition, the other terms can be bounded as follows

$$\begin{aligned} \int_{-T}^{T}\int_{\Omega} e^{2s\phi}|\mu^* h_0^{-2}\nabla w|^2 &\leq \int_{-T}^{T}\int_{\Omega}(\mu^*)^2 h_0^{-4} e^{2s\varphi}|\nabla w|^2 \\ \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\nabla\cdot(\mu^* h_0^{-2}\nabla w)|^2 &\leq (12)^2\int_{-T}^{T}\int_{\Omega} e^{2s\varphi} h_0^{-2}|\nabla\cdot(\mu\nabla w)|^2 \end{aligned}$$

$$\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\nabla\wedge(\mu^* h_0^{-2}\nabla w)|^2 \leq 2\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\nabla(\mu^* h_0^{-2})|^2|\nabla w|^2,$$

where in the last inequality we used $\nabla\wedge(\nabla w)=0$.

By the absorption of terms in (20) and the previous bounds we obtain the estimate

$$\begin{aligned}
&sr\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi\left(\left|\frac{\partial\theta}{\partial t}\right|^2+|\nabla\theta|^2+\left|\frac{\partial(\nabla\cdot\theta)}{\partial t}\right|^2+|\nabla(\nabla\cdot\theta)|^2+\left|\frac{\partial(\nabla\wedge\theta)}{\partial t}\right|^2+|\nabla(\nabla\wedge\theta)|^2\right)\\
&+s^3r^3\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi^3(|\theta|^2+|\nabla\cdot\theta|^2+|\nabla\wedge\theta|^2)+\sum_{i=1}^{4}\int_{-T}^{T}\int_{\Omega}(|P_i^1\zeta_i|^2+|P_i^2\zeta_i|^2) \qquad (21)\\
&\leq C_\theta\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}(|\mathcal{L}^f|^2+|\nabla\cdot\mathcal{L}^f|^2+|\nabla\wedge\mathcal{L}^f|^2)+C_\theta\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}(|\nabla w|^2+|\mathrm{div}(\mu\nabla w)|^2)\\
&+C_\theta sr\int_{-T}^{T}\int_{\Gamma} e^{2s\varphi}\varphi\left(\left|\frac{\partial\theta^1}{\partial n}\right|^2+\left|\frac{\partial\theta^2}{\partial n}\right|^2\right)(x-x_0)\cdot n\\
&+C_\theta sr\int_{-T}^{T}\int_{\Gamma} e^{2s\varphi}\varphi\left(|\partial_t(\nabla\cdot\theta)|^2+\left|\frac{\partial(\nabla\cdot\theta)}{\partial n}\right|^2+\left|\frac{\partial(\nabla\cdot\theta)}{\partial\tau}\right|^2\right.\\
&\left.+|\partial_t(\nabla\wedge\theta)|^2+\left|\frac{\partial(\nabla\wedge\theta)}{\partial n}\right|^2+\left|\frac{\partial(\nabla\wedge\theta)}{\partial\tau}\right|^2\right)\\
&+C_\theta s^3r^3\int_{-T}^{T}\int_{\Gamma} e^{2s\varphi}\varphi^3(|\nabla\cdot\theta|^2+|\nabla\wedge\theta|^2).
\end{aligned}$$

In the above expression, there are two terms involving w at the right hand side of the inequality, so the next step in the proof is to try to absorb them, thus we will need two more Carleman estimates which are related with w and w_t respectively.

2.- The definition of $\mathcal{L}^g$ implies that w and $z:=w_t$ satisfy

$$\begin{aligned}
&w_{tt}-\mathrm{div}(\mu\nabla w)+qw=\mathcal{L}^g-\mu\nabla\cdot\theta+\nabla\mu\cdot\theta \qquad (22)\\
&w|_{\Gamma\times(0,T)}=0,\quad w(\pm T)=0,\ \text{in }\Omega
\end{aligned}$$

$$\begin{aligned}
&z_{tt}-\mathrm{div}(\mu\nabla z)+qz=\partial_t\mathcal{L}^g-\mu\nabla\cdot\theta_t+\nabla\mu\cdot\theta_t \qquad (23)\\
&z|_{\Gamma\times(0,T)}=0,\quad z(\pm T)=0,\ \text{in }\Omega.
\end{aligned}$$

Therefore, there exists two positive constants C_w and C_{w_t} such that

$$\begin{aligned}
&sr\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\left(\left|\frac{\partial w}{\partial t}\right|^2+|\nabla w|^2\right)\varphi+s^3r^3\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi^3|w|^2+\int_{-T}^{T}\int_{\Omega}(|P_5^1\zeta_5|^2+|P_5^2\zeta_5|^2)\\
&\leq C_w\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}(|\mathcal{L}^g|^2+\|\mu\|^2_{C^0(\bar\Omega)}|\nabla\cdot\theta|^2+\|\mu\|^2_{W^{1,\infty}(\Omega)}|\theta|^2) \qquad (24)\\
&+C_w sr\int_{-T}^{T}\int_{\Gamma} e^{2s\varphi}\varphi\left|\frac{\partial w}{\partial n}\right|^2(x-x_0)\cdot n
\end{aligned}$$

and

$$sr\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\left(\left|\frac{\partial w_t}{\partial t}\right|^2+|\nabla w_t|^2\right)\varphi+s^3r^3\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi^3|w_t|^2+\int_{-T}^{T}\int_{\Omega}(|P_6^1\zeta_6|^2+|P_6^2\zeta_6|^2)$$

$$\leq C_{w_t} \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} (|\partial_t \mathcal{L}^g|^2 + \|\mu\|^2_{C^0(\bar{\Omega})} |\nabla \cdot \theta_t|^2 + \|\mu\|^2_{W^{1,\infty}(\Omega)} |\theta_t|^2) \tag{25}$$
$$+ C_{w_t} sr \int_{-T}^{T} \int_{\Gamma} e^{2s\varphi} \varphi \left| \frac{\partial w_t}{\partial n} \right|^2 (x - x_0) \cdot n$$

with

$$\zeta_5 = e^{s\varphi} w, \quad \zeta_6 = e^{s\varphi} w_t. \tag{26}$$

Analogously, the operators P_i^j, $j \in \{1,2\}$, $i \in \{5,6\}$ come from the decomposition of the hyperbolic operators (22), (23) and they are also defined as in (18).

3.- If we sum the left hand sides of the inequalities (21), (24) and (25), we can absorb the terms involving θ on (24) and (25), but we can not absorb the divergence of $\mu\nabla w$ which appears in (21). Nevertheless, by using (22) we obtain for some $C > 0$ the following bound:

$$\int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |\mathrm{div}(\mu \nabla w)|^2 \leq C \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} (|\mathcal{L}^g|^2 + \|\mu\|^2_{C^0(\bar{\Omega})} |\nabla \cdot \theta|^2 + \|\mu\|^2_{C^1(\bar{\Omega})} |\theta|^2)$$
$$+ C \|q\|_\infty \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |w|^2 + C \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |w_{tt}|^2. \tag{27}$$

Finally, if we join (21), with (24), (25) and (27) we can absorb all the non-source and non-boundary terms. Then, using the first hypothesis **H1** we obtain the desire estimate.

3.2. Stability estimate for RM

In this section we will prove the stability estimate (10) of Theorem 2 for the RM-system.

For $(\theta(p), w(p))$ and $(\theta(q), w(q))$, two solutions of (1) with potentials p and q respectively, both in $\mathcal{U}_M$, let us define

$$(\xi, y) := (\theta(q) - \theta(p), w(q) - w(p)).$$

Subtracting both systems we get that (ξ, y) satisfies

$$\begin{cases} \mathcal{L}^f(\xi, y) = 0 & \text{in } \Omega \times (0,T) \\ \mathcal{L}^g(\xi, y) = d(x) R(x,t) & \text{in } \Omega \times (0,T) \\ \xi = (0,0)^{\mathrm{T}}, \ y = 0 & \text{on } \Gamma \times (0,T) \\ \xi(0) = (0,0)^{\mathrm{T}}, \ y(0) = 0 & \text{in } \Omega \\ \xi_t(0) = (0,0)^{\mathrm{T}}, \ y_t(0) = 0 & \text{in } \Omega \end{cases}$$

with $d(x) := p(x) - q(x)$ and $R(x,t) := w(p)(x,t)$.

If we take $\xi' = \dfrac{\partial \xi}{\partial t}$, $y' = \dfrac{\partial y}{\partial t}$, $R' = \dfrac{\partial R}{\partial t}$, and if we extend these functions on $(-T, 0)$

in a odd way, then (ξ', y') satisfies

$$\begin{cases} \mathcal{L}^f(\xi', y') = 0 & \text{in } \Omega \times (-T, T) \\ \mathcal{L}^g(\xi', y') = d(x)R'(x,t) & \text{in } \Omega \times (-T, T) \\ \xi' = (0,0)^{\mathrm{T}},\ y' = 0 & \text{on } \Gamma \times (-T, T) \\ \xi'(0) = (0,0)^{\mathrm{T}},\ y'(0) = 0 & \text{in } \Omega \\ \xi'_t(0) = (0,0)^{\mathrm{T}},\ y'_t(0) = d(x)R(x,0) & \text{in } \Omega, \end{cases}$$

Remark 3 *From* **H4** *we have* $R(x,t) \in \mathcal{R}$, $\partial_t^k R'(x,t) \in L^2(0,T; H^{2-k}(\Omega))$ *with* $k = 0, 1, 2$ *and since* $R(x,t) \in C^0(0,T; W^{1,\infty}(\Omega)) \cap C^1(0,T; L^\infty(\Omega))$, *we deduce that* $R(x,0) \in W^{1,\infty}(\Omega)$, $R'(x,0) \in L^\infty(\Omega)$.

Remark 4 *By the previous remark and the Theorem 6,* $\partial_t^k(\xi', y') \in L^\infty(0,T; (H^{3-k}(\Omega))^3)$ *with* $k = 0, 1, 2, 3$. *Note that* **H6** *was used to satisfy the compatibility condition.*

For $\delta > 0$ (small), let $\eta(t) \in C_0^\infty(-T, T)$ be a cut-off function such that

$$0 \le \eta \le 1, \quad \eta(t) = 1 \ \ \forall t \in (-T + \delta, T - \delta)$$

and let us set

$$\phi = \eta \xi', \quad v = \eta y'.$$

It is easy to note that (ϕ, v) solve the following problem

$$\begin{cases} \mathcal{L}^f(\phi, v) = 2\eta_t y'_t + \eta_{tt} y' & \text{in } \Omega \times (0, T) \\ \mathcal{L}^g(\phi, v) = \eta d(x) R'(x,t) + 2\eta_t \xi'_t + \eta_{tt} \xi' & \text{in } \Omega \times (0, T) \\ \phi = (0,0)^{\mathrm{T}},\ v = 0 & \text{on } \Gamma \times (0, T) \\ \phi(0) = (0,0)^{\mathrm{T}}, v(0) = 0 & \text{in } \Omega \\ \phi_t(0) = (0,0)^{\mathrm{T}}, v_t(0) = d(x)R(x,0) & \text{in } \Omega \\ \phi(\pm T) = \phi_t(\pm T) = 0, v(\pm T) = v_t(\pm T) = v_{tt}(\pm T) = 0 & \text{in } \Omega, \end{cases} \tag{28}$$

and we are able to apply the first theorem over (ϕ, v).

By **H3** we have that $\psi(x,t) < M_0$ near $-T$ and T which means that $\varphi(x,t) < e^{rM_0}$ in the extremes of the interval $[-T, T]$. Therefore, by taking $\varepsilon > 0$ small enough we can choose a sufficiently small δ (depending on ε) such that

$$\varphi(x,t) < b - 2\varepsilon, \quad \forall\, t \in [-T, -T + \delta] \cup [T - \delta, T],$$

where we define $b := e^{rM_0}$. Moreover,

$$b \le \varphi(x, 0) \quad \text{and} \quad \varphi(x,t) < \varphi(x,0) < \tau,\ \forall\, (x,t) \in \Omega \times [-T, T],$$

with

$$\tau := \sup_{\bar{\Omega} \times [-T,T]} \varphi(x,t).$$

Lemma 5 *There exists a constant $C_{\mathcal{L}} > 0$, depending on M, T, $\|R\|_{\mathcal{R}}$ and the constants of the energy estimates, such that*

$$\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\left(\left|\mathcal{L}^f(\phi,v)\right|^2 + \left|\nabla\cdot\mathcal{L}^f(\phi,v)\right|^2 + \left|\nabla\wedge\mathcal{L}^f(\phi,v)\right|^2 + |\mathcal{L}^g(\phi,v)|^2 + \left|\frac{\partial\mathcal{L}^g}{\partial t}(\phi,v)\right|^2\right)$$
$$\leq C_{\mathcal{L}}\left(\int_{\Omega} e^{2s\varphi(x,0)}|d(x)|^2 + \int_{\Omega} e^{2s(b-2\varepsilon)}|\nabla d(x)|^2_{L^2(\Omega)} + e^{2s\tau}\int_0^T \mathcal{T}(t)dt\right). \quad (29)$$

Proof: As the derivatives of η vanishes on $[-T+\delta, T-\delta]$, it is possible to obtain the following bound

$$\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\mathcal{L}^g(\phi,v)|^2 \leq \|R\|^2_{H^1(-T,T;L^\infty(\Omega))}\int_{\Omega} e^{2s\varphi(x,0)}|d(x)|^2$$
$$+ C\left(\int_{-T}^{-T+\delta} + \int_{T-\delta}^{T}\right) e^{2sb}E_{\xi',y'}(t),$$

where $E_{\xi',y'}(t)$ is the energy defined in (11). Then, by the Lemma 1 applied to (ξ', y') it is easy to see that there exists a constant $K > 0$ depending on the norms of μ and λ, h_0 and M, such that

$$\sqrt{E_{\xi',y'}(t)} \leq K\left(\sqrt{E_{\xi',y'}(0)} + \|R\|_{H^1(0,T;L^\infty(\Omega))}\|d(x)\|_{L^2(\Omega)}\right), \quad (30)$$

and the initial energy can be bounded by

$$E_{\xi',y'}(0) = \frac{1}{2}\int_{\Omega}|y'_t(x,0)|^2 \leq \frac{1}{2}\|d(x)\|^2_{L^2}\|R(x,0)\|^2_{L^\infty}. \quad (31)$$

Hence we get the following estimate

$$\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\mathcal{L}^g(\phi,v)|^2 \leq C_g\int_{\Omega} e^{2s\varphi(x,0)}|d(x)|^2, \quad (32)$$

where the constant $C_g > 0$ depends on $\|R\|_{\mathcal{R}}$, T and the constant of the first energy estimate.

Analogously, we can also obtain an estimate for the operator $\mathcal{L}^f$,

$$\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\mathcal{L}^f(\phi,v)|^2 = \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|2\eta_t\xi'_t + \eta_{tt}\xi'|^2$$
$$\leq C\left(\int_{-T}^{-T+\delta} + \int_{T-\delta}^{T}\right) e^{2sb}E_{\xi',y'}(t)$$

then

$$\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\mathcal{L}^f(\phi,v)|^2 \leq C_f\int_{\Omega} e^{2s\varphi(x,0)}|d(x)|^2 \quad (33)$$

for some constant $C_f > 0$ which also depends on $\|R\|_{\mathcal{R}}$, T and the constant of the first energy estimate.

Let us see now how to bound the operators $\nabla\cdot\mathcal{L}^f$, $\nabla\wedge\mathcal{L}^f$ and $\partial_t\mathcal{L}^g$. Recalling the notation of the second energy estimate in (13) we write

$$\alpha=\nabla\cdot\xi',\quad \beta=\nabla\wedge\xi',\quad \gamma=\partial_t y',$$

and we proceed in the same way as in the previous estimates but now by using the Lemma 2. If we take divergence on the first equation of (28) and use an interpolation inequality given by

$$\|u\|_{L^2(\Omega)}\leq C(\|\nabla u\|_{L^2(\Omega)}+\|u\|_{L^2(\Gamma)}),\quad \forall u\in H^1(\Omega),$$

we have

$$\begin{aligned}\int_{-T}^{T}\int_\Omega e^{2s\varphi}|\nabla\cdot\mathcal{L}^f(\phi,v)|^2 &= \int_{-T}^{T}\int_\Omega e^{2s\varphi}|2\eta_t\nabla\cdot\xi'_t+\eta_{tt}\nabla\cdot\xi'|^2\\ &\leq C\left(\int_{-T}^{-T+\delta}+\int_{T-\delta}^{T}\right)e^{2s(b-2\varepsilon)}(\|\alpha_t\|^2_{L^2(\Omega)}+\|\nabla\alpha\|^2_{L^2(\Omega)}+\|\alpha\|^2_{L^2(\Gamma)}).\\ &\leq C\left(\int_{-T}^{-T+\delta}+\int_{T-\delta}^{T}\right)e^{2s(b-2\varepsilon)}(E_{\alpha,\beta,\gamma}(t)+\|\nabla\cdot\xi'\|^2_{2,\Gamma})\end{aligned}$$

Now, from the Lemma 2 we can deduce that

$$E_{\alpha,\beta,\gamma}(t)\leq K\left(E_{\xi',y'}(0)+E_{\alpha,\beta,\gamma}(0)+\|R\|^2_{H^2(0,T;L^\infty(\Omega))}\|d(x)\|^2_{L^2(\Omega)}+\int_0^T\mathcal{T}(t)dt\right),$$

with

$$\begin{aligned}\mathcal{T}(t)=&\ \|\nabla\cdot\xi'\|^2_{L^2(\Gamma)}+\|\nabla\cdot\xi'_t\|^2_{L^2(\Gamma)}+\left\|\frac{\partial(\nabla\cdot\xi')}{\partial n}\right\|^2_{L^2(\Gamma)}\\ &+\|\nabla\wedge\xi'\|^2_{L^2(\Gamma)}+\|\nabla\wedge\xi'_t\|^2_{L^2(\Gamma)}+\left\|\frac{\partial(\nabla\wedge\xi')}{\partial n}\right\|^2_{L^2(\Gamma)}.\end{aligned}$$

The first energy at $t=0$ was already bounded previously while the second energy at $t=0$ satisfies the following estimate

$$\begin{aligned}E_{\alpha,\beta,\gamma}(0)\quad &=\frac{1}{2}\int_\Omega(|\gamma_t(x,0)|^2+\mu|\nabla\gamma(x,0)|^2)\\ \leq\frac{1}{2}(\|R'(x,0)\|^2_{L^\infty(\Omega)}&+\|\mu\|_{C^0(\bar\Omega)}\|R(x,0)\|^2_{W^{1,\infty}(\Omega)})\left(\|d(x)\|^2_{L^2(\Omega)}+\|\nabla d(x)\|^2_{L^2(\Omega)}\right),\end{aligned}\tag{34}$$

thus we obtain

$$\begin{aligned}&\int_{-T}^{T}\int_\Omega e^{2s\varphi}|\nabla\cdot\mathcal{L}^f(\phi,v)|^2\\ &\quad\leq C_f^{div}\left(\int_\Omega e^{2s\varphi(x,0)}|d(x)|^2+\int_\Omega e^{2s(b-2\varepsilon)}|\nabla d(x)|^2_{L^2(\Omega)}+e^{2s\tau}\int_0^T\mathcal{T}(t)dt\right)\end{aligned}\tag{35}$$

where C_f^{det} depends on M, T, $\|R\|_{\mathcal{R}}$ and the constants of both energy estimates.

Similarly, it is possible to obtain an estimate for the curl of $\mathcal{L}^f$ as follows

$$\begin{aligned}\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\nabla\wedge\mathcal{L}^f(\phi,v)|^2 &= \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|2\eta_t\nabla\wedge\xi'_t+\eta_{tt}\nabla\wedge\xi'|^2\\ &\leq C\left(\int_{-T}^{-T+\delta}+\int_{T-\delta}^{T}\right)e^{2s(b-2\varepsilon)}(\|\beta_t\|^2_{L^2(\Omega)}+\|\nabla\beta\|^2_{L^2(\Omega)}+\|\beta\|^2_{L^2(\Gamma)})\\ &\leq C\left(\int_{-T}^{-T+\delta}+\int_{T-\delta}^{T}\right)e^{2s(b-2\varepsilon)}(E_{\alpha,\beta,\gamma}(t)+\|\nabla\wedge\xi'\|^2_{L^2(\Gamma)})\\ &\leq Ce^{2s(b-2\varepsilon)}\left(\|d(x)\|^2_{L^2(\Omega)}+\|\nabla d(x)\|^2_{L^2(\Omega)}+\int_{-T}^{T}\mathcal{T}(t)dt\right)\end{aligned}$$

then

$$\begin{aligned}&\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\nabla\wedge\mathcal{L}^f(\phi,v)|^2 \\ &\quad\leq C_f^{rot}\left(\int_{\Omega} e^{2s\varphi(x,0)}|d(x)|^2+\int_{\Omega} e^{2s(b-2\varepsilon)}|\nabla d(x)|^2+e^{2s\tau}\int_0^T\mathcal{T}(t)dt\right)\end{aligned}\tag{36}$$

with C_f^{rot} depending on the same parameters as C_f^{det}.

The last estimate we need is about $\partial_t\mathcal{L}^g$. We proceed in the same way as what we did before.

$$\begin{aligned}&\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\partial_t\mathcal{L}^g(\phi,v)|^2\\ &\quad= \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|d(x)R''(x,t)\eta+d(x)R'(x,t)\eta_t+2\eta_t y'_{tt}+3\eta_{tt}y'_t+\eta_{ttt}y'|^2\\ &\quad\leq C\left(\int_{-T}^{-T+\delta}+\int_{T-\delta}^{T}\right)e^{2s(b-2\varepsilon)}(\|\gamma_t\|^2_{L^2(\Omega)}+\|\gamma\|^2_{L^2(\Omega)}+\|y'\|^2_{L^2(\Omega)})\\ &\qquad+2C\,\|R\|^2_{H^2(0,T;L^\infty(\Omega))}\int_{\Omega} e^{2s\varphi(x,0)}|d(x)|^2\\ &\quad\leq C\left(\int_{-T}^{-T+\delta}+\int_{T-\delta}^{T}\right)e^{2s(b-2\varepsilon)}(E_{\alpha,\beta,\gamma}(t)+E_{\xi',y'}(t))\\ &\qquad+2C\,\|R\|^2_{H^2(0,T;L^\infty(\Omega))}\int_{\Omega} e^{2s\varphi(x,0)}|d(x)|^2\end{aligned}$$

By both energy estimates, (31) and (34), we get

$$\begin{aligned}&\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\partial_t\mathcal{L}^g(\phi,v)|^2\\ &\quad\leq C_g^t\left(\int_{\Omega} e^{2s\varphi(x,0)}|d(x)|^2+\int_{\Omega} e^{2s(b-2\varepsilon)}|\nabla d(x)|^2_{L^2(\Omega)}+e^{2s\tau}\int_0^T\mathcal{T}(t)dt\right)\end{aligned}\tag{37}$$

again with a constant $C_g^t>0$ depending on M, T, $\|R\|_{\mathcal{R}}$ and the constants of the energy estimates.

Adding the inequalities (32), (33), (35), (36) and (37) we deduce (29) and we conclude the proof of Lemma 5. □

Lemma 6 *Let P_i^j be the operators defined in Theorem 1 for $j = 1$ and $i = 5, 6$, let $\zeta := \zeta_5 = e^{s\varphi}v$ and $\bar{\zeta} := \zeta_6 = e^{s\varphi}v_t$, then we have the lower bounds*

$$\int_{-T}^{0}\int_{\Omega}(P_1^5\zeta)\frac{\partial\zeta}{\partial t} \geq \frac{1}{2}\int_{\Omega} e^{2s\varphi(x,0)}|d(x)|^2 a_0^2 - \frac{1}{2}Cs^2r^3\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}\varphi^2|v|^2, \tag{38}$$

$$\begin{aligned}\int_{-T}^{0}\int_{\Omega}(P_1^6\tilde{\zeta})\tilde{\zeta}_t \ \geq\ & \frac{1}{2}\int_{\Omega} e^{2s\varphi(x,0)}\left(|R'(x,0)|^2 - \|\mu\|_{C^0(\bar{\Omega})}|\nabla R(x,0)|^2\right)|d(x)|^2 \\ & + \frac{1}{4}\int_{\Omega}\mu_0 e^{2s\varphi(x,0)}a_0^2|\nabla d(x)|^2 - Cs^3r^3\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}|v_t|^2 - Csr\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}(|v_{tt}|^2 + |\nabla v_t|^2).\end{aligned} \tag{39}$$

Proof: By integration by parts in space and time, we have

$$\begin{aligned}\int_{-T}^{0}\int_{\Omega}(P_1^5\zeta)\frac{\partial\zeta}{\partial t} =\ & \int_{-T}^{0}\int_{\Omega}\left(\zeta_{tt} - \mu\Delta\zeta + s^2r^2\varphi^2(|\psi_t|^2 - \mu|\nabla\psi|^2)\zeta - \nabla\mu\cdot\nabla\zeta\right)\zeta_t \\ =\ & \frac{1}{2}\int_{\Omega}(|\zeta_t(0)|^2 - |\zeta_t(-T)|^2) + \frac{1}{2}\int_{\Omega}\mu(|\nabla\zeta(0)|^2 - |\nabla\zeta(-T)|^2) \\ & + \int_{-T}^{0}\int_{\Omega}\zeta_t\nabla\zeta\cdot\nabla\mu - \int_{-T}^{0}\int_{\Gamma}\mu\zeta_t\frac{\partial\zeta}{\partial n} + \int_{\Omega} s^2r^2(\frac{1}{2}|\zeta|^2\varphi^2(|\psi_t|^2 - \mu|\nabla\psi|^2))\Big|_{-T}^{0} \\ & - \frac{1}{2}s^2r^2\int_{-T}^{0}\int_{\Omega}|\zeta|^2\frac{\partial}{\partial t}(\varphi^2(|\psi_t|^2 - \mu|\nabla\psi|^2)) - \int_{-T}^{0}\int_{\Omega}\zeta_t\nabla\zeta\cdot\nabla\mu,\end{aligned}$$

and since

$$\zeta_t = e^{s\varphi}(v_t + sr\varphi\psi_t v), \quad \text{and} \quad \nabla\zeta = e^{s\varphi}(\nabla v + sr\varphi v\nabla\psi),$$

by evaluating at $t = 0$, $t = -T$ we have that

$$\int_{-T}^{0}\int_{\Omega}(P_1^5\zeta)\frac{\partial\zeta}{\partial t} = \frac{1}{2}\int_{\Omega} e^{2s\varphi(x,0)}|d(x)|^2|R(x,0)|^2 - \frac{1}{2}s^2r^2\int_{-T}^{0}\int_{\Omega}|\zeta|^2\frac{\partial}{\partial t}(\varphi^2(|\psi_t|^2 - |\nabla\psi|^2))$$

from where, using **H4** to bound from below, we obtain the inequality (38).

On the other hand, we can do something similar with $\tilde{\zeta} = e^{s\varphi}v_t$ and we obtain

$$\begin{aligned}\int_{-T}^{0}\int_{\Omega}(P_1^6\tilde{\zeta})\frac{\partial\tilde{\zeta}}{\partial t} =\ & \int_{-T}^{0}\int_{\Omega}\left(\tilde{\zeta}_{tt} - \mu\Delta\tilde{\zeta} + s^2r^2\varphi^2(|\psi_t|^2 - \mu|\nabla\psi|^2)\tilde{\zeta} - \nabla\mu\cdot\nabla\tilde{\zeta}\right)\tilde{\zeta}_t \\ =\ & \frac{1}{2}\int_{\Omega}(|\tilde{\zeta}_t(0)|^2 - |\tilde{\zeta}_t(-T)|^2) - \int_{-T}^{T}\int_{\Omega}\mu\Delta\tilde{\zeta}\tilde{\zeta}_t - \int_{-T}^{0}\int_{\Omega}\zeta_t\nabla\zeta\cdot\nabla\mu \\ & + (sr)^2\int_{-T}^{0}\int_{\Omega}\varphi^2(|\psi_t|^2 - \mu|\nabla\psi|^2)\tilde{\zeta}\tilde{\zeta}_t.\end{aligned}$$

Denoting the three last terms in the previous equality by I_1, I_2 and I_3 respectively, we can bound those integrals using integration by parts and the boundary conditions for

(ϕ, v).

$$
\begin{aligned}
I_1 \;=\; & -\int_{-T}^{0}\int_{\Omega} \mu\left[\nabla\cdot(e^{s\varphi}\nabla v_t) + \nabla\cdot(e^{s\varphi} sr\varphi v_t \nabla\psi)\right]\left[e^{s\varphi}(v_{tt} + sr\varphi\psi_t v_t)\right] \\
= \; & \underbrace{\int_{-T}^{0}\int_{\Omega} \mu s^{s\varphi}\nabla v_t \cdot \nabla(e^{s\varphi} v_{tt})}_{J_1} + \underbrace{\int_{-T}^{0}\int_{\Omega} \mu e^{s\varphi}\nabla v_t \cdot \nabla(e^{s\varphi} sr\varphi\psi_t v_t)}_{J_2} \\
& \underbrace{-\int_{-T}^{0}\int_{\Omega} \mu\left[sre^{s\varphi}\varphi(v_t(\Delta\psi + r(s\psi+1)|\nabla\psi|^2) + \nabla v_t\cdot\nabla\psi)\right]\left[e^{s\varphi}(v_{tt} + sr\varphi\psi_t v_t)\right]}_{J_3}
\end{aligned}
$$

Since $\nabla(e^{s\varphi} v_{tt}) = e^{s\varphi}\nabla v_{tt} + sre^{s\varphi}\varphi v_{tt}\nabla\psi$, we have

$$
\begin{aligned}
J_1 \;=\; & \frac{1}{2}\int_{-T}^{0}\int_{\Omega} \mu e^{2s\varphi}\frac{d}{dt}|\nabla v_t|^2 + sr\int_{-T}^{0}\int_{\Omega} \mu e^{2s\varphi}\varphi v_{tt}\nabla\psi\cdot\nabla v_t \\
= \; & \frac{1}{2}\int_{\Omega} \mu e^{2s\varphi}|\nabla v_t|^2\Big|_{-T}^{0} - sr\int_{-T}^{0}\int_{\Omega} \mu e^{2s\varphi}\varphi\psi_t|\nabla v_t|^2 + sr\int_{-T}^{0}\int_{\Omega} \mu e^{2s\varphi}\varphi v_{tt}\nabla\psi\cdot\nabla v_t, \\
\geq \; & \frac{1}{2}\int_{\Omega} \mu e^{2s\varphi}|\nabla v_t|^2\Big|_{-T}^{0} - Csr\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}\left(|\nabla v_t|^2 + |v_{tt}|^2\right).
\end{aligned}
$$

Moreover, using that $\nabla(sre^{s\varphi}\varphi\psi_t v_t) = sre^{s\varphi}\varphi\psi_t[rv_t(s\varphi+1)\nabla\psi + \nabla v_t]$ and the inequality

$$
(sr)^2 ab = [(sr)^{3/2}a][(sr)^{1/2}b] \leq \frac{1}{2}(sr)^3 a^2 + \frac{1}{2}(sr)b^2
$$

we get

$$
\begin{aligned}
J_2 = \; & sr\int_{-T}^{0}\int_{\Omega} \mu e^{2s\varphi}\varphi\psi_t[rv_t(s\varphi+1)\nabla\psi\cdot\nabla v_t + |\nabla v_t|^2] \\
\geq \; & -C(sr)^2\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}|v_t||\nabla\psi\cdot\nabla v_t| - Csr\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}|\nabla v_t|^2 \\
\geq \; & -C(sr)^3\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}|v_t|^2 - Csr\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}|\nabla v_t|^2,
\end{aligned}
$$

and also we obtain

$$
\begin{aligned}
J_3 = \; & -(sr)^2\int_{-T}^{0}\int_{\Omega} \mu e^{2s\varphi}\varphi^2\psi_t(\Delta\psi + r(s\varphi+1)|\nabla\psi|^2)|v_t|^2 \\
& - sr\int_{-T}^{0}\int_{\Omega} \mu e^{2s\varphi}\varphi(\Delta\psi + r(s\varphi+1)|\nabla\psi|^2)v_t v_{tt} \\
& - (sr)^2\int_{-T}^{0}\int_{\Omega} \mu e^{2s\varphi}\varphi^2\psi_t v_t(\nabla\psi\cdot\nabla v_t) \\
& - sr\int_{-T}^{0}\int_{\Omega} \mu e^{2s\varphi}\varphi v_{tt}(\nabla\psi\cdot\nabla v_t) \\
\geq \; & -C(sr)^3\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}|v_t|^2 - Csr\int_{-T}^{0}\int_{\Omega} e^{2s\varphi}(|v_{tt}|^2 + |\nabla v_t|^2).
\end{aligned}
$$

Summing up the above inequalities for J_i, $i = 1, 2, 3$, we obtain the following lower bound

$$I_1 \geq \frac{1}{2} \int_\Omega \mu e^{2s\varphi} |\nabla v_t|^2 \Big|_{-T}^{0} - C(sr)^3 \int_{-T}^{0} \int_\Omega e^{2s\varphi} |v_t|^2 - Csr \int_{-T}^{0} \int_\Omega e^{2s\varphi} \left(|\nabla v_t|^2 + |v_{tt}|^2 \right). \tag{40}$$

The terms I_2 and I_3 can be also bounded from below as follows

$$\begin{aligned} I_2 &= (sr)^2 \int_{-T}^{0} \int_\Omega e^{2s\varphi} \varphi^2 (|\psi_t|^2 + \mu |\nabla \psi|^2)(v_{tt} + sr\varphi\psi_t v_t) v_t \\ &\geq - C(sr)^3 \int_{-T}^{0} \int_\Omega e^{2s\varphi} |v_t|^2 - Csr \int_{-T}^{0} \int_\Omega e^{2s\varphi} |v_{tt}|^2, \end{aligned} \tag{41}$$

and

$$\begin{aligned} I_3 =& - \int_{-T}^{0} \int_\Omega [e^{s\varphi}(v_{tt} + sr\varphi\psi_t v_t)] \, [e^{s\varphi}(\nabla v_t + sr\varphi v_t \nabla \psi)] \cdot \nabla \mu \\ =& - (sr)^2 \int_{-T}^{0} \int_\Omega e^{2s\varphi} \varphi^2 \psi_t (\nabla \psi \cdot \nabla \mu) |v_t|^2 - \int_{-T}^{0} \int_\Omega e^{2s\varphi} v_{tt} (\nabla v_t \cdot \nabla \mu) \\ & - sr \int_{-T}^{0} \int_\Omega e^{2s\varphi} \varphi v_t (\psi_t \nabla v_t \cdot \nabla \mu + v_{tt} \nabla \psi \cdot \nabla \mu) \\ \geq& - C(sr)^2 \int_{-T}^{0} \int_\Omega e^{2s\varphi} |v_t|^2 - Csr \int_{-T}^{0} \int_\Omega e^{2s\varphi} (|v_{tt}|^2 + |\nabla v_t|^2). \end{aligned} \tag{42}$$

By using (40), (41) and (42), it is posible to obtain the next bound

$$\begin{aligned} \int_{-T}^{0} \int_\Omega (P_1^6 \tilde{\zeta}) \tilde{\zeta}_t \geq& \frac{1}{2} \int_\Omega \left(|\tilde{\zeta}_t(0)|^2 - |\tilde{\zeta}_t(-T)|^2 \right) - \frac{1}{2} \int_\Omega \mu e^{2s\varphi} \left(|\nabla v_t(0)|^2 - |\nabla v_t(-T)|^2 \right) \\ & - C(sr)^3 \int_{-T}^{0} \int_\Omega e^{2s\varphi} |v_t|^2 - Csr \int_{-T}^{0} \int_\Omega e^{2s\varphi} (|v_{tt}|^2 + |\nabla v_t|^2), \end{aligned}$$

and after evaluating at $t = 0$, $t = -T$ and since

$$\tilde{\zeta} = e^{s\varphi} v_t, \quad \tilde{\zeta}_t = e^{s\varphi}(v_{tt} + sr\varphi\psi_t v_t), \quad \nabla \tilde{\zeta} = e^{s\varphi}(\nabla v + sr\varphi v \nabla \psi),$$

therefore

$$\begin{aligned} v_t(0) &= d(x) R(x,0), & v_t(-T) &= 0 \\ \tilde{\zeta}(0) &= e^{s\varphi(x,0)} d(x) R(x,0), & \tilde{\zeta}(-T) &= 0 \\ \tilde{\zeta}_t(0) &= e^{s\varphi(x,0)} d(x) R'(x,0), & \tilde{\zeta}_t(-T) &= 0, \end{aligned}$$

from (43) we deduce (39) and we conclude the proof of Lemma 6. □

Using (38) and (39), and applying the Carleman inequality from Theorem 1 on (ϕ, v), we get

$$\frac{1}{2} \int_\Omega e^{2s\varphi(x,0)} (\sqrt{s} a_0^2 + |R'(x,0)|^2 - \|\mu\|_{C(\bar{\Omega})} |\nabla R(x,0)|^2) |d(x)|^2 + \frac{1}{4} \int_\Omega e^{2s\varphi(x,0)} a_0^2 |\nabla d(x)|^2$$

$$
\begin{aligned}
&\le \sqrt{s}\int_{-T}^{0}\int_\Omega (P_1^5\zeta)\frac{\partial \zeta}{\partial t} + Cs^{5/2}r^2\int_{-T}^{0}\int_\Omega e^{2s\varphi}\varphi^2|v|^2\\
&\quad + \int_{-T}^{0}\int_\Omega (P_1^6\tilde{\zeta})\frac{\partial \tilde{\zeta}}{\partial t} + C(sr)^3\int_{-T}^{0}\int_\Omega e^{2s\varphi}|v_t|^2 + Csr\int_{-T}^{0}\int_\Omega e^{2s\varphi}(|v_{tt}|^2+|\nabla v_t|^2)\\
&\le C\left(\int_{-T}^{T}\int_\Omega \left|P_1^5\zeta\right|^2 + \int_{-T}^{T}\int_\Omega \left|P_1^6\tilde{\zeta}\right|^2\right.\\
&\quad \left. +(sr)^3\int_{-T}^{T}\int_\Omega e^{2s\varphi}\varphi^2(|v|^2+|v_t|^2) + sr\int_{-T}^{0}\int_\Omega e^{2s\varphi}(|v_{tt}|^2+|\nabla v_t|^2)\right)\\
&\le C\int_{-T}^{T}\int_\Omega e^{2s\varphi}\left(\left|\mathcal{L}^f(\phi,v)\right|^2 + \left|\nabla\cdot\mathcal{L}^f(\phi,v)\right|^2 + \left|\nabla\wedge\mathcal{L}^f(\phi,v)\right|^2\right.\\
&\quad \left. + |\mathcal{L}^g(\phi,v)|^2 + \left|\frac{\partial\mathcal{L}^g(\phi,v)}{\partial t}\right|^2\right) + C\,\mathcal{B}_{RM}(\phi,v).
\end{aligned}
$$

In order to bound the right hand side in the previous inequality we use (29) and for the lef hand side we recall the Remark 3, then we get

$$
\begin{aligned}
&\sqrt{s}\int_\Omega e^{2s\varphi(x,0)}|d(x)|^2 + \int_\Omega e^{2sb}|\nabla d(x)|^2\\
&\qquad \le C\left[\int_\Omega e^{2s\varphi(x,0)}\,|d(x)|^2 + e^{2s\tau}\int_0^T \mathcal{T}(t)dt + \int_\Omega e^{2s(b-2\varepsilon)}|\nabla d(x)|^2 + \mathcal{B}_{RM}(\phi,v)\right],
\end{aligned}
$$

for an apropiated constant $C > 0$.

Finally, we need to write the integrals over Γ in terms of ξ' and y', thus there exists $C(r^3) > 0$ such that

$$
\begin{aligned}
&\sqrt{s}\int_\Omega e^{2sb}|d(x)|^2 + \int_\Omega e^{2sb}(1-e^{-4s\varepsilon})|\nabla d(x)|^2\\
&\qquad \le + Ce^{5s\tau}\left[\int_{-T}^{T}\int_{\Gamma_0}\left(|(\nabla\xi').n|^2 + \left|\frac{\partial y_t'}{\partial n}\right|^2 + \left|\frac{\partial y'}{\partial n}\right|^2\right)(x-x_0)\cdot n\right.\\
&\qquad\quad + \int_{-T}^{T}\int_\Gamma\left(\left|\frac{\partial(\nabla\cdot\xi')}{\partial t}\right|^2 + \left|\frac{\partial(\nabla\cdot\xi')}{\partial n}\right|^2 + \left|\frac{\partial(\nabla\cdot\xi')}{\partial \tau}\right|^2\right)\\
&\qquad\quad + \int_{-T}^{T}\int_\Gamma\left(\left|\frac{\partial(\nabla\wedge\xi')}{\partial t}\right|^2 + \left|\frac{\partial(\nabla\wedge\xi')}{\partial n}\right|^2 + \left|\frac{\partial(\nabla\wedge\xi')}{\partial \tau}\right|^2\right)\\
&\qquad\quad \left. + \int_{-T}^{T}\int_\Gamma\left(|\nabla\cdot\xi'|^2 + |\nabla\wedge\xi'|^2\right)\right],
\end{aligned}
$$

where we used that $b \le \varphi(x,0)$, $1 < s$ and $s^3e^{2s\tau} \le e^{5s\tau}$.

To conclude, we have to come back to the original functions: $d(x) = p(x) - q(x)$, $\xi' = \theta_t(q) - \theta_t(p)$ and $y' = w_t(q) - w_t(p)$.

3.3. Carleman estimate for KL

In this section we will prove the global Carleman inequality (9) of Theorem 3.

Let us note first that we just need to prove the inequality with L in the right hand side since

$$\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|Lu|^2 \leq 2\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|L_q u|^2 + 2\|q\|^2_{L^\infty(\Omega)}\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|u|^2,$$

and we can obsorb the last term by choosing $s_0 > 0$ large enough.
As we did in the proof of the energy estimate for the Kirchhoff-Love equation, it is possible to split the operator L into an ellyptic and a wave operator, therefore we get the following system:

$$\begin{cases} L_1 u := -\kappa_0\Delta u + u = v & \text{in } \Omega,\ \forall\ -T \leq t \leq T, \\ L_2 v := v_{tt} - \kappa_0^{-1}\Delta v - \kappa_0^{-2} v = g - \kappa_0^{-2} u & \text{in } \Omega \times (-T,T), \end{cases}$$

where the three operators are related under the following equality

$$L_2(L_1 u) = Lu - \kappa_0^{-1} u.$$

Since $u \in H^1(-T,T;H^3(\Omega))$ then $L_1 u = v \in H^1(-T,T;H_0^1(\Omega))$. Also, L_1 satisfies a Carleman estimate which can be probed similarly as in the wave operator case, thus we have that for $\Gamma_0 \supset \Gamma_{x_0}$ there exists $r_0 > 0$, $s_0 > 0$ and a positive constant $C = C(r_0, s_0, \Omega, \beta, x_0)$ such that, for every $r \geq r_0$ and $s \geq s_0$:

$$sr\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi|\nabla u|^2 + s^3r^3\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi^3|u|^2 \leq C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|L_1 u|^2 + Csr\int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi}\left|\frac{\partial u}{\partial n}\right|^2. \tag{43}$$

In addition, we have the same inequality if we differentiate on time,

$$sr\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi|\nabla u_t|^2 + s^3r^3\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi^3|u_t|^2 \leq C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|L_1 u_t|^2 + Csr\int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi}\left|\frac{\partial u_t}{\partial n}\right|^2, \tag{44}$$

where the boundary terms are well defined since $u, u_t \in L^2(-T,T;H^3(\Omega))$.

Furthermore, $L_2 v \in L^2(-T,T;L^2(\Omega))$, $v = 0$ on $\Gamma\times(-T,T)$ and $v(\pm T) = v_t(\pm T) = 0$ over Ω. From the Lemma 4 and its remark, $\forall M > 0$ there exists a positive constant $\tilde{C} = \tilde{C}(s_0, r_0, \Omega, \beta, x_0, M)$ such that for every $r \geq r_0$ and $s \geq s_0$ we have:

$$sr\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi(|v_t|^2 + |\nabla v|^2) + s^3r^3\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\varphi^3|v|^2 + \int_{-T}^{T}\int_{\Omega}|P_1\zeta|^2 + \int_{-T}^{T}\int_{\Omega}|P_2\zeta|^2 \leq \tilde{C}\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|L_2 v|^2 + \tilde{C}sr\int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi}\left|\frac{\partial v}{\partial n}\right|^2. \tag{45}$$

The boundary integral has sense due to the hidden regularity which states that $\frac{\partial v}{\partial n} \in L^2(-T,T;L^2(\Gamma))$ (Teorema 1.1.4 en [14]).

Multipliying (43) by s^3r^3 and adding that estimate with (44) and (45) we can deduce the Carleman estimate. We also use the following inequalities in order to bound from below

$$|\nabla\Delta u|^2 \leq 2|\nabla v|^2 + 2|\nabla u|^2, \quad |\Delta u|^2 \leq 2|v|^2 + 2|u|^2, \quad |\Delta u_t|^2 \leq 2|v_t|^2 + 2|u_t|^2,$$

and let us note that

$$\left\|\frac{\partial \Delta u}{\partial n}\right\|^2_{L^2(-T,T;L^2(\Gamma))} \leq \left\|\frac{\partial v}{\partial n}\right\|^2_{L^2(-T,T;L^2(\Gamma))} + \left\|\frac{\partial u}{\partial n}\right\|^2_{L^2(-T,T;L^2(\Gamma))}.$$

Therefore, all the boundary integrals in (3) are well defined thank to the regularity of u and the hidden regularity of v.

3.4. Stability estimate for KL

In this section we will prove the stability inequality (10) of Theorem 4. We will apply again the Bukhgeim-Klibanov method which works simpler here than for the proof of Theorem 2. Indeed, the proof is quite similar to the one for the wave equation. Nevertheless, for the sake of completeness, we will give an sketch of the proof.

Let us write

$$y := u(q) - u(p), \quad d(x) := p(x) - q(x),$$

and

$$y' = \frac{\partial y}{\partial t}, \;\; R' = \frac{\partial R}{\partial t}$$

where we extend these last two functions to $(-T, 0)$ in a odd way. It is easy to see that y' satisfies

$$\begin{cases} L_q y' = d(x)R'(x,t) & \text{en } \Omega \times (-T,T) \\ y' = 0,\; \Delta y' = 0 & \text{en } \Gamma \times (-T,T) \\ y'(0) = 0,\; (-\kappa_0 \Delta y' + y')_t(0) = d(x)R(x,0) & \text{en } \Omega. \end{cases}$$

Now, for $\delta > 0$ small enough and a cut-off function $\eta \in C_0^\infty(-T,T)$ such that

$$0 \leq \eta \leq 1, \quad \eta(t) = 1 \; \forall t \in (-T+\delta, T-\delta)$$

we define

$$z(x,t) := \eta(t) y'(x,t),$$

which solves

$$\begin{cases} L_q z = \eta(t)d(x)R'(x,t) + 2\eta_t(y'_t - \kappa_0\Delta y'_t) + \eta_{tt}(y' - \kappa_0 \Delta y') & \text{in } \Omega \times (0,T) \\ z = 0,\; \Delta z = 0 & \text{on } \Gamma \times (0,T) \\ z(0) = 0,\; (-\kappa_0\Delta z + z)_t(0) = d(x)R(x,0) & \text{in } \Omega \\ (-\kappa_0 \Delta z + z)(\pm T) = (-\kappa_0\Delta z + z)_t(\pm T) = 0 & \text{in } \Omega. \end{cases}$$

Due to the regularity that we assume for $u(p)$, $u(q)$ and since $d(x)R'(x,t)$ belongs to $L^2(-T,T;L^\infty(\Omega))$, then $z \in H^1(-T,T;H^3(\Omega))$, $Lz \in L^2(-T,T;L^2(\Omega))$ and we can apply the Theorem 3 over z.

Since

$$\frac{\rho^2}{T^2} < \beta < 1,$$

choosing δ small enough, we have

$$\psi(x,t) < M_0 < \psi(x,0), \quad \forall\, t \in [-T+\delta, -T] \cup [T-\delta, T],$$

and we obtain the following inequality

$$\begin{aligned} \int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|L_q z|^2 &\leq C\|R\|_{L^2(0,T;L^\infty(\Omega))}\int_{-T}^{T}\int_{\Omega} e^{2s\varphi(0)}|d(x)|^2 \\ &\quad + C\left(\int_{-T}^{-T+\delta} + \int_{T-\delta}^{T}\right)\int_{\Omega} e^{2sM_0}E(t) \\ &\leq C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi(0)}|d(x)|^2, \end{aligned} \tag{46}$$

where we used the energy estimate from the Lemma (3) with $E(t)$ the energy of y'.

On the other hand, writing $\zeta = e^{s\varphi}(-\kappa_0\Delta z + z)$ and by integration by parts we obtain that

$$\int_{-T}^{0}\int_{\Omega}(P_1\zeta)\zeta_t \geq \frac{1}{2}\int_{\Omega}|\zeta_t(0)|^2 - Cs^2r^3\int_{-T}^{T}\int_{\Omega}\varphi^3|\zeta|^2,$$

and evaluating at $t=0$ we deduce

$$\sqrt{s}\int_{-T}^{0}\int_{\Omega}(P_1\zeta)\zeta_t + s^3r^3\int_{-T}^{0}\int_{\Omega}\varphi^3|\zeta|^2 \geq C\sqrt{s}\int_{\Omega} e^{2s\varphi(0)}|d(x)|^2 a_0^2. \tag{47}$$

Finally, combining (46) and (47) plus the Carleman estimate applied to z, we have

$$\begin{aligned} \sqrt{s}\int_{\Omega} e^{2s\varphi(0)}|d(x)|^2 &\leq \sqrt{s}\int_{-T}^{0}\int_{\Omega}(P_1\zeta)\zeta_t + s^3r^3\int_{-T}^{0}\int_{\Omega}\varphi^3|\zeta|^2 \\ &\leq C\,\mathcal{I}_{KL}(z) \\ &\leq Cs^4r^4\int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi}\left(\left|\frac{\partial z_t}{\partial n}\right|^2 + \left|\frac{\partial \Delta z}{\partial n}\right|^2 + \left|\frac{\partial z}{\partial n}\right|^2\right) \\ &\quad + C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi(0)}|d(x)|^2. \end{aligned}$$

Then, we conclude by writing the above inequality in terms of the original functions $u(q)$ and $u(p)$.

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References

[1] Albin P, Guillarmou C, Tzou L and Uhlmann G 2012 Inverse boundary problems for systems in two dimensions *Annales IHP* to appear

[2] Bukhgeim A L and Klibanov M V 1981 Global uniqueness of a class of multi-dimensional inverse problems *Soviet Math. Dokl.* **24** 244–247

[3] Contreras-Reyes E and Osses A 2010 Lithospheric flexure modeling seaward of the chile trench: implications for oceanic plate weakening in the trench outer rise region *Geophysical Journal International* **182**(1), 97–112

[4] Evans L C 1998 *Partial Differential Equations* Graduate Studies in Mathematics vol 19 (Providence: American Mathematical Society) p 662

[5] Fu X, Zhang X and Zuazua E 2006 On the optimality of some observability inequalities for plate systems with potentials In *Phase space analysis of partial differential equations Progr. Nonlinear Differential Equations Appl.* vol 69 (Boston: Birkhäuser Boston MA) 117–132

[6] Fursikov A V and Imanuvilov O Y 1996 *Controllability of evolution equations* (Seoul: Seoul National University)

[7] Imanuvilov O Y 2002 On Carleman estimates for hyperbolic equations *Asymptot. Anal.* **32**(3-4) 185–220

[8] Imanuvilov O Y and Yamamoto M 2001 Global uniqueness and stability in determining coefficients of wave equations *Comm. Partial Differential Equations* **26**(7-8) 1409–1425

[9] Imanuvilov O Y and Yamamoto M 2003 Determination of a coefficient in an acoustic equation with a single measurement *Inverse Problems* **19**(1) 157–171

[10] Imanuvilov O Y and Yamamoto M 2005 Carleman estimates for the non-stationary Lamé system and the application to an inverse problem *ESAIM Control Optim. Calc. Var.* **11**(1) 1–56

[11] Krupchyk K, Lassas M and Uhlmann G 2012 Determining a first order perturbation of the biharmonic operator by partial boundary measurements *J. Funct. Anal.* **262**(4) 1781–1801

[12] Krupchyk K and Päivärinta L 2012 A Borg-Levinson theorem for higher order elliptic operators *Int. Math. Res. Not. IMRN*, **6** 1321–1351

[13] Lagnese J E 1989 *Boundary Stabilization of Thin Plates* SIAM studies in applied mathematics vol 10 (Philadelphia: SIAM) p 176

[14] Puel J P 2004 Global carleman inequalities for the wave equations and applications to controllability and inverse problems *Preprint course in Control of Solids and Structures: Mathematical Modeling and Engineering Applications, Udine (Italy), June 2004*

[15] Wu S R 2004 Reissner-Mindlin plate theory for elastodynamics *J. Appl. Math.*, **3** 179–189

[16] Zhang X and Zuazua E 2006 A sharp observability inequality for Kirchhoff plate systems with potentials. *Comput. Appl. Math.* **25**(2-3) 353–373

[17] Zhang X and Zuazua E 2008 On the optimality of the observability inequalities for Kirchhoff plate systems with potentials in unbounded domains In *Hyperbolic problems: theory, numerics, applications* 233–243 (Berlin: Springer)