IOP PUBLISHING INVERSE PROBLEMS

Inverse Problems **24** (2008) 015017 (18pp) doi:10.1088/0266-5611/24/1/015017

Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights

Alberto Mercado[1], **Axel Osses**[1] **and Lionel Rosier**[1,2,3]

[1] Centro de Modelamiento Matemático (CMM) and Departamento de Ingeniería Matemática, Universidad de Chile (UMI CNRS 2807), Avenida Blanco Encalada 2120, Casilla 170-3, Correo 3, Santiago, Chile
[2] Institut Élie Cartan, UMR 7502 UHP/CNRS/INRIA, BP 239, 54506 Vandœuvre-lès-Nancy Cedex, France

E-mail: amercado@dim.uchile.cl, axosses@dim.uchile.cl and rosier@iecn.u-nancy.fr

Received 10 August 2007, in final form 21 November 2007
Published 16 January 2008
Online at stacks.iop.org/IP/24/015017

Abstract
Baudouin and Puel (2002 *Inverse Problems* **18** 1537–54), investigated some inverse problems for the evolution Schrödinger equation by means of Carleman inequalities proved under a strict pseudoconvexity condition. We show here that new Carleman inequalities for the Schrödinger equation may be derived under a relaxed pseudoconvexity condition, which allows us to use degenerate weights with a spatial dependence of the type $\psi(x) = x \cdot e$, where e is some fixed direction in $\mathbb{R}^N$. As a result, less restrictive boundary or internal observations are allowed to obtain the stability for the inverse problem consisting in retrieving a stationary potential in the Schrödinger equation from a single boundary or internal measurement.

1. Introduction

Carleman inequalities constitute a very efficient tool to derive observability estimates, both in control theory or in inverse problems.

In [1], Baudouin and Puel established some global Carleman inequalities for the evolution Schrödinger equation $\mathrm{i}y' + \Delta y + q(x)y = 0$, and applied them to prove the stability in some inverse problems. The global Carleman estimates in [1] were derived under a (strict) pseudoconvexity condition (namely (3.12), see below), whereas the local Carleman estimates for the Schrödinger equation in [7, 8, 16] required a *strong* pseudoconvexity condition, which turned out to be equivalent to the strong convexity of the weight in the space variable. Here, we show that the pseudoconvexity condition in [1] may be relaxed into a *weak* pseudoconvexity condition (see below (2.2), and [5, 15] in a different context), which may degenerate at some points. Accordingly, the corresponding Carleman weights will be said to be *degenerate*. The

[3] Corresponding author.

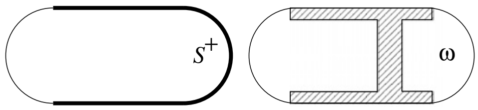

Figure 1. Observation regions in a stadium for the Schrödinger equation using a single (or a combination of) weight(s) with a spatial dependence of the type $|x - x_0|^2$, x_0 denoting a point located far away on the horizontal axis of symmetry of the stadium. The boundary case (left) was considered in [1] and the internal case (right) was studied in [13] in a controllability context using multipliers.

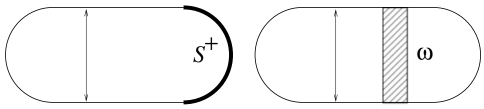

Figure 2. Observation regions in a stadium for the Schrödinger equation using degenerate Carleman weights with spatial dependence of the type $x \cdot e_1$ as given in proposition 2.2. A trapped ray avoiding the observation region is represented by an arrow.

degenerate Carleman weights considered here are not necessarily *limiting* Carleman weights in the sense of [9], the pseudoconvexity condition degenerating everywhere for the latter. However, the idea of relaxing the pseudoconvexity condition into a weak one was essentially present in [9, 19]. The weak pseudoconvexity condition considered here allows us to use Carleman weights containing a spatial dependence of the type $\psi(x) = x \cdot e$, where e is some fixed direction in $\mathbb{R}^N$. The global Carleman inequalities proved here have important applications both in inverse and control problems for the evolution Schrödinger equation. The focus being in this paper on inverse problems, we shall obtain less restrictive boundary or internal observation regions than in [1] in order to obtain the Lipschitz stability for the inverse problem consisting in retrieving a stationary potential in the evolution Schrödinger equation from either a single boundary or an internal measurement. With the new Carleman inequalities at hand, we shall derive the stability estimate in following the general method developed in [2, 10] for the analysis of inverse problems based upon global Carleman inequalities (see also [6]). Note that this study could probably be extended to unbounded domains as in [4].

It should be noted that in the resulting global Carleman inequalities obtained with the degenerate weights mentioned above, only a part of the weighted-H^1 energy, namely the energy in the direction of the gradient of the weight (corresponding to the term $|\nabla q \cdot \nabla\psi|^2$ in the left-hand side of (2.4) and (3.5)), may be bounded by observations. This is an important difference with respect to classical global Carleman inequalities for the Schrödinger equation (see for instance [1, 12]). That 'intermediate' energy is sufficient to obtain the Lipschitz stability in $L^2(\Omega)$, but not in $H_0^1(\Omega)$ as in [1].

In most of the previous articles related to this topic, both for the controllability issue [13, 14] and for Carleman inequalities [12], the explicit examples of observation regions for the Schrödinger equation worked also for the wave equation. Nevertheless, there are evidences that we can observe or control in less restrictive regions for the Schrödinger equation. See, e.g., [3, 17] for recent results in this direction and [11, 20] for a general discussion. With the Carleman inequalities proved in this paper, the observation region may be dramatically reduced in certain cases including the ball, the rectangle and the stadium (see figures 1 and 2).

We therefore believe that the weak pseudoconvexity condition introduced here could be an important future direction of research for the investigation of the inverse and controllability properties of the Schrödinger equation.

The article is organized as follows. In section 2, we state a global Carleman inequality for boundary observations on some part $S^+ \subset \partial\Omega$ as in figure 2 left (see proposition 2.1) under the weak pseudoconvexity condition (2.2). Some geometrical examples are given in examples 2.4. Then we apply the Carleman inequality to the Lipschitz stability in the inverse problem of retrieving a stationary potential in the evolution Schrödinger equation from a single boundary measurement (see theorem 2.7). The same analysis is done in section 3 for internal measurements in $\omega \subset \Omega$ as in figure 2 right (see proposition 3.1, examples 3.4 and theorem 3.5).

2. Boundary observations

2.1. A Carleman inequality

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz boundary $\partial\Omega$, and let $Q := \Omega \times (0, T)$, where $T > 0$. Let $S^+ \subset \partial\Omega$ be an open subset of $\partial\Omega$, and let us set

$$S^- := \partial\Omega\backslash S^+, \qquad \Sigma := \partial\Omega \times (0, T), \qquad \text{and} \qquad \Sigma^\pm := S^\pm \times (0, T).$$

We assume that there exists a function $\psi \in C^4(\overline{\Omega})$ such that

$$\psi > 0 \quad \text{in } \overline{\Omega}, \qquad \nabla\psi \neq 0 \quad \text{in } \overline{\Omega}, \qquad \partial_n\psi \leqslant 0 \quad \text{on } S^-, \qquad \partial_n\psi > 0 \quad \text{on } S^+, \tag{2.1}$$

and that

$$|\nabla\psi(x)\cdot\xi|^2 + \sum_{i,j=1}^N (\partial_i\partial_j\psi(x))\xi_i\xi_j \geqslant 0 \qquad \forall x \in \overline{\Omega}, \forall \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N. \tag{2.2}$$

A function ψ fulfilling (2.2) will be said to satisfy a *weak pseudoconvexity condition*, and to be a *degenerate Carleman weight*.

Replacing ψ by $\psi + C$, where $C > 0$ is a large enough number, we may also assume that

$$\psi(x) > \tfrac{2}{3}||\psi||_{L^\infty(\Omega)} \qquad \forall x \in \Omega. \tag{2.3}$$

That property will be used later. Set $C_\psi = 2||\psi||_{L^\infty(\Omega)}$ and

$$\theta(x,t) := \frac{e^{\lambda\psi(x)}}{t(T-t)}, \qquad \varphi(x,t) := \frac{e^{\lambda C_\psi} - e^{\lambda\psi(x)}}{t(T-t)}, \qquad \forall (x,t) \in \Omega \times (0,T)$$

where λ denotes some positive number which will be specified later. We also introduce the set

$$\mathcal{Z} := \{q \in C^{2,1}(\overline{\Omega} \times [0,T]);\, q = 0 \text{ on } \Sigma\}.$$

Note that the weight φ blows up at $t = 0$ and $t = T$. As a consequence, the functions $\exp(-2s\varphi)$, $\theta\exp(-2s\varphi)$, etc. are smooth, and they vanish at $t = 0$ and $t = T$. The later property is related to the fact that no information about the function at $t = 0$ or $t = T$ is required in the Carleman estimate given in the next result.

Proposition 2.1. *Assume that there exists a function $\psi \in C^4(\overline{\Omega})$ such that (2.1), (2.2) and (2.3) hold for some $S^+, S^- \subset \partial\Omega$. Then, there exist constants $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$ and $C_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, all $s \geqslant s_0$, and all $q \in \mathcal{Z}$, it holds*

$$\begin{aligned}\int_0^T\!\!\int_\Omega &[\lambda^2 s\theta|\nabla q\cdot\nabla\psi|^2 + \lambda^4(s\theta)^3|q|^2 + |\widetilde{M}_1 q|^2 + |\widetilde{M}_2 q|^2]\, e^{-2s\varphi}\, dx\, dt \\ &+ \int_0^T\!\!\int_{S^-} \lambda s\theta \left|\frac{\partial\psi}{\partial n}\right| \left|\frac{\partial q}{\partial n}\right|^2 e^{-2s\varphi}\, d\sigma\, dt \\ &\leqslant C_0\left(\int_0^T\!\!\int_\Omega |\partial_t q + i\Delta q|^2 e^{-2s\varphi}\, dx\, dt + \int_0^T\!\!\int_{S^+} \lambda s\theta \left|\frac{\partial\psi}{\partial n}\right| \left|\frac{\partial q}{\partial n}\right|^2 e^{-2s\varphi}\, d\sigma\, dt\right),\end{aligned} \tag{2.4}$$

where i stands for the imaginary unit and $\widetilde{M}_1$ and $\widetilde{M}_2$ denote the operators

$$\widetilde{M}_1 q := [s(\varphi_t + \mathrm{i}\Delta\varphi) - 2\mathrm{i}s^2|\nabla\varphi|^2]q + 2\mathrm{i}s\nabla\varphi\cdot\nabla q, \tag{2.5}$$

$$\widetilde{M}_2 q := [-s(\varphi_t + \mathrm{i}\Delta\varphi) + 2\mathrm{i}s^2|\nabla\varphi|^2]q + q_t - 2\mathrm{i}s\nabla\varphi\cdot\nabla q + \mathrm{i}\Delta q. \tag{2.6}$$

Proof. In what follows, the letter C will denote a constant (independent of s, λ, q) which may vary from line to line. Let $q \in \mathcal{Z}$ be given, and let $u = \mathrm{e}^{-s\varphi}q$ and $w = \mathrm{e}^{-s\varphi}L(q) = \mathrm{e}^{-s\varphi}L(\mathrm{e}^{s\varphi}u)$, where L denotes the operator

$$L = \partial_t + \mathrm{i}\Delta.$$

Straightforward computations show that

$$w = Mu := u_t + s\varphi_t u + \mathrm{i}(\Delta u + 2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u + s^2|\nabla\varphi|^2 u)$$

with the convention that

$$z\cdot z' = \sum_{i=1}^{N} z_i z_i' \qquad \text{for all} \quad z = (z_1, \ldots, z_N) \in \mathbb{C}^N, \quad z' = (z_1', \ldots, z_N') \in \mathbb{C}^N.$$

Let M_1 and M_2 denote respectively the (formal) adjoint and skew-adjoint parts of the operator M. We readily obtain that

$$M_1 u := \mathrm{i}(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u) + s\varphi_t u \tag{2.7}$$

$$M_2 u := u_t + \mathrm{i}(\Delta u + s^2|\nabla\varphi|^2 u). \tag{2.8}$$

Defining the operators $\widetilde{M}_1 q := \mathrm{e}^{s\varphi}M_1 u$ and $\widetilde{M}_2 q := \mathrm{e}^{s\varphi}M_2 u$, we easily check that (2.5) and (2.6) hold true.

On the other hand,

$$||w||^2 = ||M_1 u + M_2 u||^2 = ||M_1 u||^2 + ||M_2 u||^2 + 2\mathrm{Re}(M_1 u, M_2 u) \tag{2.9}$$

where $(u, v) := \int_0^T\int_\Omega u(x,t)\overline{v(x,t)}\,\mathrm{d}x\,\mathrm{d}t$ and $||w||^2 = (w, w)$. From now on, for the sake of brevity, we write $\iint u$ (resp. $\iint_{\Sigma^\pm} u$) instead of $\int_0^T\int_\Omega u(x,t)\,\mathrm{d}x\,\mathrm{d}t$ (resp. $\int_0^T\int_{S^\pm} u(x,t)\,\mathrm{d}\sigma\,\mathrm{d}t$). ∂_i will stand for $\partial/\partial x_i$.

The proof of the Carleman estimate follows the same pattern as in [18] for the Ginzburg–Landau equation. The first step provides an exact computation of a scalar product in L^2, whereas the second step gives the estimates obtained thanks to the weak pseudoconvexity condition.

Step 1. Exact computation of the scalar product in (2.9).

The scalar product in (2.9) is decomposed into the sum of three integral terms, namely

$$2\mathrm{Re}(M_1 u, M_2 u) = I_1 + I_2 + I_3 \tag{2.10}$$

with

$$I_1 := 2\mathrm{Re}\iint \mathrm{i}(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u)(\overline{u}_t - \mathrm{i}(\Delta\overline{u} + s^2|\nabla\varphi|^2\overline{u})) \tag{2.11}$$

$$I_2 := 2\mathrm{Re}\iint s\varphi_t u(\overline{u}_t - \mathrm{i}\Delta\overline{u}) \tag{2.12}$$

$$I_3 := 2\mathrm{Re}\iint s\varphi_t u(-\mathrm{i}s^2|\nabla\varphi|^2\overline{u}). \tag{2.13}$$

Let us begin with the computation of I_1.

$$I_1 = 2\mathrm{Re}\left\{\iint (2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u)(\Delta\overline{u} + s^2|\nabla\varphi|^2\overline{u})\right\}$$
$$+2\mathrm{Re}\left\{\mathrm{i}\iint (2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u)\overline{u}_t\right\} =: I_1^1 + I_1^2.$$

To calculate I_1^1, we first need to evaluate the integral term $J := \iint \Delta\overline{u}(\nabla\varphi\cdot\nabla u)$. Using the convention of repeated indices, we obtain that

$$J = \iint \left(\partial_j^2\overline{u}\right)\partial_i\varphi\partial_i u = -\iint \partial_j\overline{u}(\partial_j\partial_i\varphi\partial_i u + \partial_i\varphi\partial_j\partial_i u) + \iint_\Sigma (\partial_j\overline{u})n_j\partial_i\varphi\partial_i u.$$

Since $u = 0$ on Σ, $\nabla u = (\partial u/\partial n)n$, so $\nabla\varphi\cdot\nabla u = (\partial\varphi/\partial n)(\partial u/\partial n)$ and

$$\iint_\Sigma (\partial_j\overline{u})n_j\partial_i\varphi\partial_i u = \iint_\Sigma \frac{\partial\varphi}{\partial n}\left|\frac{\partial u}{\partial n}\right|^2.$$

On the other hand

$$\begin{aligned}2\mathrm{Re}\iint \partial_j\overline{u}\partial_i\varphi\partial_j\partial_i u &= \iint \partial_i\varphi\partial_i(\partial_j u\partial_j\overline{u})\\ &= -\iint \partial_i^2\varphi|\partial_j u|^2 + \iint_\Sigma (\partial_i\varphi)n_i\partial_j u\partial_j\overline{u}\\ &= -\iint \Delta\varphi|\nabla u|^2 + \iint_\Sigma \frac{\partial\varphi}{\partial n}\left|\frac{\partial u}{\partial n}\right|^2.\end{aligned}$$

We conclude that

$$2\mathrm{Re}J = \iint \Delta\varphi|\nabla u|^2 - 2\iint \partial_j\partial_i\varphi\partial_j u\partial_i\overline{u} + \iint_\Sigma \frac{\partial\varphi}{\partial n}\left|\frac{\partial u}{\partial n}\right|^2. \tag{2.14}$$

Expanding I_1^1, performing integrations by parts and using (2.14), we find that

$$\begin{aligned}I_1^1 &= 2\mathrm{Re}\left\{2sJ + \iint s(\Delta\varphi)u\Delta\overline{u} + \iint 2s^3(\nabla\varphi\cdot\nabla u)|\nabla\varphi|^2\overline{u} + \iint s^3\Delta\varphi|\nabla\varphi|^2|u|^2\right\}\\ &= 2s(2\mathrm{Re}J) - 2s\ \mathrm{Re}\iint (\nabla(\Delta\varphi)u + \Delta\varphi\nabla u)\cdot\nabla\overline{u}\\ &\quad + 2s^3\iint |\nabla\varphi|^2\nabla\varphi\cdot\nabla|u|^2 + 2s^3\iint \Delta\varphi|\nabla\varphi|^2|u|^2 \qquad (2.15)\\ &= 2s\left(\iint (\Delta\varphi|\nabla u|^2 - 2\partial_j\partial_i\varphi\partial_j u\partial_i\overline{u}) + \iint_\Sigma \frac{\partial\varphi}{\partial n}\left|\frac{\partial u}{\partial n}\right|^2\right)\\ &\quad + s\left(\iint \Delta^2\varphi|u|^2 - 2\iint \Delta\varphi|\nabla u|^2\right) - 2s^3\iint (\nabla|\nabla\varphi|^2\cdot\nabla\varphi)|u|^2. \qquad (2.16)\end{aligned}$$

On the other hand, integrating by parts with respect to t or x in I_1^2, we obtain

$$\begin{aligned}I_1^2 &= \mathrm{i}\iint (2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u)\overline{u}_t - \mathrm{i}\iint (2s\nabla\varphi\cdot\nabla\overline{u} + s(\Delta\varphi)\overline{u})u_t\\ &= -\mathrm{i}\iint (2s\nabla\varphi_t\cdot\nabla u + 2s\nabla\varphi\cdot\nabla u_t + s(\Delta\varphi_t)u + s(\Delta\varphi)u_t)\overline{u}\\ &\quad +\mathrm{i}\iint (2s(\Delta\varphi)\overline{u}u_t + 2s(\nabla\varphi\cdot\nabla u_t)\overline{u} - s(\Delta\varphi)\overline{u}u_t)\end{aligned}$$

$$= \mathrm{i} \iint (s\nabla\varphi_t \cdot \nabla|u|^2 - 2s\nabla\varphi_t \cdot (\nabla u)\overline{u})$$

$$= \mathrm{i} \iint s\nabla\varphi_t \cdot (u\nabla\overline{u} - \overline{u}\nabla u). \tag{2.17}$$

Gathering together (2.16) and (2.17), we arrive to

$$I_1 = s\left(\iint -4\partial_j\partial_i\varphi\partial_j u\partial_i\overline{u} + 2\iint_\Sigma \partial_n\varphi|\partial_n u|^2 + \iint \Delta^2\varphi|u|^2\right)$$
$$- 2s^3\iint (\nabla|\nabla\varphi|^2\cdot\nabla\varphi)|u|^2 + 2\mathrm{Re}\left\{\mathrm{i}s\iint \nabla\varphi_t\cdot(u\nabla\overline{u})\right\}. \tag{2.18}$$

On the other hand,

$$I_2 = \iint s\varphi_t(u\overline{u}_t + u_t\overline{u}) + \iint s\varphi_t(-\mathrm{i}u\Delta\overline{u} + \mathrm{i}\overline{u}\Delta u) =: I_2^1 + I_2^2.$$

Integrating by parts with respect to t in I_2^1 yields

$$I_2^1 = -\iint s\varphi_{tt}|u|^2. \tag{2.19}$$

On the other hand

$$I_2^2 = -\iint s\nabla\varphi_t(-\mathrm{i}u\nabla\overline{u} + \mathrm{i}\overline{u}\nabla u) = 2\mathrm{Re}\left\{\mathrm{i}s\iint \nabla\varphi_t\cdot(u\nabla\overline{u})\right\}. \tag{2.20}$$

Finally,

$$I_3 = 0. \tag{2.21}$$

Gathering together (2.18), (2.19), (2.20) and (2.21), we infer that

$$2\mathrm{Re}(M_1u, M_2u) = s\left(\iint -4\partial_j\partial_i\varphi\partial_j u\partial_i\overline{u} + 2\iint_\Sigma \frac{\partial\varphi}{\partial n}\left|\frac{\partial u}{\partial n}\right|^2 + \iint \Delta^2\varphi|u|^2\right)$$
$$- 2s^3\iint(\nabla|\nabla\varphi|^2\cdot\nabla\varphi)|u|^2 + 2\mathrm{Re}\left\{\mathrm{i}s\iint\nabla\varphi_t\cdot(u\nabla\overline{u})\right\}$$
$$- \iint s\varphi_{tt}|u|^2 + 2\mathrm{Re}\left\{\mathrm{i}s\iint\nabla\varphi_t\cdot(u\nabla\overline{u})\right\} \tag{2.22}$$
$$= 2s\left(-2\iint \partial_j\partial_i\varphi\partial_j u\partial_i\overline{u} + \iint_\Sigma \frac{\partial\varphi}{\partial n}\left|\frac{\partial u}{\partial n}\right|^2\right) + 4\,\mathrm{Re}\left\{\mathrm{i}s\iint\nabla\varphi_t\cdot(u\nabla\overline{u})\right\}$$
$$+ \iint |u|^2[s(\Delta^2\varphi - \varphi_{tt}) - 2s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2]. \tag{2.23}$$

Finally, (2.9) and (2.23) give

$$||w||^2 = ||M_1u||^2 + ||M_2u||^2 + 2s\left(-2\iint \partial_j\partial_i\varphi\partial_j u\partial_i\overline{u} + \iint_\Sigma \frac{\partial\varphi}{\partial n}\left|\frac{\partial u}{\partial n}\right|^2\right)$$
$$- 4s\,\mathrm{Im}\left\{\iint\nabla\varphi_t\cdot(u\nabla\overline{u})\right\} + \iint|u|^2[s(\Delta^2\varphi - \varphi_{tt}) - 2s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2]. \tag{2.24}$$

Step 2. Estimation of the terms in (2.24).

The control of the terms in the right-hand side of (2.24) is detailed in a series of claims. We begin with the

Claim 1. There exist two numbers $\lambda_1 \geqslant 1$ and $s_1 \geqslant 1$ and some constant $A > 0$ such that for all $\lambda \geqslant \lambda_1$ and all $s \geqslant s_1$,

$$A\lambda s^3\iint |u|^2|\nabla\varphi|^3 \leqslant \iint |u|^2[s(\Delta^2\varphi - \varphi_{tt}) - 2s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2]. \tag{2.25}$$

Easy computations show that

$$\partial_i\varphi = -\frac{\lambda e^{\lambda\psi(x)}\partial_i\psi}{t(T-t)}, \qquad \partial_j\partial_i\varphi = -\frac{e^{\lambda\psi(x)}}{t(T-t)}(\lambda^2\partial_i\psi\partial_j\psi + \lambda\partial_j\partial_i\psi) \tag{2.26}$$

and

$$-\nabla|\nabla\varphi|^2\cdot\nabla\varphi = -2(\partial_j\partial_i\varphi)\partial_i\varphi\partial_j\varphi = 2\left(\frac{e^{\lambda\psi(x)}\lambda}{t(T-t)}\right)^3(\lambda|\nabla\psi|^4 + \partial_j\partial_i\psi\partial_i\psi\partial_j\psi). \tag{2.27}$$

Since $\nabla\psi \neq 0$ on $\overline{\Omega}$, it follows that for λ large enough, say $\lambda \geqslant \lambda_1$, we have

$$-\nabla|\nabla\varphi|^2\cdot\nabla\varphi \geqslant C\lambda|\nabla\varphi|^3.$$

As by (2.3) $|\Delta^2\varphi| + |\varphi_{tt}| \leqslant C\lambda|\nabla\varphi|^3$, we infer that for s large enough, say $s \geqslant s_1$, and for all $\lambda \geqslant \lambda_1$, we have that for some constant $A > 0$

$$s(\Delta^2\varphi - \varphi_{tt}) - 2s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2 \geqslant A\lambda s^3|\nabla\varphi|^3. \tag{2.28}$$

Multiplying each term in (2.28) by $|u|^2$ and integrating over $\Omega\times(0,T)$ yields at once (2.25).

Claim 2. For any $\lambda \geqslant 1$, $-\iint \partial_j\partial_i\varphi\partial_j u\partial_i\overline{u} \geqslant 0$.

This follows from (2.2) and (2.26), since

$$-\iint \partial_j\partial_i\varphi\partial_j u\partial_i\overline{u} = \lambda\iint\frac{e^{\lambda\psi}}{t(T-t)}(\lambda|\nabla\psi\cdot\nabla u|^2 + \partial_j\partial_i\psi\partial_j u\partial_i\overline{u}). \tag{2.29}$$

Claim 3. There exists a constant $A' > 0$ such that

$$\left|\operatorname{Im}\left\{\iint u\nabla\varphi_t\cdot\nabla\overline{u}\right\}\right| \leqslant \iint\frac{e^{\lambda\psi}}{t(T-t)}|\nabla\psi\cdot\nabla u|^2 + A'\lambda^{-1}\iint|\nabla\varphi|^3|u|^2. \tag{2.30}$$

Indeed, using Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}\left|\operatorname{Im}\left\{\iint u\nabla\varphi_t\cdot\nabla\overline{u}\right\}\right| &\leqslant \iint|\nabla\varphi_t\cdot\nabla u||u| \\ &\leqslant \iint\frac{|2t-T|}{t^2(T-t)^2}\lambda e^{\lambda\psi}|\nabla\psi\cdot\nabla u||u| \\ &\leqslant \iint\frac{e^{\lambda\psi}}{t(T-t)}|\nabla\psi\cdot\nabla u|^2 + (T\lambda/2)^2\iint\frac{e^{\lambda\psi}}{t^3(T-t)^3}|u|^2\end{aligned}$$

and the claim follows at once.

Claim 4. There exist some numbers $\lambda_2 \geqslant \lambda_1$, $s_2 \geqslant s_1$ and $A'' > 0$, such that for all $\lambda \geqslant \lambda_2$, $s \geqslant s_2$

$$\begin{aligned}&-4s\iint\partial_j\partial_i\varphi\partial_j u\partial_i\overline{u} - 4s\operatorname{Im}\left\{\iint u\nabla\varphi_t\cdot\nabla\overline{u}\right\} \\ &\quad + \iint|u|^2[s(\Delta^2\varphi - \varphi_{tt}) - 2s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2] \\ &\quad \geqslant A''\left(\lambda^2 s\iint\frac{e^{\lambda\psi}}{t(T-t)}|\nabla\psi\cdot\nabla u|^2 + \lambda s^3\iint|\nabla\varphi|^3|u|^2\right).\end{aligned} \tag{2.31}$$

Indeed, using (2.2), (2.29), claim 1 and claim 3, we obtain that

$$
\begin{aligned}
-4s \iint \partial_j \partial_i \varphi \partial_j u \partial_i \overline{u} - 4s \,\mathrm{Im} \left\{ \iint u \nabla \varphi_t \cdot \nabla \overline{u} \right\} \\
+ \iint |u|^2 [s(\Delta^2 \varphi - \varphi_{tt}) - 2s^3 \nabla \varphi \cdot \nabla |\nabla \varphi|^2] \\
\geqslant 4s \iint \frac{\mathrm{e}^{\lambda \psi}}{t(T-t)} [(\lambda^2 - 1)|\nabla \psi \cdot \nabla u|^2 + \lambda \partial_j \partial_i \psi \partial_j u \partial_i \overline{u}] \\
+ \iint (As^3 \lambda - 4A' s \lambda^{-1}) |\nabla \varphi|^3 |u|^2 \\
\geqslant 2\lambda^2 s \iint \frac{\mathrm{e}^{\lambda \psi}}{t(T-t)} |\nabla \psi \cdot \nabla u|^2 + \frac{A}{2} \lambda s^3 \iint |\nabla \varphi|^3 |u|^2
\end{aligned}
$$

for $s \geqslant \max(s_1, \sqrt{8A'/A})$ and $\lambda \geqslant \max(\lambda_1, 1 + \sqrt{3})$ (so that $4(\lambda^2 - 1) \geqslant 2\lambda^2 + 4\lambda$). The claim follows with $A'' = \min(2, A/2)$.

We infer from (2.31) and the fact that $\frac{\partial \varphi}{\partial n} = -\lambda \mathrm{e}^{\lambda \psi(x)} \left(\frac{\partial \psi}{\partial n}\right) t^{-1}(T-t)^{-1} \geqslant 0$ on Σ^-, that for all $\lambda \geqslant \lambda_2$ and all $s \geqslant s_2$

$$
\begin{aligned}
||M_1 u||^2 + ||M_2 u||^2 + \lambda^2 s \iint \frac{\mathrm{e}^{\lambda \psi}}{t(T-t)} |\nabla \psi \cdot \nabla u|^2 + \lambda s^3 \iint |\nabla \varphi|^3 |u|^2 \\
+ s \iint_{\Sigma^-} \left| \frac{\partial \varphi}{\partial n} \right| \left| \frac{\partial u}{\partial n} \right|^2 \leqslant C \left(||w||^2 + s \iint_{\Sigma^+} \left| \frac{\partial \varphi}{\partial n} \right| \left| \frac{\partial u}{\partial n} \right|^2 \right).
\end{aligned} \tag{2.32}
$$

Replacing u by $\mathrm{e}^{-s\varphi} q$ in (2.32) yields (2.4). □

Proposition 2.2. *The assumptions (2.1) and (2.2) are fulfilled when, for some vector $e \neq 0$, we have*

$$
S^- := \{x \in \partial\Omega; n(x) \cdot e \leqslant 0\}, \qquad S^+ := \{x \in \partial\Omega; n(x) \cdot e > 0\} \tag{2.33}
$$

Proof. It is sufficient to pick $\psi(x) := e \cdot x + C$, with $C > 0$ large enough. □

Remark 2.3. Given $\psi(x) := e \cdot x + C$ as in proposition 2.2, for each $\xi \in \mathbb{R}^N$ the expression in the left-hand side of (2.2) reads

$$
|\nabla \psi(x) \cdot \xi|^2 + \sum_{i,j=1}^{N} (\partial_i \partial_j \psi(x)) \xi_i \xi_j = |e \cdot \xi|^2 \qquad \forall x \in \overline{\Omega}, \tag{2.34}
$$

hence it vanishes (degenerates) for all ξ orthogonal to e. Accordingly, a weight fulfilling (2.2) will be said to be *degenerate*.

Using the degenerate Carleman weight $\psi(x) = e \cdot x + C$, we may design dramatically less restrictive observational regions than in [1], as it can be seen in the following examples. Note that in each of them, a trapped ray avoiding the observation region can easily be drawn. It means that the wave equation fails to be controllable (or observable).

Example 2.4.

(1) Ω is a ball: $\Omega := B_R(0)$ ($R > 0$ being arbitrary). Taking $e = e_1 = (1, 0, \ldots, 0)$ in (2.33), we obtain for S^+ the half-sphere

$$
S^+ = \{x \in \mathbb{R}^N; ||x|| = R \text{ and } x_1 = x \cdot e_1 > 0\}.
$$

(2) Ω is a rectangle: $\Omega := (-L_1, L_1) \times \cdots \times (-L_N, L_N)$. Taking $e = e_1$ in (2.33), we obtain for S^+ the side

$$S^+ = \{L_1\} \times (-L_2, L_2) \times \cdots \times (-L_N, L_N).$$

(3) Ω is a stadium:

$$\Omega = \left\{x = (x_1, x_2) \in \mathbb{R}^2; x \in (-L_1, L_1) \times (-L_2, L_2) \text{ or } (x_1 \pm L_1)^2 + x_2^2 < L_2^2\right\}.$$

Taking $e = e_1$ in (2.33), we obtain once again for S^+ the half-sphere

$$S^+ = \left\{x = (x_1, x_2) \in \mathbb{R}^2; x_1 > L_1 \text{ and } (x_1 - L_1)^2 + x_2^2 = L_2^2\right\}.$$

Remark 2.5.

(1) For any $\lambda_0 > 0$, the equivalence between

$$-\iint (\partial_i \partial_j \varphi) \partial_i u \partial_j \overline{u} \geqslant 0 \qquad \forall \lambda \geqslant \lambda_0, \quad \forall q \in \mathcal{Z}$$

and

$$\lambda_0 |\nabla \psi \cdot \xi|^2 + (\partial_i \partial_j \psi) \xi_i \xi_j \geqslant 0 \qquad \forall \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \tag{2.35}$$

is easy to prove when $N = 1$. On the other hand, scaling ψ by λ_0, we see that there is no loss of generality in assuming that $\lambda_0 = 1$.

(2) A function ψ fulfilling (2.2) need not be convex (consider e.g. the function $\psi(x_1, x_2) := x_1^2 - x_2^2$ on the set $\left\{x = (x_1, x_2); x_2^2 > x_1^2 + 1\right\}$). However, the level sets

$$C_a := \{x \in \overline{\Omega}; \psi(x) \leqslant a\}$$

have to be *convex* for any $a \in \mathbb{R}^+$. Indeed, let us pick any point $x_0 \in C_a \cap \Omega$, and any regular curve $\{\gamma(t)\} \subset \{x \in \Omega; \psi(x) = a\}$ with $\gamma(t_0) = x_0$. Differentiating twice in $\psi(\gamma(t)) = a$ yields $\nabla \psi(\gamma(t)) \cdot \dot{\gamma}(t) = 0$ and $D^2 \psi(\dot{\gamma}(t), \dot{\gamma}(t)) + \nabla \psi(\gamma(t)) \cdot \ddot{\gamma}(t) = 0$, hence

$$\nabla \psi(\gamma(t)) \cdot \ddot{\gamma}(t) = -D^2 \psi(\dot{\gamma}(t), \dot{\gamma}(t)) \leqslant 0,$$

which proves the claim.

(3) Applying the above observation to example 1 (the ball $\Omega = B_R(0)$), we claim that if a function ψ fulfilling (2.1)–(2.2) exists, then the region S^+ has to contain a half-sphere, i.e. a set of the form $\{x \in \mathbb{R}^N; ||x|| = R \text{ and } x \cdot e > 0\}$ for some $e \neq 0$. To prove the claim, let a_0 denote the largest real number such that there exists $x_0 \in \partial B_R(0)$ with $\psi(x_0) = a_0$ and $\frac{\partial \psi}{\partial n}(x_0) \leqslant 0$. Then $\frac{\partial \psi}{\partial n}(x_0) = 0$ and $\frac{\partial \psi}{\partial n}(x) > 0$ for all $x \in \partial B_R(0) \backslash C_{a_0}$. Pick $e = \nabla \psi(x_0)/||\nabla \psi(x_0)||$. Then the set C_{a_0} is convex, and it is contained in the half-ball $\{x \in B_R(0); x \cdot e \leqslant 0\}$. Thus the set $\{x \in \partial B_R(0); x \cdot e > 0\}$ is contained in S^+.

(4) Proposition 2.1 has also some interesting consequences in control theory. Indeed, it follows from (2.4) that the Schrödinger equation is exactly controllable in $L^2(\Omega)$ with a Dirichlet boundary control supported in S^+ when (2.1) and (2.2) are fulfilled.

For each $p \in L^\infty(\Omega)$, we define the operator

$$L_p := \partial_t + \mathrm{i}\Delta + \mathrm{i}p$$

and the space

$$\mathcal{Z}_p := \left\{z \in L^2(Q); L_p z \in L^2(Q), z = 0 \text{ on } \Sigma, \frac{\partial z}{\partial n} \in L^2(\Sigma^+)\right\}. \tag{2.36}$$

The following Carleman estimate is a direct consequence of proposition 2.1.

Corollary 2.6. *Let ψ be as in proposition 2.1. Given $m \geqslant 0$, there exist $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$ and $C > 0$ such that for each $p \in L^\infty(\Omega)$ with $\|p\|_{L^\infty} \leqslant m$ it holds*

$$\int_0^T\int_\Omega \left(\lambda^4 s^3\theta^3|z|^2 + |\widetilde{M}_1(z)|^2 + |\widetilde{M}_2(z)|^2\right) \mathrm{e}^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t \leqslant C\left(\int_0^T\int_\Omega \mathrm{e}^{-2s\varphi}|L_p z|^2 + \int_0^T\int_{S^+} \lambda s\theta\left|\frac{\partial z}{\partial n}\right|^2 \mathrm{e}^{-2s\varphi}\,\mathrm{d}\sigma\,\mathrm{d}t\right) \tag{2.37}$$

for all $\lambda \geqslant \lambda_0$, $s \geqslant s_0$ and $z \in \mathcal{Z}_p$.

2.2. The inverse problem, boundary observations

We consider the following boundary initial-value problem

$$\begin{cases} \mathrm{i}u_t + \Delta u + p(x)u = 0 & \text{in} \quad \Omega\times(0,T),\\ u = h & \text{on} \quad \partial\Omega\times(0,T),\\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{2.38}$$

In what follows, we shall denote by $u(p)$ the solution of the system (2.38) associated with the potential p.

Theorem 2.7. *Assume that there exists a function $\psi \in C^4(\overline{\Omega})$ such that (2.1), (2.2) and (2.3) hold for some $S^+ \subset \partial\Omega$. Suppose also that $p \in L^\infty(\Omega;\mathbb{R})$, $u_0 \in L^\infty(\Omega)$ and $r > 0$ are such that*

- *$u_0(x) \in \mathbb{R}$ or $\mathrm{i}u_0(x) \in \mathbb{R}$ a.e. in Ω,*
- *$|u_0(x)| \geqslant r > 0$ a.e. in Ω, and*
- *$u(p) \in H^1(0,T;L^\infty(\Omega))$.*

Then, for any $m \geqslant 0$, there exists a constant $C = C(m, \|u(p)\|_{H^1(0,T;L^\infty(\Omega))}, r) > 0$ such that for any $q \in B_m(0) \subset L^\infty(\Omega;\mathbb{R})$ satisfying

$$\frac{\partial u(p)}{\partial n} - \frac{\partial u(q)}{\partial n} \in H^1(0,T;L^2(S^+))$$

we have that

$$\|p-q\|_{L^2(\Omega)} \leqslant C\left\|\frac{\partial u(p)}{\partial n} - \frac{\partial u(q)}{\partial n}\right\|_{H^1(0,T;L^2(S^+))}.$$

Proof. Pick any p, q as in the statement of the theorem, and introduce the difference $y := u(p) - u(q)$ of the corresponding solutions of (2.38).

Then y fulfills the system

$$\begin{cases} \mathrm{i}y_t + \Delta y + q(x)y = f(x)R(x,t) & \text{in} \quad \Omega\times(0,T),\\ y = 0 & \text{on} \quad \partial\Omega\times(0,T),\\ y(0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.39}$$

with $f := q - p$ (real valued) and $R := u(p)$. To complete the proof of theorem 2.7, we need the following result. □

Proposition 2.8. *Assume that there exists a function* $\psi \in C^4(\overline{\Omega})$ *such that* (2.1), (2.2) *and* (2.3) *hold for some* $S^+ \subset \partial\Omega$. *Suppose that*

- $R(x,0) \in \mathbb{R}$ *or* $\mathrm{i}R(x,0) \in \mathbb{R}$ *a.e. in* Ω,
- $|R(x,0)| \geqslant r > 0$ *a.e. in* Ω,
- $R \in H^1(0,T;L^\infty(\Omega))$, *and*
- $\frac{\partial y}{\partial n} \in H^1(0,T;L^2(S^+))$.

Then for any $m \geqslant 0$ *there exists a constant* $C > 0$ *such that for any* $q \in L^\infty(\Omega;\mathbb{R})$ *with* $\|q\|_{L^\infty(\Omega)} \leqslant m$ *and for all* $f \in L^2(\Omega;\mathbb{R})$, *the solution* y *of* (2.39) *satisfies*

$$\|f\|_{L^2(\Omega)} \leqslant C \left\| \frac{\partial y}{\partial n} \right\|_{H^1(0,T;L^2(S^+))}. \tag{2.40}$$

Proof of proposition 2.8. Let $f \in L^2(\Omega;\mathbb{R})$ and $R \in H^1(0,T;L^\infty(\Omega))$ be such that $R(x,0) \in \mathbb{R}$ a.e. in Ω, and let y be the solution of (2.39). We take the even-conjugate extensions of y and R to the interval $(-T,T)$; i.e., we set $y(x,t) = \overline{y(x,-t)}$ for $t \in (-T,0)$ and similarly for R. Since $R(x,0) \in \mathbb{R}$ a.e. in Ω, we have that $R \in H^1(-T,T;L^\infty(\Omega))$, and y satisfies the system (2.39) in $\Omega \times (-T,T)$. In the case when $R(x,0) \in \mathrm{i}\mathbb{R}$, the proof is still valid by taking odd-conjugate extensions.

Changing t into $t+T$, we may assume that y and R are defined on $\Omega \times (0,2T)$, instead of $\Omega \times (-T,T)$.

Let $z(x,t) = y_t(x,2T-t)$. Then z satisfies the following system:

$$\begin{cases} z_t + \mathrm{i}\Delta z + \mathrm{i}qz = \mathrm{i}f(x)R_t(x,t) & \text{in} \quad \Omega \times (0,2T), \\ z = 0 & \text{on} \quad \partial\Omega \times (0,2T), \\ z(T) = -\mathrm{i}f(x)R(x,T) & \text{in} \quad \Omega. \end{cases} \tag{2.41}$$

We shall apply corollary 2.6, with $2T$ instead of T. Therefore, here we consider

$$\theta(x,t) = \frac{\mathrm{e}^{\lambda\psi(x)}}{t(2T-t)}, \qquad \varphi(x,t) = \frac{\mathrm{e}^{\lambda C_\psi} - \mathrm{e}^{\lambda\psi(x)}}{t(2T-t)}, \qquad \forall (x,t) \in \Omega \times (0,2T).$$

As in the proof of proposition 2.1, we introduce $w = \mathrm{e}^{-s\varphi}z$, $\widetilde{M}_2 z = \mathrm{e}^{s\varphi}M_2 w$ and $M_2 w = w_t + \mathrm{i}(\Delta w + s^2|\nabla\varphi|^2 w)$.

Now, set

$$J = \int_0^T \int_\Omega \mathrm{e}^{-2s\varphi} \widetilde{M}_2(z)\overline{z}\,\mathrm{d}x\,\mathrm{d}t.$$

Then we have

$$\begin{aligned} J &= \int_0^T \int_\Omega M_2(w)\overline{w}\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega w_t\overline{w}\,\mathrm{d}x\,\mathrm{d}t + \mathrm{i}\int_0^T \int_\Omega (-|\nabla w|^2 + s^2|\nabla\varphi|^2|w|^2)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{2.42}$$

hence

$$\mathrm{Re}(J) = \frac{1}{2}\int_\Omega |w(x,T)|^2\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_\Omega \mathrm{e}^{-2s\varphi(x,T)}|f(x)|^2|R(x,T)|^2\,\mathrm{d}x.$$

Using the hypothesis on $R(x,T)$, it follows that

$$\mathrm{Re}(J) \geqslant \frac{r^2}{2}\int_\Omega \mathrm{e}^{-2s\varphi(x,T)}|f(x)|^2\,\mathrm{d}x. \tag{2.43}$$

On the other hand, we have that

$$|J| \leqslant \left(\int_0^T\int_\Omega \mathrm{e}^{-2s\varphi}|\widetilde{M}_2(z)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int_\Omega \mathrm{e}^{-2s\varphi}|z|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}$$

$$\leqslant \lambda^{-2}s^{-3/2}\int_0^T\int_\Omega \mathrm{e}^{-2s\varphi}|\widetilde{M}_2(z)|^2\,\mathrm{d}x\,\mathrm{d}t + \lambda^2 s^{3/2}\int_0^T\int_\Omega \mathrm{e}^{-2s\varphi}|z|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant C\lambda^{-2}s^{-3/2}\left(\int_0^T\int_\Omega \mathrm{e}^{-2s\varphi}|\widetilde{M}_2(z)|^2\,\mathrm{d}x\,\mathrm{d}t + \lambda^4 s^3\int_0^T\int_\Omega \theta^3\mathrm{e}^{-2s\varphi}|z|^2\,\mathrm{d}x\,\mathrm{d}t\right). \tag{2.44}$$

The last inequality comes from the fact that θ is bounded from below. From (2.44), the Carleman inequality (2.37) (applied with $2T$ instead of T) and the fact that $\varphi(x,T) \leqslant \varphi(x,t)$ for all $(x,t) \in \Omega \times (0,2T)$, that $\theta\,\mathrm{e}^{-2s\varphi}$ is bounded from above in $\Omega \times (0,2T)$ and that $R_t \in L^2(0,2T;L^\infty(\Omega))$, we obtain

$$|J| \leqslant C\lambda^{-2}s^{-3/2}\left(\int_0^{2T}\int_\Omega \mathrm{e}^{-2s\varphi}|fR_t|^2\,\mathrm{d}x\,\mathrm{d}t + \lambda s\int_0^{2T}\int_{S^+}\theta\,\mathrm{e}^{-2s\varphi}\left|\frac{\partial z}{\partial n}\right|^2\mathrm{d}\sigma\,\mathrm{d}t\right)$$

$$\leqslant C\lambda^{-2}s^{-3/2}\int_\Omega \mathrm{e}^{-2s\varphi(x,T)}|f(x)|^2\,\mathrm{d}x + C\lambda^{-1}s^{-1/2}\int_0^{2T}\int_{S^+}\left|\frac{\partial z}{\partial n}\right|^2\mathrm{d}\sigma\,\mathrm{d}t. \tag{2.45}$$

From (2.43) and (2.45) it follows that, for s and λ large enough,

$$\int_\Omega \mathrm{e}^{-2s\varphi(x,T)}|f(x)|^2\,\mathrm{d}x \leqslant C\int_0^{2T}\int_{S^+}\left|\frac{\partial z}{\partial n}\right|^2\mathrm{d}\sigma\,\mathrm{d}t. \tag{2.46}$$

Then (2.43) follows from (2.46) since

$$\mathrm{e}^{-2s\varphi(x,T)} \geqslant \mathrm{e}^{-2sM} > 0,$$

where $M = \frac{1}{T^2}(\mathrm{e}^{\lambda C_\psi} - 1)$. This completes the proof of proposition 2.8 and of theorem 2.7. □

Remark 2.9. The previous stability theorem shows that in the three cases of pairs (Ω, S^+) given in example 2.4, by observing the time derivative of the normal derivative of the solution of the Schrödinger equation (2.38) on $S^+ \times (0,T)$, we can locally recover its potential in $L^2(\Omega)$.

3. Internal observations

A stability result corresponding to an internal observation is derived along the same lines as in section 2. We first establish a Carleman estimate.

3.1. An internal Carleman inequality

Once again, Ω denotes a bounded open set in $\mathbb{R}^N$ with a Lipschitz boundary. Let $\omega \subset \Omega$ be any given open subset. We shall assume that there exists a function $\psi \in C^4(\overline{\Omega})$ such that

$$\nabla\psi \neq 0 \quad \text{in} \quad \overline{\Omega\backslash\omega}, \tag{3.1}$$

$$\frac{\partial\psi}{\partial n} \leqslant 0 \quad \text{on} \quad \partial\Omega, \tag{3.2}$$

$$|\nabla\psi(x)\cdot\xi|^2 + \sum_{i,j=1}^N(\partial_i\partial_j\psi(x))\xi_i\xi_j \geqslant 0 \qquad \forall x \in \overline{\Omega\backslash\omega}, \forall\xi = (\xi_1,\ldots,\xi_N) \in \mathbb{R}^N, \tag{3.3}$$

$$\psi(x) > \frac{2}{3}||\psi||_{L^\infty(\Omega)} \qquad \forall x \in \Omega. \tag{3.4}$$

Set $C_\psi = 2||\psi||_{L^\infty(\Omega)}$ and

$$\theta(x,t) := \frac{e^{\lambda\psi(x)}}{t(T-t)}, \qquad \varphi(x,t) := \frac{e^{\lambda C_\psi} - e^{\lambda\psi(x)}}{t(T-t)}, \qquad \forall (x,t) \in \Omega \times (0,T),$$

where λ denotes some positive number which will be specified later. We again introduce the set

$$\mathcal{Z} := \{q \in C^{2,1}(\overline{\Omega} \times [0,T]); q = 0 \text{ on } \Sigma\}.$$

Then we may derive the following Carleman estimate.

Proposition 3.1. *Assume that there exists a function* $\psi \in C^4(\overline{\Omega})$ *such that (3.1), (3.2), (3.3) and (3.4) hold for some* $\omega \subset \Omega$. *Then there exist constants* $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$ *and* $C_0 > 0$ *such that for all* $\lambda \geqslant \lambda_0$, *all* $s \geqslant s_0$, *and all* $q \in \mathcal{Z}$, *it holds*

$$\begin{aligned}
&\int_0^T\int_\Omega [\lambda^2 s\theta|\nabla q\cdot\nabla\psi|^2 + \lambda^4(s\theta)^3|q|^2 + |\widetilde{M}_1 q|^2 + |\widetilde{M}_2 q|^2]\, e^{-2s\varphi}\, dx\, dt \\
&\qquad + \int_0^T\int_{\partial\Omega} \lambda s\theta \left|\frac{\partial\psi}{\partial n}\right| \left|\frac{\partial q}{\partial n}\right|^2 e^{-2s\varphi}\, d\sigma\, dt \\
&\quad \leqslant C_0 \left(\int_0^T\int_\Omega |\partial_t q + i\Delta q|^2\, e^{-2s\varphi}\, dx\, dt \right. \\
&\qquad \left. + \int_0^T\int_\omega [\lambda s\theta|\nabla q|^2 + \lambda^4(s\theta)^3|q|^2]\, e^{-2s\varphi}\, dx\, dt \right),
\end{aligned} \tag{3.5}$$

where $\widetilde{M}_1$ *and* $\widetilde{M}_2$ *denote the operators defined in (2.5)–(2.6).*

Proof. The computations performed in the first step of the proof of proposition 2.1 are still valid, hence (2.24) still holds true. The control of the terms in the right-hand side of (2.24) is detailed in a series of claims.

Claim 5. There exist two numbers $\lambda_1 \geqslant 1$ and $s_1 \geqslant 1$ and some constants $A > 0$, $B > 0$ such that for all $\lambda \geqslant \lambda_1$ and all $s \geqslant s_1$,

$$A\lambda^4 s^3 \iint |u|^2\theta^3 - B\lambda^4 s^3 \iint_{\omega\times(0,T)} |u|^2\theta^3 \leqslant \iint |u|^2[s(\Delta^2\varphi - \varphi_{tt}) - 2s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2]. \tag{3.6}$$

Proof of claim 5. Since $\nabla\psi \neq 0$ on $\overline{\Omega\backslash\omega}$, it follows from (2.27) that for λ large enough, say $\lambda \geqslant \lambda_1$, we have

$$-\nabla|\nabla\varphi|^2 \cdot \nabla\varphi \geqslant C\lambda^4\theta^3 \qquad \text{on} \quad \Omega\backslash\omega.$$

On the other hand, we infer from (2.27) that for any $\lambda \geqslant \lambda_1$ and some constant $C' > 0$

$$|\nabla|\nabla\varphi|^2 \cdot \nabla\varphi| \leqslant C'\lambda^4\theta^3 \qquad \text{on} \quad \omega.$$

Therefore, for $\lambda \geqslant \lambda_1$,

$$\iint |u|^2(-2s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2) \geqslant 2C\lambda^4 s^3 \iint_{(\Omega\backslash\omega)\times(0,T)} |u|^2\theta^3 - 2C'\lambda^4 s^3 \iint_{\omega\times(0,T)} |u|^2\theta^3$$

$$\geqslant 2C\lambda^4 s^3 \iint |u|^2\theta^3 - 2(C+C')\lambda^4 s^3 \iint_{\omega\times(0,T)} |u|^2\theta^3. \tag{3.7}$$

On the other hand, by (3.4) and (2.26), $|\Delta^2\varphi| \leqslant \text{const}\,\lambda^4\theta$ and $|\varphi_{tt}| \leqslant \text{const}\,\theta^3$ on $\Omega\times(0,T)$, hence

$$\iint |u|^2[s(\Delta^2\varphi - \varphi_{tt})] \leqslant C''\lambda^4 s \iint |u|^2\theta^3. \tag{3.8}$$

Combining (3.7) and (3.8) yields (3.6) for s large enough ($s \geqslant s_1$) and some positive constants A, B. □

It follows from the proof of claim 3 that there exists some constant $A' > 0$ such that

$$\left|\text{Im}\left\{\iint u\nabla\varphi_t\cdot\nabla\overline{u}\right\}\right| \leqslant \iint \theta|\nabla\psi\cdot\nabla u|^2 + A'\lambda^2 \iint \theta^3|u|^2. \tag{3.9}$$

We proceed with the last claim.

Claim 6. There exist some numbers $\lambda_2 \geqslant \lambda_1$, $s_2 \geqslant s_1$, $A'' > 0$ and $B'' > 0$, such that for all $\lambda \geqslant \lambda_2$, all $s \geqslant s_2$

$$\begin{aligned} -4s \iint \partial_j\partial_i\varphi\partial_j u\partial_i\overline{u} &- 4s\,\text{Im}\left\{\iint u\nabla\varphi_t\cdot\nabla\overline{u}\right\} \\ &+ \iint |u|^2[s(\Delta^2\varphi - \varphi_{tt}) - 2s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2] \\ &\geqslant A'' \iint [\lambda^2 s\theta|\nabla\psi\cdot\nabla u|^2 + \lambda^4(s\theta)^3|u|^2] \\ &- B'' \iint_{\omega\times(0,T)} [\lambda s\theta|\nabla u|^2 + \lambda^4(s\theta)^3|u|^2]. \end{aligned} \tag{3.10}$$

Proof of claim 6. Using (3.6), (3.9) and (2.29), we obtain that

$$\begin{aligned} -4s \iint \partial_j\partial_i\varphi\partial_j u\partial_i\overline{u} &- 4s\,\text{Im}\left\{\iint u\nabla\varphi_t\cdot\nabla\overline{u}\right\} \\ &+ \iint |u|^2[s(\Delta^2\varphi - \varphi_{tt}) - 2s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2] \\ &\geqslant 4s \iint \theta[(\lambda^2-1)|\nabla\psi\cdot\nabla u|^2 + \lambda\partial_j\partial_i\psi\partial_j u\partial_i\overline{u}] \\ &+ \iint (A\lambda^4 s^3 - 4A'\lambda^2 s)\theta^3|u|^2 - B\lambda^4 s^3 \iint_{\omega\times(0,T)} \theta^3|u|^2 \\ &\geqslant \lambda^2 s \iint \theta|\nabla\psi\cdot\nabla u|^2 + \frac{A}{2}\lambda^4 s^3 \iint \theta^3|u|^2 \\ &- 4N\lambda s||\psi||_{W^{2,\infty}(\omega)} \iint_{\omega\times(0,T)} \theta|\nabla u|^2 - B\lambda^4 s^3 \iint_{\omega\times(0,T)} \theta^3|u|^2 \end{aligned}$$

for s and λ large enough. The claim is proved. □

We infer from (2.24), (3.10) and the fact that $\frac{\partial \varphi}{\partial n} \geqslant 0$ on Σ that for all $\lambda \geqslant \lambda_2$ and all $s \geqslant s_2$

$$||M_1 u||^2 + ||M_2 u||^2 + \iint [\lambda^2 s\theta |\nabla\psi \cdot \nabla u|^2 + \lambda^4 (s\theta)^3 |u|^2] + s \iint_\Sigma \left|\frac{\partial \varphi}{\partial n}\right| \left|\frac{\partial u}{\partial n}\right|^2$$

$$\leqslant C \left(||w||^2 + \iint_{\omega\times(0,T)} [\lambda s\theta |\nabla u|^2 + \lambda^4 (s\theta)^3 |u|^2] \right). \tag{3.11}$$

Replacing u by $\mathrm{e}^{-s\varphi} q$ in (3.11) yields (3.5). □

Note that the term ∇q does not appear in the same way in both sides of (3.5). The symmetry may however be restored in the following two situations: (i) the assumption (3.3) is strengthened; (ii) the function ψ depends only on one variable in Ω.

Corollary 3.2. *Assume the existence of a function $\psi \in C^4(\overline{\Omega})$ such that (3.1), (3.2), (3.4) hold for some $\omega \subset \Omega$, and that for some constant $\mu > 0$*

$$|\nabla\psi(x) \cdot \xi|^2 + \sum_{i,j=1}^N (\partial_i \partial_j \psi(x)) \xi_i \xi_j \geqslant \mu |\xi|^2 \qquad \forall x \in \overline{\Omega \backslash \omega}, \forall \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N, \tag{3.12}$$

Then there exist constants $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$ and $C_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, all $s \geqslant s_0$, and all $q \in \mathcal{Z}$, it holds

$$\int_0^T \int_\Omega [\lambda s\theta |\nabla q|^2 + \lambda^4 (s\theta)^3 |q|^2 + |\widetilde{M}_1 q|^2 + |\widetilde{M}_2 q|^2]\, \mathrm{e}^{-2s\varphi}\, \mathrm{d}x\, \mathrm{d}t$$

$$\leqslant C_0 \left(\int_0^T \int_\Omega |\partial_t q + \mathrm{i}\Delta q|^2\, \mathrm{e}^{-2s\varphi}\, \mathrm{d}x\, \mathrm{d}t \right.$$

$$\left. + \int_0^T \int_\omega [\lambda s\theta |\nabla q|^2 + \lambda^4 (s\theta)^3 |q|^2]\, \mathrm{e}^{-2s\varphi}\, \mathrm{d}x\, \mathrm{d}t \right). \tag{3.13}$$

Indeed,

$$4s \iint_{(\Omega\backslash\omega)\times(0,T)} \theta[(\lambda^2 - 1)|\nabla\psi \cdot \nabla u|^2 + \lambda \partial_j \partial_i \psi \partial_j u \partial_i \overline{u}] \geqslant C\lambda s \iint_{(\Omega\backslash\omega)\times(0,T)} \theta |\nabla u|^2.$$

Corollary 3.3. *Assume the existence of a function $\psi \in C^4(\overline{\Omega})$ such that (3.1), (3.2), (3.3) and (3.4) hold for some $\omega \subset \Omega$, and such that $\psi(x)$ depends only on x_1 on Ω. Then there exist some constants $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$ and $C_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, all $s \geqslant s_0$ and all $q \in \mathcal{Z}$, it holds*

$$\int_0^T \int_\Omega \left[\lambda^2 s\theta \left| \frac{\partial q}{\partial x_1} \right|^2 + \lambda^4 (s\theta)^3 |q|^2 + |\widetilde{M}_1 q|^2 + |\widetilde{M}_2 q|^2 \right] \mathrm{e}^{-2s\varphi}\, \mathrm{d}x\, \mathrm{d}t$$

$$\leqslant C_0 \left(\int_0^T \int_\Omega |\partial_t q + i\Delta q|^2 \mathrm{e}^{-2s\varphi}\, \mathrm{d}x\, \mathrm{d}t \right.$$

$$\left. + \int_0^T \int_\omega \left[\lambda^2 s\theta \left| \frac{\partial q}{\partial x_1} \right|^2 + \lambda^4 (s\theta)^3 |q|^2 \right] \mathrm{e}^{-2s\varphi}\, \mathrm{d}x\, \mathrm{d}t \right). \tag{3.14}$$

Example 3.4. Below we list some examples where proposition 3.1 may be applied.

(i) Ω is a ball: $\Omega = B_R(0)$.

 (a) $\omega = B_R(0) \cap \{x; 0 < x_1 < \varepsilon\}$, for an arbitrarily small $\varepsilon > 0$; or

 (b) ω is a neighborhood of a half-sphere.

 Moreover, one may take $\psi(x) = \pm x \cdot e_1 + C$ on each connected component of $\Omega\backslash\omega$.

(ii) Ω is a rectangle and ω is a 'strip':

$$\omega := (a, a+\varepsilon) \times (-L_2, L_2) \times \cdots \times (-L_N, L_N).$$

 Actually, Corollary 3.3 may be applied as well.

(iii) Ω is a stadium:

 (a) ω is a strip $\omega := (a, a+\varepsilon) \times (-L_2, L_2)$; or

 (b) ω is a neighborhood of a half-sphere.

(iv) There exist open sets $\Omega_j \subset \Omega$ with Lipschitz boundary $\partial\Omega_j$ and vectors $\xi^j \in \mathbb{R}^N\backslash\{0\}$, $j = 1, 2, \ldots, J$, such that $\Omega_i \cap \Omega_j = \emptyset$ for $1 \leqslant i < j \leqslant J$ and

$$\mathcal{N}_\varepsilon\left[\left(\cup_{j=1}^J \Gamma_j\right) \cup \left(\Omega\backslash \cup_{j=1}^J \Omega_j\right)\right] \subset \omega$$

 for some $\varepsilon > 0$, where

$$\mathcal{N}_\varepsilon[S] := \cup_{x\in S} B_\varepsilon(x) \qquad \text{for} \quad S \subset \mathbb{R}^N,$$

 and

$$\Gamma_j := \{x \in \partial\Omega_j; \xi^j \cdot n^j(x) > 0\},$$

 n^j denoting the unit outward normal vector to $\partial\Omega_j$. A function ψ fulfilling (3.1)–(3.4), and of the form $\psi(x) = \xi^j \cdot x + C_j$ on $\Omega_j\backslash\omega$ for $j = 1, 2, \ldots, J$, may be constructed by means of a partition of unity. The details are left to the reader. Recall that the exact controllability of the wave equation has been proved in [13, theorem 4.1] under similar assumptions. Note however that, in the definition of Γ_j, the set $\{x \in \partial\Omega | \xi^j \cdot n^j(x) > 0\}$ is often strictly included in the set $\left\{x \in \partial\Omega \,\middle|\, \left(x - x_0^j\right) \cdot n^j(x) > 0\right\}$ used for instance in [13].

3.2. The inverse problem, internal observations

As in section 2.2, we denote by $u(p)$ the solution of the system (2.38) associated with the potential p.

Theorem 3.5. *Assume that there exists a function $\psi \in C^4(\overline{\Omega})$ such that (3.1), (3.2), (3.3) and (3.4) hold for some $\omega \subset \Omega$. Suppose also that $p \in L^\infty(\Omega; \mathbb{R})$, $u_0 \in L^\infty(\Omega)$ and $r > 0$ satisfy*

- *$u_0(x) \in \mathbb{R}$ or $\mathrm{i}u_0(x) \in \mathbb{R}$ a.e. in Ω,*
- *$|u_0(x)| \geqslant r > 0$ a.e. in Ω, and*
- *$u(p) \in H^1(0, T; L^\infty(\Omega))$.*

Then, for any $m \geqslant 0$, there exists a constant $C = C(m, \|u(p)\|_{H^1(0,T;L^\infty(\Omega))}, r) > 0$ such that for any $q \in B_m(0) \subset L^\infty(\Omega; \mathbb{R})$ satisfying

$$u(p) - u(q) \in H^1(0, T; H^1(\omega))$$

we have that

$$\|p - q\|_{L^2(\Omega)} \leqslant C \, \|u(p) - u(q)\|_{H^1(0,T;H^1(\omega))} . \tag{3.15}$$

Proof. We follow the same steps as in the proof of theorem 2.7 until (2.44). More precisely, let $f = q - p$ and let R and y be the even-conjugate extensions to $(-T, T)$ of $u(p)$ and $u(p) - u(q)$

respectively in the case $u_0 \in \mathbb{R}$ (or the odd-conjugate extensions if $iu_0 \in \mathbb{R}$). Changing t into $t + T$, we consider y and R defined on $\Omega \times (0, 2T)$ and we put $z(x, t) = y_t(x, 2T - t)$ so that (2.41) holds. Inequalities (2.43) and (2.44) are still valid in the present case of internal observations. Nevertheless, at this point we use the internal Carleman inequality (3.5) to obtain

$$\begin{aligned}\frac{r^2}{2}\int_\Omega e^{-2s\varphi(x,T)}|f|^2\,dx \leqslant{}& C\lambda^{-2}s^{-3/2}\left(\int_0^{2T}\int_\Omega e^{-2s\varphi}|fR_t|^2\,dx\,dt\right.\\ &\left.+\lambda s\int_0^{2T}\int_\omega \theta e^{-2s\varphi}\,|\nabla z|^2\,dx\,dt + \lambda^4 s^3\int_0^{2T}\int_\omega \theta^3 e^{-2s\varphi}\,|z|^2\,dx\,dt\right)\\ \leqslant{}& C\lambda^{-2}s^{-3/2}\int_\Omega e^{-2s\varphi(x,T)}|f|^2\,dx\\ &+C\lambda^{-1}s^{-1/2}\int_0^{2T}\int_\omega |\nabla z|^2\,dx\,dt + C\lambda^2 s^{3/2}\int_0^{2T}\int_\omega |z|^2\,dx\,dt. \end{aligned} \tag{3.16}$$

Therefore, for s and λ sufficiently large, we deduce that

$$\begin{aligned}\|f\|_{L^2(\Omega)} &\leqslant C(\|\nabla z\|_{L^2(\omega\times(0,2T))} + \|z\|_{L^2(\omega\times(0,2T))})\\ &\leqslant C\|z\|_{L^2(0,2T;H^1(\omega))}\\ &\leqslant 2C\|y_t\|_{L^2(0,T;H^1(\omega))}\end{aligned}$$

from which the stability inequality (3.15) follows. □

Remark 3.6. The previous stability theorem shows that in the four cases of pairs (Ω, ω) given in example 3.4, by observing the time derivative of the solution of the Schrödinger equation (2.38) and its gradient in $\omega \times (0, T)$, we can locally recover its potential in $L^2(\Omega)$.

Acknowledgments

This work was performed when the third author was visiting the Centro de Modelamiento Matematico at the Universidad de Chile, UMI CNRS 2807. The third author wants to thank that institution for its hospitality, and the CNRS for its support. The second author was partially supported by FONDECYT grant 1061263 and STIC-AMSUD grants. The first and second authors acknowledge ECOS C04E08 grant.

References

[1] Baudouin L and Puel J-P 2002 Uniqueness and stability in an inverse problem for the Schrödinger equation *Inverse Problems* **18** 1537–54
[2] Bukhgeim A and Klibanov M 1982 Global uniqueness of a class of inverse problems *Sov. Math. Dokl.* **24** 244–7
[3] Burq N and Zworski M 2004 Geometric control in the presence of a black box. *J. Am. Math. Soc.* **17** 443–71
[4] Cardoulis L, Cristofol M and Gaitan P 2006 *Inverse Problem for the Schrödinger Operator in an Unbounded Strip* http://arxiv.org/abs/math/0612539v1
[5] Egorov Yu V 1986 Linear differential equations of principal type *Contemporary Soviet Mathematics* (New York: Consultants Bureau)
[6] Imanuvilov O and Yamamoto M 2003 Determination of a coefficient in an acoustic equation with a single measurement *Inverse Problems* **19** 157–71
[7] Isakov V 1993 Carleman type estimates in an anisotropic case and applications *J. Diff. Eqns* **105** 217–38
[8] Isakov V 1998 *Inverse Problems for Partial Differential Equations* (Berlin: Springer)
[9] Kenig C, Sjöstrand J and Uhlmann G 2007 The Calderón problem with partial data *Ann. Math.* **165** 567–91
[10] Klibanov M and Malinsky J 1991 Newton–Kantorovich method for three-dimensional potential inverse scattering problem and stability of the hyperbolic Cauchy problem with time-dependent data *Inverse Problems* **7** 577–95

[11] Komornik V and Loreti P 2005 *Fourier Series in Control Theory (Springer Monographs in Mathematics)* (New York: Springer)

[12] Lasiecka I, Triggiani R and Zhang X 2004 Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates: Part I. H1-estimates *J. Inv. Ill-posed Problems* **11** 43–123

[13] Liu K 1997 Locally distributed control and damping for the conservative systems *SIAM J. Control Optim.* **35** 1574–90

[14] Machtyngier E 1994 Exact controllability for the Schrödinger equation *SIAM J. Control Optim.* **32** 24–34

[15] Sakai A 1989 A characterization of weak pseudoconvexity *Proc. Am. Math. Soc.* **105** 314–6

[16] Tataru D 1997 Carleman estimates, unique continuation and controllability for anizotropic PDE's *Contemp. Math.* **209** 267–79

[17] Ramdani K, Takahashi T, Tenenbaum G and Tucsnak M 2005 A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator *J. Funct. Anal.* **226** 193–229

[18] Rosier L and Zhang B-Y *Null controllability of the complex Ginzburg–Landau equation Ann. Inst. H. Poincaré Anal. Non Linéaire* to appear

[19] Saint Raymond X 1986 Résultats d'unicité de Cauchy instable dans des situations où la condition de pseudo-convexité dégénère *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **13** 661–87

[20] Zuazua E 2003 Remarks on the controllability of the Schrödinger equation *Quantum Control: Mathematical and Numerical Challenges (CRM Proc. Lect. Notes Ser.)* ed A Bandrauk, M C Delfour and C Le Bris (Providence, RI: American Mathematical Society) pp 181–99