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Four variations in global Carleman weights applied to inverse and controllability problems*

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Abstract. This paper reviews four variants of global Carleman weights that are especially adapted to some singular controllability and inverse problems in partial differential equations. These variants arise when studying: i) one measurement stationary source inverse problems for the heat equation with discontinuous coefficients, ii) one measurement stationary potential inverse problems for the heat equation with discontinuous coefficients, iii) null controllability for fluid-structure problems in mobile domains and iv) recovering coefficients from locally supported boundary observations for the wave equation. In all the case we explain how to explicitly construct the Carleman weight.

Mathematical subject classification: 74G75, 76D05, 93B05.

Key words: Carleman inequalities, exact controllability, inverse problems, Navier-Stokes equations.

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1 Introduction

Let Ω be a regular domain in $\mathbb{R}^n$ and let us consider a second order adjoint operator of the form $P_q^* z = f$ evolving in $Q = \Omega \times I$, where I is a time interval. We suppose that P_q depends on some stationary parameter $q \in L^\infty(\Omega)$. Given

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some suitable weight function Φ defined in Q, we perform the following change of variables or *conjugation*:

$$w = \rho z, \quad \rho = \exp(-s\Phi(x,t)), \quad s > 0, \tag{1}$$

$$P_q^* z = f \Leftrightarrow \rho P_q^*(\rho^{-1} w) = \rho f. \tag{2}$$

For a given parameter $\lambda > 0$ and α larger enough, typical weights functions Φ are of the form:

Heat equation: $P_q^* = -\delta_t - \Delta + q$, $Q = \Omega \times (0,T)$

$$\Phi(x,t) = \frac{\exp(\lambda\alpha) - \exp(\lambda\psi(x))}{T-t} \tag{3}$$

where $\psi(x)$ is some suitable continuous function to be precised later (see for instance Table 1 for some conditions on ψ and Figure 1 for typical shapes of ψ).

Wave equation: $P_q^* = \delta_{tt} - \Delta + q$, $Q = \Omega \times (-T,T)$

$$\Phi(x,t) = -\exp(\lambda(\psi(x) - \beta t^2)), \ \psi(x) = |x - x_0|^2, \tag{4}$$

where x_0 is some given point outside $\overline{\Omega}$ and $\beta \in (0,1)$ is suitably chosen.

Schrödinger equation: $P_q^* = i\partial_t + \Delta + q$, $Q = \Omega \times (-T,T)$

$$\Phi(x,t) = \frac{\exp(\lambda\alpha) - \exp(\lambda\psi(x))}{(T-t)(T+t)}, \ \psi(x) = |x - x_0|^2, \tag{5}$$

where x_0 is some given point outside $\overline{\Omega}$.

We also introduce a function $\varphi(x,t)$ such that

$$\nabla\Phi = -\lambda\nabla\psi\,\varphi.$$

We consider an *internal observational or control region* $\omega \subset\subset \Omega$ and a *boundary observational or control region* $\Gamma_0 \subset \partial\Omega$. Under some assumptions, we will work with *global Carleman inequalities* of the form

$$\begin{aligned} & p_1(s,\lambda)\|\varphi^{3/2}\rho\,\nabla z\|^2_{L^2(Q)} + p_0(s,\lambda)\|\varphi^{1/2}\rho\, z\|^2_{L^2(Q)} \\ & \leq C\left(\|\rho f\|^2_{L^2(Q)} + p_1(s,\lambda)\|\varphi^{3/2}\rho\,\nabla z\cdot n\|^2_{L^2(\Gamma_0\times I)} + p_0(s,\lambda)\|\varphi^{1/2}\rho\, z\|^2_{L^2(\omega\times I)}\right), \end{aligned} \tag{6}$$

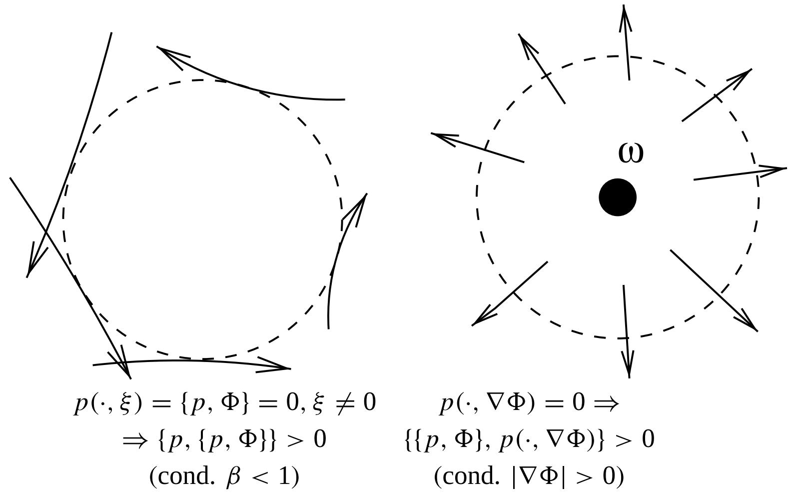

Figure 1 – Graphical interpretation of pseudoconvexity for $\Phi = |x - x_0|^2 - \beta t^2$ (waves) with increasing velocity from the outer to the inner levels of Φ. $p = \xi_0^2 - |\xi|^2$ is the principal symbol of P^*. Left: rays are bicharacteristics, right: arrows are $\nabla\Phi$.

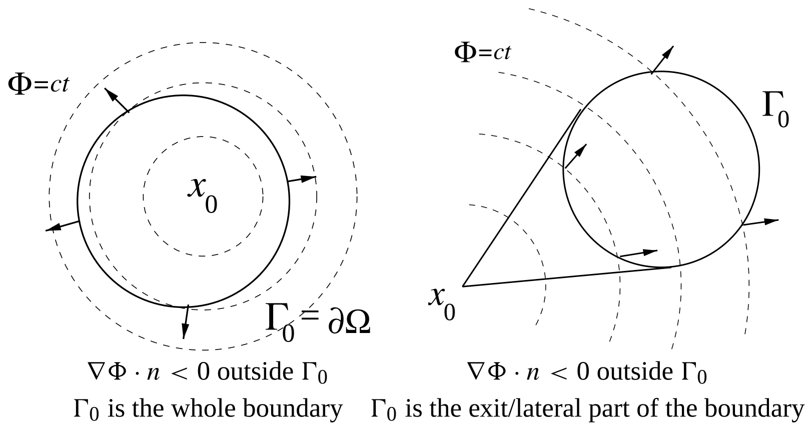

Figure 2 – Graphical interpretation of the strong Lopatinskii condition for $\Phi = |x - x_0|^2 - \beta t^2$ (waves) in the cases that x_0 is inside and outside of the domain. The level sets of the weight Φ are represented by the dotted lines.

where n is the unit exterior normal to Ω, p_i are polynomial weights (see Table 2) and ρ is the weight function given by (1). Notice that $\rho \to 0$ exponentially as $s\Phi \to +\infty$.

The internal observational or control region ω appearing at the right hand side of the global Carleman inequality is such that the pseudoconvexity of Φ with respect to P_q^* holds *outside* ω (for pseudoconvexity notion see [20], [43]). The boundary observational or control region Γ_0 is such that a strong Lopatinskii condition holds *outside* Γ_0 (see Table 1). If the condition of pseudoconvexity is satisfied in all $\Omega \times I$ then ω can be empty. Also, if the strong Lopatinskii condition holds on all $\partial\Omega$ then Γ_0 can be empty (see [43] for a much more general statement of global Carleman inequalities in this cases).

	outside ω	outside Γ_0
Condition	$\lvert\nabla\psi(x)\rvert > 0$	$\nabla\psi(x) \cdot n < 0$
	(necessary to pseudoconvexity)	(strong Lopatinskii)

Table 1 – Pseudoconvexity and strong Lopatinskii conditions could not be satisfied in the internal and boundary observational/control regions.

Equation	p_1	p_0
Heat	$s\lambda^2$	$s^3\lambda^4$
Wave	$s\lambda$	$s^3\lambda^3$
Schrödinger	$s\lambda$	$s^3\lambda^4$

Table 2 – Polynomial weights in global Carleman inequalities.

Some variants we consider in this review appear when considering operators with *discontinuous coefficients* in the principal part. In this case, the function ψ has to be well adapted to this new situation and specific global Carleman estimates can be derived. In both cases, some spatial monotonicity of the coefficients is needed. As an application of these inequalities, we study one measurement inverse problems for the heat and wave equations using the general Bukhgeim-Klibanov approach [11]. The results explained here have been collected from the articles [13], [6], [7] and the preprint [2].

Other interesting variants we consider here arise in the case of *mobile domains* in fluid-structure problems, when studying the boundary null controllability of an immersed solid into a viscous Navier-Stokes fluid. In this case, the function ψ depends on time, and the global Carleman inequality is much more complicated than (6) because on one hand of incompressibility in Navier-Stokes and on the other hand due to the presence of the structure. The results we present in this review were adapted from the articles [9], [10].

The last variant is concerned with one measurement inverse problems from *local boundary observations* for the wave equation. In this case, the function ψ is modified in order to obtain some strong Lopatinskii condition of the form $(x - x_0) \cdot Tn < 0$, where T is some linear transformation of the normal field. For further details we refer to the article [14] where the Carleman weights were introduced for two dimensional domains. Here we explain how to deal with the three dimensional case.

Although this is a reduced selection of variants, this collection of Carleman weights and applications illustrate the wealth of the extent of Carleman inequalities when they are applied to the study of some singular inverse and controllability problems.

2 Inverse source problem for heat transmission problems

Given $\Omega \subset \mathbb{R}^n$ be a bounded and regular subset and let Ω_1 be a subdomain such that $\overline{\Omega}_1 \subset \Omega$ and let us set $\Omega_0 = \Omega \setminus \overline{\Omega}_1$. Let S be the interface between Ω_0 and Ω_1 with unit normal n exterior to Ω_1. Let us denote by S^+ and S^- the outer and inner sides of the interface S with respect to n and $\Sigma^+ = S^+ \times (0, T)$, $\Sigma^- = S^- \times (0, T)$.

Let us consider the heat transmission problem

$$\begin{cases} y_t - \operatorname{div}(a_0(x)\nabla y) = f(x)g(x,t) & \text{in } \Omega_0 \times (0,T) \\ y_t - \operatorname{div}(a_1(x)\nabla y) = f(x)g(x,t) & \text{in } \Omega_1 \times (0,T) \\ y|_{\Sigma^+} = y|_{\Sigma^-}, \quad a_0 \dfrac{\partial y}{\partial n}|_{\Sigma^+} = a_1 \dfrac{\partial y}{\partial n}|_{\Sigma^-}, & y = 0 \text{ on } \partial\Omega \times (0,T) \ (*) \end{cases} \tag{7}$$

with $a_i \geq c_0 > 0$ a.e. in Ω. Let us introduce the space

$$V = \left\{ y \in C^2(\overline{\Omega}_i \times [0,T]),\ i = 0,1,\ y \text{ satisfies } (*) \right\}. \tag{8}$$

The inverse source problem consists in retrieving the source $f(x)$ from the knowledge of $g(x,t)$, the local trace of the solution y in $\omega_0 \times (0,T)$, where $\overline{\omega}_0 \subset \Omega_0$ and from some time slice $y(\cdot, T_0)$ for some $T_0 \in (0,T)$, but without any knowledge of the initial condition $y(\cdot, 0)$ of the system. We have to assume also that some isotopy type condition is satisfied (see Figure 2 and the details in the article [13]). The inverse stability result is

Theorem 2.1 ([6], [7]). *Let $T_0 \in (0, T)$ and $\omega_0 \subset \Omega_0$ and let us assume that Ω_1 and Ω_0 satisfy the isotopy type conditions of* [13]. *Assume that y solution of* (7) *such that $y, y_t \in V$. Assume that $a_1|_{S^-} - a_0|_{S^+} \geq 0$ and that $g \in C^2(\overline{\Omega} \times [0, T])$, $|g(\cdot, T_0)| \geq r_0 > 0$ a.e. in Ω. Then there exists a constant $C = C(g, \omega_0, T_0)$ such that for all $f \in L^2(\Omega)$*

$$\|f\|_{L^2(\Omega)} \leq C \left(\|y(\cdot, T_0)\|_{H^2(\Omega_0)} + \|y(\cdot, T_0)\|_{L^2(\Omega_1)} + \|y\|_{H^1(0,T;L^2(\omega_0))} \right). \quad (9)$$

This result has as main ingredient a global Carleman estimate for the system (7) stated in [13]. This inequality was firstly used in order to prove the exact controllability to trajectories for a semilinear system similar to (7) that is controlled in $\omega_0 \times (0, T)$. In the general case when Ω_1 is not simply connected, and in order to construct the weight functions, an isotopy type condition between S and the boundary of two disjoint open subsets O_i, $i = 1, 2$ of Ω_1 is used. Two weights similar to (3) are then constructed of the form

$$\Phi_i(x, t) = \frac{\exp(\lambda\alpha) - \exp(\lambda\psi_i(x))}{T - t}, \; i = 1, 2, \quad (10)$$

where $\psi_i \in V$ and $\nabla\psi = 0$ only in O_i (see Figure 1 left). Notice that you can also consider the opposite case when $\overline{\Omega}_0 \subset \Omega$ and $\Omega_1 = \Omega \setminus \overline{\Omega}_0$, and always $\overline{\omega}_0 \subset \Omega_0$. In this case, an isotopy type condition between $\partial\Omega$ and S is a sufficient condition. See Figure 1 right).

3 Controllability problems in mobile domain for fluid-structure interaction

Let $\Omega \subset \mathbb{R}^2$ be a fixed bounded connected open subset with regular boundary. We denote respectively by $\Omega_S(t)$ and $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ the domains occupied by the structure (we consider here only one connected component of solid but the results shown here are still valid for a finite number of solids) and by the fluid respectively. Let n be the unit exterior normal to $\partial\Omega_S(t)$. The time evolution for $t \in (0, T)$ of the fluid eulerian velocity u and pressure p is governed by

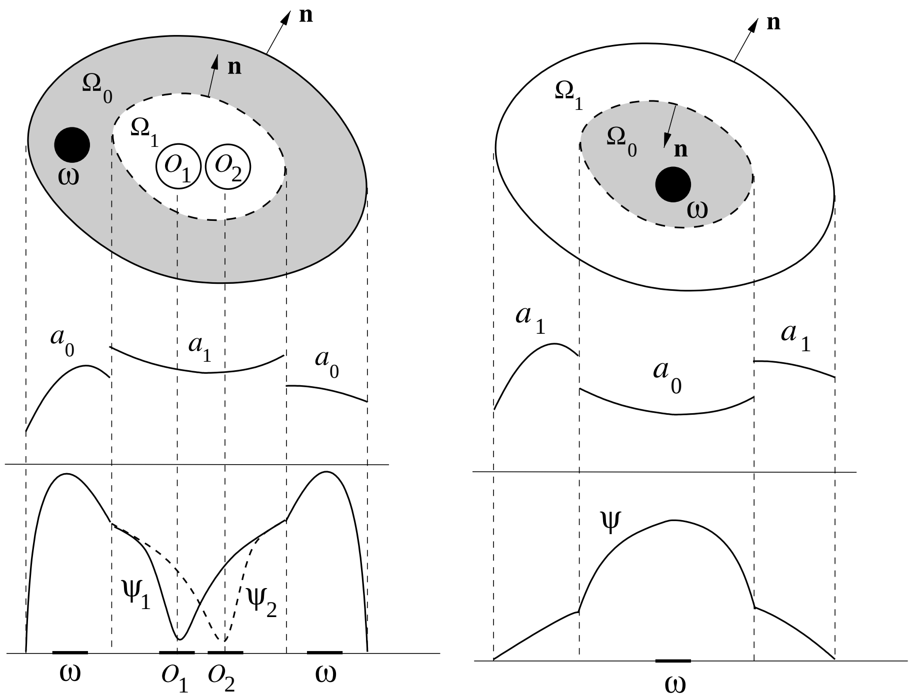

Figure 3 – Construction of the global Carleman weight (bottom curves) for the heat equation with discontinuous coefficients such that $a_1(S^-) - a_0(S^+) > 0$ (middle curves). In the case $\overline{\Omega}_0 \subset \Omega$ (left) two combined weights are used and in the case $\overline{\Omega}_1 \subset \Omega$ (right) one weight suffices. In both cases the observation zone ω is represented by a black dot.

the incompressible Navier-Stokes equations where the Cauchy tensor $\sigma(u, p) = \nu(\nabla u + \nabla u^t) - p\,\mathrm{Id}$ with viscosity $\nu > 0$ will appear. The movement of the rigid solid with mass $m > 0$ and moment of inertia $J > 0$ is described by the velocity of its center of mass $a(t) \in \mathbb{R}^2$ and by its angular velocity $r(t) \in \mathbb{R}$. The system is

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \operatorname{div}\sigma(u, p) = f 1_\omega, \ \operatorname{div} u = 0 \text{ in } \ \Omega_F(t) \\ m\ddot{a} = \displaystyle\int_{\partial\Omega_S(t)} \sigma(u, p) n d\sigma, \ J\dot{r} = \int_{\partial\Omega_S(t)} (\sigma(u, p)n) \cdot (x - a)^\perp d\sigma, \\ u = \dot{a} + r(x - a)^\perp \text{ on } \ \partial\Omega_S(t), \ u = 0 \text{ on } \ \partial\Omega, \\ u(0, \cdot) = u_0 \text{ in } \ \Omega_F(0), \ a(0) = a_0, \ \dot{a}(0) = a_1, \ r(0) = r_0, \end{cases} \tag{11}$$

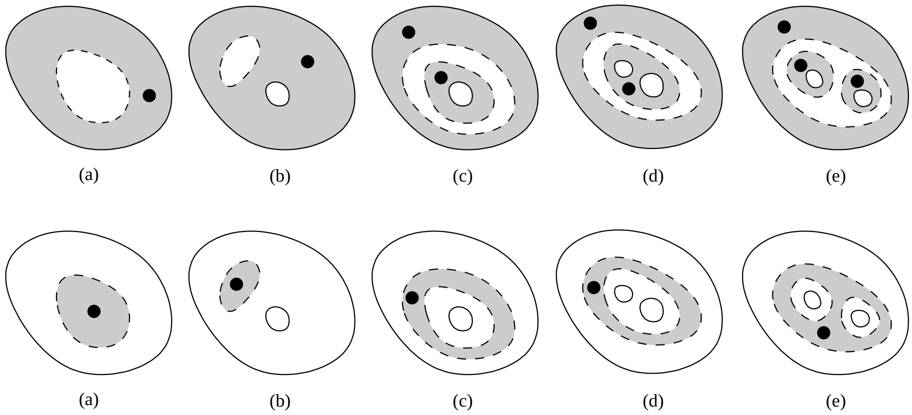

Figure 4 – In all the figures Ω_0 is filled in gray and the observation region ω is represented by a black dot. The interface S between Ω_0 and Ω_1 is represented by a dashed line and the boundary $\partial\Omega$ by a solid line. In the lower line, all the examples except for (b) and (d) satisfy the isotopy type condition of [13] between S and $\partial\Omega$. In the upper line, all the examples except for (e) satisfy the isotopy type condition between S and the boundaries of two disjoints subsets O_1 and O_2 of Ω_1. In all the exceptions, the Carleman weight can not be constructed with the method of [13].

Here the function f is the *control* function which acts over a fixed small non-empty open subset ω (with characteristic function 1_ω). See Figure 5.

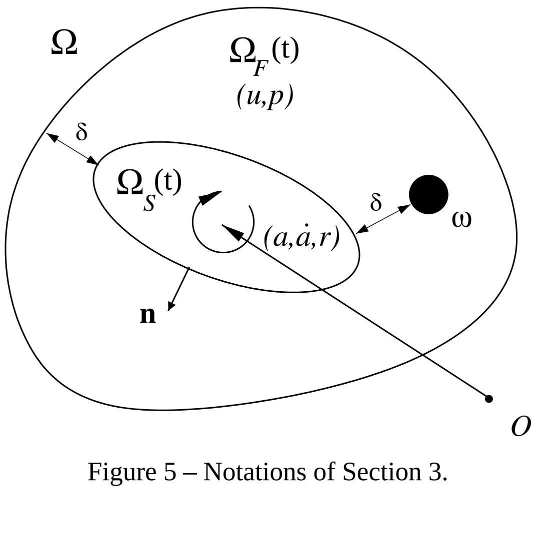

Figure 5 – Notations of Section 3.

We have used the notation $x^\perp = (x_1, x_2)^\perp = (-x_2, x_1)$. The total angle θ associated to the angular velocity r is defined by $\theta(t) = \theta_0 + \int_0^t r(s)\, ds$, where $\theta_0 \in \mathbb{R}$ complements the initial data. The existence of solutions and regularity for this system has been recently studied in several papers (see [39], [41] and the references therein).

The controllability result is the following, saying that it is possible to drive the structure and the fluid at rest and the immersed solid up to its reference position in arbitrarily small time with a localized control f, provided the initial conditions are sufficiently small.

Theorem 3.1 ([9]). *Suppose that: $i)$ the initial body solid shape satisfies*

$$\Omega_S(0) \subset \Omega \setminus \omega,\ d(\Omega_S(0), \partial(\Omega \setminus \omega)) > 0,\ \int_{\partial\Omega_S(0)} (y - a_0)\, d\sigma = 0; \tag{12}$$

$ii)$ the initial conditions $u_0 \in H^3(\Omega_F(0))^2$, $a_0 \in \mathbb{R}^2$, $a_1 \in \mathbb{R}^2$, $\theta_0 \in \mathbb{R}$ and $r_0 \in \mathbb{R}$ satisfy the compatibility conditions

$$\begin{aligned} &\operatorname{div} u_0 = 0 \ \ in\ \ \Omega_F(0), \\ &u_0 = a_1 + r_0(x - a_0)^\perp \ \ on\ \ \partial\Omega_S(0) \quad and \\ &u_0 = 0 \ \ on\ \ \partial\Omega; \end{aligned} \tag{13}$$

$iii)$ the acceleration u_1 of the fluid and the accelerations a_2 and r_1 of the structure at initial time (determined by the equations of the motion (11) and by the boundary conditions taken at initial time, well defined thanks to Helmholtz decomposition) satisfy

$$\begin{aligned} u_1 &= 0 \ \ on\ \ \partial\Omega, \\ u_1 &= a_2 + r_1(x - a_0)^\perp - r_0^2(x - a_0) \\ &\qquad - \nabla u_0\big(a_1 + r_0(x - a_0)^\perp\big) \ \ on\ \ \partial\Omega_S(0). \end{aligned} \tag{14}$$

Under the above assumptions, for all $T > 0$ there exists $\varepsilon > 0$ and $f \in L^2((0, T) \times \omega)^2$ such that if

$$\|u_0\|_{H^3(\Omega_F(0))^2} + |a_0| + |a_1| + |\theta_0| + |r_0| \le \varepsilon$$

then

$$u(T, \cdot) = 0 \ \ in\ \Omega_F(T),\ a(T) = 0,\ \dot{a}(T) = 0,\ \theta(T) = 0,\ r(T) = 0.$$

The last condition in (12) is a symmetry restriction over the shape of the solid needed in the proof of the Carleman inequality satisfied by the adjoint problem of the linearized system to get estimates on the structure motion from estimates on the fluid velocity on the interface.

The result only holds for small initial data because we want to keep the non-collision condition on the whole interval $(0, T)$

$$\inf_{t\in(0,T)} \mathrm{d}(\Omega_S(t), \partial(\Omega \setminus \omega)) > 0.$$

The first result of this kind using global Carleman estimates was obtained in [12] for a one-dimensional Burgers-particle system studied in [45]. Also, similar results to the one presented here has been simultaneously and independently obtained in the preprint [12] (see the preprint version of [10]).

The idea of the proof is the following and follows ideas for the controllability of the Navier-Stokes equations recently used in [15], [21] and the ideas of [12]. We first consider a linearized problem. Let $(\tilde{a}, \tilde{r})$ be given in $H^2(0, T)^2 \times H^1(0, T)$. We define $\tilde{\theta}$ the angle associated to the rotation velocity $\tilde{r}$ defined up to a constant. For any $t \in (0, T)$, we define the structure domain

$$\widetilde{\Omega}_S(t) = \left\{\tilde{a}(t) + R_{\tilde{\theta}(t)-\theta_0}(y - a_0),\ y \in \Omega_S(0)\right\}.$$

We suppose that $\tilde{a}(0) = a_0$, $\tilde{\theta}(0) = \theta_0$ and $\inf_{t\in(0,T)} \mathrm{d}(\widetilde{\Omega}_S(t), \partial(\Omega \setminus \omega)) > 0$. Thus, we can define the fluid domain $\widetilde{\Omega}_F(t) = \Omega \setminus \overline{\widetilde{\Omega}_S(t)}$. We also consider a given velocity $\tilde{u}$ satisfying regularity properties and compatibility conditions with $(\tilde{a}, \tilde{r})$. The linearized problem associated to (11) is the one where we have replaced in (11) $\Omega_S(t)$ by $\widetilde{\Omega}_S(t)$, $\Omega_F(t)$ by $\widetilde{\Omega}_F(t)$ and the nonlinear term $(u \cdot \nabla)u$ in the first equation by $(\tilde{u} \cdot \nabla)u$. We prove a null controllability result for the linearized problem with the help of a Carleman inequality shown on the adjoint system associated to a linearized system. Finally, Theorem 3.1 is proved by applying Kakutani's fixed point theorem.

We will give the Carleman inequality here for two reasons. First, once Navier-Stokes equations are involved, the corresponding global Carleman inequality is different from (6) because of the pressure (or because of incompressibility). Indeed, the exponential weights appearing at the left and right hand sides of the inequality are no more the same. Secondly, since we are working with variable domains, this is in fact a Carleman inequality in moving domains.

Let us first introduce the corresponding adjoint operator P_q^* on this case given by the solution (v, q, b, γ) of the linear system

$$\begin{cases} -\partial_t v - (\tilde{u}\cdot\nabla)v - \operatorname{div}\sigma(v,q) = 0,\ \operatorname{div} v = 0 \text{ in } \widetilde{\Omega}_F(t), \\ m\ddot{b}(t) = -\int_{\partial\widetilde{\Omega}_S(t)} \sigma(v,q)n\,d\sigma, \\ J\dot{\gamma}(t) = -\int_{\partial\widetilde{\Omega}_S(t)} (\sigma(v,q)n)\cdot(x-\tilde{a}(t))^{\perp} d\sigma, \\ v = \dot{b} + \gamma(x-\tilde{a})^{\perp} \text{ on } \partial\widetilde{\Omega}_S(t),\ v = 0 \text{ on } \partial\Omega, \\ v(T,\cdot) = v_0^T \text{ in } \widetilde{\Omega}_F(T),\ b(T) = 0,\ \dot{b}(T) = b_1^T, \gamma(T) = \gamma_0^T, \end{cases} \tag{15}$$

with

$$\begin{aligned} &\operatorname{div} v_0^T = 0 \ \text{ in } \ \widetilde{\Omega}_F(T), \\ &v_0^T = b_1^T + \gamma_0^T(x-\tilde{a}(T))^{\perp} \ \text{ on } \ \partial\widetilde{\Omega}_S(T) \quad \text{and} \\ &v_0^T = 0 \ \text{ on } \ \partial\Omega \end{aligned} \tag{16}$$

and $\tilde{u}$ regular enough and conveniently chosen.

The following extremal weights appear when eliminating the explicit dependence on the pressure q of the Carleman inequality. For a given field $v(x,t)$ we take the notation:

$$\overline{v}(t) = \sup_{x\in\widetilde{\Omega}_F(t)} v(x,t), \qquad \underline{v}(t) = \inf_{x\in\widetilde{\Omega}_F(t)} v(x,t). \tag{17}$$

Theorem 3.2 ([9]). *There exists $\hat{s}$, $\hat{\lambda}$ and C depending on Ω, ω and T such that, for every regular solution (v, b, γ) of (15), for all $\lambda > \hat{\lambda}$ and $s > \hat{s}$ we have:*

$$\begin{aligned} &\int_0^T\int_{\widetilde{\Omega}_F(t)} \rho^2 \left(\frac{1}{s\varphi}\big(|\Delta v|^2 + |\partial_t v|^2\big) + s\lambda^2\varphi|\nabla v|^2 + s^3\lambda^4\varphi^3|v|^2\right) dxdt \\ &\quad + s\lambda\int_0^T \underline{\rho}^2\underline{\varphi}\left(\big|\ddot{b}\big|^2 + |\dot{\gamma}|^2\right) dt \\ &\int_0^T\int_{\partial\widetilde{\Omega}_S(t)} \underline{\rho}^2\left(s^3\lambda^3\underline{\varphi}^3|v|^2\nabla\psi\cdot n + 2s\lambda\underline{\varphi}|\nabla v\, n|^2\right) d\sigma dt \\ &\quad \le Cs^{19/2}\lambda^{13}\int_0^T\int_\omega (\overline{\rho}/\underline{\rho})^2\,\overline{\rho}^2\,\overline{\varphi}^{10}|v|^2 dxdt. \end{aligned} \tag{18}$$

The Carleman inequality is expressed on the *moving* domains $\widetilde{\Omega}_S(t)$ and $\widetilde{\Omega}_F(t)$ and the transport theorem is used on its deduction. More precisely, the weight

Φ used here is of the form (compare with (3))

$$\Phi(x) = \frac{\exp(\lambda\alpha_0) - \exp(\alpha_1\lambda\psi(x,t))}{(T-t)^4} \tag{19}$$

where α_0, α_1 are suitable constants. The time dependent weight function $\psi(x,t)$ is chosen as the standard weight for the heat equation but it follows the shape of $\widetilde{\Omega}_S(t)$. More precisely, $\psi(x,t)$ is a regular function such that $\frac{\partial\psi}{\partial n} < 0$ on Σ, $|\nabla\psi| > 0$ outside ω and it satisfies the time dependent conditions $\frac{\partial\psi}{\partial n} > 0$ on $\partial\widetilde{\Omega}_S(t)$, ψ constant on $\partial\widetilde{\Omega}_S(t)$. Notice that an interesting property is that the spatial gradient of ψ is related to $1/\delta(t)$, where $\delta(t) > 0$ is the distance between $\widetilde{\Omega}_S(t)$ and $\partial(\Omega \setminus \omega)$. This could be useful when doing explicit calculations of constants in the Carleman inequality as the *collision parameter* $\delta(t) \to 0^+$ as $t \to t_0^-$ for some collision time $t_0 > 0$.

4 Inverse problem in wave equation with partial boundary data

The main idea is to modify the weight function Φ given in (4) in such a way that its gradient $\nabla\Phi$ a rotation of the original field $(x - x_0)$ with a radially dependent magnitude. This concept come up from multipliers technique and controllability [19], [34].

Let Ω be a domain in $\mathbb{R}^n$, $n = 2, 3$. In order to solve the Dirichlet to Neumann one measurement inverse problem, it suffices to measure on a *rotated exit part* of the boundary Γ_r. If $n = 2$, this region depends on a point $x_0 \in \mathbb{R}^n$ and on a rotation T_θ in an angle $\theta \in (-\pi/2, \pi/2)$. If $n = 3$, it depends also on a unit direction $\alpha \in \mathbb{R}^3$ and the rotation T_θ is considered on the orthogonal plane to α denoted here by $\alpha^\perp$. We will use the notation $v^\perp = v - (v\cdot\alpha)\alpha$ for the projection of the field v on $\alpha^\perp$. More precisely

$$(n=2)\ \Gamma(x_0,\theta) = \{x \in \partial\Omega \mid (x - x_0)\cdot T_\theta n > 0\} \tag{20}$$

$$(n=3)\ \Gamma(x_0,\alpha,\theta) = \{x \in \partial\Omega \mid (x - x_0)\cdot(\cos\theta(n - n^\perp) + T_\theta n^\perp > 0\} \tag{21}$$

We can always reduce the two-dimensional case to the bi-dimensional one by considering $\mathbb{R}^2$ immersed in $\mathbb{R}^3$ with $\alpha = (0,0,1)$, so in the following we will only refer to the three-dimensional case. Notice that, as θ approaches $\pi/2$, the rotated exit part tends to be the union of the boundary points x where the dot product between the vector $x - x_0$ and the projection $n^\perp$ rotated in $\pi/2$ is positive (see Figure 6). There are of course an infinite number of intermediate cases depending on the localization of x_0, the direction of α and the angle θ. The case $\theta = 0$ corresponds to the standard exit condition previously used for the same inverse problem [24]. Remark also that the rotated exit condition is a particular case of the geometrical optics BLR condition [1], [33].

The main stability result is the following (we present here the case $x_0 \notin \overline{\Omega}$, the other case in which two arbitrarily near interior points are used can be found in [14]).

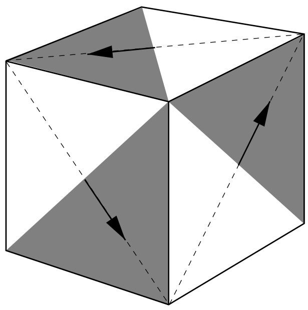

Figure 6 – Rotated exit boundary region in a cube $(-a, a)^3$ with respect to the origin $x_0 = 0$ and a diagonal axis α parallel to $(1, 1, 1)$ with orthogonal plane $\alpha^\perp$ (dotted lines). The exit region (shaded) are the boundary points whose position vectors have a positive dot product with the $\pi/2$ rotation of the projection of the normal $n^\perp$ on $\alpha^\perp$ (arrows). By taking a suitable time-dependent Neumann measure on a little bit more than this region, the inverse problem of recovering a time-independent potential in the wave equation is locally stable (not really in a cube but in a regularized C^2 cube).

Theorem 4.1 ([14]). *Let $P_q^* = \partial_{tt} - \Delta + q$ and let $u(q)$ and $u(\overline{q})$ be the respective solutions of $P_q^* u = 0$ with Dirichlet boundary conditions associated to $q, \overline{q} \in L^\infty(\Omega)$ and with Neumann measurements ξ and $\overline{\xi}$ on $\Gamma_r \times (0, T)$ respectively. There exists a time $\overline{T} > 0$ such that if $T > \overline{T}$, if $u(\overline{q}) \in H^1(0, T; L^\infty(\Omega))$ and if $|u(0)| \geq \alpha_0 > 0$ a.e. in Ω, then there exists a positive constant C_M depending on $M = \|q\|_{L^\infty(\Omega)}$ such that*

$$||\overline{q} - q||_{L^2(\Omega)} \leq C_M \left\| \overline{\xi} - \xi \right\|_{H^1(0,T;L^2(\Gamma_r))} \quad \forall q \text{ with } \|q\|_{L^\infty(\Omega)} \leq M . \quad (22)$$

If $\theta \to \pm\frac{\pi}{2}$ *then* $\overline{T} \to +\infty$ *and it behaves asymptotically as*

$$\overline{T} \approx \frac{R_0}{\sqrt{\beta}} \exp\left(\frac{2}{\cos\theta}\theta_0\right), \tag{23}$$

where $R_0 = \sup_{x\in\Omega} |x - x_0|$ *and* $\theta_0 = \sup_{x\in\Omega} \arg(x - x_0)^\perp - \inf_{x\in\Omega} \arg(x - x_0)^\perp$, *i.e., the angle of view of* Ω *with respect to the axis of direction* α *passing by* x_0.

The proof of this Theorem is based on a global Carleman estimate using the variant of the Carleman weight for the wave equation (compare with (4))

$$\Phi(x, t) = -\lambda \exp\left(\cos\theta |x - x_0|^2 \exp\left(2\tan\theta \arg(x - x_0)^\perp\right) - \beta\, t^2\right) \tag{24}$$

for some suitable constant $\beta \in (0, 1)$. This weight was constructed in order to make appear in the gradient $\nabla\Phi$ the matrix

$$\begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{25}$$

written in a basis attached to $\alpha, \alpha^\perp$. The product of this matrix and a vector field v is exactly $\cos\theta(v - v^\perp) + T_\theta v^\perp$, expression which appears in the definition (21) of Γ_r.

The main steps in the deduction of such inequality are taken from [35, 23] following a well known technique due to Bukhgeim and Klibanov [11], [29]. Roughly speaking, the technique in this case consists in reducing the problem to a source inverse problem for the perturbed equation around $u(\overline{q})$ and then take its time derivative in order to obtain a quasi-observability inequality after use of the Carleman inequality. Quasi, since the initial condition is bounded not only by the observation but also by the unknown source term, but a sufficiently large time and the properties of the Carleman weight allow to get rid of this source term.

There are also geometrical exit type conditions for the analogous inverse problem in the case of the Scrhödinger equation (see [3]). The present method should also work in this case because the spatial part used in Carleman weights for wave and Schödinger equations are the same (compare (4) with (5)). Nevertheless, in the case of the Scrhödinger operator, it is strongly possible that the geometrical optic condition are not necessary to solve the one measurement inverse problem (see [27]). Other completely different problem is the case when you have Dirichlet to Newmann map measurements. In this case, a suitable arbitrarily small boundary of measurements is enough to solve the inverse problem both for Schödinger and wave equations (see [28]). But recently, it has been shown that you can solve the one measurement problem for the wave equation with an arbitrarily boundary measurement region (in the case of Neumann boundary conditions and Dirichlet measurements), but the corresponding inequality analogous to (22) is logarithmic [5].

5 Inverse coefficient problem for wave transmission problems

Notice that recently, Global Carleman estimates and application to one measurement inverse problems for the wave equation were obtained in the case of variable but still regular coefficients [4], [26]. The inverse problem of retrieving coefficients from a wave equation with discontinuous coefficients from boundary measurements arise naturally in geophysics and more precisely, in seismic prospection of earth inner layers [44].

Let Ω and $\Omega_1 \subset \Omega$ be two open subsets of $\mathbb{R}^2$ with smooth boundaries Γ and Γ_1 respectively and let $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. To fix ideas we assume that Ω_1 is simply connected. We set:

$$\bar{a}(x) = \begin{cases} a_1 & x \in \Omega_1 \\ a_2 & x \in \Omega_2 \end{cases} \tag{26}$$

with $a_j > 0$ for $j = 1, 2$, for each $q \in L^\infty(\Omega)$, we consider $u(q)$ as the solution of the following wave transmission equation

$$\begin{cases} u_{tt} - \operatorname{div}(\bar{a}(x)\nabla u) + q(x)u = 0 & \text{in } \ Q = \Omega \times (0, T) \\ u = g \quad \text{on } \ \Sigma = \Gamma \times (0, T) \\ u(0) = u_0 \quad \text{in } \ \Omega \\ u_t(0) = u_1 \quad \text{in } \Omega. \end{cases} \tag{27}$$

The following inverse stability result holds (see the preprint [2]):

Theorem 5.1 ([2]). *Assume* Ω_1 *is* strictly convex *and* $a_1 > a_2 > 0$. *Let* $\mathcal{U}$ *be a bounded subset of* $L^\infty(\Omega)$, $\overline{q} \in L^\infty(\Omega)$ *and* $r > 0$. *If* $|u_0(x)| \geq r > 0$ *a. e. in* Ω

and $u(\overline{q}) \in H^1(0, T; L^\infty(\Omega))$, *then there exists* $C = C(\Omega, T, \|q\|_{L^\infty(\Omega)}, \mathcal{U}) > 0$ *such that:*

$$\|\overline{q} - q\|_{L^2(\Omega)} \leq C \left\| \frac{\partial u(\overline{q})}{\partial n} - \frac{\partial u(q)}{\partial n} \right\|_{H^1(0,T;L^2(\Gamma))}$$

for all $u_0 \in H_0^1(\Omega)$ *and* $q \in \mathcal{U}$.

This Theorem is proved by combining the Carleman inequality for the wave equation with discontinuous coefficients proved in [2] and the method of Bukhgeim-Klibanov explained in section 4. To this end, system (27) is viewed as two wave equations with constant coefficients coupled with transmission conditions (see [31]). Then, a global Carleman inequality is found out for this transmission problem by working with variants of Carleman weights of the form (compare with (4))

$$\Phi(x, t) = -\exp(\lambda\varphi(x)), \tag{28}$$

$$\varphi(x) = \begin{cases} \eta(x)\dfrac{a_2}{r(x)^2}|x - x_0|^2 - \beta t^2 + M_1 & \text{in } \Omega_1 \times (-T, T) \\ \dfrac{a_1}{r(x)^2}|x - x_0|^2 - \beta t^2 + M_2 & \text{in } \Omega_2 \times (-T, T) \end{cases} \tag{29}$$

where M_1 and M_2 are constants such that $M_1 - M_2 = a_1 - a_2$, $r(x) = |x_0 - y(x)|$, $y(x) = \Gamma_1 \cap [x_0, x]$ and η is some cut-off function with support in Ω_1 centered at x_0. We also combine the Carleman inequalities obtained from two different interior points as we did in section 2, see also Figure 1, left. The convexity hypothesis on Ω_1 comes from the fact that the positiveness of the Hessian of the weight Φ is related with the curvature of Γ_1 with respect to x_0.

There are a lot of important works concerning this inverse problem in the case that a wide class of measurements are available. In these cases, microlocal analysis has been used and it gives positive answer to the problem of retrieving coefficients and discontinuity interfaces without restrictive hypothesis of convexity of the interfaces or monotonicity of the speed of waves. These kinds of results are fundamental for seismic prospection. For an overview on this subject see [44] and the references therein, in particular related to thie study are the works of [8], [17], [37], [38], [40].

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