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Rotated weights in global Carleman estimates applied to an inverse problem for the wave equation

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Abstract
In this paper, we establish geometrical conditions in order to solve an inverse problem of retrieving a stationary potential for the wave equation with Dirichlet data from a single time-dependent Neumann boundary measurement on a suitable part of the boundary. We prove the uniqueness and the stability results for this problem when a Neumann measurement is only located on a part of the boundary satisfying a rotated exit condition. The strategy consists of introducing an angle-type dependence in the weight functions used to obtain global Carleman estimates for the wave equation and combination of several of these estimates and then apply it to the inverse problem.

1. Introduction

The direct methods used to obtain global Carleman estimates for hyperbolic-type equations (see, for instance, [7, 11, 22, 23, 25]) implicitly involve a multiplier technique, which is a common tool in controllability theory (see [9, 17, 18]). Indeed, after splitting the corresponding partial differential operator, in the deduction of Carleman inequalities, there appears, among other things, an important term multiplied by the gradient of the solution of the corresponding equation following some direction $\nabla\Phi$, where Φ is a specific weight function adapted to the nature of the equation. In the case of a global Carleman estimate for the wave equation, the direction $\nabla\Phi$ is a radial field of type $(x - x_0)$, where x_0 is some exterior point to the domain.

Our first main idea in this work is to modify this weight function Φ in such a way that the gradient $\nabla\Phi$ is essentially a rotation of the original field $(x - x_0)$, but with a magnitude radially depending on space. Using that modified weight function we are able to obtain a modified global Carleman estimate. This idea is directly taken from the rotated multipliers used in exact controllability of hyperbolic-type equations [20].

Our second main contribution is that in the Carleman estimates that we prove here, we can take a point x_0 not only inside the domain (see, for instance, [12, 22]) but also outside it.

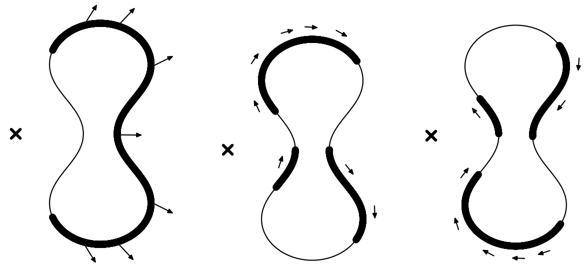

Figure 1. From left to right: '*exit*', '*outgoing*' and '*incoming*' boundary regions in a domain with respect to some exterior point x_0 marked here with the symbol $\times$. Of course, the last two cases are exactly the opposite if we invert the sense we go round the boundary. If we consider the wave equation with Dirichlet boundary conditions in this domain and if we take a single time-dependent Neumann measurement on some of these regions, then the inverse problem of recovering a time-independent potential in the equation can be locally solved.

To this end, we use a combination of several Carleman estimates. This idea was already used in [6] in the context of Carleman inequality for the heat equation with discontinuous diffusion coefficients.

It was shown (see, for example, [12, 24]) that global Carleman estimates can be applied in order to study the stability (also called Lipschitz stability) for some specific kind of inverse problem for the wave equation. The problem is to locally retrieve a time-independent bounded potential in the wave equation by knowing only a single time-dependent over-determining measurement on the whole boundary (or some part of it) of the domain where the wave equation evolves. If the boundary condition is Dirichlet or Neumann, then the complementary (over-determining) measurement is Neumann or Dirichlet respectively. The case of Dirichlet boundary conditions and Neumann over-determining measurements has been treated in [22, 24] and the other complementary case in [12].

The stability and uniqueness results were obtained (see [12, 22]) if the over-determining measurements are taken on the part of the boundary which satisfies a geometrical hypothesis saying that it is sufficient that the measure region includes the so-called *exit boundary region* with respect to some exterior point x_0 (see (8) for $x_0 = x_1$, $\Gamma_0 = \Gamma_1$ and $b = 0$).

In this paper, we extend the previous geometrical hypothesis. More precisely, we will see that in order to solve the inverse problem (in the case of Dirichlet data and Neumann measurement), it suffices to measure on a *rotated exit part* of the boundary. This region can be in fact described by an angle parameter $\theta \in (-\pi/2, \pi/2)$. The case $\theta = 0$ corresponds to the standard exit condition ('exit' boundary region, see figure 1, left). If θ approaches $\pi/2$, the corresponding rotated part of the boundary is the one on which the distance to the point x_0 is increasing if we follow the boundary in the counterclockwise sense ('outgoing' boundary region, see figure 1, centre). If θ is near $-\pi/2$, the rotated part is the one in which the distance from the point x_0 is decreasing if we follow the boundary in the same sense ('incoming' boundary region, see figure 1, right). There are, of course, an infinite number of intermediate cases.

Our generalization is indeed not optimal. In fact, a reasonable conjecture is that the measurements should be placed in order to capture the geometrical optic rays coming from the support of the unknown potential (see [3]). However, we are far from proving it with the techniques presented here.

As mentioned before, the essential point in our proofs is a global Carleman estimate with modified weights. The main steps in the obtention of this inequality are taken from [11, 22]. Nevertheless, all our results related to both global Carleman estimates and stability of the inverse problem are stated here in two-dimensional domains. In fact, we are able to construct the modified weight function only in two dimensions. The analysis in the case of higher dimensions could be the subject of another work.

On the other hand, in order to obtain this Carleman inequality we combine two Carleman inequalities with different—but similar—weight functions. This argument allows us to take the mentioned point x_0 also inside the domain and our argument does not depend on the dimension.

Let us remark that extending our result to the Schrödinger equation may be interesting by itself but the final stability result is not as strong as for the wave equation. In fact, there are also geometrical exit-type conditions for the analogous inverse problem in the case of the Schrödinger equation with Dirichlet data and Neumann measurements (see [4]). Nevertheless, in the case of the Schrödinger operator, it is highly possible that the geometrical optics condition, and particularly the exit-type condition, is far from being necessary to solve the inverse problem considered here. Firstly, it is a known fact in control theory that the geometrical optics condition is not necessary for the internal exact controllability of the Schrödinger equation [8] or plates [15]. Secondly, there are some recent results in other similar kinds of inverse problems for the Schrödinger equation, showing that it is sufficient to measure the whole Dirichlet to Neumann map on an arbitrarily small strategical part of the boundary in order to have uniqueness (see [2, 16]).

2. Main results

Let Ω be a bounded connected open subset of $\mathbb{R}^2$ with boundary $\partial\Omega$ of class C^2. We will denote by $\nu = \nu(x)$ the unit outward normal vector to Ω at the point $x \in \partial\Omega$. Let $\mathcal{U}_M$ be a M-bounded set in $L^\infty(\Omega)$, where M is some positive constant, which is the following set:

$$\mathcal{U}_M = \{q \in L^\infty(\Omega) : \|q\|_{L^\infty(\Omega)} \leqslant M\}. \tag{1}$$

Let γ be a nonempty open subset of $\partial\Omega$ and $T > 0$. In this paper, we will deal with the following inverse problem:

Given u_0, u_1, h and ξ in appropriate spaces, we look for sufficient conditions under γ and T such that we can find $q \in \mathcal{U}_M$ such that the solution u of the wave equation

$$\begin{cases} \partial_{tt}u - \Delta u + q(x)u = 0 & \text{in} \quad \Omega \times (0,T), \\ u = h & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0, \quad \partial_t u(x,0) = u_1 & \text{in} \quad \Omega \end{cases} \tag{2}$$

satisfies the additional condition

$$\frac{\partial u}{\partial \nu} = \xi \qquad \text{on} \quad \gamma \times (0,T). \tag{3}$$

Note that the dependence of u on q is nonlinear. Let us denote by $u(q)$ and $u(\overline{q})$ the solutions of (2) associated with q and with some known potential $\overline{q}$ respectively. If we set $f = q - \overline{q}$ and $y = u(q) - u(\overline{q})$, then $qu(q) - \overline{q}u(\overline{q}) = qy - fu(\overline{q})$. Therefore, we obtain the following equivalent inverse problem for the function y:

Given $\overline{q} \in \mathcal{U}_M$, $R = -u(\overline{q})$ and $\overline{\xi} = \frac{\partial u(\overline{q})}{\partial \nu}$ in appropriate spaces, we look for sufficient conditions for γ and T such that we can find a time-independent function f such that the solution y of the wave equation

$$\begin{cases} \partial_{tt}y - \Delta y + \overline{q}(x)y = f(x)R(x,t) & \text{in} \quad \Omega \times (0,T), \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ y(x,0) = \partial_t y(x,0) = 0 & \text{in} \quad \Omega \end{cases} \tag{4}$$

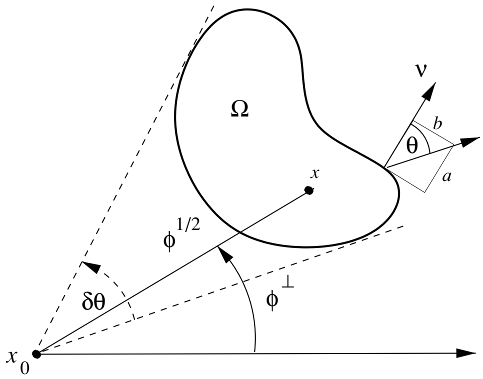

Figure 2. Main notation in the case $x_0 = x_1 \in \mathbb{R}^2\backslash\overline{\Omega}$.

satisfies the additional condition

$$\frac{\partial y}{\partial \nu} = \xi - \overline{\xi} \qquad \text{on} \quad \gamma \times (0, T). \tag{5}$$

Let x_0 and x_1 be two points either both in Ω or both in $\mathbb{R}^2\backslash\overline{\Omega}$ such that

$$\begin{cases} x_0 = x_1 & \text{if} \quad x_0, x_1 \in \mathbb{R}^2\backslash\overline{\Omega} \\ x_0 \neq x_1 & \text{if} \quad x_0, x_1 \in \Omega. \end{cases} \tag{6}$$

The main novelty of this work is that we will consider that γ is sufficiently large in such a way that

$$\gamma = \Gamma_0 \cup \Gamma_1, \tag{7}$$

where

$$\Gamma_i \supseteq \{x \in \partial\Omega : (x - x_i) \cdot (aI - bA)\nu(x) > 0\}, \qquad i = 0, 1 \tag{8}$$

with x_i, $i = 0, 1$, verifying (6) and for some $a > 0$ and $b \in \mathbb{R}$ such that

$$a^2 + b^2 = 1. \tag{9}$$

Here I is the identity matrix and

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that $(aI - bA)\nu(x)$ corresponds to a clockwise rotation of the normal field in an angle (see figure 2)

$$\theta = \tan^{-1}(b/a) \in (-\pi/2, \pi/2), \tag{10}$$

where

$$a = \cos\theta \qquad \text{and} \qquad b = \sin\theta. \tag{11}$$

Let us introduce the following radial ϕ_i and angular $\phi_i^{\perp}$-dependent scalar fields:

$$\phi_i(x) = |x - x_i|^2, \qquad i = 0, 1, \qquad x \in \overline{\Omega} \tag{12}$$

$$\phi_i^{\perp}(x) = \arg(x - x_i), \qquad i = 0, 1, \qquad x \in \overline{\Omega}, \tag{13}$$

the last one being the argument of the vector $x - x_i$, that is to say, the angle of the vector $x - x_i$ measured counterclockwise with respect to the horizontal axis (see figure 2 for the case

$x_0 = x_1 \in \mathbb{R}^2\backslash\overline{\Omega}$). We define the following constants only depending on x_i, θ and Ω:

$$\underline{\theta}^i = \inf_{x\in\Omega} \phi_i^{\perp}(x), \qquad \overline{\theta}^i = \sup_{x\in\Omega} \phi_i^{\perp}(x), \qquad \delta\theta^i = \overline{\theta}^i - \underline{\theta}^i, \qquad i = 0, 1, \tag{14}$$

$$\theta_{\min} = \min_{i=0,1} \underline{\theta}^i, \qquad \theta_{\max} = \max_{i=0,1} \overline{\theta}^i, \qquad \delta\theta = \theta_{\max} - \theta_{\min} \tag{15}$$

and

$$R_{\min} = \inf_{x\in\Omega, i=0,1} \phi_i^{1/2}(x), \qquad R_{\max} = \sup_{x\in\Omega, i=0,1} \phi_i^{1/2}(x). \tag{16}$$

Let us consider $\beta \in (0, \beta_2)$ with $\beta_2 > 0$ given by

$$\beta_2 = \min\{\beta_2^i, \text{for } i = 0, 1, \text{where } \beta_2^i \text{ is defined in (129)}\}, \tag{17}$$

where β_2^i, $i = 0, 1$, are two positive constants that will be introduced later on the paper and that will depend on $x_i, a, b, \overline{\theta}^i$ and $\underline{\theta}^i$, $i = 0, 1$. Given $\beta \in (0, \beta_2)$, let us introduce the following principal weight functions involved in this paper:

$$\Phi^i(x, t) = a\phi_i(x) \exp\left(2\rho\phi_i^{\perp}(x)\right) - \beta t^2, \qquad x \in \overline{\Omega}, \qquad t \in \mathbb{R}, \qquad i = 0, 1, \tag{18}$$

where we have introduced the parameter

$$\rho = \frac{b}{a}, \qquad \rho \in (-\infty, +\infty). \tag{19}$$

We also introduce the following parameters that will be used throughout the paper in order to simplify the expressions:

$$D_\theta = \left(R_{\max}^2 \exp(2|\rho|\delta\theta) - R_{\min}^2\right)^{1/2} \tag{20}$$

$$\alpha = \frac{\beta}{\beta_2}, \qquad 0 < \alpha < 1, \tag{21}$$

$$m = \max\{1, 2 - 4|\rho|\}, \qquad 1 \leqslant m \leqslant 2, \tag{22}$$

$$F^i = \exp(2|\rho|\delta\theta^i) - \frac{ma^2}{4}, \qquad F^i \geqslant \frac{1}{2}, \qquad i = 0, 1, \tag{23}$$

$$F_{\min} = \min_{i=0,1} F^i. \tag{24}$$

Obviously, when $x_0 = x_1 \in \mathbb{R}^2\backslash\overline{\Omega}$ all these constants and the both weight functions introduced before are the same for $i = 0$ and $i = 1$.

Our first main result is on the stability for the inverse problem (2) and (3):

Theorem 2.1. *Suppose that γ satisfies (8) for some $x_0, x_1 \in \mathbb{R}^2$ verifying (6), $a > 0$ and $b \in \mathbb{R}$ with $a^2 + b^2 = 1$. Let $u(q)$ and $u(\overline{q})$ be the respective solutions of (2) associated with $q \in \mathcal{U}_M$ and $\overline{q} \in L^\infty(\Omega)$ and with the measurements ξ given by (3) and $\overline{\xi} = \frac{\partial u(\overline{q})}{\partial \nu}$. There exists a time $\overline{T} = \overline{T}(\Omega, \theta, x_0, x_1) > 0$ such that if $T > \overline{T}$, $u(\overline{q}) \in H^1(0, T; L^\infty(\Omega))$ and $|u_0| \geqslant \alpha_0 > 0$ a.e. in Ω, then there exists a positive constant C depending on Ω, $T, x_0, x_1, a, b, M, \|u(\overline{q})\|_{H^1(L^\infty)}, \|\overline{q}\|_{L^\infty}, \alpha_0, u_0, u_1$ and h such that*

$$\|\overline{q} - q\|_{L^2(\Omega)} \leqslant C\|\overline{\xi} - \xi\|_{H^1(0,T;L^2(\gamma))} \qquad \forall q \in \mathcal{U}_M. \tag{25}$$

Furthermore, $\overline{T}$ is given by

$$\overline{T} = \left(\frac{F_{\min} + \sqrt{F_{\min}^2 + ma^2}}{m}\right)^{1/2} \frac{D_\theta}{\sqrt{\alpha}}, \tag{26}$$

where $F_{\min}, m, D_\theta$ and α are respectively defined in (24), (22), (20) and (21).

Note that this stability result is of a local nature, since for a given $\overline{q} \in L^\infty(\Omega)$ the stability holds in an L^∞-ball of radius M and centre $\overline{q}$. The stability constant does not depend only on the radius of the ball but also on its centre.

An immediate consequence of theorem 2.1 is the following uniqueness result:

Corollary 2.2. *Under the hypothesis of theorem 2.1, if $u(\overline{q})$ and $u(q)$ are the solutions of (2) associated with $\overline{q} \in L^\infty(\Omega)$ and $q \in L^\infty(\Omega)$, respectively, such that*

$$\frac{\partial u(\overline{q})}{\partial \nu} = \frac{\partial u(q)}{\partial \nu} \qquad on \qquad \gamma \times (0, T),$$

then $\overline{q} = q$ and $u(\overline{q}) = u(q)$.

The proof of theorem 2.1, in view of the previous discussions, is a direct consequence of the following result concerning the linearized inverse problem (4) and (5), which is our second main stability result:

Theorem 2.3. *Assume that the same hypothesis under γ, x_0, x_1, a, b and T of theorem 2.1 holds. Let be given $f \in L^2(\Omega)$, $R \in H^1(0, T; L^\infty(\Omega))$ with $\|\partial_t R\|_{L^2(L^\infty)} \leqslant M_1$, where M_1 is a positive constant and with $|R(x, 0)| \geqslant \alpha_0 > 0$ a.e. in Ω. Then, for each bounded subset $\mathcal{U}_M$ of $L^\infty(\Omega)$ there exists a positive constant C depending on Ω, $T, x_0, x_1, a, b, M, M_1$ and α_0 such that if y is the solution of (4) we have*

$$\|f\|_{L^2(\Omega)} \leqslant C \left\| \frac{\partial y}{\partial \nu} \right\|_{H^1(0,T;L^2(\gamma))} \qquad \forall q \in \mathcal{U}_M. \tag{27}$$

The proof of this theorem will be given in section 5. The essential ingredient in the proof is a global Carleman estimate for the wave equation, with the constants depending on q only via M, which we will deduce in section 3 (see theorem 3.1). We will also use the ideas introduced in [21] at later used, for example, in [4, 12, 24].

Remark 2.1. In our main results (theorems 2.1 and 2.3) the respective source terms q and f can only depend on x and not on t. These restrictive hypotheses are imposed by the arguments used in section 5. The stability result for the source terms depending on x and t (even when the over-determining measurements are taken in the whole boundary) is, to our knowledge, an open problem.

Remark 2.2. Note that the constant C in (27) depends on $q \in \mathcal{U}_M$ only via M; therefore, the result of theorem 2.3 can be applied directly to the nonlinear inverse problem, in order to deduce theorem 2.1.

Remark 2.3. If we take $a = 1$ and $b = 0$ in (18), this corresponds to the case in which no rotation is made, we recover the standard expression for the weight function $\Phi^i(x, t) = |x - x_i|^2 - \beta t^2$, $i = 0, 1$, which gives the standard 'exit' boundary region corresponding to $\theta = 0$ (see figure 1, left). In addition, if $x_0 = x_1 \in \mathbb{R}^2 \backslash \overline{\Omega}$, we have a more simpler and already known situation (see [12, 22]) in which $\varphi_0 = \varphi_1, \theta = 0$ and then $\Phi^0 = \Phi^1$. In this case, we also recover the standard inversion time. Indeed, if $a = 1$ and $b = 0$, then $F_{\min} = 1/2$, $\rho = 0$, $\beta_2 = 1$ and the left factor on the right-hand side of (26) is equal to 1, we have

$$\overline{T} = \frac{D}{\sqrt{\beta}}, \qquad D = \left(R^2_{\max} - R^2_{\min}\right)^{1/2}.$$

See [14] for the idea of obtaining this time as a difference.

Remark 2.4. If $a \to 0$ and $b \to \pm 1$, then $\theta \to \pm\frac{\pi}{2}$. This corresponds to a maximal rotation (see figure 1, centre and right). Note that in this case the inversion time $\overline{T} \to +\infty$ and behaves asymptotically as

$$\overline{T} \approx R_{\max}\sqrt{\frac{2}{\alpha}}\exp\left(\frac{2}{\cos\theta}\delta\theta\right) \qquad \text{as} \quad \theta \to \pm\frac{\pi}{2}.$$

This fact recalls the one encountered in [20] where the control time for the wave equation with a rotated boundary control was of order $\frac{2R_{\max}}{\cos\theta}$ as $\theta \to \pm\frac{\pi}{2}$.

Remark 2.5. Expression (26) for the inversion time $\overline{T}$ does not come from the deduction of the Carleman inequality in section 3 nor from the hypothesis in the technical lemma of section 4. It appears in order to have inequality (173) of section 5 in the last step of the analysis.

Remark 2.6. The constant we found in the stability inequality (25) is proportional to

$$C \approx \overline{s}^{1/2}\exp(\overline{s}(e^{\overline{\lambda}k_1} - e^{\overline{\lambda}k_2})) \tag{28}$$

where $\overline{\lambda}$ and $\overline{s}$ are the constants of theorem 3.1 and $k_1 > k_2$ are given by

$$k_1 = \sup_{\gamma\times(0,T)}\{\Phi^0(x,t), \Phi^1(x,t)\}, \qquad k_2 = \inf_{\Omega}\{\Phi^0(x,0), \Phi^1(x,0)\}. \tag{29}$$

Remark 2.7. If $x_0, x_1 \in \Omega$ in theorem 2.1, we have interest in taking x_0 and x_1 closer in order to have a smaller set γ. Nevertheless, if $\sigma = |x_0 - x_1| \to 0$, then $C \to \infty$ in the stability inequality (25). This is due to the dependence of $\overline{s}$ in σ in theorem 3.1.

Remark 2.8. Our result is in two dimensions, but it could be generalized to three dimensions by considering a rotation with respect to a fixed axis passing by x_0.

Finally, the following uniqueness result is also an immediate consequence of theorem 2.3.

Corollary 2.4. *Under the hypothesis of theorem 2.3, if y is the solution of (4) such that*

$$\frac{\partial y}{\partial \nu} = 0 \qquad \text{on} \quad \gamma \times (0,T),$$

then $f = 0$ and $y = 0$.

The rest of the paper is organized as follows. In section 3, we prove the global Carleman estimate which we use in the proof of theorem 2.3. Section 4 deals with the proof of some technical results. Finally, in section 5, we give the proof of theorem 2.3.

3. Global Carleman inequality with a rotated condition

In this section, we will deduce a global Carleman inequality that we need for the proof of theorem 2.3. In the following, we will take the following notation $Q = \Omega \times (-T, T)$, $\Sigma = \partial\Omega \times (-T, T)$ and $\Sigma_i = \Gamma_i \times (-T, T)$, where Γ_i is defined in (8), depending on the constants $a > 0$, $b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and x_i verifying (6), $i = 0, 1$.

Let us note that we have chosen the radial and angular fields ϕ_i and $\phi_i^{\perp}$, $i = 0, 1$, respectively introduced in (12) and (13) in such a way that they satisfy for all $x \in \Omega$ the following:

$$\nabla\phi_i = 2(x - x_i), \tag{30}$$

$$\nabla\phi_i^{\perp} = \frac{1}{|x-x_i|^2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}(x-x_i) = \frac{1}{\phi}A(x-x_i). \tag{31}$$

Then, we can write that the weight functions Φ^i defined in (18) satisfy for all $x \in \Omega$

$$\begin{aligned}\nabla\Phi^i &= \left(a\nabla\phi_i + 2b\phi_i\nabla\phi_i^{\perp}\right)\exp\left(2\rho\phi_i^{\perp}\right)\\ &= 2(aI+bA)(x-x_i)\exp\left(2\rho\phi_i^{\perp}\right), \qquad i=0,1.\end{aligned} \tag{32}$$

Let λ be sufficiently large positive constant which will be fixed later on. For $i = 0, 1$ we introduce weight functions

$$\varphi_i(x,t) = \exp(\lambda\Phi^i(x,t)) = \exp\left(\lambda\left(a\phi_i(x)\exp\left(2\rho\phi_i^{\perp}(x)\right) - \beta t^2\right)\right). \tag{33}$$

Note that the following calculations are quite simple:

$$\partial_t\varphi_i = \lambda\varphi_i\partial_t\Phi^i = -2\lambda\varphi_i\beta t, \tag{34}$$

$$\partial_{tt}\varphi_i = \lambda\varphi_i\partial_{tt}\Phi^i + \lambda\partial_t\varphi_i\partial_t\Phi^i = \lambda\varphi_i\partial_{tt}\Phi^i + \lambda^2\varphi_i|\partial_t\Phi^i|^2, \tag{35}$$

$$\nabla\varphi_i = \lambda\varphi_i\nabla\Phi^i, \tag{36}$$

$$\Delta\varphi_i = \lambda\varphi_i\Delta\Phi^i + \lambda^2\varphi_i|\nabla\Phi^i|^2. \tag{37}$$

Let us set

$$\begin{aligned}Z = \{&v : v \in C^2(\overline{\Omega}\times[-T,T]), v = 0 \text{ on } \Sigma,\\ &v(x,-T) = v(x,T) = 0, \partial_t v(x,-T) = \partial_t v(x,T) = 0 \text{ in } \Omega\}.\end{aligned}$$

We have the following global Carleman estimate.

Theorem 3.1. *Let $x_0, x_1 \in \mathbb{R}^2$. We distinguish two cases: either $x_0 = x_1 \in \mathbb{R}^2\backslash\overline{\Omega}$ (case 1) or $x_0, x_1 \in \Omega$, $|x_0 - x_1| = \sigma > 0$ (case 2). Let $a > 0$, $b \in \mathbb{R}$ with $a^2 + b^2 = 1$, $\gamma = \Gamma_0 \cup \Gamma_1$ such that (8) holds and $\beta \in (0, \beta_2)$ where β_2 was defined in (17). Then for all $M > 0$, there exist positive constants $\overline{\lambda}$, $\overline{\mu}$ and C only depending on a, b, Ω, x_0 and x_1, and independent of σ and there exists $\delta > 0$ such that for any $q \in L^\infty(Q)$ with $\|q\|_{L^\infty(Q)} \leqslant M$, for any $\lambda \geqslant \overline{\lambda}$, any $s \geqslant \overline{s} = \overline{\mu}\delta^{-3/2}M(1+T^2)\exp(3\lambda\beta T^2)$ and any $v \in Z$ we have*

$$\begin{aligned}s\lambda&\iint_Q(\varphi_0\,\mathrm{e}^{2s\varphi_0} + \varphi_1\,\mathrm{e}^{2s\varphi_1})(|\partial_t v|^2 + s\lambda|\nabla v|^2)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + s^3\lambda^3\iint_Q\left(\varphi_0^3\,\mathrm{e}^{2s\varphi_0} + \varphi_1^3\,\mathrm{e}^{2s\varphi_1}\right)|v|^2)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant C\iint_Q(\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1})|\partial_{tt}v - \Delta v + q(x,t)v|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + Cs\lambda\iint_{\gamma\times(-T,T)}(\varphi_0\,\mathrm{e}^{2s\varphi_0} + \varphi_1\,\mathrm{e}^{2s\varphi_1})\left|\frac{\partial v}{\partial\nu}\right|^2\,\mathrm{d}\sigma\,\mathrm{d}t,\end{aligned} \tag{38}$$

where either $\delta = 1$ in case 1 or $\delta = \delta_\sigma = 1 - \exp(-\overline{\lambda}\kappa_0\sigma^2)$ with κ_0 a constant only depending on a, b, Ω, x_0 and x_1 but independent of σ in case 2.

Remark 3.1. In the statement of the theorem, *independent of σ* means *independent of σ as $\sigma \to 0$*.

Remark 3.2. Note that in the Carleman estimate (38) the source term $q = q(x,t)$ can depend on x and t. Moreover, (38) is uniform in q when q lies in a bounded set of $L^\infty(Q)$. Obviously, when $q = q(x)$ and lies in a bounded set $\mathcal{U}_M$ of $L^\infty(\Omega)$ the above estimate (uniform in q) remains true. This is important for the application we have in mind for the proof of theorem 2.1 (see remark 2.2).

Remark 3.3. The estimate (38) does not require any sufficiently large size for the time interval $2T$.

We will give the proof of theorem 3.1 in two cases. First we consider the simpler situation in which $x_0 = x_1 \in \mathbb{R}^2\backslash\overline{\Omega}$ and where we need only one weight function $\Phi^0 = \Phi^1$. This case is reduced to the global Carleman estimate of theorem 3.2. Then, we consider the case in which $x_0, x_1 \in \Omega$ with $x_0 \neq x_1$. We use a variant of the Carleman estimate of theorem 3.2 valid in the case $x_i \in \Omega$ in proposition 3.3, and then we prove theorem 3.1 by adding the corresponding Carleman inequalities obtained for $i = 0$ and $i = 1$. This last idea was taken from [6].

3.1. Proof of theorem 3.1 in the case $x_0 = x_1 \in \mathbb{R}^2\backslash\overline{\Omega}$

Let us assume that $x_0 = x_1 \in \mathbb{R}^2\backslash\overline{\Omega}$. As we mentioned before all functions and constants introduced until now coincide for $i = 0$ and $i = 1$, in particular $\varphi_0(x, t) = \varphi_1(x, t)$ for all $x \in \Omega, t \in \mathbb{R}$. For simplicity of notation, in this section, we will omit the corresponding subscripts and superscripts, that is to say instead of $\Phi^0, \varphi_0, \phi_0, \beta_2^0$, etc, we will simply write $\Phi, \varphi, \phi, \beta_2$, etc.

Theorem 3.2. *Assume that $x_0 \in \mathbb{R}^2\backslash\overline{\Omega}$, $a > 0, b \in \mathbb{R}$ with $a^2 + b^2 = 1$, Γ_0 is such that (8) holds and $\beta \in (0, \beta_2)$, where β_2 is defined in (129). Then for all $M > 0$, there exist positive constants $\overline{\lambda}_0, \overline{\mu}_0$ and C only depending on a, b, Ω and x_0, such that for any $q \in L^\infty(Q)$ with $\|q\|_{L^\infty(Q)} \leqslant M$, for any $\lambda \geqslant \overline{\lambda}_0$, any $s \geqslant \overline{s}_0 = \overline{\mu}_0 M(1 + T^2)\exp(3\lambda\beta T^2)$ and any $v \in Z$ we have*

$$s\lambda \iint_Q \varphi\, \mathrm{e}^{2s\varphi}(|\partial_t v|^2 + |\nabla v|^2)\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^3 \iint_Q \varphi^3\, \mathrm{e}^{2s\varphi}|v|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C\left(\iint_Q \mathrm{e}^{2s\varphi}|\partial_{tt}v - \Delta v + q(x,t)v|^2\,\mathrm{d}x\,\mathrm{d}t + s\lambda \iint_{\Sigma_0} \varphi\, \mathrm{e}^{2s\varphi}\left|\frac{\partial v}{\partial \nu}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t\right). \tag{39}$$

Proof of theorem 3.2. We will essentially follow the method from [11]. We will also need a technical result given in lemma 4.1, whose proof we will present in section 4.

In the following, C will stand for a generic positive constant, whose value can change from line to line and we will frequently indicate the data on which it depends. We will also use the usual convention of repeated indices.

Let us assume $v \in Z$ and $s > 0$. We set $g = \partial_{tt}v - \Delta v + q(x,t)v$ and make the change of variables

$$w = \mathrm{e}^{s\varphi}v. \tag{40}$$

Note that thanks to the fact that $v \in Z$, we have vanishing boundary conditions in time

$$w(x, -T) = w(x, T) = \partial_t w(x, -T) = \partial_t w(x, T) = 0. \tag{41}$$

We have the following equality:

$$\mathrm{e}^{s\varphi}(\partial_{tt}(\mathrm{e}^{-s\varphi}w) - \Delta(\mathrm{e}^{-s\varphi}w) + q(x,t)\,\mathrm{e}^{-s\varphi}w) = \mathrm{e}^{s\varphi}g. \tag{42}$$

Using the formulas (34)–(37), we can write (42) in the form

$$P_1 w + P_2 w = \mathrm{e}^{s\varphi}g - q(x,t)w + Ks\lambda\varphi(\partial_{tt}\Phi - \Delta\Phi)w \equiv g_s, \tag{43}$$

where

$$P_1 w = \partial_{tt}w - \Delta w + s^2\lambda^2\varphi^2(|\partial_t\Phi|^2 - |\nabla\Phi|^2)w \tag{44}$$

and

$$P_2 w = (K-1)s\lambda\varphi(\partial_{tt}\Phi - \Delta\Phi)w - s\lambda^2\varphi(|\partial_t\Phi|^2 - |\nabla\Phi|^2)w - 2s\lambda\varphi(\partial_t\Phi\partial_t w - \nabla\Phi\cdot\nabla w) \tag{45}$$

with K a positive constant whose value will be fixed later on (see lemma 4.1).

From (43), we obtain

$$\|P_1 w\|_2^2 + \|P_2 w\|_2^2 + 2(P_1 w, P_2 w) = \|g_s\|_2^2, \tag{46}$$

where $(\cdot,\cdot)$ and $\|\cdot\|_2$ respectively denote the standard scalar product and norm in $L^2(Q)$. Let us split the scalar product on the left-hand side of (46) as follows:

$$(P_1 w, P_2 w) = I_{11} + I_{12} + I_{13} + I_{21} + I_{22} + I_{23} + I_{31} + I_{32} + I_{33}. \tag{47}$$

In (47), all the integrals denote the respective scalar products for the terms of $P_1 w$ and $P_2 w$. Now we begin the explicit calculation of each of these integrals. We have for the first one

$$\begin{aligned} I_{11} &= s\lambda \iint_Q \partial_{tt} w (K-1)\varphi(\partial_{tt}\Phi - \Delta\Phi) w \,\mathrm{d}x\,\mathrm{d}t \\ &= (1-K)s\lambda \iint_Q \varphi(\partial_{tt}\Phi - \Delta\Phi)|\partial_t w|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{(1-K)}{2}s\lambda^2 \iint_Q \varphi\partial_t\Phi(\partial_{tt}\Phi - \Delta\Phi)\partial_t |w|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= (1-K)s\lambda \iint_Q \varphi(\partial_{tt}\Phi - \Delta\Phi)|\partial_t w|^2 \,\mathrm{d}x\,\mathrm{d}t + X_1, \end{aligned} \tag{48}$$

where

$$X_1 = -\frac{(1-K)}{2}s\lambda^2 \iint_Q \varphi(\partial_{tt}\Phi + \lambda|\partial_t\Phi|^2)(\partial_{tt}\Phi - \Delta\Phi)|w|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{49}$$

Making the scalar product between the first term of $P_1 w$ and the second one of $P_2 w$, we obtain

$$\begin{aligned} I_{12} &= -s\lambda^2 \iint_Q \partial_{tt} w \varphi(|\partial_t\Phi|^2 - |\nabla\Phi|^2) w \,\mathrm{d}x\,\mathrm{d}t \\ &= s\lambda^2 \iint_Q \varphi(|\partial_t\Phi|^2 - |\nabla\Phi|^2)|\partial_t w|^2 \,\mathrm{d}x\,\mathrm{d}t + s\lambda^3 \iint_Q \varphi\partial_t\Phi(|\partial_t\Phi|^2 - |\nabla\Phi|^2) w\partial_t w \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + 2s\lambda^2 \iint_Q \varphi\partial_t\Phi\partial_{tt}\Phi w\partial_t w \,\mathrm{d}x\,\mathrm{d}t \\ &= s\lambda^2 \iint_Q \varphi(|\partial_t\Phi|^2 - |\nabla\Phi|^2)|\partial_t w|^2 \,\mathrm{d}x\,\mathrm{d}t - \frac{s\lambda^3}{2} \iint_Q \varphi\partial_{tt}\Phi(|\partial_t\Phi|^2 - |\nabla\Phi|^2)|w|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \frac{s\lambda^4}{2} \iint_Q \varphi|\partial_t\Phi|^2(|\partial_t\Phi|^2 - |\nabla\Phi|^2)|w|^2 \,\mathrm{d}x\,\mathrm{d}t - s\lambda^3 \iint_Q \varphi|\partial_t\Phi|^2\partial_{tt}\Phi|w|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - s\lambda^2 \iint_Q \varphi(|\partial_{tt}\Phi|^2 + \lambda|\partial_t\Phi|^2\partial_{tt}\Phi)|w|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= s\lambda^2 \iint_Q \varphi(|\partial_t\Phi|^2 - |\nabla\Phi|^2)|\partial_t w|^2 \,\mathrm{d}x\,\mathrm{d}t + X_2 + X_3 + X_4 + X_5 \end{aligned} \tag{50}$$

with X_i for $i = 2, \ldots, 5$ given by

$$X_2 = -\frac{5s\lambda^3}{2} \iint_Q \varphi\partial_{tt}\Phi|\partial_t\Phi|^2|w|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{51}$$

$$X_3 = \frac{s\lambda^3}{2} \iint_Q \varphi \partial_{tt}\Phi |\nabla\Phi|^2 |w|^2 \, \mathrm{d}x \, \mathrm{d}t, \tag{52}$$

$$X_4 = -\frac{s\lambda^4}{2} \iint_Q \varphi |\partial_t\Phi|^2 (|\partial_t\Phi|^2 - |\nabla\Phi|^2)|w|^2 \, \mathrm{d}x \, \mathrm{d}t \tag{53}$$

and

$$X_5 = -s\lambda^2 \iint_Q \varphi |\partial_{tt}\Phi|^2 |w|^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{54}$$

Let us compute the scalar product of the first term of $P_1 w$ and the third one of $P_2 w$.

$$\begin{aligned}
I_{13} &= -2s\lambda \iint_Q \partial_{tt} w \varphi (\partial_t\Phi \partial_t w - \nabla\Phi \cdot \nabla w) \, \mathrm{d}x \, \mathrm{d}t \\
&= -s\lambda \iint_Q \varphi \partial_t \Phi \partial_t |w|^2 \, \mathrm{d}x \, \mathrm{d}t + 2s\lambda \iint_Q \varphi \partial_{tt} w \nabla\Phi \cdot \nabla w \, \mathrm{d}x \, \mathrm{d}t \\
&= s\lambda \iint_Q (\varphi \partial_{tt}\Phi + \lambda\varphi |\partial_t\Phi|^2) |\partial_t w|^2 \, \mathrm{d}x \, \mathrm{d}t - 2s\lambda \iint_Q \varphi \partial_t w \nabla\Phi \cdot \nabla(\partial_t w) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - 2s\lambda^2 \iint_Q \varphi \partial_t \Phi \partial_t w \nabla\Phi \cdot \nabla w \, \mathrm{d}x \, \mathrm{d}t \\
&= s\lambda \iint_Q \varphi \partial_{tt}\Phi |\partial_t w|^2 \, \mathrm{d}x \, \mathrm{d}t + s\lambda^2 \iint_Q \varphi |\partial_t\Phi|^2 |\partial_t w|^2 \, \mathrm{d}x \, \mathrm{d}t + s\lambda \iint_Q \varphi \Delta\Phi |\partial_t w|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + s\lambda^2 \iint_Q \varphi |\nabla\Phi|^2 |\partial_t w|^2 \, \mathrm{d}x \, \mathrm{d}t - 2s\lambda^2 \iint_Q \varphi \partial_t\Phi \partial_t w \nabla\Phi \cdot \nabla w \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{55}$$

The scalar product of the second term of $P_1\psi$ with the first one of $P_2\psi$ gives

$$\begin{aligned}
I_{21} &= -s\lambda \iint_Q \Delta w (K-1) \varphi (\partial_{tt}\Phi - \Delta\Phi) w \, \mathrm{d}x \, \mathrm{d}t \\
&= (K-1)s\lambda \iint_Q \varphi (\partial_{tt}\Phi - \Delta\Phi) |\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + (K-1)s\lambda^2 \iint_Q \varphi \nabla\Phi (\partial_{tt}\Phi - \Delta\Phi) \nabla w w \, \mathrm{d}x \, \mathrm{d}t \\
&= -(1-K)s\lambda \iint_Q \varphi (\partial_{tt}\Phi - \Delta\Phi) |\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}t + X_6 + X_7,
\end{aligned} \tag{56}$$

where

$$X_6 = \frac{(1-K)s\lambda^2}{2} \iint_Q \varphi \Delta\Phi (\partial_{tt}\Phi - \Delta\Phi) |w|^2 \, \mathrm{d}x \, \mathrm{d}t \tag{57}$$

and

$$X_7 = \frac{(1-K)s\lambda^3}{2} \iint_Q \varphi |\nabla\Phi|^2 (\partial_{tt}\Phi - \Delta\Phi) |w|^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{58}$$

The scalar product between the second term of $P_1 w$ with the second one of $P_2 w$ gives:

$$\begin{aligned}
I_{22} &= s\lambda^2 \iint_Q \Delta w \varphi (|\partial_t\Phi|^2 - |\nabla\Phi|^2) w \, \mathrm{d}x \, \mathrm{d}t \\
&= -s\lambda^2 \iint_Q \varphi (|\partial_t\Phi|^2 - |\nabla\Phi|^2) |\nabla w|^2 \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}$$

$$- s\lambda^3 \iint_Q \varphi \nabla\Phi(|\partial_t\Phi|^2 - |\nabla\Phi|^2)\nabla w w \,\mathrm{d}x\,\mathrm{d}t + 2s\lambda^2 \iint_Q \varphi \frac{\partial^2\Phi}{\partial x_k \partial x_j}\frac{\partial\Phi}{\partial x_j}\frac{\partial w}{\partial x_k} w \,\mathrm{d}x\,\mathrm{d}t$$

$$= -s\lambda^2 \iint_Q \varphi(|\partial_t\Phi|^2 - |\nabla\Phi|^2)|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t + X_8 + X_9 + X_{10} + X_{11}, \tag{59}$$

where X_i for $i = 8, \ldots, 11$, are the following integrals:

$$X_8 = \frac{s\lambda^3}{2} \iint_Q \varphi\Delta\Phi(|\partial_t\Phi|^2 - |\nabla\Phi|^2)|w|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{60}$$

$$X_9 = \frac{s\lambda^4}{2} \iint_Q \varphi|\nabla\Phi|^2(|\partial_t\Phi|^2 - |\nabla\Phi|^2)|w|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{61}$$

$$X_{10} = -2s\lambda^3 \iint_Q \varphi H(\Phi)\nabla\Phi\cdot\nabla\Phi|w|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{62}$$

and

$$X_{11} = -s\lambda^2 \iint_Q \varphi \left(\frac{\partial^2\Phi}{\partial x_k \partial x_j}\right)^2 |w|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{63}$$

In (62), $H(\Phi)$ denotes the Hessian matrix of Φ.

Now we consider the second and the third terms of $P_1 w$ and $P_2 w$, respectively.

$$\begin{aligned}
I_{23} &= 2s\lambda \iint_Q \Delta w \varphi(\partial_t\Phi\partial_t w - \nabla\Phi\cdot\nabla w) \,\mathrm{d}x\,\mathrm{d}t \\
&= 2s\lambda \iint_Q \varphi \frac{\partial^2 w}{\partial x_k^2}\left(\partial_t\Phi\partial_t w - \frac{\partial\Phi}{\partial x_j}\frac{\partial w}{\partial x_j}\right) \mathrm{d}x\,\mathrm{d}t \\
&= -2s\lambda \iint_Q \varphi\partial_t\Phi \frac{\partial w}{\partial x_k}\frac{\partial}{\partial x_k}(\partial_t w)\,\mathrm{d}x\,\mathrm{d}t - 2s\lambda^2 \iint_Q \varphi\partial_t\Phi\frac{\partial\Phi}{\partial x_k}\frac{\partial w}{\partial x_k}\partial_t w \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2s\lambda^2 \iint_Q \varphi \frac{\partial\Phi}{\partial x_k}\frac{\partial w}{\partial x_k}\frac{\partial\Phi}{\partial x_j}\frac{\partial w}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t + 2s\lambda \iint_Q \varphi\frac{\partial^2\Phi}{\partial x_k\partial x_j}\frac{\partial w}{\partial x_k}\frac{\partial w}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2s\lambda \iint_Q \varphi\frac{\partial\Phi}{\partial x_j}\frac{\partial^2 w}{\partial x_k\partial x_j}\frac{\partial w}{\partial x_k}\,\mathrm{d}x\,\mathrm{d}t - 2s\lambda \iint_\Sigma \varphi\frac{\partial w}{\partial x_k}\nu_k\frac{\partial\Phi}{\partial x_j}\frac{\partial w}{\partial x_j}\,\mathrm{d}\sigma\,\mathrm{d}t \\
&= s\lambda \iint_Q (\varphi\partial_{tt}\Phi + \lambda|\partial_t\Phi|^2\varphi)|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - 2s\lambda^2 \iint_Q \varphi\partial_t\Phi\nabla\Phi\cdot\nabla w\partial_t w \,\mathrm{d}x\,\mathrm{d}t + 2s\lambda^2 \iint_Q \varphi|\nabla\Phi\cdot\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2s\lambda \iint_Q \varphi H(\Phi)\nabla w\cdot\nabla w \,\mathrm{d}x\,\mathrm{d}t - s\lambda \iint_Q \varphi\Delta\Phi|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - s\lambda^2 \iint_Q \varphi|\nabla\Phi|^2|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t - s\lambda \iint_\Sigma \varphi(\nabla\Phi\cdot\nu)\left|\frac{\partial w}{\partial\nu}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t \\
&= s\lambda \iint_Q \varphi(\partial_{tt}\Phi - \Delta\Phi)|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t + 2s\lambda^2 \iint_Q \varphi|\nabla\Phi\cdot\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + s\lambda^2 \iint_Q \varphi(|\partial_t\Phi|^2 - |\nabla\Phi|^2)|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t + 2s\lambda \iint_Q \varphi H(\Phi)\nabla w\cdot\nabla w \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - 2s\lambda^2 \iint_Q \varphi\partial_t\Phi\nabla\Phi\cdot\nabla w\partial_t w \,\mathrm{d}x\,\mathrm{d}t - s\lambda \iint_\Sigma \varphi(\nabla\Phi\cdot\nu)\left|\frac{\partial w}{\partial\nu}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t.
\end{aligned} \tag{64}$$

Here we have used that $\frac{\partial w}{\partial x_j}\nu_k = \frac{\partial w}{\partial x_k}\nu_j$, since $w = 0$ on Σ.

The last integrals give

$$I_{31} = -(1-K)s^3\lambda^3 \iint_Q \varphi^3(|\partial_t\Phi|^2 - |\nabla\Phi|^2)(\partial_{tt}\Phi - \Delta\Phi)|w|^2 \,\mathrm{d}x\,\mathrm{d}t, \quad (65)$$

$$I_{32} = -s^3\lambda^4 \iint_Q \varphi^3(|\partial_t\Phi|^2 - |\nabla\Phi|^2)^2|w|^2 \,\mathrm{d}x\,\mathrm{d}t \quad (66)$$

and

$$\begin{aligned}
I_{33} &= -2s^3\lambda^3 \iint_Q \varphi^3(|\partial_t\Phi|^2 - |\nabla\Phi|^2)w(\partial_t\Phi\partial_t w - \nabla\Phi\cdot\nabla w)\,\mathrm{d}x\,\mathrm{d}t \\
&= s^3\lambda^3 \iint_Q \varphi^3(|\partial_t\Phi|^2 - |\nabla\Phi|^2)(-\partial_t\Phi\partial_t|w|^2 + \nabla\Phi\nabla|w|^2)\,\mathrm{d}x\,\mathrm{d}t \\
&= s^3\lambda^3 \iint_Q \varphi^3[(|\partial_t\Phi|^2 - |\nabla\Phi|^2)(\partial_{tt}\Phi + 3\lambda|\partial_t\Phi|^2) + 2|\partial_t\Phi|^2\partial_{tt}\Phi]|w|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - s^3\lambda^3 \iint_Q \varphi^3[(|\partial_t\Phi|^2 - |\nabla\Phi|^2)(\Delta\Phi + 3\lambda|\nabla\Phi|^2) - 2H(\Phi)\nabla\Phi\cdot\nabla\Phi]|w|^2\,\mathrm{d}x\,\mathrm{d}t \\
&= s^3\lambda^3 \iint_Q \varphi^3(\partial_{tt}\Phi - \Delta\Phi)(|\partial_t\Phi|^2 - |\nabla\Phi|^2)|w|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2s^3\lambda^3 \iint_Q \varphi^3(|\partial_t\Phi|^2\partial_{tt}\Phi + H(\Phi)\nabla\Phi\cdot\nabla\Phi)|w|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 3s^3\lambda^4 \iint_Q \varphi^3(|\partial_t\Phi|^2 - |\nabla\Phi|^2)^2|w|^2\,\mathrm{d}x\,\mathrm{d}t. \quad (67)
\end{aligned}$$

Finally from (47), taking into account (48), (50), (55), (56), (59) and (64)–(67) we deduce that

$$\begin{aligned}
(P_1w, P_2w) &= 2s\lambda \iint_Q \varphi\partial_{tt}\Phi|\partial_t w|^2\,\mathrm{d}x\,\mathrm{d}t - Ks\lambda \iint_Q \varphi(\partial_{tt}\Phi - \Delta\Phi)|\partial_t w|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2s\lambda^2 \iint_Q \varphi(|\partial_t\Phi|^2|\partial_t w|^2 - 2\partial_t\Phi\partial_t w\nabla\Phi\cdot\nabla w + |\nabla\Phi\cdot\nabla w|^2)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2s\lambda \iint_Q \varphi H(\Phi)\nabla w\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t + Ks\lambda \iint_Q \varphi(\partial_{tt}\Phi - \Delta\Phi)|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2s^3\lambda^4 \iint_Q \varphi^3(|\partial_t\Phi|^2 - |\nabla\Phi|^2)^2)|w|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2s^3\lambda^3 \iint_Q \varphi^3[|\partial_t\Phi|^2\partial_{tt}\Phi + H(\Phi)\nabla\Phi\cdot\nabla\Phi]|w|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + Ks^3\lambda^3 \iint_Q \varphi^3(|\partial_t\Phi|^2 - |\nabla\Phi|^2)(\partial_{tt}\Phi - \Delta\Phi)|w|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - s\lambda \iint_\Sigma \varphi(\nabla\Phi\cdot\nu)\left|\frac{\partial w}{\partial\nu}\right|^2\,\mathrm{d}\sigma\,\mathrm{d}t + \sum_{i=1}^{11} X_i, \quad (68)
\end{aligned}$$

where $X_i, i = 1, \ldots, 11$, are given by (49), (51)–(54), (57), (58) and (60)–(63), respectively.

Note that in (68), the third term on the left-hand side can be written as

$$\begin{aligned}
&2s\lambda^2 \iint_Q \varphi(|\partial_t\Phi|^2|\partial_t w|^2 - 2\partial_t\Phi\partial_t w\nabla\Phi\cdot\nabla w + |\nabla\Phi\cdot\nabla w|^2)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = 2s\lambda^2 \iint_Q \varphi(\partial_t\Phi\partial_t w - \nabla\Phi\cdot\nabla w)^2\,\mathrm{d}x\,\mathrm{d}t \geqslant 0. \quad (69)
\end{aligned}$$

Then, from (46) taking into account (68) and (69) we obtain

$$\begin{aligned}
&\|P_1w\|_2^2+\|P_2w\|_2^2+s\lambda\iint_Q\varphi[4\partial_{tt}\Phi-2K(\partial_{tt}\Phi-\Delta\Phi)]|\partial_tw|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+s\lambda\iint_Q\varphi[4H(\Phi)\nabla w\cdot\nabla w+K(\partial_{tt}\Phi-\Delta\Phi)|\nabla w|^2]\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+4s^3\lambda^4\iint_Q\varphi^3(|\partial_t\Phi|^2-|\nabla\Phi|^2)^2|w|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+4s^3\lambda^3\iint_Q\varphi^3[|\partial_t\Phi|^2\partial_{tt}\Phi+H(\Phi)\nabla\Phi\cdot\nabla\Phi]|w|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+2Ks^3\lambda^3\iint_Q\varphi^3(|\partial_t\Phi|^2-|\nabla\Phi|^2)(\partial_{tt}\Phi-\Delta\Phi)|w|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-2s\lambda\iint_\Sigma\varphi(\nabla\Phi\cdot\nu)\left|\frac{\partial w}{\partial\nu}\right|^2\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\quad\leqslant 2\left|\sum_{i=1}^{11}X_i\right|+\|g_s\|_2^2.
\end{aligned}\tag{70}$$

On the other hand, from (53) and (61) we get

$$\begin{aligned}
|X_4+X_9|&\leqslant s\lambda^4\iint_Q\varphi(|\partial_t\Phi|^2-|\nabla\Phi|^2)^2|w|^2\,\mathrm{d}x\,\mathrm{d}t\\
&=s\lambda^4\iint_Q(\varphi^2\varphi)\varphi^{-2}(|\partial_t\Phi|^2-|\nabla\Phi|^2)^2|w|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant s^2\lambda^4\iint_Q\varphi^3(|\partial_t\Phi|^2-|\nabla\Phi|^2)^2|w|^2\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{71}$$

where we have used that

$$\varphi^{-2}=\exp(-2\lambda\Phi)\leqslant\exp(2\lambda(\beta T^2-a\phi\exp(2\rho\phi^\perp)))\leqslant s,\tag{72}$$

for $s\geqslant s_1(\lambda)=\exp(2\lambda\beta T^2)$. Thus, from (70) and (71) we get

$$\begin{aligned}
&\|P_1w\|_2^2+\|P_2w\|_2^2+s\lambda\iint_Q\varphi[4\partial_{tt}\Phi-2K(\partial_{tt}\Phi-\Delta\Phi)]|\partial_tw|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+s\lambda\iint_Q\varphi[4H(\Phi)\nabla w\cdot\nabla w+2K(\partial_{tt}\Phi-\Delta\Phi)|\nabla w|^2]\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+3s^3\lambda^4\iint_Q\varphi^3(|\partial_t\Phi|^2-|\nabla\Phi|^2)^2|w|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+4s^3\lambda^3\iint_Q\varphi^3[|\partial_t\Phi|^2\partial_{tt}\Phi+H(\Phi)\nabla\Phi\cdot\nabla\Phi]|w|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+2Ks^3\lambda^3\iint_Q\varphi^3(|\partial_t\Phi|^2-|\nabla\Phi|^2)(\partial_{tt}\Phi-\Delta\Phi)|w|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-2s\lambda\iint_\Sigma\varphi(\nabla\Phi\cdot\nu)\left|\frac{\partial w}{\partial\nu}\right|^2\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\quad\leqslant 2\sum_{i=1,i\neq4,9}^{11}|X_i|+\|g_s\|_2^2,
\end{aligned}\tag{73}$$

for $s\geqslant s_1(\lambda)\geqslant 1$. We remark that until now we have not used that $x_0\in\mathbb{R}^2\backslash\overline{\Omega}$.

We will choose now the constants β and K in such a way that on the left-hand side of (73) all the terms on $|\partial_t w|^2$, $|\nabla w|^2$ and $|w|^2$ have positive sign. More precisely, we need the following:

$$4\partial_{tt}\Phi - 2K(\partial_{tt}\Phi - \Delta\Phi) \geqslant N_1 > 0, \tag{74}$$

$$4H(\Phi)\xi\cdot\xi + 2K(\partial_{tt}\Phi - \Delta\Phi)|\xi|^2 \geqslant N_2 > 0, \qquad \text{for all} \quad \xi \in \mathbb{R}^2\backslash\{0\} \tag{75}$$

and

$$3\lambda(|\partial_t\Phi|^2 - |\nabla\Phi|^2)^2 + 4[|\partial_t\Phi|^2\partial_{tt}\Phi + H(\Phi)\nabla\Phi\cdot\nabla\Phi] \\ + 2K(|\partial_t\Phi|^2 - |\nabla\Phi|^2)(\partial_{tt}\Phi - \Delta\Phi) \geqslant N_3 > 0. \tag{76}$$

Let us use lemma 4.1 for $i = 0$. According to it, if $\beta \in \left(0, \beta_2^0\right)$ and $K \equiv K^0$ is such that (121) holds for $i = 0$, then for $\lambda \geqslant \overline{\lambda}_0$ with $\overline{\lambda}_0 = \overline{\lambda}_0(a, b, \Omega, x_0) \geqslant 1$ sufficiently large, conditions (74)–(76) are fulfilled with $N_i = N_i(a, b, \Omega, x_0)$, $i = 1, 2, 3$. Consequently, from (73) we deduce that there exists a constant $C = C(a, b, \Omega, x_0)$ such that

$$s\lambda\iint_Q \varphi(|\partial_t w|^2 + |\nabla w|^2)\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^3\iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant C\left(\|g_s\|_2^2 + s\lambda\iint_{\Sigma_0}\varphi(\nabla\Phi\cdot\nu)\left|\frac{\partial w}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t + \sum_{i=1, i\neq 4,9}^{11}|X_i|\right) \tag{77}$$

for all $\lambda \geqslant \overline{\lambda}_0$ and $s \geqslant s_1(\lambda)$.

Now, we will see that all the terms X_i, $i = 1, \ldots, 11$, $i \neq 4, 9$, on the right-hand side of (77) can be controlled by the terms on the left-hand side. More precisely, using (18) and (30) we obtain

$$|\partial_t\Phi| = |-2\beta t| \leqslant CT, \tag{78}$$

$$|\partial_{tt}\Phi| = |-2\beta| \leqslant C, \tag{79}$$

$$\Delta\Phi = \frac{4}{a}\,\mathrm{e}^{2\rho\phi^\perp}, \tag{80}$$

$$||\partial_t\Phi|^2 - |\nabla\Phi|^2| = \left|4\beta^2t^2 - 4\phi\,\mathrm{e}^{4\rho\phi^\perp}\right| \leqslant C(a, b, \Omega, x_0)T^2, \tag{81}$$

$$|\partial_{tt}\Phi - \Delta\Phi| = \left|2\left(\beta + \frac{2}{a}\,\mathrm{e}^{2\rho\phi^\perp}\right)\right| \leqslant C(a, b, \Omega, x_0). \tag{82}$$

Moreover, using the computation from lemma 4.1 for $i = 0$, thanks to (136), (137) and (150) we get

$$H(\Phi)\nabla\Phi\cdot\nabla\Phi = 8\frac{1+b^2}{a}\phi\,\mathrm{e}^{6\rho\phi^\perp} \leqslant C(a, b, \Omega, x_0), \tag{83}$$

$$\left|\frac{\partial^2\Phi}{\partial x_k\partial x_j}\right| \leqslant C(a, b, \Omega, x_0). \tag{84}$$

Recall that here we use the notation $\Phi = \Phi^0$, $\phi = \phi_0, \ldots$. Thus, from (49), (51), (52), (54)–(60), (62) and (63) and taking into account (78)–(82), we deduce the following:

$$|X_1| \leqslant CT^2s\lambda^3\iint_Q \varphi|w|^2\,\mathrm{d}x\,\mathrm{d}t = CT^2s\lambda^3\iint_Q (\varphi^2\varphi)\varphi^{-2}|w|^2\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant CT^2s^2\lambda^3\iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{85}$$

where we have used again (72). Similarly

$$|X_2| \leqslant CT^2 s\lambda^3 \iint_Q \varphi|w|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant CT^2 s^2\lambda^3 \iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{86}$$

$$|X_3| \leqslant Cs\lambda^3 \iint_Q \varphi|w|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant Cs^2\lambda^3 \iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{87}$$

$$|X_5| \leqslant Cs\lambda^2 \iint_Q \varphi|w|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant Cs^2\lambda^2 \iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{88}$$

$$|X_6| \leqslant Cs\lambda^2 \iint_Q \varphi|w|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant Cs^2\lambda^2 \iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{89}$$

$$|X_7| \leqslant Cs\lambda^3 \iint_Q \varphi|w|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant Cs^2\lambda^3 \iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{90}$$

$$|X_8| \leqslant CT^2 s\lambda^3 \iint_Q \varphi|w|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant CT^2 s^2\lambda^3 \iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{91}$$

$$|X_{10}| \leqslant Cs\lambda^3 \iint_Q \varphi|w|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant Cs^2\lambda^3 \iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{92}$$

$$|X_{11}| \leqslant Cs\lambda^2 \iint_Q \varphi|w|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant Cs^2\lambda^2 \iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{93}$$

and consequently since $\lambda \geqslant \overline{\lambda}_0 \geqslant 1$ there exists a constant $C = C(a, b, \Omega, x_0)$ such that

$$\sum_{i=1, i\neq 4,9}^{11} |X_i| \leqslant C(1+T^2)s^2\lambda^3 \iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{94}$$

On the other hand, from (43) and due to (72) we get

$$\begin{aligned}\|g_s\|_2^2 &\leqslant \|\mathrm{e}^{s\varphi}g - q(x,t)w\|_2^2 + Cs\lambda \iint_Q \varphi|w|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \|\mathrm{e}^{s\varphi}g - q(x,t)w\|_2^2 + Cs^2\lambda \iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{95}$$

Therefore, using (85)–(93) and (95) in (77) we obtain

$$\begin{aligned}&s\lambda \iint_Q \varphi(|\partial_t w|^2 + |\nabla w|^2)\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^3 \iint_Q \varphi^3|w|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leqslant C\left(\|\mathrm{e}^{s\varphi}g - q(x,t)w\|_2^2 + s\lambda \iint_{\Sigma_0} \varphi(\nabla\Phi\cdot\nu)\left|\frac{\partial w}{\partial\nu}\right|^2\,\mathrm{d}\sigma\,\mathrm{d}t\right)\end{aligned} \tag{96}$$

for all $\lambda \geqslant \overline{\lambda}_0$ and $s \geqslant s_2(\lambda) = \mu_2(1+T^2)\,\mathrm{e}^{2\lambda\beta T^2}$ for some $\mu_2 = \mu_2(a, b, \Omega, x_0) \geqslant 1$.

Let us deduce from (96) that inequality (39) holds for all $\lambda \geqslant \overline{\lambda}_0$ and $s \geqslant \overline{s}_0(\lambda)$, where $\overline{s}_0(\lambda)$ is as in theorem 3.2. Recall that $w = \mathrm{e}^{s\varphi}v$. Then, using (34) and (36) we have

$$\begin{aligned}\partial_t w &= \mathrm{e}^{s\varphi}(\partial_t v + s\partial_t\varphi v) = \mathrm{e}^{s\varphi}(\partial_t v - 2s\lambda\varphi\beta t v), \\ \nabla w &= \mathrm{e}^{s\varphi}(\nabla v + s\nabla\varphi v) = \mathrm{e}^{s\varphi}(\nabla v + s\lambda\varphi\nabla\Phi v).\end{aligned}$$

Then, we can write that

$$\mathrm{e}^{s\varphi}\partial_t v = \partial_t w + 2s\lambda\beta t\,\mathrm{e}^{s\varphi}v = \partial_t w + 2s\lambda\varphi\beta t w, \tag{97}$$

$$\mathrm{e}^{s\varphi}\nabla v = \nabla w - s\lambda\varphi\nabla\Phi\,\mathrm{e}^{s\varphi}v = \nabla w - s\lambda\varphi\nabla\Phi w. \tag{98}$$

Consequently, we find the following:

$$s\lambda \iint_Q \varphi\, \mathrm{e}^{2s\varphi} |\partial_t v|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant Cs\lambda \iint_Q \varphi |\partial_t w|^2 \,\mathrm{d}x\,\mathrm{d}t + Cs^3\lambda^3 T^2 \iint_Q \varphi^3 |w|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{99}$$

and

$$\begin{aligned} s\lambda \iint_Q \varphi\, \mathrm{e}^{s\varphi} |\nabla v|^2 \mathrm{d}x\,\mathrm{d}t &\leqslant Cs\lambda \iint_Q \varphi |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t + Cs^3\lambda^3 \iint_Q \varphi^3 |\nabla\Phi|^2 |w|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant Cs\lambda \iint_Q \varphi |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t + Cs^3\lambda^3 \iint_Q \varphi^3 |w|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{100}$$

In (100), we have used that $|\nabla\Phi| \leqslant C(a, b, \Omega, x_0)$ in $\Omega \times (-T, T)$ as can be seen from (30). On the other hand, on Σ we have

$$\frac{\partial w}{\partial \nu} = \mathrm{e}^{s\varphi} \left(\frac{\partial v}{\partial \nu} + s \frac{\partial \varphi}{\partial \nu} v \right) = \mathrm{e}^{s\varphi} \frac{\partial v}{\partial \nu}, \tag{101}$$

since $v = 0$ on Σ. Then, taking into account (99)–(101) in (96) we deduce that there exists a positive constant C only depending on a, b, Ω and x_0 such that

$$\begin{aligned} &s\lambda \iint_Q \varphi\, \mathrm{e}^{2s\varphi} (|\partial_t v|^2 + |\nabla v|^2) \,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^3 \iint_Q \varphi^3\, \mathrm{e}^{2s\varphi} |v|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leqslant C \Bigg(\|\mathrm{e}^{s\varphi} g\|_2^2 + \|q\|_{L^\infty(Q)}^2 \iint_Q \mathrm{e}^{2s\varphi} |v|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + s\lambda \iint_{\Sigma_0} \varphi\, \mathrm{e}^{2s\varphi} (\nabla\Phi \cdot \nu) \left| \frac{\partial v}{\partial \nu} \right|^2 \mathrm{d}\sigma\,\mathrm{d}t \Bigg), \end{aligned} \tag{102}$$

where we have used that $C/(1 + T^2) \leqslant C$. Now since

$$\varphi^{-3} = \exp(-3\lambda\Phi) \leqslant \exp(3\lambda(\beta T^2 - a\phi \exp(2\rho\phi^\perp))) \leqslant s, \tag{103}$$

for $s \geqslant \exp(3\lambda\beta T^2)$, then we have that

$$\|q\|_{L^\infty(Q)}^2 \iint_Q \mathrm{e}^{2s\varphi} |v|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant sM^2 \iint_Q \varphi^3\, \mathrm{e}^{2s\varphi} |v|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{104}$$

Finally, taking $\lambda \geqslant \overline{\lambda}_0$ and $s \geqslant \overline{s}_0(\lambda)$, where $\overline{s}_0$ can be taken of the form $\overline{s}_0(\lambda) = \overline{\mu}_0 M(1 + T^2) \exp(3\lambda\beta T^2)$ for some constant $\overline{\mu}_0 = \overline{\mu}_0(a, b, \Omega, x_0)$, we get the estimate (39) by absorbing the term in (104) on the left-hand side of (102). This ends the proof of theorem 3.2. □

Remark 3.4. Let us see that Φ is a pseudoconvex function in the sense of Hörmander (see [10], p 239) if $x_0 \in \mathbb{R}^2 \backslash \overline{\Omega}$ and $\beta \in (0, \beta_2)$. In our case, if $z = (z_0, z_1, z_2) = (z_0, z')$, $t = z_0$, $\xi = (\xi_0, \xi_1, \xi_2) = (\xi_0, \xi')$, $p(z, \xi) = p(\xi) = \xi_0^2 - (\xi_1^2 + \xi_2^2)$ and $\{p, q\} = \sum_{j=0}^n \frac{\partial p}{\partial \xi_i} \frac{\partial q}{\partial z_i} - \frac{\partial q}{\partial \xi_i} \frac{\partial p}{\partial z_i}$ is the usual Poisson bracket. In our case, pseudoconvexity corresponds to checking that (see, for instance, [1])

$$\{p, \{p, \Phi\}\}(z, \xi) > 0 \qquad \text{on} \quad \{\xi \neq 0 : p(z, \xi) = \{p, \Phi\}(z, \xi) = 0\} \tag{105}$$

$$\{\{p, \Phi\}, p(z, \nabla\Phi)\} > 0 \qquad \text{on} \quad \{p(z, \nabla\Phi) = 0\}. \tag{106}$$

Using inequality (140) and that $p(z, \xi) = 0$, i.e. $\xi_0^2 = |\xi'|^2 \neq 0$ we get

$$\begin{aligned} \{p, \{p, \Phi\}\} &= 4H(\Phi)\xi' \cdot \xi' - 8\beta\xi_0^2 \\ &\geqslant 4ma\, \mathrm{e}^{2\rho\phi^\perp} |\xi'|^2 - 8\beta\xi_0^2 \\ &= 4\xi_0^2 \left(ma\, \mathrm{e}^{2\rho\phi^\perp} - 2\beta \right) > 0, \end{aligned} \tag{107}$$

which is strictly positive in $\overline{\Omega}$ since we assume that $\beta \in (0, \beta_2)$. Indeed, β_2 defined later in (129) is chosen in order to have a nonempty interval in (126) that is

$$\sup_{x\in\Omega} \frac{2\beta a}{\beta a + 2\,e^{2\rho\phi_i^\perp}} = \frac{2\beta a}{\beta a + 2\,e^{2\rho\theta_-^i}} < \frac{a^2 m\, e^{2\rho\theta_-^i}}{\beta a + 2\,e^{2\rho\theta_+^i}} = \inf_{x\in\Omega} \frac{a^2 m\, e^{2\rho\phi_i^\perp}}{\beta a + 2\,e^{2\rho\phi_i^\perp}},$$

and this implies that

$$ma\, e^{2\rho\phi(x)^\perp} - 2\beta > 0, \qquad \forall x \in \overline{\Omega}.$$

On the other hand, using again (140) and that $p(z, \nabla\Phi) = 0$, i.e. $4\beta^2 t^2 = |\nabla\Phi|^2$ we obtain

$$\begin{aligned}\{\{p, \Phi\}, p(z, \nabla\Phi)\} &= 4H(\Phi)\nabla'\Phi \cdot \nabla'\Phi - 32\beta^2 t^2\\ &\geqslant 4ma\, e^{2\rho\phi^\perp}|\nabla'\Phi|^2 - 32\beta^2 t^2\\ &= 4|\nabla'\Phi|^2\big(ma\, e^{2\rho\phi^\perp} - 2\beta\big) > 0,\end{aligned} \tag{108}$$

which is strictly positive in $\overline{\Omega}$ for $\beta \in (0, \beta_2)$ and $x_0 \in \mathbb{R}^2\backslash\overline{\Omega}$ as we can see from (32) since in this case $|x - x_0| > 0$ for each $x \in \overline{\Omega}$.

Remark 3.5. Since Φ is pseudoconvex, a Carleman estimate like (39) without explicit dependence on λ can be obtained (see [23]). We have homogeneous Dirichlet boundary conditions in $\Omega \times (-T, T)$ for $v \in Z$ and the region Σ_0 corresponds to the one where the strong Lopatinskii condition (see [23]) is not verified.

3.2. Proof of theorem 3.1 in the case $x_0, x_1 \in \Omega$, $x_0 \neq x_1$

We note that if $x_i \in \Omega$, then $\nabla\Phi^i$ vanishes only for $x = x_i$. By considering a small ball centred at x_i we have the following variant of the Carleman estimate (39).

Proposition 3.3. *Let $i = 0$ or $i = 1$. Assume that $x_i \in \Omega$, and let B_i be a ball of centre x_i and positive radius. Let $a > 0$, $b \in \mathbb{R}$ with $a^2 + b^2 = 1$, Γ_i such that (8) holds and $\beta \in \big(0, \beta_2^i\big)$ where β_2^i is defined in (129). Then for all $M > 0$, there exist positive constants $\overline{\lambda}_i$, $\overline{\mu}_i$ and C only depending on a, b, Ω and x_i, such that for any $q \in L^\infty(Q)$ with $\|q\|_{L^\infty(Q)} \leqslant M$, for any $\lambda \geqslant \overline{\lambda}_i$, any $s \geqslant \overline{s}_i = \overline{\mu}_i M(1 + T^2)\exp(3\lambda\beta T^2)$ and any $v \in Z$ we have*

$$\begin{aligned}&s\lambda \iint_Q \varphi_i\, e^{2s\varphi_i}(|\partial_t v|^2 + |\nabla v|^2)\,dx\,dt + s^3\lambda^3 \iint_Q \varphi_i^3\, e^{2s\varphi_i}|v|^2\,dx\,dt\\ &\quad\leqslant C\Bigg(\iint_Q e^{2s\varphi_i}|\partial_{tt}v - \Delta v + q(x,t)v|^2\,dx\,dt\\ &\qquad + s\lambda \iint_{\Gamma_i\times(-T,T)} \varphi_i\, e^{2s\varphi_i}\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma\,dt\\ &\qquad + s^3\lambda^3 \iint_{(B_i\cap\Omega)\times(-T,T)} \varphi_i^3\, e^{2s\varphi_i}|v|^2\,dx\,dt\Bigg).\end{aligned} \tag{109}$$

Proof of proposition 3.3. In the proof of theorem 3.2, the hypothesis $x_i \in \mathbb{R}^2\backslash\overline{\Omega}$ is only used to apply lemma 4.1 in order to have condition (76). The other two conditions (74) and (75) can be obtained without this hypothesis. Therefore, the proof can be performed until (75) without changes even if $x_i \in \Omega$. After (75), the same arguments of the proof of theorem 3.2 can be continued in $Q_i = Q\backslash((B_i \cap \Omega) \times (-T, T))$ since $x_i \in \Omega\backslash\overline{B}_i$ and then (76) holds in Q_i. In this way, the terms X_i on the right-hand side of (73) can be absorbed in Q_i. For the remaining terms supported in $(B_i \cap \Omega) \times (-T, T)$, note that the term with factor $s^3\lambda^4$ on the left-hand side of (73) is non-negative and that the two integrals with factor $s^3\lambda^3$ on the left-hand side of

(73) can be now bounded together with X_i in $(B_i \cap \Omega) \times (-T, T)$. Inequality (109) is then obtained. □

We have now all the ingredients in order to prove theorem 3.1 in the case $x_0, x_1 \in \Omega$, $x_0 \neq x_1$. Let us put $\sigma = |x_0 - x_1| > 0$ and for some $r > 2$ that will be chosen later; let us define $B_0 = B(x_0, \sigma/r)$ and $B_1 = B(x_1, \sigma/r)$, and $D_0 = (B_0 \cap \Omega) \times (-T, T)$ and $D_1 = (B_1 \cap \Omega) \times (-T, T)$.

Let Φ^i and φ_i for $i = 0, 1$ be the corresponding weight functions given by (18) and (33), respectively. Let us first justify the following properties of the weight functions: for each $C > 0$, there exists $s_5 > 0$ such that

$$\varphi_1^3 \,\mathrm{e}^{2s\varphi_1} \geqslant 2C\varphi_0^3 \,\mathrm{e}^{2s\varphi_0} \qquad \text{in} \quad D_0, \tag{110}$$

$$\varphi_0^3 \,\mathrm{e}^{2s\varphi_0} \geqslant 2C\varphi_1^3 \,\mathrm{e}^{2s\varphi_1} \qquad \text{in} \quad D_1, \tag{111}$$

for all $s \geqslant s_5$ where the constant s_5 can be taken of the form $s_5 = \mu_5 \delta_\sigma^{-3/2} \exp(3\lambda\beta T^2)$ for some positive constant $\mu_5 = \mu_5(a, b, \Omega, x_0, x_1)$.

To see this, note that

$$\Phi^1 - \Phi^0 \geqslant a\left(\frac{r-1}{r}\right)\sigma^2 \,\mathrm{e}^{2\rho\theta_-^1} - \frac{a}{r^2}\sigma^2 \,\mathrm{e}^{2\rho\theta_+^0} \qquad \text{in} \quad D_0, \tag{112}$$

where θ_-^1 and θ_+^0 are constants only depending on a, b, Ω, x_0 and x_1 introduced in (120). For $r > \exp\left(\rho\left(\theta_+^0 - \theta_-^1\right)\right) + 1 \geqslant 2$ we have then

$$\Phi^1 - \Phi^0 \geqslant \kappa_0 \sigma^2 \qquad \text{in} \quad D_0, \tag{113}$$

where $\kappa_0 = \kappa_0(a, b, \Omega, x_0, x_1) > 0$ and can be chosen independent of σ as $\sigma \to 0$. Therefore, using the fact that $\lambda \geqslant \overline{\lambda} \geqslant 1$ we obtain that

$$\frac{\varphi_1}{\varphi_0} \geqslant \exp(\overline{\lambda}\kappa_0\sigma^2) \geqslant 1 \qquad \text{in} \quad D_0, \tag{114}$$

and

$$\varphi_1 - \varphi_0 \geqslant \varphi_1(1 - \exp(-\overline{\lambda}\kappa_0\sigma^2)) = \delta_\sigma \varphi_1 \qquad \text{in} \quad D_0. \tag{115}$$

Using (114) and (115) we obtain that

$$\frac{\varphi_1^3 \,\mathrm{e}^{2s\varphi_1}}{\varphi_0^3 \,\mathrm{e}^{2s\varphi_0}} \geqslant \exp(2s\delta_\sigma\varphi_1) \qquad \text{in} \quad D_0. \tag{116}$$

Since $\varphi_1 \geqslant \exp(-\lambda\beta T^2)$, given $C > 0$ there exists $\mu_3 = \mu_3(a, b, \Omega, x_0, x_1)$ such that

$$\exp(2s\delta_\sigma\varphi_1) \geqslant \exp(2s^{2/3}\delta_\sigma) \leqslant 2C \tag{117}$$

for all $s \geqslant s_3 = \mu_3 \delta_\sigma^{-3/2} \exp(3\lambda\beta T^2)$. This is exactly inequality (110) for $s \geqslant s_3$. Using symmetrical arguments we obtain (111) in D_1 for $s \geqslant s_4$ where s_4 has the same form of s_3. Then we take $s_5 = \max\{s_3, s_4\}$ in order to obtain (110) and (111) simultaneously.

Now, adding the two Carleman inequalities (109) for $i = 0, 1$, we obtain

$$\begin{aligned}
s\lambda \iint_Q &(\varphi_0 \,\mathrm{e}^{2s\varphi_0} + \varphi_1 \,\mathrm{e}^{2s\varphi_1})(|\partial_t v|^2 + |\nabla v|^2)\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^3 \iint_Q \left(\varphi_0^3 \,\mathrm{e}^{2s\varphi_0} + \varphi_1^3 \,\mathrm{e}^{2s\varphi_1}\right)|v|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant C\left(\iint_Q \left(\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1}\right)|\partial_{tt} v - \Delta v + q(x,t)v|^2\,\mathrm{d}x\,\mathrm{d}t\right) \\
&\quad + Cs\lambda \iint_{\gamma\times(-T,T)} (\varphi_0 \,\mathrm{e}^{2s\varphi_0} + \varphi_1 \,\mathrm{e}^{2s\varphi_1})\left|\frac{\partial v}{\partial \nu}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t \\
&\quad + C\left(s^3\lambda^3 \iint_{D_0} \varphi_0^3 \,\mathrm{e}^{2s\varphi_0}|v|^2\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^3 \iint_{D_1} \varphi_1^3 \,\mathrm{e}^{2s\varphi_1}|v|^2\,\mathrm{d}x\,\mathrm{d}t\right)
\end{aligned} \tag{118}$$

for all $s \geqslant s_6 = \max\{\overline{s}_0, \overline{s}_1\}$ and $\lambda \geqslant \overline{\lambda} = \max\{\overline{\lambda}_0, \overline{\lambda}_1\}$.

On the other hand, taking into account (110) and (111), we get

$$
\begin{aligned}
&Cs^3\lambda^3 \iint_{D_0} \varphi_0^3 \,\mathrm{e}^{2s\varphi_0} |v|^2 \,\mathrm{d}x\,\mathrm{d}t + Cs^3\lambda^3 \iint_{D_1} \varphi_1^3 \,\mathrm{e}^{2s\varphi_1} |v|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant \frac{s^3\lambda^3}{2} \iint_{D_0} \varphi_1^3 \,\mathrm{e}^{2s\varphi_1} |v|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{s^3\lambda^3}{2} \iint_{D_1} \varphi_0^3 \,\mathrm{e}^{2s\varphi_0} |v|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant \frac{s^3\lambda^3}{2} \iint_{Q} \left(\varphi_0^3 \,\mathrm{e}^{2s\varphi_0} + \varphi_1^3 \,\mathrm{e}^{2s\varphi_1}\right) |v|^2 \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{119}
$$

for all $s \geqslant s_5$. Combining (119) and (118) we find (38) for all $s \geqslant \overline{s} = \max\{s_5, s_6\}$. Moreover $\overline{s}$ can be taken of the form $\overline{s} = \overline{\mu}\delta_\sigma^{-3/2} M(1+T^2)\exp(3\lambda\beta T^2)$, with $\overline{\mu}$ a positive constant only depending on a, b, Ω, x_0 and x_1. This ends the proof of theorem 3.1.

4. A technical lemma

In this section, we precise the conditions over β and K in order to fulfil inequalities (74)–(76), which are necessary for the Carleman estimate of theorem 3.2.

Lemma 4.1. *For $i = 0, 1$, let the functions Φ^i be given by (18), $a > 0$ and $b \in \mathbb{R}$ with $a^2 + b^2 = 1$. Let us define for $i = 0, 1$*

$$
\theta_+^i = \begin{cases} \overline{\theta}^i & \text{if } b \geqslant 0 \\ \underline{\theta}^i & \text{if } b < 0 \end{cases} \qquad \theta_-^i = \begin{cases} \underline{\theta}^i & \text{if } b \geqslant 0 \\ \overline{\theta}^i & \text{if } b < 0. \end{cases}
\tag{120}
$$

Then there exists a positive constant β_2^i depending on a, b, θ_+^i and θ_-^i such that for all $\beta \in (0, \beta_2^i)$ and for all K^i in the nonempty interval

$$
\frac{2\beta a}{\beta a + 2\,\mathrm{e}^{2\rho\theta_-^i}} < K^i < \frac{ma^2\,\mathrm{e}^{2\rho\theta_-^i}}{\beta a + 2\,\mathrm{e}^{2\rho\theta_+^i}}, \qquad i = 0, 1
\tag{121}
$$

(where ρ and m are defined in (19) and (22), respectively) there exists $N_j = N_j(a, b, \Omega, x_i)$, $j = 1, 2$, such that we have the following inequalities for all $i = 0, 1$, $x \in \overline{\Omega}$ and $t \in \mathbb{R}$:

$$
4\partial_{tt}\Phi^i - 2K(\partial_{tt}\Phi^i - \Delta\Phi^i) \geqslant N_1 > 0,
\tag{122}
$$

$$
4H(\Phi^i)\xi\cdot\xi + 2K(\partial_{tt}\Phi^i - \Delta\Phi^i)|\xi|^2 \geqslant N_2 > 0 \qquad \forall \xi \in \mathbb{R}^2\backslash\{0\}.
\tag{123}
$$

Moreover, if $x_i \in \mathbb{R}^2\backslash\overline{\Omega}$ then there exist $\overline{\lambda}_i(a, b, \Omega, x_i) > 0$ and $N_3 = N_3(a, b, \Omega, x_i)$ such that for all $i = 0, 1$, $\lambda \geqslant \overline{\lambda}_i$, $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ we have

$$
\begin{aligned}
&3\lambda(|\partial_t\Phi^i|^2 - |\nabla\Phi^i|^2)^2 + 4[|\partial_t\Phi^i|^2\partial_{tt}\Phi^i + H(\Phi^i)\nabla\Phi^i\cdot\nabla\Phi^i] \\
&\qquad + 2K(|\partial_t\Phi^i|^2 - |\nabla\Phi^i|^2)(\partial_{tt}\Phi^i - \Delta\Phi^i) \geqslant N_3 > 0.
\end{aligned}
\tag{124}
$$

Remark 4.1. Note that for $a = 1$ and $b = 0$ (this corresponds to the case $\theta = 0$ when no rotation is made) inequality (121) becomes

$$
\frac{2\beta}{\beta+2} < K^i < \frac{2}{\beta+2}, \qquad i = 0, 1
\tag{125}
$$

which is nonempty for all $\beta \in (0, 1)$ and $\beta_2^i = 1$ in this case.

Proof of lemma 4.1. In the following, we will assume that $i = 0, 1$. Let us first see that the interval in (121) is nonempty. We must have

$$
\frac{2\beta a}{\beta a + 2\,\mathrm{e}^{2\rho\theta_-^i}} < \frac{ma^2\,\mathrm{e}^{2\rho\theta_-^i}}{\beta a + 2\,\mathrm{e}^{2\rho\theta_+^i}}.
\tag{126}
$$

Inequality (126) is equivalent to the following one (recall that $a > 0$):

$$2\beta^2 a + 4\beta F^i\, \mathrm{e}^{2\rho\theta_-^i} - 2am\, \mathrm{e}^{4\rho\theta_-^i} < 0, \tag{127}$$

where F was defined in (23). The equation $2\beta^2 a + 4\beta F^i\, \mathrm{e}^{2\rho\theta_-^i} - 2am\, \mathrm{e}^{4\rho\theta_-^i} = 0$ has two solutions

$$\beta_1^i = \frac{-F^i - \sqrt{(F^i)^2 + ma^2}}{a}\, \mathrm{e}^{2\rho\theta_-^i} \tag{128}$$

and

$$\beta_2^i = \frac{-F^i + \sqrt{(F^i)^2 + ma^2}}{a}\, \mathrm{e}^{2\rho\theta_-^i}. \tag{129}$$

Clearly $\beta_1^i < 0$ and $\beta_2^i > 0$. This proves that for $\beta \in \big(0, \beta_2^i\big)$ we have (126).

In order to simplify the expressions that follow in this proof, let us introduce the notation

$$\overline{x}_i = x - x_i \qquad \text{and} \qquad \overline{x}_i = (\overline{x}_{i1}, \overline{x}_{i2}). \tag{130}$$

Note that using the hypothesis that $x_i \notin \overline{\Omega}$ we have that

$$\Phi^i(x) \geqslant \inf_{x\in\Omega} |\overline{x}_i|^2 > 0, \qquad \forall x \in \overline{\Omega}. \tag{131}$$

Let us now verify (123). Using (32) we can compute for $x \in \overline{\Omega}$ the first derivatives

$$\frac{\partial \Phi^i}{\partial x_1} = 2\,\mathrm{e}^{2\rho\phi_i^\perp}(a\overline{x}_{i1} - b\overline{x}_{i1}), \tag{132}$$

$$\frac{\partial \Phi^i}{\partial x_2} = 2\,\mathrm{e}^{2\rho\phi_i^\perp}(a\overline{x}_{i2} + b\overline{x}_{i1}), \tag{133}$$

and the second derivatives which are well defined thanks to (131),

$$\frac{\partial^2 \Phi^i}{\partial x_1^2} = 2\,\mathrm{e}^{2\rho\phi_i^\perp}\left(a - 2\rho\frac{\big(a\overline{x}_{i1}\overline{x}_2 - b\overline{x}_{i2}^2\big)}{\phi^i}\right), \tag{134}$$

$$\frac{\partial^2 \Phi^i}{\partial x_2^2} = 2\,\mathrm{e}^{2\rho\phi_i^\perp}\left(a + 2\rho\frac{\big(a\overline{x}_{i1}\overline{x}_{i2} + b\overline{x}_{i1}^2\big)}{\phi^i}\right), \tag{135}$$

$$\frac{\partial^2 \Phi^i}{\partial x_2 \partial x_1} = 2\,\mathrm{e}^{2\rho\phi_i^\perp}\left(-b + 2\rho\frac{\big(a\overline{x}_{i1}^2 - b\overline{x}_{i2}\overline{x}_{i1}\big)}{\phi^i}\right), \tag{136}$$

$$\frac{\partial^2 \Phi^i}{\partial x_1 \partial x_2} = 2\,\mathrm{e}^{2\rho\phi_i^\perp}\left(b - 2\rho\frac{\big(a\overline{x}_{i2}^2 + b\overline{x}_{i1}\overline{x}_{i2}\big)}{\phi^i}\right). \tag{137}$$

The reader can easily verify that the last two derivatives (136) and (137) are of course equal, but it is convenient to write them in this form in order to have the following expression for the second derivative matrix or Hessian matrix:

$$H(\Phi^i) = 2\,\mathrm{e}^{2\rho\phi_i^\perp}\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \frac{2b}{a\phi_i} D(\Phi^i)\right), \tag{138}$$

with

$$D(\Phi^i) = \begin{bmatrix} -a\overline{x}_{i1}\overline{x}_{i2} + b\overline{x}_{i2}^2 & a\overline{x}_{i1}^2 - b\overline{x}_{i1}\overline{x}_{i2} \\ -a\overline{x}_{i2}^2 - b\overline{x}_{i1}\overline{x}_{i2} & a\overline{x}_{i1}\overline{x}_{i2} + b\overline{x}_{i1}^2 \end{bmatrix}. \tag{139}$$

Let us first see that

$$H(\Phi^i)\xi \cdot \xi \geqslant am\, \mathrm{e}^{2\rho\phi_i^\perp}|\xi|^2 \qquad \text{for all} \quad \xi \in \mathbb{R}^2, \quad \xi = (\xi_1, \xi_2), \quad \xi \neq 0, \tag{140}$$

where m is given in (22).

First it is clear that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = a|\xi|^2. \tag{141}$$

Using now (139) and making the computations we get

$$D(\Phi^i) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = a(\xi_1\overline{x}_{i1} + \xi_2\overline{x}_{i2})(\xi_2\overline{x}_{i1} - \xi_1\overline{x}_{i2}) + b(\xi_2\overline{x}_{i1} - \xi_1\overline{x}_{i2})^2.$$

Using this equality and (138) in (141) we deduce that

$$H(\Phi^i)\xi \cdot \xi = 2\,\mathrm{e}^{2\rho\phi_i^\perp}\left(a|\xi|^2 + \frac{2b^2(\xi_2\overline{x}_{i1} - \xi_1\overline{x}_{i2})^2}{a\phi_i} + \frac{2b(\xi_2\overline{x}_{i1} - \xi_1\overline{x}_{i2})(\xi_1\overline{x}_{i1} + \xi_2\overline{x}_{i2})}{\phi}\right). \tag{142}$$

There are at least two ways for finding a lower bound of expression (142). Indeed, taking into account that

$$\left|\frac{2b\sqrt{a}(\xi_2\overline{x}_{i1} - \xi_1\overline{x}_{i2})(\xi_1\overline{x}_{i1} + \xi_2\overline{x}_{i2})}{\sqrt{a}\phi_i}\right| \leqslant \frac{2b^2(\xi_2\overline{x}_{i1} - \xi_1\overline{x}_{i2})^2}{a\phi_i} + \frac{a}{2}|\xi|^2$$

and

$$\left|\frac{2b(\xi_2\overline{x}_{i1} - \xi_1\overline{x}_{i2})(\xi_1\overline{x}_{i1} + \xi_2\overline{x}_{i2})}{\phi_i}\right| \leqslant 2|b||\xi|^2,$$

these bounds are respectively

$$H(\Phi^i)\xi \cdot \xi \geqslant a\,\mathrm{e}^{2\rho\phi_i^\perp}|\xi|^2$$

and

$$H(\Phi^i)\xi \cdot \xi \geqslant (2a - 4|b|)\,\mathrm{e}^{2\rho\phi_i^\perp}|\xi|^2.$$

Taking the best of the two lower bounds and using the definition of the positive constant m given in (22) we obtain (140).

On the other hand, we have

$$\partial_t\Phi^i = -2\beta t, \tag{143}$$

$$\partial_{tt}\Phi^i = 2\beta, \tag{144}$$

$$\Delta\Phi^i = \frac{4}{a}\,\mathrm{e}^{2\rho\phi_i^\perp}, \tag{145}$$

$$\partial_{tt}\Phi^i - \Delta\Phi^i = -2\beta - \frac{4}{a}\,\mathrm{e}^{2\rho\phi_i^\perp}, \tag{146}$$

$$|\partial_t\Phi^i|^2 - |\nabla\Phi^i|^2 = 4\beta^2t^2 - 4\phi_i\,\mathrm{e}^{4\rho\phi_i^\perp}. \tag{147}$$

In order to verify (123), taking into account (140) and (146), it is sufficient to have the following:

$$4am\,\mathrm{e}^{2\rho\phi_i^\perp} - 2K^i\left(2\beta + \frac{4}{a}\,\mathrm{e}^{2\rho\phi_i^\perp}\right) > 0. \tag{148}$$

Since

$$K^i < \frac{a^2 m\, e^{2\rho\theta^i_-}}{\beta a + 2\, e^{2\rho\theta^i_+}} = \inf_{x\in\Omega} \frac{a^2 m\, e^{2\rho\phi_i^\perp}}{\beta a + 2\, e^{2\rho\phi_i^\perp}},$$

inequality (148) holds and then (123) is also true.

Now, we verify condition (122). Using (144) and (146), (122) holds if

$$-4\beta + 2K^i\left(\beta + \frac{2}{a}\, e^{2\rho\phi_i^\perp}\right) > 0. \tag{149}$$

By (121), K^i is such that

$$K^i > \frac{2\beta a}{\beta a + 2\, e^{2\rho\theta^i_-}} = \sup_{x\in\Omega} \frac{2\beta a}{\beta a + 2\, e^{2\rho\phi_i^\perp}}.$$

So, we have (149). This proves that (122) holds.

Let us now check condition (124). Taking $\xi \equiv \nabla\Phi^i$ with $\nabla\Phi^i$ given by (132) and (133), we obtain

$$|\xi| = 2\phi_i\, e^{2\rho\phi_i^\perp},$$
$$\xi_1\overline{x}_{i1} + \xi_2\overline{x}_{i2} = 2a\phi_i\, e^{2\rho\phi_i^\perp},$$
$$\xi_2\overline{x}_{i1} - \xi_1\overline{x}_{i2} = 2b\phi_i\, e^{2\rho\phi_i^\perp},$$

thus, from (142) we easily obtain

$$H(\Phi^i)\nabla\Phi^i\cdot\nabla\Phi^i = 8\phi_i\, e^{6\rho\phi_i^\perp}\left(a + \frac{2b^4}{a} + 2ab^2\right) = 8\frac{1+b^2}{a}\phi_i\, e^{6\rho\phi_i^\perp}, \tag{150}$$

where we have used that $a^2 + b^2 = 1$.

Taking into account (143), (144), (146), (147) and (150), in order to prove inequality (124) it suffices to show that for $x \in \overline{\Omega}$ and $t \in \mathbb{R}$ we have

$$3\lambda\left(\beta^2t^2 - \phi_i\, e^{4\rho\phi_i^\perp}\right)^2 - 2\beta^3t^2 + 2\frac{1+b^2}{a}\phi_i\, e^{6\rho\phi_i^\perp} - K^i\left(\beta^2t^2 - \phi_i\, e^{4\rho\phi_i^\perp}\right)\left(\beta + \frac{2}{a}\, e^{2\rho\phi_i^\perp}\right) > 0.$$

By adding and subtracting $2\beta\phi_i\, e^{4\rho\phi_i^\perp}$ and after introducing the auxiliary function

$$Y(x,t) = \beta^2t^2 - \phi_i\, e^{4\rho\phi_i^\perp},$$

we can rewrite the previous condition as

$$3\lambda Y^2 - c_1(x)Y + c_2(x)\phi_i(x) > 0, \tag{151}$$

where

$$c_1(x) = (2+K^i)\beta + \frac{2K^i}{a}\, e^{2\rho\phi_i^\perp}, \qquad c_2(x) = 2\left(\frac{1+b^2}{a}\, e^{2\rho\phi_i^\perp} - \beta\right) e^{4\rho\phi_i^\perp}. \tag{152}$$

Note that

$$(2+K^i)\beta + \frac{2K^i}{a}\, e^{2\rho\theta^i_-} \leqslant c_1(x) \leqslant (2+K^i)\beta + \frac{2K^i}{a}\, e^{2\rho\theta^i_+}, \qquad \forall x \in \overline{\Omega} \tag{153}$$

and if

$$0 < \beta < \beta^i_3 \qquad \text{with} \quad \beta^i_3 = \frac{1+b^2}{a}\, e^{2\rho\theta^i_-}$$

and using that $x_i \in \mathbb{R}^2\backslash\overline{\Omega}$, we would have that there exists a constant C such that

$$c_2(x)\phi_i(x) \geqslant C > 0, \qquad \forall x \in \overline{\Omega}. \tag{154}$$

This is true for $\beta \in \left(0, \beta_2^i\right)$ since in fact $\beta_2^i \leqslant \beta_3^i$. To see this, note that $a \leqslant 1$ and from the definitions of m and F^i in (22) and (23) we have $m \leqslant 2$ and $F^i \leqslant \frac{1}{2}$; thus

$$a^2 m \leqslant 2 \leqslant 2(1+b^2)F^i + (1+b^2)^2, \tag{155}$$

that is

$$(F^i)^2 + a^2 m \leqslant (F^i)^2 + 2(1+b^2)F^i + (1+b^2)^2 = (F^i + (1+b^2))^2 \tag{156}$$

so we obtain

$$\beta_2^i = \frac{-F^i + \sqrt{(F^i)^2 + ma^2}}{a}\, \mathrm{e}^{2\rho\theta_-^i} \leqslant \frac{1+b^2}{a}\, \mathrm{e}^{2\rho\theta_-^i} \equiv \beta_3^i. \tag{157}$$

In conclusion, if $\beta \in \left(0, \beta_2^i\right)$ and $x_i \notin \overline{\Omega}$, the coefficients $c_1(x)$ and $c_2(x)\phi_i(x)$ in (151) are respectively uniformly bounded and uniformly positive for $x \in \Omega$; therefore, it is sufficient to take $\lambda \geqslant \overline{\lambda}_i$ with $\overline{\lambda}_i = \overline{\lambda}_i(a, b, \Omega, x_i)$ large enough in order to obtain (151) and consequently the desired condition (124). This ends the proof of this lemma. □

5. Inverse problem for linearized systems

In this section, we will prove theorem 2.3. We will follow the ideas already used in [12, 13, 22]. We recall that the global Carleman estimate (38) will be crucial in our proof. We will also need a regularity result for the solution to the wave equation with homogeneous Dirichlet boundary conditions. More precisely, let us consider the system

$$\begin{cases} \partial_{tt}k - \Delta k + q(x,t)k = h(x,t) & \text{in} \quad \Omega \times (0,T), \\ k = 0 & \text{in} \quad \partial\Omega \times (0,T), \\ k(x,0) = k_0, \quad \partial_t k(x,0) = k_1 & \text{in} \quad \Omega. \end{cases} \tag{158}$$

We have the following result.

Lemma 5.1. *Assume that* $q \in L^\infty(\Omega \times (0,T)), h \in L^1(0,T; L^2(\Omega)), k_0 \in H_0^1(\Omega)$ *and* $k_1 \in L^2(\Omega)$. *There exists only one solution* k *of (158) such that* $k \in C([0,T]; H_0^1(\Omega))$ *with* $k_t \in C^0([0,T]; L^2(\Omega))$ *and* $\frac{\partial k}{\partial \nu} \in L^2(\partial\Omega \times (0,T))$. *Furthermore, there exists a positive constant* $C = C(\Omega, T, \|q\|_{L^\infty})$ *such that*

$$\|k\|_{L^\infty(H_0^1)} + \|\partial_t k\|_{L^\infty(L^2)} + \left\|\frac{\partial k}{\partial \nu}\right\|_{L^2(L^2)} \leqslant C\left(\|k_0\|_{H_0^1} + \|k_1\|_{L^2} + \|h\|_{L^1(L^2)}\right). \tag{159}$$

The existence and uniqueness of the finite energy solution for (158) is a classical well-known result, whose proof can be found for instance in [5]. The regularity result for the normal derivative of this solution is known as a result of 'hidden regularity' (see, for example, [18]). It is clear that the result of this lemma continues being true if $q = q(x)$ with $q \in L^\infty(\Omega)$.

Proof of theorem 2.3. In order to obtain the estimate (27), we will proceed in four steps. In the first step, we write a system similar to (4) for $\psi = \partial_t y$, where f is as an initial condition. In order to apply the global Carleman estimate (38), we extend the solution $\partial_t y$ of the obtained system for $(x,t) \in \Omega \times (-T,T)$. In the second step, we introduce a cut-off function which vanishes in $t = -T$ and $t = T$ and we perform a suitable change of variables and we will see that equality (180) holds. In the third step, we will estimate the right-hand side of (180) getting (181). Finally, in the last step, we conclude the proof of this theorem. In all the proof, we will fix $\lambda = \overline{\lambda}$, where $\overline{\lambda} = \overline{\lambda}(a, b, \Omega, x_0, x_1)$ is the constant of theorem 3.1.

Step 1. Let $q \in \mathcal{U}_M$, where $\mathcal{U}_M$ is given in (1). Let us set $\psi = \partial_t y$. Then, we have

$$\begin{cases} \partial_{tt}\psi - \Delta\psi + q(x)\psi = f(x)\partial_t R(x,t) & \text{in} \quad \Omega \times (0,T), \\ \psi = 0 & \text{in} \quad \partial\Omega \times (0,T), \\ \psi(x,0) = 0, \quad \partial_t \psi(x,0) = f(x)R(x,0) & \text{in} \quad \Omega. \end{cases} \tag{160}$$

Note that from the hypothesis of theorem 2.3 we have that $q \in L^\infty(\Omega)$, $f\partial_t R \in L^2(0,T; L^2(\Omega))$ and $f(\cdot)R(\cdot,0) \in L^2(\Omega)$ since $R(\cdot,0) \in L^\infty(\Omega)$ as $H^1(0,T;L^\infty(\Omega)) \subset C([0,T];L^\infty(\Omega))$. Then, using lemma 5.1 we deduce that there exists only one solution ψ of (160) such that

$$\psi \in C\big([0,T]; H_0^1(\Omega)\big) \cap C^1([0,T];L^2(\Omega)), \qquad \frac{\partial\psi}{\partial\nu} \in L^2(0,T;L^2(\partial\Omega)).$$

So, we have $\partial_{tt} y \in C^0([0,T];L^2(\Omega))$ (remember that $\psi = \partial_t y$). Therefore, from the first equation of (4) we can write that $\Delta y = \partial_{tt} y - qy - f(x)R(x,t) \in C([0,T];L^2(\Omega))$. Thus using again the result of lemma 5.1 we obtain that the solution y of (4) is such that

$$y \in C\big([0,T]; H_0^1(\Omega) \cap H^2(\Omega)\big) \cap C^1\big([0,T]; H_0^1(\Omega)\big) \cap C^2([0,T];L^2(\Omega)).$$

Let us extend the functions $\partial_t R$ and ψ on $\Omega \times (-T,0)$ as even functions by setting $\partial_t R(x,t) = -\partial_t R(x,-t)$ and $\psi(x,t) = -\psi(x,-t)$ for all $(x,t) \in \Omega \times (-T,0)$. We denote the extensions by the same symbols. It is clear that

$$\partial_t R \in L^2(-T,T;L^\infty(\Omega)), \tag{161}$$

$$\psi \in C\big([-T,T]; H_0^1(\Omega)\big) \cap C^1([-T,T];L^2(\Omega)), \tag{162}$$

$$\frac{\partial\psi}{\partial\nu} \in L^2(-T,T;L^2(\partial\Omega)). \tag{163}$$

Moreover, using lemma 5.1, we deduce that there exists a positive constant $C = C(\Omega, T, M)$ such that

$$\|\psi\|_{L^\infty(H_0^1)} + \|\partial_t\psi\|_{L^\infty(L^2)} + \left\|\frac{\partial\psi}{\partial\nu}\right\|_{L^2(L^2)} \leqslant C(\|f(\cdot)R(\cdot,0)\|_{L^2} + \|f\partial_t R\|_{L^2(L^2)}). \tag{164}$$

Consequently, we have

$$y \in C\big([-T,T]; H^2(\Omega) \cap H_0^1(\Omega)\big) \cap C^1\big([-T,T]; H_0^1(\Omega)\big) \cap C^2([-T,T];L^2(\Omega)), \tag{165}$$

$$\frac{\partial y}{\partial\nu} \in H^1(-T,T;L^2(\partial\Omega)) \tag{166}$$

and

$$\|\partial_t y\|_{L^\infty(H_0^1)} + \|\partial_{tt} y\|_{L^\infty(L^2)} + \left\|\frac{\partial y}{\partial\nu}\right\|_{H^1(L^2)} \leqslant C(\|f(\cdot)R(\cdot,0)\|_{L^2} + \|f\partial_t R\|_{L^2(L^2)}) \tag{167}$$

with $C = C(\Omega, T, M)$ a positive constant.

Step 2. In this step, we will introduce a suitable cut-off function in order to apply the Carleman estimate (38). We will also choose a sufficiently large time $\overline{T}$ which appears in theorem 2.1 in order to have $\varphi_i(x, \pm T) < \varphi_i(x,0)$, $i = 0, 1$, uniformly for all $x \in \overline{\Omega}$ whenever $T > \overline{T}$. To this end, we will use the argument from [14].

Let us recall that the weight functions which appear in (38) are of the form (33), that is to say

$$\varphi_i(x,t) = \exp\big(\overline{\lambda}\big(a\phi_i \exp\big(2\rho\phi_i^\perp\big) - \beta t^2\big)\big), \qquad i = 0, 1,$$

where $\beta \in (0, \beta_2)$ with β_2 given by

$$\beta_2 = \min\left\{\beta_2^0, \beta_2^1\right\} = \frac{-F_{\min} + \sqrt{F_{\min}^2 + ma^2}}{a}\, \mathrm{e}^{2\rho \min\{\theta_-^0, \theta_-^1\}}, \tag{168}$$

where $F_{\min}$ was defined in (24). Here we have used (129) for the expression of β_2^i, $i = 0, 1$.

Let us set

$$\overline{c} = \min\{c_0, c_1\}, \qquad \text{where} \quad c_i = \inf_{x\in\Omega} \Phi^i(x, 0), \qquad i = 0, 1. \tag{169}$$

First note that for all $x \in \Omega$ and $i = 0, 1$ we have

$$\begin{aligned}\varphi_i(x, 0) &= \exp(\overline{\lambda}\Phi^i(x, 0)) \geqslant \exp(\overline{\lambda} \inf_{x\in\Omega} \Phi^i(x, 0)) = \exp(\overline{\lambda} c_i)\\ &\geqslant \exp(\overline{\lambda}\overline{c}).\end{aligned} \tag{170}$$

Let us now take $T > \overline{T}$, where $\overline{T}$ is given by (26). Then we can choose a sufficiently small $\varepsilon > 0$ such that

$$T^2 = \overline{T}^2 + \frac{2\varepsilon}{\beta}, \tag{171}$$

where $\beta = \alpha\beta_2$ with β_2 given by (168). Therefore, we can write

$$\begin{aligned}\beta T^2 &= \alpha\beta_2\overline{T}^2 + 2\varepsilon\\ &= \frac{\left(-F_{\min} + \sqrt{F_{\min}^2 + ma^2}\right)\left(F_{\min} + \sqrt{F_{\min}^2 + ma^2}\right)}{ma}\, \mathrm{e}^{2\rho\min\{\theta_-^0,\theta_-^1\}}\left(R_{\max}^2\, \mathrm{e}^{2|\rho|\delta\theta} - R_{\min}^2\right) + 2\varepsilon\\ &= a\left(R_{\max}^2\, \mathrm{e}^{2\rho\max\{\theta_+^0,\theta_+^1\}} - R_{\min}^2\, \mathrm{e}^{2\rho\min\{\theta_-^0,\theta_-^1\}}\right) + 2\varepsilon\\ &= a\max\left\{\sup_{x\in\Omega}\left(\phi_0 \exp\left(2\rho\phi_0^\perp\right)\right), \sup_{x\in\Omega}\left(\phi_1\exp\left(2\rho\phi_1^\perp\right)\right)\right\}\\ &\quad - a\min\left\{\inf_{x\in\Omega}\left(\phi_0\exp\left(2\rho\phi_0^\perp\right)\right), \inf_{x\in\Omega}\left(\phi_1\exp\left(2\rho\phi_1^\perp\right)\right)\right\} + 2\varepsilon\\ &= \max\left\{\sup_{x\in\Omega}\Phi^0(x, 0), \sup_{x\in\Omega}\Phi^1(x, 0)\right\} - \overline{c} + 2\varepsilon,\end{aligned} \tag{172}$$

where we have used (169) and that $|\rho|\delta\theta = \rho\left(\max\left\{\theta_+^0, \theta_+^1\right\} - \min\left\{\theta_-^0, \theta_-^1\right\}\right)$.

Thus, according to (172) we have

$$\begin{aligned}\varphi_i(x, \pm T) &= \mathrm{e}^{\overline{\lambda}(\Phi^i(x,0) - \beta T^2)}\\ &\leqslant \mathrm{e}^{\overline{\lambda}(\sup_{x\in\Omega}\Phi^i(x,0) - \beta T^2)}\\ &\leqslant \mathrm{e}^{\overline{\lambda}(\overline{c} - 2\varepsilon)} \qquad \forall x \in \Omega, \qquad i = 0, 1.\end{aligned} \tag{173}$$

Therefore, we can choose $\delta(\varepsilon) > 0$ such that

$$\begin{cases}\varphi_i(x, t) = \mathrm{e}^{\overline{\lambda}\Phi^i(x,t)} \leqslant \mathrm{e}^{\overline{\lambda}(\overline{c} - \varepsilon)}, \qquad i = 0, 1\\ \text{for } (x, t) \in \overline{\Omega} \times ([-T, -T + \delta] \cup [T - \delta, T]).\end{cases} \tag{174}$$

Let us introduce a cut-off function $\chi \in C^\infty([-T, T])$, $0 \leqslant \chi \leqslant 1$, satisfying

$$\chi(t) = \begin{cases}1 & \text{if } t \in [-T + \delta, T - \delta],\\ 0 & \text{if } t \in [-T, -T + \delta/2] \cup [T - \delta/2, T].\end{cases} \tag{175}$$

Now, we set

$$z = (\mathrm{e}^{s\varphi_0} + \mathrm{e}^{s\varphi_1})\chi\psi \equiv (\mathrm{e}^{s\varphi_0} + \mathrm{e}^{s\varphi_1})\chi\partial_t y. \tag{176}$$

From (162) and (176), we deduce that $z \in C\big([-T,T]; H_0^1(\Omega)\big) \cap C^1([-T,T]; L^2(\Omega))$ and verifies

$$z(x,-T) = z(x,T) = \partial_t z(x,-T) = \partial_t z(x,T) = 0, \tag{177}$$

$$z = 0 \qquad \text{on} \quad \Sigma, \tag{178}$$

$$\partial_t z(x,0) = f(x)R(x,0)(\mathrm{e}^{s\varphi_0(x,0)} + \mathrm{e}^{s\varphi_1(x,0)}). \tag{179}$$

Let us set

$$Lz = \partial_{tt} z - \Delta z + q(x)z.$$

Making the integration by parts and using (177) and (178), we can write that

$$\int_{-T}^{0}\int_\Omega (Lz)\partial_t z \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_{-T}^{0}\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}(|\partial_t z|^2 + |\nabla z|^2 + q(x)|z|^2)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \frac{1}{2}\int_\Omega |\partial_t z(x,0)|^2\,\mathrm{d}x.$$

Thus, from (179) we obtain

$$\frac{1}{2}\int_\Omega (\mathrm{e}^{s\varphi_0(x,0)} + \mathrm{e}^{s\varphi_1(x,0)})^2 |f(x)|^2 |R(x,0)|^2\,\mathrm{d}x = \int_{-T}^{0}\int_\Omega (Lz)\partial_t z\,\mathrm{d}x\,\mathrm{d}t. \tag{180}$$

Recall that our goal is to estimate $\|f\|_{L^2(\Omega)}$ as appears in (27). Thanks to the last identity, we will be able to do this by getting a suitable estimate on the right-hand side of (180).

Step 3. Recall that we take $\lambda = \overline{\lambda}$ fixed, where $\overline{\lambda} = \overline{\lambda}(a, b, \Omega, x_0, x_1)$ is the constant of theorem 3.1. This being the case, the functions $\varphi_i, i = 0, 1$, are uniformly bounded with positive constants that depend only on a, b, Ω, x_0 and x_1 from above and below. We will see in this step that there exists a positive constant $C = C(a, b, \Omega, T, x_0, x_1)$ such that

$$\int_{-T}^{0}\int_\Omega (Lz)\partial_t z\,\mathrm{d}x\,\mathrm{d}t \leqslant C\iint_Q (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1})\chi^2 |f(x)|^2 |\partial_t R(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ C\iint_Q (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1})|2\partial_t\chi\,\partial_t\psi + \partial_{tt}\chi\,\psi|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ Cs\iint_{\gamma\times(0,T)} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1})\left|\frac{\partial\psi}{\partial\nu}\right|^2\,\mathrm{d}\sigma\,\mathrm{d}t \tag{181}$$

for all $s \geqslant \overline{s}$ with $\overline{s}(\overline{\lambda})$ given as in theorem 3.1.

Let us first compute Lz, where z is given by (176). For simplicity of the notation let us put

$$w = \chi\psi. \tag{182}$$

It is not difficult to see that $w \in C\big([-T,T]; H_0^1(\Omega)\big) \cap C^1([-T,T]; L^2(\Omega))$, $w = 0$ on Σ and satisfies

$$Lw \equiv \partial_{tt} w - \Delta w + q(x)w = \chi f(x)\partial_t R(x,t) + 2\partial_t\chi\,\partial_t\psi + \partial_{tt}\chi\,\psi \quad \text{in } Q. \tag{183}$$

Thanks to (175), we have

$$w(x,-T) = w(x,T) = \partial_t w(x,-T) = \partial_t w(x,T) = 0.$$

Consequently, by density we can claim that the Carleman inequality (38) is true for w. More precisely, taking into account (183) we have that there exists a positive constant $C = C(a, b, \Omega, x_0, x_1)$ such that

$$\int\int_Q (e^{2s\varphi_0} + e^{2s\varphi_1})(s|\partial_t w|^2 + s|\nabla w|^2 + s^3|w|^2)\,dx\,dt$$
$$\leqslant C\left(\int\int_Q (e^{2s\varphi_0} + e^{2s\varphi_1})|\chi f(x)\partial_t R(x,t) + 2\partial_t\chi\partial_t\psi + \partial_{tt}\chi\psi|^2\,dx\,dt\right.$$
$$\left. + s\int\int_{\gamma\times(0,T)} (e^{2s\varphi_0} + e^{2s\varphi_1})\left|\frac{\partial w}{\partial\nu}\right|^2 d\sigma\,dt\right) \tag{184}$$

for all $s \geqslant \bar{s}$ with $\bar{s}(\bar{\lambda})$ as in theorem 3.1. We have used here that the functions $\varphi_i, i = 0, 1$, are uniformly bounded with positive constants from above and below, since we have supposed that $\lambda = \bar{\lambda}$ is fixed.

We will use the estimate (184) to deduce (181). From (176) and (182) we have

$$z = (e^{s\varphi_0} + e^{s\varphi_1})w. \tag{185}$$

Then, we can write that

$$\partial_t z = (e^{s\varphi_0} + e^{s\varphi_1})\partial_t w + s(\partial_t\varphi_0\, e^{s\varphi_0} + \partial_t\varphi_1\, e^{s\varphi_1})w, \tag{186}$$

$$\partial_{tt} z = (e^{s\varphi_0} + e^{s\varphi_1})\partial_{tt} w + 2s(\partial_t\varphi_0\, e^{s\varphi_0} + \partial_t\varphi_1\, e^{s\varphi_1})\partial_t w$$
$$+ s(\partial_{tt}\varphi_0\, e^{s\varphi_0} + \partial_{tt}\varphi_1\, e^{s\varphi_1})w + s^2(|\partial_t\varphi_0|^2 e^{s\varphi_0} + |\partial_t\varphi_1|^2 e^{s\varphi_1})w, \tag{187}$$

$$\Delta z = (e^{s\varphi_0} + e^{s\varphi_1})\Delta w + 2s(\nabla\varphi_0\, e^{s\varphi_0} + \nabla\varphi_1\, e^{s\varphi_1})\cdot\nabla w$$
$$+ s(\Delta\varphi_0\, e^{s\varphi_0} + \Delta\varphi_1\, e^{s\varphi_1})w + s^2(|\nabla\varphi_0|^2 e^{s\varphi_0} + |\nabla\varphi_1|^2 e^{s\varphi_1})w. \tag{188}$$

From (187), (188) and using (183) we obtain

$$Lz = (e^{s\varphi_0} + e^{s\varphi_1})(\chi f(x)\partial_t R(x,t) + 2\partial_t\chi\partial_t\psi + \partial_{tt}\chi\psi)$$
$$+ 2s(\partial_t\varphi_0\, e^{s\varphi_0} + \partial_t\varphi_1\, e^{s\varphi_1})\partial_t w - 2s(\nabla\varphi_0\, e^{s\varphi_0} + \nabla\varphi_1\, e^{s\varphi_1})\cdot\nabla w$$
$$+ s((\partial_{tt}\varphi_0 - \Delta\varphi_0)\, e^{s\varphi_0} + (\partial_{tt}\varphi_1 - \Delta\varphi_1)\, e^{s\varphi_1})w$$
$$+ s^2((|\partial_t\varphi_0|^2 - |\nabla\varphi_0|^2)\, e^{s\varphi_0} + (|\partial_t\varphi_1|^2 - |\nabla\varphi_1|^2)\, e^{s\varphi_1})w. \tag{189}$$

Taking into account (186) and (189) we deduce that

$$\int_{-T}^{0}\int_\Omega (Lz)\partial_t z\,dx\,dt = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{190}$$

where

$$I_1 = \int_{-T}^{0}\int_\Omega (e^{s\varphi_0} + e^{s\varphi_1})\chi f(x)\partial_t R(x,t)((e^{s\varphi_0} + e^{s\varphi_1})\partial_t w + s(\partial_t\varphi_0\, e^{s\varphi_0} + \partial_t\varphi_1\, e^{s\varphi_1})w)\,dx\,dt, \tag{191}$$

$$I_2 = \int_{-T}^{0}\int_\Omega (e^{s\varphi_0} + e^{s\varphi_1})(2\partial_t\chi\partial_t\psi + \partial_{tt}\chi\psi)$$
$$\times((e^{s\varphi_0} + e^{s\varphi_1})\partial_t w + s(\partial_t\varphi_0\, e^{s\varphi_0} + \partial_t\varphi_1\, e^{s\varphi_1})w)\,dx\,dt, \tag{192}$$

$$I_3 = 2s\int_{-T}^{0}\int_\Omega (\partial_t\varphi_0\, e^{s\varphi_0} + \partial_t\varphi_1\, e^{s\varphi_1})\partial_t w((e^{s\varphi_0} + e^{s\varphi_1})\partial_t w + s(\partial_t\varphi_0\, e^{s\varphi_0} + \partial_t\varphi_1\, e^{s\varphi_1})w)\,dx\,dt, \tag{193}$$

$$I_4 = -2s \int_{-T}^{0} \int_{\Omega} (\nabla\varphi_0\, e^{s\varphi_0} + \nabla\varphi_1\, e^{s\varphi_1}) \cdot \nabla w \times ((e^{s\varphi_0} + e^{s\varphi_1})\partial_t w + s(\nabla\varphi_0\, e^{s\varphi_0} + \nabla\varphi_1\, e^{s\varphi_1})w)\, dx\, dt, \tag{194}$$

$$I_5 = s \int_{-T}^{0} \int_{\Omega} ((\partial_{tt}\varphi_0 - \Delta\varphi_0)\, e^{s\varphi_0} + (\partial_{tt}\varphi_1 - \Delta\varphi_1)\, e^{s\varphi_1})w \times ((e^{s\varphi_0} + e^{s\varphi_1})\partial_t w + s(\partial_t\varphi_0\, e^{s\varphi_0} + \partial_t\varphi_1\, e^{s\varphi_1})w)\, dx\, dt, \tag{195}$$

$$I_6 = s^2 \int_{-T}^{0} \int_{\Omega} ((|\partial_t\varphi_0|^2 - |\nabla\varphi_0|^2)\, e^{s\varphi_0} + (|\partial_t\varphi_1|^2 - |\nabla\varphi_1|^2)\, e^{s\varphi_1})w \times ((e^{s\varphi_0} + e^{s\varphi_1})\partial_t w + s(\partial_t\varphi_0\, e^{s\varphi_0} + \partial_t\varphi_1\, e^{s\varphi_1})w)\, dx\, dt. \tag{196}$$

Let us estimate all the terms $I_k, k = 1, \dots, 6$. Applying Cauchy–Schwartz and Young's inequalities and using (34) and that the functions φ_i are bounded since $\overline{\lambda}$ is fixed, we have

$$\begin{aligned} |I_1| &\leqslant C \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})\chi |f(x)||\partial_t R(x,t)||\partial_t w + s\overline{\lambda} w|\, dx\, dt \\ &\leqslant C \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})\chi^2 |f(x)|^2 |\partial_t R(x,t)|^2\, dx\, dt \\ &\quad + C \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})(|\partial_t w|^2 + s^2|w|^2)\, dx\, dt \end{aligned} \tag{197}$$

and

$$\begin{aligned} |I_2| &\leqslant C \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})|2\partial_t\chi\,\partial_t\psi + \partial_{tt}\chi\,\psi||\partial_t w + s\overline{\lambda} w|\, dx\, dt \\ &\leqslant C \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})|2\partial_t\chi\,\partial_t\psi + \partial_{tt}\chi\,\psi|^2\, dx\, dt \\ &\quad + C \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})(|\partial_t w|^2 + s^2|w|^2)\, dx\, dt \end{aligned} \tag{198}$$

with C depending on a, b, Ω, x_0, x_1 and T. From (193) and (34) we obtain

$$\begin{aligned} |I_3| &\leqslant Cs \int_{-T}^{0} \int_{\Omega} (e^{s\varphi_0} + e^{s\varphi_1})^2 |\partial_t w|^2\, dx\, dt + Cs^2 \int_{-T}^{0} \int_{\Omega} (e^{s\varphi_0} + e^{s\varphi_1})^2 |\partial_t w w|\, dx\, dt \\ &\leqslant C \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})(s|\partial_t w|^2 + s^3|w|^2)\, dx\, dt. \end{aligned} \tag{199}$$

Using (194) we deduce

$$\begin{aligned} |I_4| &\leqslant Cs \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})|\nabla w|^2\, dx\, dt + Cs \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})|\partial_t w + s\overline{\lambda} w|^2\, dx\, dt \\ &\leqslant C \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})(s|\partial_t w|^2 + s|\nabla w|^2 + s^3|w|^2)\, dx\, dt. \end{aligned} \tag{200}$$

Since (30) and (34) hold. Now, taking into account (195), (35) and (37) we can write that

$$\begin{aligned} |I_5| &\leqslant Cs \int_{-T}^{0} \int_{\Omega} (e^{s\varphi_0} + e^{s\varphi_1})^2 |w(\partial_t w + s\overline{\lambda} w)|\, dx\, dt \\ &\leqslant Cs \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})|\partial_t w|^2\, dx\, dt + C(s^2 + s) \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})|w|^2\, dx\, dt \\ &\leqslant C \int_{-T}^{0} \int_{\Omega} (e^{2s\varphi_0} + e^{2s\varphi_1})(s|\partial_t w|^2 + s^2|w|^2)\, dx\, dt. \end{aligned} \tag{201}$$

By (196), (34) and (36) we deduce that

$$
\begin{aligned}
|I_6| &\leqslant C \int_{-T}^{0} \int_{\Omega} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1})(s^2 |w \partial_t w| + s^3 |w|^2) \, \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C \int_{-T}^{0} \int_{\Omega} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1})(s \partial_t w|^2 + s^3 |w|^2) \, \mathrm{d}x \, \mathrm{d}t. \qquad (202)
\end{aligned}
$$

Then, from (190) and (197)–(202) we get

$$
\begin{aligned}
\int_{-T}^{0} \int_{\Omega} (Lz) \partial_t z \, \mathrm{d}x \, \mathrm{d}t &\leqslant C \int_{-T}^{0} \int_{\Omega} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1})(s|\partial_t w|^2 + s|\nabla w|^2 + s^3 |w|^2) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + C \int_{-T}^{0} \int_{\Omega} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1}) \chi^2 |f(x)|^2 |\partial_t R(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + C \int_{-T}^{0} \int_{\Omega} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1}) |2 \partial_t \chi \partial_t \psi + \partial_{tt} \chi \psi|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C \iint_{Q} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1})(s|\partial_t w|^2 + s|\nabla w|^2 + s^3 |w|^2) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + C \int_{-T}^{0} \int_{\Omega} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1}) \chi^2 |f(x)|^2 |\partial_t R(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + C \int_{-T}^{0} \int_{\Omega} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1}) |2 \partial_t \chi \partial_t \psi + \partial_{tt} \chi \psi|^2 \, \mathrm{d}x \, \mathrm{d}t \qquad (203)
\end{aligned}
$$

with C a positive constant only depending on a, b, Ω, x_0, x_1 and T. Now, using the Carleman inequality (184) in (203), we deduce that the following holds:

$$
\begin{aligned}
\int_{-T}^{0} \int_{\Omega} (Lz) \partial_t z \, \mathrm{d}x \, \mathrm{d}t &\leqslant C \iint_{Q} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1}) \chi^2 |f(x)|^2 |\partial_t R(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + C \iint_{Q} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1}) |2 \partial_t \chi \partial_t \psi + \partial_{tt} \chi \psi|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + Cs \iint_{\gamma \times (0,T)} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1}) \left| \frac{\partial \psi}{\partial \nu} \right|^2 \, \mathrm{d}\sigma \, \mathrm{d}t
\end{aligned}
$$

for all $s \geqslant \overline{s}$ with a positive constant $C = C(a, b, \Omega, x_0, x_1, T)$. Here, we also have used that

$$
\left| \frac{\partial w}{\partial \nu} \right| = \left| \frac{\partial \psi}{\partial \nu} \chi \right|, \qquad (204)
$$

since $w = \chi \psi$ with χ given by (175) and that

$$
\iint_{\gamma \times (0,T)} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1}) \left| \frac{\partial \psi}{\partial \nu} \right|^2 \, \mathrm{d}\sigma \, \mathrm{d}t = \iint_{\gamma \times (-T,0)} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1}) \left| \frac{\partial \psi}{\partial \nu} \right|^2 \, \mathrm{d}\sigma \, \mathrm{d}t. \qquad (205)
$$

The last inequality is exactly the estimate (181).

Step 4. We will conclude the proof of theorem 2.3. Let us consider the second term on the right-hand side of (181). Using that $\partial_t \chi = \partial_{tt} \chi = 0$ in $[-T, -T + \delta/2] \cup [-T + \delta, T - \delta] \cup [T - \delta/2, T]$ and taking into account (174) and (164) we have

$$
\begin{aligned}
&\iint_{Q} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1}) |2 \partial_t \chi \partial_t \psi + \partial_{tt} \chi \psi|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad \leqslant C \int_{-T+\delta/2}^{-T+\delta} \int_{\Omega} (\mathrm{e}^{2s\varphi_0} + \mathrm{e}^{2s\varphi_1})(|\partial_t \psi|^2 + |\psi|^2) \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_{T-\delta}^{T-\delta/2}\int_\Omega (\mathrm{e}^{2s\varphi_0}+\mathrm{e}^{2s\varphi_1})(|\partial_t\psi|^2+|\psi|^2)\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C\,\mathrm{e}^{2s\exp(\bar\lambda(\bar c-\varepsilon))}\big(\|\partial_t\psi\|^2_{L^2(L^2)}+\|\psi\|^2_{L^2(L^2)}\big)\\
&\leqslant C\,\mathrm{e}^{2s\exp(\bar\lambda(\bar c-\varepsilon))}\big(\|f(\cdot)R(\cdot,0)\|^2_{L^2}+\|f\partial_t R\|^2_{L^2(L^2)}\big)\\
&\leqslant \mathrm{e}^{2s\exp(\bar\lambda(\bar c-\varepsilon))}\|f\|^2_{L^2}\big(\|R(\cdot,0)\|^2_{L^\infty}+\|\partial_t R\|^2_{L^2(L^\infty)}\big),
\end{aligned}
\tag{206}
$$

where C is a positive constant depending on Ω, T and M.

Let us now consider the first term on the right-hand side of (181). Since $R \in H^1(-T,T;L^\infty(\Omega))$ and $|R(x,0)| \geqslant \alpha_0 > 0$, then we define $g_0 \in L^2(-T,T)$ by

$$
g_0(t)=\sup_{x\in\Omega}\frac{|\partial_t R(x,t)|}{|R(x,0)|}.
$$

Then, taking into account (33) we have

$$
\begin{aligned}
&\iint_Q (\mathrm{e}^{2s\varphi_0}+\mathrm{e}^{2s\varphi_1})\chi^2|f(x)|^2|\partial_t R(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leqslant C\iint_Q (\mathrm{e}^{2s\varphi_0}+\mathrm{e}^{2s\varphi_1})|f(x)|^2|R(x,0)|^2|g_0(t)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leqslant C\int_\Omega \mathrm{e}^{2s\varphi_0(x,0)}|f(x)|^2|R(x,0)|^2G^0(x)\,\mathrm{d}x\\
&\qquad\quad+C\int_\Omega \mathrm{e}^{2s\varphi_1(x,0)}|f(x)|^2|R(x,0)|^2G^1(x)\,\mathrm{d}x,
\end{aligned}
\tag{207}
$$

where

$$
G^i(x)=\int_{-T}^{T}|g_0(t)|^2\,\mathrm{e}^{2s\varphi_i(x,0)(\mathrm{e}^{-\bar\lambda\beta t^2}-1)}\,\mathrm{d}t,\qquad i=0,1.
$$

Thanks to Lebesgue's theorem we deduce that for s sufficiently large

$$
G^i\to 0\quad\text{in }\Omega,\qquad i=0,1. \tag{208}
$$

On the other hand, since $|R(x,0)| \geqslant \alpha_0 > 0$ holds and using (170) we obtain

$$
\begin{aligned}
&\frac12\int_\Omega(\mathrm{e}^{s\varphi_0(x,0)}+\mathrm{e}^{s\varphi_1(x,0)})^2|f(x)|^2|R(x,0)|^2\,\mathrm{d}x\\
&\qquad\geqslant C\int_\Omega(\mathrm{e}^{2s\varphi_0(x,0)}+\mathrm{e}^{2s\varphi_1(x,0)})|f(x)|^2\,\mathrm{d}x\\
&\qquad\geqslant C\,\mathrm{e}^{2s\exp(\bar\lambda\bar c)}\|f\|_{L^2}.
\end{aligned}
\tag{209}
$$

From (180), (209) and (181), (206)–(208), we get

$$
\begin{aligned}
&\mathrm{e}^{2s\exp(\bar\lambda\bar c)}\|f\|^2_{L^2}\leqslant C(a,b,\Omega,T,x_0,x_1,M,M_1,\alpha_0)\,\mathrm{e}^{2s\exp(\bar\lambda(\bar c-\varepsilon))}\|f\|^2_{L^2}\\
&\qquad+C(a,b,\Omega,x_0,x_1,T)s\,\mathrm{e}^{2s\exp(\overline{\lambda C})}\iint_{\gamma\times(0,T)}\left|\frac{\partial}{\partial\nu}(\partial_t y)\right|^2\,\mathrm{d}\sigma\,\mathrm{d}t
\end{aligned}
\tag{210}
$$

for all $s \geqslant \bar s$, where $\overline{C}=\sup_{\gamma\times(0,T)}\{\Phi_0,\Phi_1\}$. Thus, for s large enough inequality (210) implies (25). □

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