

# Approximate Controllability and Homogenization of a Semilinear Elliptic Problem\*

CARLOS CONCA, AXEL OSSES<sup>†</sup> AND JEANNINE SAINT JEAN PAULIN<sup>‡</sup>

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## Abstract

The  $L^2$  and  $H^1$ -approximate controllability and homogenization of a semilinear elliptic boundary-value problem is studied in this paper. The principal term of the state equation has rapidly oscillating coefficients and the control region is locally distributed. The observation region is a subset of codimension 1 in the case of  $L^2$ -approximate controllability or is locally distributed in the case of  $H^1$ -approximate controllability. By using the classical Fenchel-Rockafellar's duality theory, the existence of an approximate control of minimal norm is established by means of a fixed point argument. We consider its asymptotic behavior as the rapidly oscillating coefficients  $H$ -converge. We prove its convergence to an approximate control of minimal norm for the homogenized problem.

**Key words:** approximate controllability, homogenization, semilinear elliptic equation.

## 1 Introduction

### 1.1 Setting of the problem

In this paper, we consider a non-periodic, nonlinear homogenization problem where the control is distributed in a relatively compact subdomain. Our goal is to study the approximate controllability of this problem when the operators in the state equation (given by a second order elliptic boundary-value problem) and in the cost functional (involving a Dirichlet type integral of the state function) both have rapidly oscillating coefficients.

Let  $\Omega$  be a connected bounded open set in  $\mathbb{R}^N$ ,  $N \geq 2$ , with a smooth boundary  $\partial\Omega$ . We consider two non-empty subdomains of  $\Omega$  which are the observable region  $\omega$  and the region where the error between the obtained and the desired state is minimized, that we denote by  $S$ .

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<sup>†</sup>Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 170/3 - Correo 3, Santiago, CHILE, and Centro de Modelamiento Matemático, UMR 2071 CNRS-Uchile (cconca@dim.uchile.cl and axosses@dim.uchile.cl).

<sup>‡</sup>UFR Mathématiques-Mécanique-Informatique, Département de Mathématiques, Université de Metz, Ile du Saulcy, 57045 Metz Cedex 01, FRANCE (sjpaulin@poncelet.univ-metz.fr).

For given constants  $0 < \alpha_m \leq \alpha_M$ , we denote by  $\mathcal{M}(\alpha_m, \alpha_M)$  the set of all  $N \times N$  matrices  $A = A(x)$  such that

$$(1.1) \quad A \in L^\infty(\Omega)^{N \times N}$$

$$(1.2) \quad \alpha_m I \leq A(x) \quad \text{and} \quad |A(x)\xi| \leq \alpha_M |\xi| \quad \forall \xi \in \mathbb{R}^N, \quad \text{and for a.e. } x \in \Omega,$$

where  $I$  is the  $N \times N$  identity matrix. (It is well-known that if  $A$  is symmetric, then the second condition in (1.2) is equivalent to  $A(x) \leq \alpha_M I$ ).

For each  $\varepsilon > 0$ , we consider a matrix  $A_\varepsilon \in \mathcal{M}(\alpha_m, \alpha_M)$  and the state equation

$$(1.3) \quad \begin{cases} -\operatorname{div}(A_\varepsilon \nabla y_\varepsilon(v)) + f(y_\varepsilon(v)) = \chi_\omega v & \text{in } \Omega \\ y_\varepsilon(v) = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\chi_\omega$  is the characteristic function of  $\omega$ ,  $v$  is the control, and  $y_\varepsilon(v)$  the associated state. Here  $f$  is a real-valued continuous function for which we assume that

$$(1.4) \quad f(0) = 0 \quad \text{and} \quad \exists \gamma > 0; \quad 0 \leq \frac{f(s)}{s} \leq \gamma \quad \forall s \in \mathbb{R} \setminus \{0\}.$$

There are two possible locations of the observation zone  $S$  that allow different kinds of approximate controllability. One is the case where  $S$  is an open subset of  $\Omega$  which is compactly contained in  $\omega$ . In this case the  $H^1$ -approximate controllability is studied. The other case occurs when the observation zone  $S$  is a smooth subset of  $\Omega$  of codimension 1 non intersecting the control zone  $\omega$ . In this case the  $L^2$ -approximate controllability can be considered.

The study of the  $H^1$ -approximate controllability involves a more general analysis and we will consider it in this paper. The analysis of the  $L^2$ -approximate controllability is simpler, and we have included a number of remarks at each step of the paper with the necessary changes to recover the  $L^2$ -case from the  $H^1$ -case.

Given  $y_1 \in H^1(S)$ , a constant  $\alpha \geq 0$ , and a *symmetric positive definite* matrix  $B$ , our aim is to find a control  $v_\varepsilon \in L^2(\omega)$  such that

$$(1.5) \quad \|y_\varepsilon(v_\varepsilon) - y_1\|_{B,S} \leq \alpha,$$

where, by definition,

$$\|y_\varepsilon(v_\varepsilon) - y_1\|_{B,S} \stackrel{\text{def}}{=} \left( \int_S B \nabla(y_\varepsilon(v_\varepsilon) - y_1) \cdot \nabla(y_\varepsilon(v_\varepsilon) - y_1) dx + \int_S |y_\varepsilon(v_\varepsilon) - y_1|^2 dx \right)^{1/2}.$$

This means that the error between  $y_\varepsilon(v_\varepsilon)$  and  $y_1$  is bounded from above by  $\alpha$  when using a norm equivalent to the  $H^1$ -norm and defined in terms of the matrix  $B$ .

**Remark 1.1** *In the case of the  $L^2$ -approximate controllability, the corresponding error condition is obtained by taking  $y_1 \in L^2(S)$  and the  $L^2$ -norm  $\|\cdot\|_{0,S}$  in (1.5), which formally corresponds to the case  $B = 0$ . ■*

Notice that the case  $\alpha = 0$  is the extreme situation of exact controllability. In this paper, we will just be concerned by *approximate* controllability, that is  $\alpha > 0$ .

**Remark 1.2** If  $S \subset\subset \omega$ , then the case of  $L^2$ -approximate controllability can be treated as the  $H^1$ -case. Conversely, if  $\overline{S} \cap \overline{\omega} = \emptyset$  and  $S$  is a nonempty open set, we can show by contradiction that the  $L^2$ -approximate controllability is not possible. Indeed, take  $\sigma$  a relatively compact open subset of  $S$  and define  $y_1 = 0$  in  $S \setminus \overline{\sigma}$  and  $y_1 = 1$  in  $\sigma$ . The approximate controllability of the problem

$$(1.6) \quad \begin{aligned} -\Delta y(v) &= \chi_\omega v && \text{in } \Omega \\ y(v) &= 0 && \text{on } \partial\Omega \end{aligned}$$

implies that, for each  $n \in \mathbb{N}$ , there exists  $v_n \in L^2(\Omega)$  such that

$$(1.7) \quad \|y(v_n) - 1\|_{0,\sigma} \leq 1/n \quad \text{and} \quad \|y(v_n)\|_{0,S \setminus \overline{\sigma}} \leq 1/n.$$

From this we derive that  $y(v_n) \rightarrow y^*$  strongly in  $L^2(S)$ , where

$$(1.8) \quad y^* = 1 \text{ in } \sigma \quad \text{and} \quad y^* = 0 \text{ in } S \setminus \overline{\sigma}.$$

We write (1.6) for  $v = v_n$  and we take the restriction to  $S$ . Passing to the limit, we derive

$$(1.9) \quad -\Delta y^* = 0 \quad \text{in } S$$

which contradicts (1.8). ■

## 1.2 Presentation of the main results

Our aim is to establish the approximate controllability for each  $\varepsilon > 0$  and to study the  $H$ -convergence of minimal norm controls to an approximate control linked to homogenized problems.

We notice that problem (1.3), (1.5) does not generally have a unique solution. We are therefore interested in the optimal control  $v_\varepsilon^*$  which minimizes, over all  $v \in L^2(\omega)$ , the cost functional

$$(1.10) \quad I_\varepsilon(v) \stackrel{\text{def}}{=} \frac{1}{2} \|v\|_{0,\omega}^2 + \begin{cases} 0 & \text{if } \|y_\varepsilon(v)|_S - y_1\|_{B,S} \leq \alpha \\ \infty & \text{otherwise.} \end{cases}$$

In order to do so, we first develop the fixed point strategy introduced by Fabre, Puel, and Zuazua [1]. Secondly, we pass to the limit as  $\varepsilon \rightarrow 0$  using  $H$ -convergence methods (see Murat and Tartar [6] or Tartar [8]). Our main results are Theorems 2.5 and 3.2 below.

**Remark 1.3** All the results of Sections 2 and 3 (including Lemma 2.3, Theorems 2.5 and 3.2) are also valid in the case of the  $L^2$ -approximate controllability and under the following hypothesis: (i) each point on  $S$  can be connected by an arc included in  $\Omega$  to some point in  $\omega$  without intersecting  $S$ . (ii) The coefficients of  $A_\varepsilon$  are of class  $C^1(\overline{\Omega})$  or  $L^\infty(\Omega)$  under some geometrical restrictions that allow a certain unique continuation property (see Remark 2.4).

To adapt the results and proofs to this case, it suffices to take all the variables with subindex 1 (like  $y_1, \varphi_1$ ) in  $L^2(S)$ , to replace  $\|\cdot\|_{B,S}$  and  $(\cdot, \cdot)_{B,S}$  by the usual norm  $\|\cdot\|_{0,S}$  and inner product  $(\cdot, \cdot)_{0,S}$  in  $L^2(S)$  (that is with  $B = 0$ ), and to replace  $\chi_S$  by a Dirac mass on  $S$ . ■

## 2 Existence of an optimal control

### 2.1 The linearized problem

For technical reasons, and without loss of generality, we assume that  $f \in \mathcal{C}^1(\mathbb{R})$ . (Otherwise, we can argue by density, approximating  $f$  by a sequence of smooth functions). This allows us to introduce the function

$$(2.1) \quad g(s) \stackrel{\text{def}}{=} \begin{cases} f(s)/s & \text{if } s \neq 0 \\ f'(0) & \text{if } s = 0. \end{cases}$$

The assumptions on  $f$  imply

$$(2.2) \quad g \in \mathcal{C}^0(\mathbb{R}) \quad \text{and} \quad 0 \leq g(s) \leq \gamma \quad \forall s \in \mathbb{R}.$$

We associate with  $g$  the linear problem

$$(2.3) \quad \begin{cases} -\operatorname{div}(A_\varepsilon \nabla y_\varepsilon(z, v)) + g(z) y_\varepsilon(z, v) = \chi_\omega v & \text{in } \Omega \\ y_\varepsilon(z, v) = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $z$  is a given function in  $L^2(\Omega)$ . We consider the cost functional

$$(2.4) \quad I_\varepsilon^z(v) \stackrel{\text{def}}{=} \frac{1}{2} \|v\|_{0,\omega}^2 + \begin{cases} 0 & \text{if } \|y_\varepsilon(z, v)|_S - y_1\|_{B,S} \leq \alpha \\ \infty & \text{otherwise.} \end{cases}$$

By classical linear control theory (see, e.g. Lions [4]), it is well-known that for any given  $z \in L^2(\Omega)$  there exists a unique minimal norm control  $v_\varepsilon^*(z)$  such that

$$(2.5) \quad I_\varepsilon^z(v_\varepsilon^*(z)) = \min_{v \in L^2(\omega)} I_\varepsilon^z(v) < +\infty.$$

We denote by  $y_\varepsilon^* = y_\varepsilon(z, v_\varepsilon^*(z))$  the corresponding state.

With the help of the minimal norm control  $v_\varepsilon^*(z)$ , we introduce the operator

$$(2.6) \quad \begin{aligned} \mathcal{F}_\varepsilon: L^2(\Omega) &\longrightarrow L^2(\Omega) \\ z &\longmapsto y_\varepsilon^*(z, v_\varepsilon^*(z)). \end{aligned}$$

Our goal is to find a fixed point of  $\mathcal{F}_\varepsilon$ , which will obviously solve problem (1.3).

### 2.2 Adjoint problem and dual formulation

It is useful to work with the adjoint problem in a dual formulation. To this end, we introduce the operator  $L$  defined by

$$(2.7) \quad \begin{aligned} L: L^2(\omega) &\longrightarrow H^1(S) \\ v &\longmapsto y_\varepsilon(z, v)|_S, \end{aligned}$$

where  $y_\varepsilon(z, v)$  is the solution of (2.3). Its adjoint  $L^*$  is given by

$$(2.8) \quad \begin{aligned} L^*: H^1(S) &\longrightarrow L^2(\omega) \\ \varphi_1 &\longmapsto \varphi_\varepsilon(z)|_\omega, \end{aligned}$$

where  $\varphi_\varepsilon$  is the solution of the so-called adjoint problem, which is obtained by solving the following Dirichlet problem

$$(2.9) \quad \begin{cases} -\operatorname{div}({}^t A_\varepsilon \nabla \varphi_\varepsilon(z, \varphi_1)) + g(z) \varphi_\varepsilon(z, \varphi_1) = -\operatorname{div}(\chi_S B \nabla \varphi_1) + \chi_S \varphi_1 & \text{in } \Omega \\ \varphi_\varepsilon(z, \varphi_1) = 0 & \text{on } \partial\Omega, \end{cases}$$

which defines  $\varphi_\varepsilon(z, \varphi_1)$  uniquely.

**Remark 2.1** *In this paper, we are concerned with approximate controllability in the sense of inequality (1.5). There is an alternative approach to approximate controllability which consists in proving that the set  $\{y_\varepsilon(z, v) \mid v \in L^2(\omega)\}$  is dense in  $H^1(S)$ . An equivalent condition to establish this density is to prove that  $\operatorname{Ker}(L^*) = 0$ . In our present case, this can be proved as follows. Given  $h \in H^{-1}(\Omega)$ , let us introduce  $\varphi_1 \in H_0^1(\Omega)$ , the unique solution of*

$$\begin{cases} -\operatorname{div}(B \nabla \varphi_1) + \varphi_1 = h & \text{in } \Omega \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly

$$\{\varphi_1|_S \mid h \in H^{-1}(\Omega)\} = H^1(S).$$

Therefore, if  $L^*(\varphi_1) = 0$  in  $L^2(\omega)$ , using as test functions  $\xi \in H_0^1(S)$  and  $\xi \in H_0^1(\omega)$  successively in (2.9), we obtain

$$(2.10) \quad \begin{cases} -\operatorname{div}(B \nabla \varphi_1) + \varphi_1 = 0 & \text{in } S \\ \int_{\partial S} B \nabla \varphi_1 \cdot n \xi ds = 0 & \forall \xi \in H_0^1(\omega), \end{cases}$$

since  $S \subset\subset \omega$ . Here  $n$  denotes the unit outward normal to both boundaries that of  $\omega$  and that of  $S$ . It follows that  $B \nabla \varphi_1 \cdot n = 0$  on  $\partial S$ , and hence  $\varphi_1 = 0$  in  $S$ . ■

**Remark 2.2** *In the case of an  $L^2$ -approximate controllability, the corresponding definitions (2.7)-(2.8) of  $L$  and  $L^*$  respectively can be given with  $H^1(S)$  replaced by  $L^2(S)$ . The corresponding adjoint problem is the same as in (2.10) taking  $B = 0$  and replacing  $\chi_S$  by a Dirac mass concentrated on  $S$ . A direct proof of the approximate controllability can be done as in the previous remark under the geometrical hypothesis mentioned before in Remark 1.3 (see [7]). ■*

The approximate controllability of the nonlinear problem (1.3) is obtained here by using a more constructive approach, which provides an explicit method to find a control of minimal norm. This method was introduced by Lions [5] (see also Osses and Puel [7]), and is based on the classical Fenchel-Rockafellar's duality theory.

We can write down the functional  $I_\varepsilon^z$  under the form

$$(2.11) \quad I_\varepsilon^z(v) = F(v) + G(Lv)$$

with

$$(2.12) \quad F(v) = \frac{1}{2} \|v\|_{0,\omega}^2 \quad \text{and} \quad G(Lv) = \begin{cases} 0 & \text{if } \|Lv - y_1\|_{B,S} \leq \alpha \\ \infty & \text{otherwise.} \end{cases}$$

Denoting by  $F^*$  and  $G^*$  the conjugate functions of  $F$  and  $G$  respectively, the duality theory states that

$$(2.13) \quad \inf_{v \in L^2(\omega)} I_\varepsilon^z(v) = - \inf_{\varphi_1 \in H^1(S)} J_\varepsilon^z(\varphi_1) = - \inf_{\substack{h \in H^{-1}(\Omega), \\ \varphi_1 \text{ solution of (2.10)}}} J_\varepsilon^z(\varphi_{1\varepsilon})$$

where

$$(2.14) \quad \begin{cases} J_\varepsilon^z(\varphi_{1\varepsilon}) = F^*(L^*\varphi_{1\varepsilon}) + G^*(-\varphi_{1\varepsilon}) \\ F^*(L^*\varphi_{1\varepsilon}) = \frac{1}{2} \|\varphi_\varepsilon(z, \varphi_{1\varepsilon})\|_{0,\omega}^2 \\ G^*(L^*\varphi_{1\varepsilon}) = \alpha \|\varphi_{1\varepsilon}\|_{B,S} + (\varphi_{1\varepsilon}, y_1)_{B,S}, \end{cases}$$

that is

$$(2.15) \quad J_\varepsilon^z(\varphi_{1\varepsilon}) = \frac{1}{2} \|\varphi_\varepsilon(z, \varphi_{1\varepsilon})\|_{0,\omega}^2 + \alpha \|\varphi_{1\varepsilon}\|_{B,S} - (\varphi_{1\varepsilon}, y_1)_{B,S}.$$

The following lemma, whose proof is given below in Section 2.4, summarizes the main properties of  $J_\varepsilon^z$ .

**Lemma 2.3** (Coercivity property of  $J_\varepsilon^z$ ). *For each  $\alpha > 0$  and  $y_1 \in H^1(S)$ , the functional  $J_\varepsilon^z$  defined in (2.14) is continuous, strictly convex, and satisfies*

$$(2.16) \quad \liminf_{\|\varphi_1\|_{B,S} \rightarrow +\infty} \frac{J_\varepsilon^z(\varphi_1)}{\|\varphi_1\|_{B,S}} \geq \alpha. \quad \blacksquare$$

Let us denote by  $\varphi_{1,\varepsilon}^*(z) \in H^1(S)$  the unique optimal element which minimizes  $J_\varepsilon^z(\varphi_1)$  over  $H^1(S)$  and let  $\varphi_\varepsilon^*$  be the corresponding element defined by (2.9). It is well-known that the duality theory provides extremal relations that the optimal controls satisfy, namely

$$(2.17) \quad \begin{cases} F(v_\varepsilon^*(z)) + F^*(L^*\varphi_{1,\varepsilon}^*(z)) - (L^*\varphi_{1,\varepsilon}^*(z), v_\varepsilon^*(z))_{0,\omega} = 0 \\ G(Lv_\varepsilon^*(z)) + G^*(-\varphi_{1,\varepsilon}^*(z)) + (\varphi_{1,\varepsilon}^*(z), Lv_\varepsilon^*(z))_{B,S} = 0. \end{cases}$$

From the first of these relations, we derive the following explicit formula for the minimal norm control

$$(2.18) \quad v_\varepsilon^*(z) = \varphi_\varepsilon(z, \varphi_{1,\varepsilon}^*(z))|_\omega.$$

**Remark 2.4** *The proof of Lemma 2.3 is based in the following unique continuation property: if the solution of problem (2.9) is zero in  $\omega$  then it is zero in the whole of  $\Omega$ . In the case of  $H^1$ -approximate controllability, this property is quite easy to prove under the regularity hypothesis (1.1) since  $S \subset\subset \omega$ . In the case of  $L^2$ -approximate controllability,  $S$  does not intersect  $\omega$  and the result is a Holmgren's unique continuation property [2]. This requires more regularity in the coefficients of  $A_\varepsilon$  (at least  $C^1$ ) and an additional geometrical hypothesis as mentioned in Remark 1.3. Nevertheless, if the coefficients of  $A_\varepsilon$  are only  $L^\infty$  but piecewise  $C^1$ , the unique continuation property remains valid because of transmission conditions on the discontinuity interfaces.  $\blacksquare$*

### 2.3 Fixed point strategy

Thanks to this dual formulation, we are now in a position to develop our fixed point strategy for  $\mathcal{F}_\varepsilon$ . It consists in three steps. First, we establish the continuity of  $\mathcal{F}_\varepsilon$  from  $L^2(\Omega)$  into itself. Next, we prove that it maps the whole of  $L^2(\Omega)$  into a bounded subset of  $L^2(\Omega)$ . Last, we check that  $\mathcal{F}_\varepsilon$  is compact, and using Schauder's fixed point theorem, we conclude the existence of a solution of problem (1.3). More precisely, we have

**Theorem 2.5** *For a given  $\varepsilon > 0$ , let  $A_\varepsilon$  be a matrix in  $\mathcal{M}(\alpha_m, \alpha_M)$ . Assume that the real-valued function  $f$  satisfies condition (1.4). Then there exists at least an element  $\bar{z}_\varepsilon \in L^2(\Omega)$  which is a fixed point of the operator  $\mathcal{F}_\varepsilon$  defined by (2.6). This element satisfies the equation  $\bar{z}_\varepsilon = y_\varepsilon^*(\bar{z}_\varepsilon, v_\varepsilon^*(\bar{z}_\varepsilon))$ , where  $y_\varepsilon^*(\bar{z}_\varepsilon, v_\varepsilon^*(\bar{z}_\varepsilon))$  is the state solution of problem (1.3) and  $v = v_\varepsilon^*(\bar{z}_\varepsilon)$  is the optimal control of the functional  $I_\varepsilon$  (see (1.10)). ■*

The remaining part of Section 2 is entirely devoted to the proof of the above theorem.

**1st step** *Continuity of  $\mathcal{F}_\varepsilon$*

Let  $z_n$  be any converging sequence in  $L^2(\Omega)$ , say

$$(2.19) \quad z_n \rightarrow z_0 \text{ strongly in } L^2(\Omega).$$

Denote  $\varphi_{\varepsilon,n} = \varphi_\varepsilon(z_n, \varphi_1)$  the solution of (2.9) corresponding to  $z = z_n$ . Taking  $\varphi_{\varepsilon,n}$  as a test function in the adjoint problem (2.9), we obtain (using (2.10))

$$\|\varphi_{\varepsilon,n}\|_{1,\Omega} \leq C \|\varphi_{1\varepsilon}\|_{B,S} \leq C,$$

where, here and in the following,  $C$  denotes different constants independent of  $z$  and  $n$ . Hence, up to a subsequence still denoted  $n$ , we have

$$(2.20) \quad \varphi_{\varepsilon,n} \rightharpoonup \varphi_{\varepsilon,0} \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega).$$

Let  $\varphi \in L^2(\Omega)$ . We have

$$(2.21) \quad \begin{aligned} & \left| \int_{\Omega} g(z_n) \varphi_{\varepsilon,n} \varphi dx - \int_{\Omega} g(z_0) \varphi_{\varepsilon,0} \varphi dx \right| \leq \\ & \leq \left| \int_{\Omega} g(z_n) (\varphi_{\varepsilon,n} - \varphi_{\varepsilon,0}) \varphi dx \right| + \left| \int_{\Omega} (g(z_n) - g(z_0)) \varphi_{\varepsilon,0} \varphi dx \right| \\ & \leq \|g\|_{\infty} \|\varphi_{\varepsilon,n} - \varphi_{\varepsilon,0}\|_{0,\Omega} \|\varphi\|_{0,\Omega} + \left| \int_{\Omega} (g(z_n) - g(z_0)) \varphi_{\varepsilon,0} \varphi dx \right|. \end{aligned}$$

The first term in the right-hand side tends to zero by (2.20). Besides, by (2.19), up to a subsequence, we have

$$z_n \rightarrow z_0 \text{ for a.e. } x \in \mathbb{R},$$

hence by (2.2), up to another subsequence, we also have

$$g(z_n) \rightharpoonup g(z_0) \text{ weakly } * \text{ in } L^\infty(\Omega).$$

Therefore, the second term in the right-hand side of (2.21) tends to zero by virtue of Lebesgue's dominated convergence theorem. Hence, up to a subsequence,

$$(2.22) \quad g(z_n)\varphi_{\varepsilon,n} \xrightarrow[n \rightarrow +\infty]{} g(z_0)\varphi_{\varepsilon,0} \quad \text{weakly in } L^2(\Omega) \text{ and strongly in } H^{-1}(\Omega).$$

Let us now pass to the limit in the adjoint problem (2.9) written for  $z_n$  and  $\varphi_{\varepsilon,n}$ . Using a test function  $\varphi \in H_0^1(\Omega)$ , integrating by parts in  $\Omega$  and passing to the limit using convergence (2.20), (2.22), we deduce

$$(2.23) \quad \begin{cases} -\operatorname{div}({}^t A_\varepsilon \nabla \varphi_{\varepsilon,0}) + g(z_0)\varphi_{\varepsilon,0} = -\operatorname{div}(\chi_S B \nabla \varphi_1) + \chi_S \varphi_1 & \text{in } \Omega \\ \varphi_{\varepsilon,0} = 0 & \text{on } \partial\Omega. \end{cases}$$

This means that

$$(2.24) \quad \varphi_{\varepsilon,0} = \varphi_\varepsilon(z_0, \varphi_1).$$

Let us now prove that the convergence in (2.20) is actually a strong one, that is

$$(2.25) \quad \varphi_\varepsilon(z_n, \varphi_1) \xrightarrow[n \rightarrow +\infty]{} \varphi_\varepsilon(z_0, \varphi_1) \quad \text{strongly in } H^1(\Omega).$$

In fact, multiplying (2.9) (written for  $\varphi_{\varepsilon,n}$ ) by  $\varphi_{\varepsilon,n}$ , integrating by parts in  $\Omega$ , and passing to the limit, we obtain

$$(2.26) \quad \lim_{n \rightarrow \infty} \int_{\Omega} {}^t A_\varepsilon \nabla \varphi_{\varepsilon,n} \cdot \nabla \varphi_{\varepsilon,n} \, dx = - \int_{\Omega} g(z_0) |\varphi_{\varepsilon,0}|^2 \, dx + \langle -\operatorname{div}(\chi_S B \nabla \varphi_1) + \chi_S \varphi_1, \varphi_{\varepsilon,0} \rangle,$$

where the bracket is the classical duality pairing between  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)$ . On the other hand, multiplying (2.23) by  $\varphi_{\varepsilon,0}$ , integrating by parts in  $\Omega$  and comparing with (2.26), we deduce

$$(2.27) \quad \lim_{n \rightarrow \infty} \int_{\Omega} {}^t A_\varepsilon \nabla \varphi_{\varepsilon,n} \cdot \nabla \varphi_{\varepsilon,n} \, dx = \int_{\Omega} {}^t A_\varepsilon \nabla \varphi_{\varepsilon,0} \cdot \nabla \varphi_{\varepsilon,0} \, dx.$$

Since the mapping  $\varphi \mapsto \int_{\Omega} {}^t A_\varepsilon \nabla \varphi \cdot \nabla \varphi \, dx$  defines a norm in  $H_0^1(\Omega)$  which is equivalent to the one induced by the usual  $H^1$ -topology, we conclude the strong convergence (2.25) from (2.27).

We now prove that the sequence of optimal elements  $\varphi_{1,\varepsilon}^*(z_n)$  remains bounded in  $H^1(S)$  as  $\varepsilon \rightarrow 0$  and  $n \rightarrow \infty$ . More precisely, we have

**Lemma 2.6** *Assume that  $z_n$  satisfies (2.19). Then there exists a constant  $C_\varepsilon$ , independent of  $n$ , such that*

$$(2.28) \quad \|\varphi_{1,\varepsilon}^*(z_n)\|_{B,S} \leq C_\varepsilon \quad \forall n \in \mathbb{N}.$$

**Proof** We argue by contradiction. Assume that there exists a subsequence, which we will still denote by  $n$ , such that

$$(2.29) \quad \|\varphi_{1,\varepsilon}^*(z_n)\|_{1,B_{\varepsilon_n},S} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Since  $\varphi_{1,\varepsilon}^*(z_n)$  minimizes  $J_\varepsilon^{z_n}$ , we have

$$(2.30) \quad J_\varepsilon^{z_n}(\varphi_{1,\varepsilon}^*(z_n)) \leq J_\varepsilon^{z_n}(\varphi_1) \quad \forall \varphi_1 \in H^1(S).$$

But

$$J_\varepsilon^{z_n}(\varphi_1) = \frac{1}{2} \|\varphi_\varepsilon(z_n, \varphi_1)\|_{0,\omega}^2 + \alpha \|\varphi_1\|_{B,S} - (\varphi_1, y_1)_{B,S}.$$

Thanks to (2.24), (2.25),  $J_\varepsilon^{z_n}(\varphi_1)$  converges, when  $n \rightarrow \infty$ , to

$$J_\varepsilon^{z_0}(\varphi_1) = \frac{1}{2} \|\varphi_0(z_0, \varphi_1)\|_{0,\omega}^2 + \alpha \|\varphi_1\|_{B,S} - (\varphi_1, y_1)_{B,S}.$$

Then, combining this result with (2.30), for any  $\delta > 0$  and for  $n$  large enough, we have

$$J_\varepsilon^{z_n}(\varphi_{1,\varepsilon}^*(z_n)) \leq J_\varepsilon^{z_0}(\varphi_1) + \delta,$$

which obviously contradicts the coercivity property of Lemma 2.3. ■

From (2.28), up to a subsequence, there exists a limiting function  $\xi_\varepsilon \in H^1(S)$  such that

$$(2.31) \quad \varphi_{1,\varepsilon}^*(z_n)|_S \xrightarrow[n \rightarrow \infty]{} \xi_\varepsilon \quad \text{weakly in } H^1(S) \text{ and strongly in } L^2(S).$$

Arguing as in the proof of (2.25), we deduce from (2.31)

$$(2.32) \quad \varphi_\varepsilon(z_n, \varphi_{1,\varepsilon}^*(z_n)) \xrightarrow[n \rightarrow \infty]{} \varphi_\varepsilon(z_0, \xi_\varepsilon) \quad \text{strongly in } H_0^1(\Omega).$$

Our next step consists in proving that

$$(2.33) \quad \xi_\varepsilon = \varphi_{1,\varepsilon}^*(z_0),$$

which is the optimal element minimizing  $J_\varepsilon^{z_0}$ , that is

$$(2.34) \quad J_\varepsilon^{z_0}(\xi_\varepsilon) \leq J_\varepsilon^{z_0}(\varphi_1) \quad \forall \varphi_1 \in H^1(S).$$

Since  $\varphi_{1,\varepsilon}^*(z_n)$  minimizes  $J_\varepsilon^{z_n}$ , we have

$$J_\varepsilon^{z_n}(\varphi_{1,\varepsilon}^*(z_n)) \leq J_\varepsilon^{z_n}(\varphi_1) \quad \forall \varphi_1 \in H^1(S),$$

which implies

$$(2.35) \quad \liminf_{n \rightarrow \infty} J_\varepsilon^{z_n}(\varphi_{1,\varepsilon}^*(z_n)) \leq \lim_{n \rightarrow \infty} J_\varepsilon^{z_n}(\varphi_1) = J_\varepsilon^{z_0}(\varphi_1).$$

Therefore, to prove (2.34), it suffices to show that

$$(2.36) \quad J_\varepsilon^{z_0}(\xi_\varepsilon) \leq \liminf_{n \rightarrow \infty} J_\varepsilon^{z_n}(\varphi_{1,\varepsilon}^*(z_n)).$$

Using convergence (2.31) and the definition of  $J_\varepsilon^{z_n}$ , we have

$$\liminf_{n \rightarrow \infty} J_\varepsilon^{z_n}(\varphi_{1,\varepsilon}^*(z_n)) \geq \liminf_{n \rightarrow \infty} \left( \frac{1}{2} \|\varphi_\varepsilon(z_n, \varphi_{1,\varepsilon}^*(z_n))\|_{0,\omega}^2 \right) + \alpha \|\xi_\varepsilon\|_{B,S} - (\xi_\varepsilon, y_1)_{B,S}.$$

Combining with (2.32), we conclude (2.36), which completes the proof of (2.33). Hence (2.32) becomes

$$(2.37) \quad \varphi_\varepsilon(z_n, \varphi_{1,\varepsilon}^*(z_n)) \xrightarrow{} \varphi_\varepsilon(z_0, \varphi_{1,\varepsilon}^*(z_0)) \quad \text{strongly in } H_0^1(\Omega).$$

Using the explicit formula (2.18) for the optimal control  $v_\varepsilon^*$  of problem (2.3), we have

$$\begin{cases} v_\varepsilon^*(z_n) = \varphi_\varepsilon(z_n, \varphi_{1,\varepsilon}^*(z_n))|_\omega \\ v_\varepsilon^*(z_0) = \varphi_\varepsilon(z_0, \varphi_{1,\varepsilon}^*(z_0))|_\omega. \end{cases}$$

Therefore, from (2.37), we derive

$$(2.38) \quad v_\varepsilon^*(z_n) \longrightarrow v_\varepsilon^*(z_0) \quad \text{strongly in } H^1(\omega).$$

Finally, arguing as we did for the adjoint problem, we can pass to the limit in problem (2.3) using convergence (2.38), and we obtain

$$(2.39) \quad y_\varepsilon(z_n, v_\varepsilon^*(z_n)) \longrightarrow y_\varepsilon(z_0, v_\varepsilon^*(z_0)) \quad \text{strongly in } H^1(\Omega).$$

This ends the proof of the continuity of  $\mathcal{F}_\varepsilon$ . ■

**Remark 2.7** *In the particular case when  $\varphi_1$  is defined on the whole of  $\Omega$  by (2.10), it is worthwhile to notice that it is merely the restriction  $\varphi_1|_S$  of  $\varphi_1$  to  $S$  which plays a role in the proof of Theorem 2.5. ■*

**2nd step**  $\mathcal{F}_\varepsilon(L^2(\Omega))$  is bounded in  $L^2(\Omega)$

Since for all  $z \in L^2(\Omega)$  we have  $\|g(z)\|_\infty \leq \gamma$ , then

$$\|\varphi_\varepsilon(z, \varphi_1)\|_{1,\Omega} \leq C\|\varphi_1\|_{B,S},$$

with  $C$  independent of  $z$  and  $\varepsilon$ . This implies the existence of a constant  $C = C(\varphi_1)$  such that

$$J_\varepsilon^z(\varphi_{1\varepsilon}) \leq C(\varphi_1) \quad \forall \varphi_1 \in H^1(S).$$

In particular, for the optimal element  $\varphi_{1,\varepsilon}^*$ , we have

$$J_\varepsilon^z(\varphi_{1,\varepsilon}^*(z)) \leq C(\varphi_1) \quad \forall \varphi_1 \in H^1(S).$$

This holds in particular for  $\varphi_1 = 0$ , thus

$$J_\varepsilon^z(\varphi_{1,\varepsilon}^*(z)) \leq C,$$

with  $C$  independent of  $z$  and  $\varepsilon$ .

Using again the coercivity of  $J_\varepsilon^z$  (see Lemma 2.3), we prove that  $\|\varphi_{1,\varepsilon}^*(z)\|_{1,B_\varepsilon,S}$  is bounded independently of  $z$  and of  $\varepsilon$ . Thus we have

$$\|\varphi_\varepsilon(z, \varphi_{1,\varepsilon}^*(z))\|_{1,\Omega} \leq C$$

with  $C$  independent of  $z$  and of  $\varepsilon$ . This clearly implies that both  $v_\varepsilon^*(z)$  and  $y_\varepsilon(z, v_\varepsilon^*(z))$  are bounded in their corresponding spaces, i.e., there exists  $C$  independent of  $z$  and  $\varepsilon$  such that

$$(2.40) \quad \|v_\varepsilon^*(z)\|_{0,\omega} \leq C$$

and

$$(2.41) \quad \|y_\varepsilon(z, v_\varepsilon^*(z))\|_{0,\Omega} \leq \|y_\varepsilon(z, v_\varepsilon^*(z))\|_{1,\Omega} \leq C,$$

which concludes the second step.

**3rd step**  $\mathcal{F}_\varepsilon$  is compact

In the second step, a stronger result than the one announced was proved. Indeed from (2.41), we see that  $\mathcal{F}_\varepsilon$  maps the whole of  $L^2(\Omega)$  into a bounded subset of  $H^1(\Omega)$ , and hence into a relatively compact subset of  $L^2(\Omega)$ . This proves the compactness of  $\mathcal{F}_\varepsilon$ , and hence completes the proof of Theorem 2.5, provided the coercivity Lemma 2.3 is established. ■

## 2.4 Proof of Lemma 2.3

To simplify matters, in this subsection we drop the index  $\varepsilon$  in the notation for  $\varphi_1$ . From (2.14), for  $\varphi_1 \in H^1(S)$  and  $\varphi_1 \neq 0$ , we have

$$\frac{J_\varepsilon^z(\varphi_1)}{\|\varphi_1\|_{B,S}} = \frac{1}{2\|\varphi_1\|_{B,S}} \int_\omega |\varphi_\varepsilon(z, \varphi_1)|^2 dx + \alpha - \int_S B \nabla \left( \frac{\varphi_1}{\|\varphi_1\|_{B,S}} \right) \cdot \nabla y_1 dx.$$

Let  $\varphi_{1,n} \in H^1(S)$  be a sequence such that  $\|\varphi_{1,n}\|_{B,S} \xrightarrow{n \rightarrow \infty} \infty$  and

$$(2.42) \quad \lim_{n \rightarrow \infty} \frac{J_\varepsilon^z(\varphi_{1,n})}{\|\varphi_{1,n}\|_{B,S}} = \liminf_{\|\varphi_1\|_{B,S} \rightarrow \infty} \frac{J_\varepsilon^z(\varphi_1)}{\|\varphi_1\|_{B,S}}.$$

We introduce the following normalizations

$$(2.43) \quad \widehat{\varphi}_{1,n} = \frac{\varphi_{1,n}}{\|\varphi_{1,n}\|_{B,S}} \quad \text{and} \quad \widehat{\varphi}_{\varepsilon,n} = \frac{\varphi_{\varepsilon,n}(z, \varphi_{1,n})}{\|\varphi_{1,n}\|_{1,B,S}}.$$

Then we have

$$(2.44) \quad \frac{J_\varepsilon^z(\varphi_{1,n})}{\|\varphi_{1,n}\|_{B,S}} = \alpha + \frac{1}{2} \|\varphi_{1,n}\|_{B,S} \int_\omega |\widehat{\varphi}_{\varepsilon,n}|^2 dx - \int_S B \nabla \widehat{\varphi}_{1,n} \cdot \nabla y_1 dx.$$

Also, since  $\|\widehat{\varphi}_{1,n}\|_{B,S} = 1$ , using  $\varphi_{\varepsilon,n}$  as a test function in the adjoint problem (2.9), we deduce

$$(2.45) \quad \|\widehat{\varphi}_{\varepsilon,n}\|_{1,\Omega} \leq C \quad (C \text{ independent of } \varepsilon \text{ and } n).$$

Therefore, up to a subsequence, we have

$$(2.46) \quad \begin{cases} \widehat{\varphi}_{1,n} \rightharpoonup \widetilde{\varphi}_1 & \text{weakly in } L^2(S) \\ \widehat{\varphi}_{\varepsilon,n} \rightharpoonup \widetilde{\varphi}_\varepsilon & \text{weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega). \end{cases}$$

Let us distinguish various cases.

*Case (i)* Assume that

$$\lim_{n \rightarrow \infty} \int_\omega |\widehat{\varphi}_{\varepsilon,n}|^2 dx \left( = \int_\omega |\widetilde{\varphi}_\varepsilon|^2 dx \right) > 0,$$

then the second term in the right-hand side of (2.44) tends to infinity while the third term has a limit. Hence (2.16) holds in this case.

*Case (ii)* If

$$\lim_{n \rightarrow \infty} \int_\omega |\widehat{\varphi}_{\varepsilon,n}|^2 dx \left( = \int_\omega |\widetilde{\varphi}_\varepsilon|^2 dx \right) = 0,$$

then  $\widetilde{\varphi}_\varepsilon = 0$  in  $\omega$ . From the smoothness hypothesis on the coefficients of the matrix  $A_\varepsilon$  (see (1.1)), since  $S \subset \subset \omega$ , we have an homogeneous problem (2.9) in  $\Omega \setminus \omega$  with homogeneous Dirichlet boundary conditions, and this implies that  $\widetilde{\varphi}_\varepsilon = 0$  in  $\Omega$ . Therefore  $\widetilde{\varphi}_1 = 0$  and so,

$$\lim_{n \rightarrow \infty} \frac{J_\varepsilon^z(\varphi_{1,n})}{\|\varphi_{1,n}\|_{B,S}} \geq \alpha + \liminf_{n \rightarrow \infty} \left( \|\varphi_{1,n}\|_{B,S} \int_\omega |\widehat{\varphi}_{\varepsilon,n}|^2 dx \right) \geq \alpha > 0,$$

which ends the proof of Lemma 2.3 and therefore that of Theorem 2.5. ■

### 3 Homogenization of the approximate controllability problem

Our goal in this section is to pass to the limit in problem (1.3) when  $v = v_\varepsilon^*$  is the optimal control constructed in Section 2.

To this general end, we begin by considering a sequence of matrices  $A_\varepsilon \in \mathcal{M}(\alpha_m, \alpha_M)$  and the corresponding state equations

$$(3.1) \quad \begin{cases} -\operatorname{div}(A_\varepsilon \nabla y_\varepsilon(v)) + f(y_\varepsilon(v)) = \chi_\omega v & \text{in } \Omega \\ y_\varepsilon(v) = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $v \in L^2(\omega)$ . The main additional assumption in this section is that

$$(3.2) \quad A_\varepsilon \text{ } H\text{-converges to } A_0$$

(see [6], [8] for details about  $H$ -convergence). It is well-known that  $A_0 \in \mathcal{M}(\alpha_m, \alpha_M^2/\alpha_m)$ .

#### 3.1 Homogenization of the state equation for a fixed control

In this subsection, we assume that the control  $v$  is a fixed element in  $L^2(\omega)$ . We prove the following homogenization result:

**Proposition 3.1** *Assume that the hypotheses of Theorem 2.5 hold and that  $A_\varepsilon$  satisfies (3.2). Then, up to a subsequence, there exists  $y_0(v)$  such that*

$$(3.3) \quad \begin{cases} y_\varepsilon(v) \rightharpoonup y_0(v) & \text{weakly in } H_0^1(\Omega) \\ A_\varepsilon \nabla y_\varepsilon(v) \rightharpoonup A_0 \nabla y_0(v) & \text{weakly in } L^2(\Omega)^N. \end{cases}$$

Moreover,  $y_0(v)$  satisfies the homogenized state equation

$$(3.4) \quad \begin{cases} -\operatorname{div}(A_0 \nabla y_0(v)) + f(y_0(v)) = \chi_\omega v & \text{in } \Omega \\ y_0(v) = 0 & \text{on } \partial\Omega. \end{cases}$$

**Proof** Since  $f$  satisfies (1.4), the first convergence in (3.3) is straightforward. We now wish to establish a convergence result on  $f(y_\varepsilon(v))$ . For all  $\varphi \in L^2(\Omega)$ , we have

$$\left| \int_{\Omega} (f(y_\varepsilon(v)) - f(y_0(v))) \varphi dx \right| \leq \left| \int_{\Omega} g(y_\varepsilon(v)) (y_\varepsilon(v) - y_0(v)) \varphi dx \right| + \left| \int_{\Omega} (g(y_\varepsilon(v)) - g(y_0(v))) y_0(v) \varphi dx \right|.$$

Arguing as we did in Section 2.3 to establish (2.22), we prove that, up to a subsequence,

$$(3.5) \quad f(y_\varepsilon(v)) \rightharpoonup f(y_0(v)) \quad \text{weakly in } L^2(\Omega) \text{ and strongly in } H^{-1}(\Omega).$$

We are now in a position to pass to the limit in problem (3.1). Thanks to (3.5) and  $H$ -convergence properties, we end the proof of Proposition 3.1. ■

### 3.2 Homogenization of the state equation for an optimal control

Denote  $\bar{z}_\varepsilon$  the fixed point of  $\mathcal{F}_\varepsilon$  constructed in Section 2 using Schauder's theorem. Since the constant in (2.40) is independent of  $z$  and  $\varepsilon$ , the sequence of optimal controls  $v_\varepsilon^*(\bar{z}_\varepsilon)$  remains bounded in  $L^2(\omega)$  as  $\varepsilon \rightarrow 0$ . Thus, up to a subsequence, there exists  $v_0 \in L^2(\omega)$  such that

$$(3.6) \quad \begin{cases} v_\varepsilon^*(\bar{z}_\varepsilon) \rightharpoonup v_0 & \text{weakly in } L^2(\omega) \text{ and strongly in } H^{-1}(\omega) \\ \chi_\omega v_\varepsilon^*(\bar{z}_\varepsilon) \rightharpoonup \chi_\omega v_0 & \text{weakly in } L^2(\Omega) \text{ and strongly in } H^{-1}(\Omega). \end{cases}$$

As in Section 2, the estimate (2.40) implies that the solution  $y_\varepsilon^* = y_\varepsilon(\bar{z}_\varepsilon, v_\varepsilon^*(\bar{z}_\varepsilon))$  of problem (1.3) satisfies

$$\|y_\varepsilon^*\|_{0,\Omega} \leq \|y_\varepsilon^*\|_{1,\Omega} \leq C,$$

where  $C$  is independent of  $\varepsilon$ . Hence there exists  $y_0 \in H_0^1(\Omega)$  such that, up to a subsequence,

$$(3.7) \quad y_\varepsilon^* \rightharpoonup y_0(v_0) \quad \text{weakly in } H_0^1(\Omega).$$

Clearly, as in Section 3.1, we derive from (3.2), (3.6), (3.7), that  $y_0(v_0)$  is solution of the homogenized problem

$$(3.8) \quad \begin{cases} -\operatorname{div}(A_0 \nabla y_0(v_0)) + f(y_0(v_0)) = \chi_\omega v_0 & \text{in } \Omega \\ y_0(v_0) = 0 & \text{on } \partial\Omega. \end{cases}$$

Our aim is to prove that  $v_0$  satisfies the following approximate controllability inequality

$$\|y_0(v_0)|_S - y_1\|_{B,S} \leq \alpha.$$

Furthermore, we will prove that  $v_0$  is optimal in the sense that it minimizes, over all  $v \in L^2(\omega)$ , the cost functional

$$(3.9) \quad I_0(v) \stackrel{\text{def}}{=} \frac{1}{2} \|v\|_{0,\omega}^2 + \begin{cases} 0 & \text{if } \|y_0(v)|_S - y_1\|_{B,S} \leq \alpha \\ +\infty & \text{otherwise,} \end{cases}$$

where  $y_0(v)$  is the solution of (3.8) corresponding to the control  $v$ .

To reach this aim, we begin by writing down the fixed point identity

$$\bar{z}_\varepsilon = y_\varepsilon(\bar{z}_\varepsilon, v_\varepsilon^*(\bar{z}_\varepsilon)) = y_\varepsilon^*.$$

Thus, from (3.7) there exists  $z_0 \in H_0^1(\Omega)$  such that, up to a subsequence

$$(3.10) \quad \bar{z}_\varepsilon \rightharpoonup z_0 \quad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega).$$

For any given control  $v \in L^2(\omega)$ , let  $y_0(z_0, v)$  be the solution of the homogenized linearized problem

$$(3.11) \quad \begin{cases} -\operatorname{div}(A_0 \nabla y_0(z_0, v)) + g(z_0, v) = \chi_\omega v & \text{in } \Omega \\ y_0(z_0, v) = 0 & \text{on } \partial\Omega. \end{cases}$$

To this state equation, we associate the cost functional

$$(3.12) \quad I_0^{z_0}(v) = \frac{1}{2} \|v\|_{0,\omega}^2 + \begin{cases} 0 & \text{if } \|y_0(z_0, v)|_S - y_1\|_{B,S} \leq \alpha \\ \infty & \text{otherwise.} \end{cases}$$

By classical linear control theory and Proposition 3.1 there exists a unique optimal control  $v_0^*(z_0)$  such that

$$(3.13) \quad I_0^{z_0}(v_0^*(z_0)) = \min_{v \in L^2(\omega)} I_0^{z_0}(v) < +\infty.$$

We denote by  $y_0^* = y_0(z_0, v_0^*(z_0))$  the corresponding state.

We are now in a position to prove our main result, namely

**Theorem 3.2** *We make the hypotheses of Theorem 2.5 and we also assume the  $H$ -convergence (3.2) of  $A_\varepsilon$  to  $A_0$ . Let  $v_0$  be the limit of the optimal controls defined in (3.6). Then*

$$(3.14) \quad v_0 = v_0^*(z_0),$$

where  $v_0^*(z_0)$  is the optimal control of the linearized problem (3.11), (3.13).

**Proof** We proceed in several steps.

**1st step** *Existence of the optimal control  $v_0^*(z_0)$*

We use again the classical Fenchel-Rockafellar's duality theory which provides an explicit control of minimal norm. Given  $\varphi_1 \in H^1(S)$ , we introduce  $\varphi_0(z_0, \varphi_1)$ , the solution of

$$(3.15) \quad \begin{cases} -\operatorname{div}({}^t A_0 \nabla \varphi_0(z_0, \varphi_1)) + g(z_0) \varphi_0(z_0, \varphi_1) = -\operatorname{div}(\chi_S B \nabla \varphi_1) + \chi_S \varphi_1 & \text{in } \Omega \\ \varphi_0(z_0, \varphi_1) = 0 & \text{on } \partial\Omega. \end{cases}$$

By duality, as in Section 2, we have

$$(3.16) \quad \inf_{v \in L^2(\omega)} I_0^{z_0}(v) = - \inf_{\varphi_1 \in H^1(S)} J_0^{z_0}(\varphi_1),$$

where

$$(3.17) \quad J_0^{z_0}(\varphi_1) = \frac{1}{2} \|\varphi_0(z_0, \varphi_1)\|_{0,\omega}^2 + \alpha \|\varphi_1\|_{B,S} - (\varphi_1, y_1)_{B,S}.$$

It is also well-known, from the extremal relations for the above optimization problem, that

$$(3.18) \quad v_0^*(z_0) = \varphi_0(z_0, \varphi_1^*(z_0))|_\omega,$$

where  $\varphi_1^*(z_0) \in H^1(S)$  is the unique optimal element which minimizes  $J_0^{z_0}$  over  $H^1(S)$ .

**2nd step** *Passage to the limit in the adjoint problem*

From system (2.9) and convergence (3.10), we derive easily that there exists a function  $\bar{\varphi}_0$  such that, up to a subsequence

$$(3.19) \quad \varphi_\varepsilon(\bar{z}_\varepsilon, \varphi_1) \rightharpoonup \bar{\varphi}_0 \quad \text{weakly in } H_0^1(\Omega).$$

By  $H$ -convergence results, we pass to the limit in (2.9) and we deduce that  $\bar{\varphi}_0$  is the solution of

$$\begin{cases} -\operatorname{div}({}^t A_0 \nabla \bar{\varphi}_0) + g(z_0) \bar{\varphi}_0 = -\operatorname{div}(\chi_S B \nabla \varphi_1) + \chi_S \varphi_1 & \text{in } \Omega \\ \bar{\varphi}_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

that is (compare with (3.15))

$$(3.20) \quad \bar{\varphi}_0 = \varphi_0(z_0, \varphi_1).$$

We are now in a position to pass to the limit in  $J_\varepsilon^{\bar{z}_\varepsilon}$  defined by (2.14). Recall that

$$J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_1) = \frac{1}{2} \|\varphi_\varepsilon(\bar{z}_\varepsilon, \varphi_1)\|_{0,\omega}^2 + \alpha \int_{\Omega} (\chi_S B \nabla \varphi_1 \cdot \nabla \varphi_1 + \chi_S |\varphi_1|^2) dx - \int_{\Omega} \chi_S B \nabla \varphi_1 \cdot \nabla y_1 dx.$$

To pass to the limit, we use convergence (3.19). We obtain

$$(3.21) \quad \lim_{\varepsilon \rightarrow 0} J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_1) = J_0^{z_0}(\varphi_1) \quad \forall \varphi_1 \in H^1(S).$$

From (3.21), we derive that between the optimal elements  $\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)$  and  $\varphi_1^*(z_0)$ , we have the following relation

$$(3.22) \quad J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)) = \min_{\varphi_1} J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_1) \longrightarrow \min_{\varphi_1} J_0^{z_0}(\varphi_1) = J_0^{z_0}(\varphi_1^*(z_0)).$$

**3rd step** *Convergence of the optimal controls for the state equation*

Using the uniform coercivity property of the functionals  $J_\varepsilon^{\bar{z}_\varepsilon}$  (see Lemma 2.3) and arguing as in the proof of Lemma 2.6, we deduce the existence of a constant  $C$  independent of  $\varepsilon$  such that

$$\|\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)\|_{1,B_\varepsilon,S} \leq C.$$

Since the matrices  $B_\varepsilon$  are equi-coercive, we derive the existence of an element  $\xi^* \in H^1(S)$  such that, up to a subsequence,

$$(3.23) \quad \varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon) \rightharpoonup \xi^* \quad \text{weakly in } H^1(S).$$

This implies that, up to another subsequence,

$$(3.24) \quad \varphi_\varepsilon(\bar{z}_\varepsilon, \varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)) \rightharpoonup \varphi_0(z_0, \xi^*) \quad \text{weakly in } H_0^1(\Omega).$$

Our next aim is to prove that  $\xi^*$  is equal to  $\varphi_1^*(z_0)$ , the unique minimizer of  $J_0^{z_0}$ , that is

$$(3.25) \quad J_0^{z_0}(\xi^*) \leq J_0^{z_0}(\varphi_1) \quad \forall \varphi_1 \in H^1(S).$$

Since  $\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)$  minimizes  $J_\varepsilon^{\bar{z}_\varepsilon}$ , we have

$$J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)) \leq J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_1) \quad \forall \varphi_1 \in H^1(S).$$

Thanks to (3.21), we deduce that

$$\liminf_{\varepsilon \rightarrow 0} J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)) \leq \lim_{\varepsilon \rightarrow 0} J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_1) = J_0^{z_0}(\varphi_1).$$

Therefore, to prove (3.25), it suffices to show that

$$(3.26) \quad J_0^{z_0}(\xi^*) \leq \liminf_{\varepsilon \rightarrow 0} J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)).$$

Using the definition of  $J_\varepsilon^{\bar{z}_\varepsilon}$ , we have

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)) &= \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{2} \|\varphi_\varepsilon(\bar{z}_\varepsilon, \varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon))\|_{0,\omega}^2 \right) + \alpha \|\xi^*\|_{B,S} - (\xi^*, y_1)_{B,S} \\ &= J_0^{z_0}(\xi^*), \end{aligned}$$

which proves (3.26) and hence (3.25). Thus (3.23), (3.24) become

$$(3.27) \quad \begin{cases} \varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon) \rightharpoonup \varphi_1^*(z_0) & \text{weakly in } H^1(S) \\ \varphi_\varepsilon(\bar{z}_\varepsilon, \varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)) \rightharpoonup \varphi_0(z_0, \varphi_1^*(z_0)) & \text{weakly in } H_0^1(\Omega). \end{cases}$$

To conclude, let us write the explicit formula (2.18) for  $z = \bar{z}_\varepsilon$ ,

$$v_\varepsilon^*(\bar{z}_\varepsilon) = \varphi_\varepsilon(\bar{z}_\varepsilon, \varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon))|_\omega.$$

From (3.6), the left-hand side converges to  $v_0$  and from (3.27), the right-hand side converges to  $\varphi_0(z_0, \varphi_1^*(z_0))|_\omega$ . Then, combining with (3.18), we deduce

$$v_0 = \varphi_0(z_0, \varphi_1^*(z_0))|_\omega = v_0^*(z_0),$$

which completes the proof of Theorem 3.2.  $\blacksquare$

## 4 Homogenization of a cost functional with rapidly oscillating coefficients. Open questions

Our aim in this section is to study the same problems when the fixed symmetric matrix  $B$  is replaced by an  $\varepsilon$ -dependent symmetric matrix  $B_\varepsilon \in \mathcal{M}(\alpha_m, \alpha_M)$  with rapidly oscillating coefficients.

The approximate controllability inequality (1.5) is now replaced by

$$(4.1) \quad \|y_\varepsilon(v) - y_1\|_{B_\varepsilon, S} \leq \alpha.$$

Since in Section 2 the parameter  $\varepsilon$  was fixed, all the results therein hold true in this new framework. Our goal is to pass to the limit as  $\varepsilon \rightarrow 0$  and to see how the results of Section 3 are modified or can be generalized. We still assume that (3.2) holds as well as a  $H$ -convergence result for  $B_\varepsilon$ , namely

$$(4.2) \quad B_\varepsilon \text{ } H\text{-converges to } B_0.$$

In what follows, we will need some kind of limiting matrix  $(\chi_S B)_\#$  whose definition requires the introduction of three auxiliary functions, namely  $X_k^\varepsilon, Y_k^\varepsilon, \psi_k^\varepsilon$ , which are defined by

$$(4.3) \quad \begin{cases} X_k^\varepsilon \rightharpoonup 0 & \text{weakly in } H^1(\Omega) \\ \operatorname{div}(A_\varepsilon \nabla(-X_k^\varepsilon + x_k)) \longrightarrow \operatorname{div}(A_0 e_k) & \text{strongly in } H^{-1}(\Omega). \end{cases}$$

$$(4.4) \quad \begin{cases} Y_k^\varepsilon \rightharpoonup 0 & \text{weakly in } H^1(\Omega) \\ \operatorname{div}(B_\varepsilon \nabla(-X_k^\varepsilon + x_k)) \longrightarrow \operatorname{div}(B_0 e_k) & \text{strongly in } H^{-1}(\Omega), \end{cases}$$

$$(4.5) \quad \begin{cases} \psi_k^\varepsilon \rightharpoonup \psi_k^0 & \text{weakly in } H^1(\Omega) \\ \operatorname{div}({}^t A_\varepsilon \nabla \psi_k^\varepsilon + B_\varepsilon \nabla(-X_k^\varepsilon + x_k)) = 0 & \text{in } \Omega. \end{cases}$$

Here,  $e_k \in \mathbb{R}^N$  is the  $k^{\text{th}}$  standard basis vector and  $x_k$  denotes the function mapping  $x \in \mathbb{R}^N$  to its  $k^{\text{th}}$  coordinate.

The matrix  $(\chi_S B)_\#$  is defined by means of the following formula

$$(4.6) \quad (\chi_S B)_\# e_k = \chi_S B_0 + \lim_{\varepsilon \rightarrow 0} ({}^t A_\varepsilon \nabla \psi_k^\varepsilon - {}^t A_0 \nabla \psi_k^0) + \chi_S \lim_{\varepsilon \rightarrow 0} (B_\varepsilon (Y_k^\varepsilon - X_k^\varepsilon)).$$

The following proposition, whose proof can be found in Kesavan and Saint Jean Paulin [3], summarizes the main properties of  $(\chi_S B)_\#$ .

**Proposition 4.1** *The matrix  $(\chi_S B)_\#$  is symmetric and there exists  $\tilde{\alpha}_M > 0$  such that  $(\chi_S B)_\# \in \mathcal{M}(\alpha_m, \tilde{\alpha}_M)$ . ■*

We use this matrix  $(\chi_S B)_\#$  in order to pass to the limit in the adjoint problem (2.9), which we now rewrite in a slightly different form. Given  $h \in H^{-1}(\Omega)$ , let  $\varphi_{1\varepsilon} \in H_0^1(\Omega)$  be the unique solution of

$$(4.7) \quad \begin{cases} -\operatorname{div}(A_\varepsilon \nabla \varphi_{1\varepsilon}) = h & \text{in } \Omega \\ \varphi_{1\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases}$$

The adjoint state  $\varphi_\varepsilon = \varphi_\varepsilon(z, \varphi_{1\varepsilon})$  is defined as the unique solution of

$$(4.8) \quad \begin{cases} -\operatorname{div}({}^t A_\varepsilon \nabla \varphi_\varepsilon(z, \varphi_{1\varepsilon}) - (\chi_S B_\varepsilon) \nabla \varphi_{1\varepsilon}) = -g(z) \varphi_\varepsilon(z, \varphi_{1\varepsilon}) + \chi_S \varphi_{1\varepsilon} & \text{in } \Omega \\ \varphi_\varepsilon(z, \varphi_{1\varepsilon}) = 0 & \text{on } \partial\Omega. \end{cases}$$

Of course, Proposition 3.1 still holds true. Furthermore, if  $v_0$  denotes the weak limit of the optimal controls  $v_\varepsilon^*(\bar{z}_\varepsilon)$  (see (3.6)), then we still have (3.7), (3.8). This means that the state equation can be homogenized as in the easier case of a constant matrix  $B$ . The homogenization of the adjoint equation is not so easy and it requires the matrix  $(\chi_S B)_\#$ . Precisely, from systems (4.7), (4.8) and convergence (3.10), we derive easily that there exist functions  $\varphi_1$  and  $\bar{\varphi}_0$  such that, up to a subsequence,

$$(4.9) \quad \begin{cases} \varphi_{1\varepsilon} \rightharpoonup \varphi_1 & \text{weakly in } H_0^1(\Omega) \\ \varphi_\varepsilon(\bar{z}_\varepsilon, \varphi_{1\varepsilon}) \rightharpoonup \bar{\varphi}_0 & \text{weakly in } H_0^1(\Omega). \end{cases}$$

Of course, by  $H$ -convergence results, it is clear that  $\varphi_1$  is the unique solution of

$$(4.10) \quad \begin{cases} -\operatorname{div}(A_0 \nabla \varphi_1) = h & \text{in } \Omega \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Besides, the right-hand side of (4.8) satisfies, up to a subsequence

$$-g(\bar{z}_\varepsilon) \varphi_\varepsilon(\bar{z}_\varepsilon, \varphi_{1\varepsilon}) + \chi_S \varphi_{1\varepsilon} \rightharpoonup -g(z_0) \bar{\varphi}_0 + \chi_S \varphi_1 \quad \text{weakly in } L^2(\Omega) \text{ and strongly in } H^{-1}(\Omega).$$

Therefore, a slight generalization of Theorem 3.1 in [3] allows us to pass to the limit in (4.8) and to deduce that  $\bar{\varphi}_0$  is the solution of

$$\begin{cases} -\operatorname{div}({}^t A_0 \nabla \bar{\varphi}_0 - (\chi_S B)_\# \nabla \varphi_1) = -g(z_0) \bar{\varphi}_0 + \chi_S \varphi_1 & \text{in } \Omega \\ \bar{\varphi}_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

that is

$$(4.11) \quad \bar{\varphi}_0 = \varphi(z_0, \varphi_1),$$

where  $\varphi(z_0, \varphi_1)$  is defined as the solution of a new homogenized adjoint problem analogous to (3.15) with  $\chi_S B$  replaced by  $(\chi_S B)_\#$ .

Our next step would be to pass to the limit in the sequence  $J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon))$  where  $\varphi_{1,\varepsilon}^*$  is the minimizer of  $J_\varepsilon^{\bar{z}_\varepsilon}$ . This is the main open question of this section. Of course the desired result would be to prove that

$$(4.12) \quad \lim_{\varepsilon \rightarrow 0} J_\varepsilon^{\bar{z}_\varepsilon}(\varphi_{1,\varepsilon}^*(\bar{z}_\varepsilon)) = J_0^{z_0}(\varphi_1^*),$$

where  $\varphi_1^*$  is the minimizer of the homogenized functional

$$J_0^{z_0}(\varphi_1) = \frac{1}{2} \|\varphi_0(z_0, \varphi_1)\|_{0,\omega}^2 + \alpha \|\varphi_1\|_{(\chi_S B)_\#, S} - (\varphi_1, y_1)_{(\chi_S B)_\#, S}.$$

This implies that Theorem 3.2 would also be true in the present case. However, we have a strong doubt about the validity of (4.12). Indeed, it is not difficult to check (using [3], Theorem 3.3) that

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon^{\tilde{z}_\varepsilon}(\varphi_1) = \tilde{J}_0^{z_0}(\varphi_1) \quad \forall \varphi_1 \in H^1(S),$$

where

$$\tilde{J}_0^{z_0}(\varphi_1) = \frac{1}{2} \|\varphi_0(z_0, \varphi_1)\|_{0,\omega}^2 + \alpha \|\varphi_1\|_{(\chi_S B)_\#, S} - (\varphi_1, y_1)_{\tilde{B}, S},$$

(compare with (3.21)). Here,  $\tilde{B}$  is another kind of limiting matrix, similar to  $(\chi_S B)_\#$  which can be explicitly constructed using  $B_\varepsilon$  and the first correctors terms associated with the  $H$ -convergence sequence  $A_\varepsilon$ . ■

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