A ROTATED MULTIPLIER APPLIED TO THE CONTROLLABILITY OF WAVES, ELASTICITY AND TANGENTIAL STOKES CONTROL*

AXEL OSSES†

Abstract. A new family of multipliers with rotated direction is introduced. This technique is applied to obtain new results concerning controllability of waves, elasticity and Stokes equations. The boundary exact controllability for the wave equation and the dynamic elasticity system is reviewed generalizing the classical exit condition in the case of explicit observability constants. Approximate controllability for the Stokes system is also studied using a boundary control acting only on the tangential component of the velocity. A geometric sufficient condition of exit generalized type is deduced.

Key words. multiplier method, exact controllability, approximate controllability, unique continuation, wave equation, elasticity, Stokes system

AMS subject classifications. 93B05, 35B37, 35B60, 35Q30, 73K50

1. Introduction. In 1940 Rellich [41] introduced a multiplier technique in order to obtain direct a priori estimates in linear partial differential equations. This method was called multiplier method since it consists in multiplying the equation by the gradient of the solution following some convenient vector field and then integrating by parts in the domain. This technique was widely used in the classical PDE development [32], [11]. Later on, in the 70' and 80', this technique was used to derive inverse estimates: the asymptotic estimates in scattering theory for unbounded domains [30], [31] and the direct study of uniform stabilization in bounded domains [5], [6], [17], [21].

In 1986, Ho [10] used this multiplier technique to prove an inverse inequality for the linear wave equation implying its exact controllability. Ho arrived to a geometrical condition called *exit condition*: the control region must contain a subset of the boundary where the scalar product between the outward normal and the vector pointing from some origin towards the normal is positive. By varying the origin, a family of control boundaries satisfying the condition are found. In a square, for instance, the condition gives control boundaries consisting in four, three or two adjacent sides. Ho's result was improved [25], [24] and adapted to other systems like vibrating plates and the elasticity system [24], [19]. Afterwards, the method gave also similar results for Maxwell [18], [16] and Schrödinger [29] equations.

Several authors have used multiplier techniques for control or stabilization of mathematical models: viscoelastic or thermoelastic beams [23], [12], semi-linear wave equations [44], wave equation with mixed boundary conditions [8], [7] or in domains with corners or cracks [9], [33], [34], Euler-Bernoulli equations [13], hybrid systems in elasticity [40], networks of membranes or beams with discontinuous coefficients [24] [20], coupled Schrödinger equations [14], Korteweg-de Vries equations [42]. See also [15] for other references.

In recent years, microlocal techniques and geometric optics analysis allow to find

*This work has been partially supported by FONDAP through its Program on Mathematical–Mechanics. The author gratefully acknowledge the Chilean and French Governments by its Scientific Committee ECOS-CONICYT.

†Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 170/3 - Correo 3, Santiago, CHILE, and Centro de Modelamiento Matemático, UMR 2071 CNRS-Uchile (axosses@dim.uchile.cl).

geometrical characterization of control location and minimal control time in the exact controllability of waves. After eventual reflection, diffraction or sliding on the boundary, every optic ray issue from the observation domain has to reach the control zone. This is a necessary and sufficient condition [4] called *BLR condition* [1]. This technique has also been applied to vibrating plates [22], the elasticity system [2] and Maxwell equations [39].

Exit condition turns to be a particular case of BLR condition. But there is a certain balance: in BLR condition, control time is optimal but the observability constants are not explicit. In exit condition, time is not optimal but observability constants can be explicit and this fact is very useful in theoretical and numerical estimations. In general [3], BLR condition assumes more regularity on coefficients and boundaries than exit condition.

In this article we introduce a family of multipliers with rotated direction as a new approach in the multiplier method. More precisely, we propose to multiply the equation by the gradient of the solution following not only a radial but also a rotated direction. This takes advantage of invariances under rotations for the differential operators considered here and leads to derive a *generalized exit condition.* For instance, in two dimensions, the condition is: the control region must contain a subset of the boundary where the scalar product between the outward normal rotated in an angle and the vector pointing from some origin towards the normal is positive. A family of control boundaries is obtained by varying the origin and the angle. The minimal time of control results to be proportional to the inverse of the cosinus of the angle of the rotation.

To show the particularities of this method we have chosen some controllability problems, but the technique could be an useful tool in other areas. We revisit the exact or approximate controllability of some linear classical models in PDE's: the wave equation, the elasticity system and the Stokes system.

The paper is organized as follows:

Section2: rotated multipliers are used to derive a generalization of classical inverse inequality for the linear wave equation conserving explicit observability constants (see Theorem 2.2 and Theorem 2.3). New boundary control geometries are found (Figures 2.1 and 2.2) which are particular cases of BLR condition and satisfy a generalized exit condition.

Section3: the method is extended to the study of the exact controllability of the elasticity system. Besides the choice of a rotated direction in a natural manner, a second multiplier formula is needed in this case. The classical inverse inequality with explicit constants is also generalized (see Theorem 3.3 and Theorem 3.4). The same geometrical conditions as for the wave equation are found.

Section 4: a different application in fluid control is developed, the study of the approximate controllability of the Stokes system with a boundary control acting only on the tangential part of the velocity. So far as we know, this is an almost untreated topic (except for references [35] and [36]). A sufficient geometric conditions is found similar to that deduced for the wave equation and the elasticity system. The final results are presented in two and three dimensions, but the technique is actually not limited by dimension.

In the case of controlling all the velocity trace on an arbitrarily small non-empty open part of the boundary, approximate controllability is easily obtained by using a unique continuation property of Holmgren's type. Secondly, approximate controllability using the normal component of the velocity is studied in [27] and [28], where the result is proved in a real analytic connected domain with a simple spectrum for the Laplacian with a control acting on an arbitrary small non-empty open part of the boundary and a counterexample in a ball is given. It is amazing to observe that the normal boundary approximate controllability also holds if the boundary has at least a rectangular corner [38]. In the tangential case that we treat in this paper, we begin by following the idea of [27] introducing a spectral decomposition to characterize the unique continuation property of the time dependent system as a unique continuation property on each frequency. Then we use rotated multipliers to obtain an inverse inequality for each eigenfrequency and a sufficient geometrical condition to have the unique continuation (see Theorem 4.3). We prove that this condition is not necessary for two dimensional connected domains with analytic boundary (see Theorem 4.4). But, as far as all the other cases are concerned, there is a lack of counterexamples for which tangential boundary approximate controllability could not hold.

In summary, multipliers with rotated direction generalize the standard multipliers in a natural way. For second order hyperbolic systems, the application of this technique provides a wider class of geometric examples with explicit observability constants which are particular cases of the BLR condition. For partially controlled Petrovskii systems, the technique reveals to be useful to find results of approximate controllability.

2. Wave equation. [1]

2.1. Control problem. Let Ω be a bounded domain of $I\!R^N$ $(N \geq 2)$ with a regular[2] boundary Γ of class C^2. Let ν be the unit exterior normal to Ω. Let $T > 0$ be given and we define $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. We consider the following classical control problem. Let $\Gamma_0 \subseteq \Gamma$ and $\Sigma_0 = \Gamma_0 \times (0,T)$. Our problem consists in finding T_0 such that for each $T > T_0$ and for every $(y_0, y_1) \in L^2(\Omega)^N \times H^{-1}(\Omega)^N$, there exists $v \in L^2(\Sigma_0)^N$ in such a way that the solution of the wave equation.

$$y'' - \Delta y = 0 \quad \text{in} \quad Q \tag{2.1a}$$
$$y = v \quad \text{on} \quad \Sigma_0 \tag{2.1b}$$
$$y = 0 \quad \text{on} \quad \Sigma \setminus \Sigma_0 \tag{2.1c}$$
$$y(0) = y_0,\ y'(0) = y_1 \quad \text{in} \quad \Omega \tag{2.1d}$$

satisfies

$$y(T) = 0,\ y'(T) = 0 \quad \text{in} \quad \Omega, \tag{2.2}$$

where the symbol prime $'$ stands for derivation with respect to time.

Following HUM method [24], the solution to this problem is equivalent to studying the observability properties of the adjoint problem. For each pair of initial conditions $(\varphi_0, \varphi_1) \in H_0^1(\Omega)^N \times L^2(\Omega)^N$, let us consider the solution φ of the wave equation as follows

$$\varphi'' - \Delta\varphi = 0 \quad \text{in} \quad Q \tag{2.3a}$$
$$\varphi = 0 \quad \text{on} \quad \Sigma \tag{2.3b}$$
$$\varphi(0) = \varphi_0,\ \varphi'(0) = \varphi_1 \quad \text{in} \quad \Omega. \tag{2.3c}$$

[1] A note about the results of this section was published in [37].

[2] The results of this section are also valid if we suppose that Ω is either a bounded polygonal of $I\!R^2$ or a bounded polyhedral of $I\!R^3$ (it suffices to apply the methods of Grisvard [8]).

More precisely, exact controllability is equivalent to demonstrate that for $T > T_0$ the following inverse inequality

$$E_0 \leq C(\Omega, T) \int_{\Sigma_0} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt$$

holds, where

$$E_0 = \frac{1}{2} \left(\int_{\Omega} |\nabla \varphi_0|^2 \, dx + \int_{\Omega} |\varphi_1|^2 \, dx \right) \tag{2.4}$$

is the initial energy of system (2.3), and $C(\Omega, T)$ is a constant depending only on geometry and final time. Multiplier methods [24] can give explicit constants, but only for large Γ_0 and T. Micro-local techniques [1], [4] characterize all Γ_0 and T for which we obtain such a result, but in this case the constant is not explicit.

Using a new choice in the classical multiplier method, we will enlarge the set of geometric examples with explicit knowledge of constants.

2.2. Inverse inequality and exact controllability.

DEFINITION 2.1. *Let $A \in I\!R^{N\times N}$ be such that $A = -A^t$ (skew-symmetric). Let $d > 0$ be a positive real number and I the identity matrix in $I\!R^{N\times N}$. We define for each $x^0 \in I\!R^N$ the set*

$$\Gamma(x^0, d, A) = \{x \in \Gamma \text{ such that } (x - x^0) \cdot (dI + A)\nu > 0\}. \tag{2.5}$$

Without loss of generality we introduce the following normalizing condition

$$d^2 + ||A||_2^2 = 1, \tag{2.6}$$

where $||A||_2 = \sup\{|Ax|, \ |x| = 1\}$ and $|\cdot|$ is the Euclidean norm in $I\!R^N$. We also define

$$r(x^0, d, A) = \max\{(x - x^0) \cdot (dI + A)\nu \text{ with } x \in \Gamma(x^0, d, A)\} \tag{2.7a}$$

$$R(x^0) = \max\{|x - x^0| \text{ with } x \in \overline{\Omega}\}. \tag{2.7b}$$

THEOREM 2.2 (inverse inequality). *Given $x^0 \in I\!R^N$, $d > 0$ and a skew-symmetric matrix A normalized as in* (2.6)*, for each $T > 2d^{-1}R(x^0)$ and for each weak solution φ of* (2.3) *the following inequality holds*

$$E_0 \leq \frac{r(x^0, d, A)}{2\,(dT - 2R(x^0))} \int_0^T \int_{\Gamma(x^0,d,A)} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt. \tag{2.8}$$

THEOREM 2.3 (exact controllability). *Suppose that we can find $x^0 \in I\!R^N$, $d > 0$ and a skew-symmetric matrix A normalized as in* (2.6) *such that $\Gamma(x^0, d, A)$ is not empty and $\Gamma(x^0, d, A) \subset \Gamma_0$, then for each $T > 2d^{-1}R(x^0)$ there exists a control $v \in L^2(\Sigma_0)^N$ such that the corresponding solution of* (2.1) *satisfies the final condition* (2.2).

Remark 1 . – In the case $d = 1$ and $A = 0$ we recover classical results (see [24]).

Remark 2 . – For $N = 2$, introducing $\theta \in]-\pi/2, \pi/2[$, taking $d = \cos\theta$ and $A_{21} = A_{12} = \sin\theta$, definition (2.5) can be replaced by

$$\Gamma(x^0, \theta) = \{x \in \Gamma \text{ such that } (x - x^0) \cdot M(\theta)\nu > 0\}, \tag{2.9}$$

where $M(\theta)$ is a rotation matrix of angle θ anti-clockwise.

Remark 3 . – For $N = 3$, if $\alpha \in I\!R^3$ and $d^2 + |\alpha|^2 = 1$ we take $A_{12} = -\alpha_3$, $A_{13} = \alpha_2$ and $A_{23} = -\alpha_1$ and the definition (2.5) can be written using the exterior product in $I\!R^3$ as

$$\Gamma(x^0, d, \alpha) = \{x \in \Gamma \text{ such that } (x - x^0) \cdot (d\nu + \alpha \times \nu) > 0\}. \tag{2.10}$$

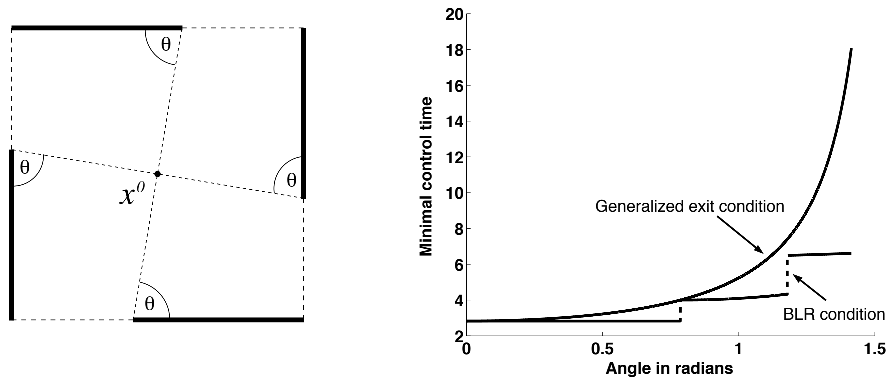

FIG. 2.1. Left. *Control region* $\Gamma(x^0, \theta)$ *(bold line) in the square* $]-1, 1[^2$ *for* x^0 *centered and* $\theta < \pi/2$. *Theorem 2.3 gives a control time* $T > 2\sqrt{2}/\cos\theta$. Right. *Comparison between BLR minimal control time and the minimal time given by Theorem 2.3 for this example.*

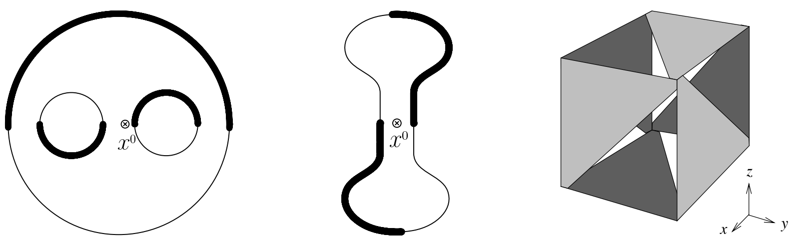

FIG. 2.2. *Other regions of control obtained by applying a rotated multiplier technique.* Left. *Control region (bold line) for the Ikawa's bowling ball and a bone-shape region for* θ *near* $\pi/2$. Right. *Control region* $\Gamma(x^0, d, \alpha)$ *(in gray) in the cube* $]-1,1[^3$ *for a centered* x^0, $d = 0.1$ *and* α *in the direction* $-(1, 1, 1)$.

2.3. Rotated multiplier: proof of Theorems 2.2 and 2.3. Let φ be a weak solution of (2.3). Multiplying (2.3) by $\nabla\varphi \cdot q$, where $q \in W^{1,\infty}(\overline{\Omega})^N$ and by φ and integrating by parts, the following classical formulas ([24], Chapter I) are deduced:

$$(\varphi'(t)\,,\, q \cdot \nabla\varphi)_{0,\Omega}\Big|_{t=0}^{T} + \frac{1}{2}\int_Q \operatorname{div} q(|\varphi'|^2 - |\nabla\varphi|^2)\,dxdt \; + \\ + \int_Q (\nabla q)\nabla\varphi \cdot \nabla\varphi\,dxdt = \frac{1}{2}\int_\Sigma q \cdot \nu \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\sigma dt, \tag{2.11}$$

and

$$(\varphi'(t)\,,\, \varphi(t))_{0,\Omega}\Big|_{t=0}^{T} = \int_Q (|\varphi'|^2 - |\nabla\varphi|^2)\,dxdt, \tag{2.12}$$

where $(\cdot\,,\,\cdot)_{0,\Omega}$ and $||\cdot||_{0,\Omega}$ denote the usual inner product and norm in $L^2(\Omega)^N$ respectively.

We consider now in (2.11) a direction of type

$$q = (dI - A)(x - x^0), \tag{2.13}$$

where $d > 0$ and $A = -A^t$ verify the normalizing condition (2.6). Note that $\operatorname{div} q = dN$, $\nabla q = dI - A$ and $(A\nabla\varphi\,,\,\nabla\varphi)_{0,\Omega} = 0$. With this choice (2.11) becomes

$$(\varphi'\,,\,q\cdot\nabla\varphi)_{0,\Omega}\Big|_{t=0}^{T} + \frac{dN}{2}\int_Q (|\varphi'|^2 - |\nabla\varphi|^2)\,dxdt + \\ + d\int_Q |\nabla\varphi|^2\,dxdt = \frac{1}{2}\int_\Sigma q\cdot\nu\left|\frac{\partial\varphi}{\partial\nu}\right|^2\,d\sigma dt.$$

If we add up this last identity to (2.12) multiplied by $d(N-1)/2$, we obtain

$$\left(\varphi'\,,\,q\cdot\nabla\varphi + \frac{d(N-1)}{2}\varphi\right)_{0,\Omega}\Bigg|_{t=0}^{T} + \frac{d}{2}\int_Q (|\varphi'|^2 + |\nabla\varphi|^2)\,dxdt = \frac{1}{2}\int_\Sigma q\cdot\nu\left|\frac{\partial\varphi}{\partial\nu}\right|^2\,d\sigma dt.$$

In virtue of the energy conservation principle it follows that

$$\left(\varphi'\,,\,q\cdot\nabla\varphi + \frac{d(N-1)}{2}\varphi\right)_{0,\Omega}\Bigg|_{t=0}^{T} + dTE_0 = \frac{1}{2}\int_\Sigma q\cdot\nu\left|\frac{\partial\varphi}{\partial\nu}\right|^2\,d\sigma dt. \tag{2.15}$$

Now, from

$$(q\cdot\nabla\varphi\,,\,\varphi)_{0,\Omega} = -\frac{1}{2}\int_\Omega \operatorname{div} q\,|\varphi|^2\,dx = -\frac{dN}{2}\,||\varphi||^2_{0,\Omega}\,,$$

we can see that

$$\begin{aligned}\left\|q\cdot\nabla\varphi + \frac{d(N-1)}{2}\varphi\right\|^2_{0,\Omega} &= ||q\cdot\nabla\varphi||^2_{0,\Omega} - \frac{d^2N(N-1)}{2}\,||\varphi||^2_{0,\Omega} + \frac{d^2(N-1)^2}{4}\,||\varphi||^2_{0,\Omega} \\ &\le ||q\cdot\nabla\varphi||^2_{0,\Omega} + \frac{d^2(1-N^2)}{4}\,||\varphi||^2_{0,\Omega} \\ &\le ||q\cdot\nabla\varphi||^2_{0,\Omega}\,.\end{aligned}$$

The above inequality implies that the first term in the left hand side of (2.15) is bounded by

$$2\left(\frac{R(x^0)}{2}\,||\varphi'||^2_{0,\Omega} + \frac{1}{2R(x^0)}\,||q\cdot\nabla\varphi||^2_{0,\Omega}\right),$$

where $R(x^0)$ was defined in (2.7). Using the normalization condition (2.6), we obtain

$$||q\cdot\nabla\varphi||_{0,\Omega} \le (d^2 + ||A||_2^2)^{1/2}R(x^0)\,||\nabla\varphi||_{0,\Omega} = R(x^0)\,||\nabla\varphi||_{0,\Omega}\,.$$

Therefore, from (2.15) we deduce that

$$-2R(x^0)E_0 + dTE_0 \le \frac{1}{2}\int_\Sigma (dI - A)(x - x^0)\cdot\nu\left|\frac{\partial\varphi}{\partial\nu}\right|^2\,d\sigma dt.$$

If we note that $(dI-A)(x-x^0)\cdot\nu = (x-x^0)\cdot(dI+A)\nu$, we have only to use definitions (2.5) of $\Gamma(x^0, d, A)$ and (2.7a) of $r(x^0, d, A)$ in order to conclude the inverse inequality (2.8) and Theorem 2.2.

The exact controllability result of Theorem 2.3 follows directly from Theorem 2.2 applying HUM method (see [24], Chapter IV). □

3. Elasticity system.

3.1. Control problem. We consider an isotropic homogeneous elastic body occupying a bounded open subset Ω of $I\!R^N$. We keep the same regularity assumptions and notations of Section 2. We introduce the boundary Γ of Ω, the control boundary Γ_0, and given a final time $T > 0$, the associated cilinders $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ and $\Sigma_0 = \Gamma_0 \times (0,T)$. We study the exact controllability of the system of linear elasticity with a control acting on a part of the boundary. More precisely, given $\mathbf{f} \in L^2(Q)^N$, a control $\mathbf{v} \in L^2(\Sigma_0)^N$ and initial conditions $\mathbf{u}^0 \in L^2(\Omega)^N$ and $\mathbf{u}^1 \in H^{-1}(\Omega)^N$, let $\mathbf{u}$ be the solution of

$$\mathbf{u}'' - \mu\Delta\mathbf{u} - (\lambda{+}\mu)\nabla \mathrm{div}\,\mathbf{u} = \mathbf{f} \quad \text{in} \quad Q \tag{3.1a}$$

$$\mathbf{u} = \mathbf{v} \quad \text{on} \quad \Sigma_0 \tag{3.1b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Sigma \setminus \Sigma_0 \tag{3.1c}$$

$$\mathbf{u}(0) = \mathbf{u}^0, \ \mathbf{u}'(0) = \mathbf{u}^1 \quad \text{in} \quad \Omega, \tag{3.1d}$$

where μ and λ are Lamé's constants with $\lambda + 2\mu > 0$. The symbol $'$ (prime) means derivation with respect to time. We take the notation $(\Delta\mathbf{u})_i = \partial^2 u_i/\partial x_j \partial x_j$ and the convention that a repeated index in some expression means implicit sum on this index.

Under the conditions described above, System (3.1) has a solution in a transposition sense. It can be shown that $\mathbf{u} \in C([0,T]; L^2(\Omega)^N)$ and also that $\mathbf{u}' \in C([0,T]; H^{-1}(\Omega)^N)$, hence the conditions (3.1d) have a sense.

We seek for a control function $\mathbf{v}$ such that

$$\mathbf{u}(T) = \mathbf{0} \quad \text{and} \quad \mathbf{u}'(T) = \mathbf{0}. \tag{3.2}$$

Now, let us consider the solution φ of the adjoint system:

$$\varphi'' - \mu\Delta\varphi - (\lambda{+}\mu)\nabla \mathrm{div}\,\varphi = \mathbf{0} \quad \text{in} \quad Q \tag{3.3a}$$

$$\varphi = \mathbf{0} \quad \text{on} \quad \Sigma \tag{3.3b}$$

$$\varphi(0) = \varphi^0, \ \varphi'(0) = \varphi^1 \quad \text{in} \quad \Omega, \tag{3.3c}$$

for each $\varphi^0 \in H_0^1(\Omega)^N$ and $\varphi^1 \in L^2(\Omega)^N$. From classical regularity results, we know that $\varphi \in C([0,T]; H_0^1(\Omega)^N)$ and $\varphi' \in C([0,T]; L^2(\Omega)^N)$.

If we define the initial energy by

$$E_0 = \frac{1}{2}\int_\Omega \left(|\varphi^1|^2 + \mu\,|\nabla\varphi^0|^2 + (\lambda{+}\mu)\,|\mathrm{div}\,\varphi^0|^2 \right) dx, \tag{3.4}$$

multiplying (3.3) by φ' we obtain the conservation of energy

$$E(t) = \frac{1}{2}\int_\Omega \left(|\varphi'(t)|^2 + \mu\,|\nabla\varphi(t)|^2 + (\lambda{+}\mu)\,|\mathrm{div}\,\varphi(t)|^2 \right) dx = E_0, \ \forall t \in [0,T]. \tag{3.5}$$

3.2. Two multiplier formulas. The following geometric property will be useful in this section and in the next section.

PROPOSITION 3.1. *Let Γ_0 be a subset of Γ with positive measure. Let $\varphi \in H^2(\Omega)^N$ be such that the trace of φ on Γ_0 is a constant vector of $I\!R^N$. Then*

$$\frac{\partial\varphi_i}{\partial x_j}\nu_k = \frac{\partial\varphi_i}{\partial x_k}\nu_j \quad \textit{on} \quad \Gamma_0 \quad \textit{for all different} \quad i,j,k \in \{1,\dots,N\}. \tag{3.6}$$

Proof. We assume that $\varphi \in C^1(\overline{\Omega})^N$ and we can deduce the general case thanks to a density argument. If the symbol $\times$ stands for the exterior product in $I\!R^N$, the condition imposed to φ on Γ_0 is equivalent to $\nabla\varphi_i \times \nu = 0$ on Γ_0 for each $i = 1, \ldots, N$, and this corresponds exactly to (3.6). $\square$

We introduce the well known tensorial product

$$e : f = e_{ij}\, f_{ij},$$

where $e = \{e_{ij}\}_{j=1}^N$ and $f = \{f_{ij}\}_{j=1}^N$ are tensorial fields defined in $\overline{\Omega}$ onto $I\!R^{N\times N}$. We suppose that this tensorial product has lower precedence than the usual matrix product in $I\!R^{N\times N}$.

Let q be a vector field defined in $\overline{\Omega}$ with $q \in W^{2,\infty}(\overline{\Omega})$.

Taking the multiplier $(\nabla\varphi)q$ for each term of the left hand side in (3.3a) the following identities are deduced.

$$\int_\Omega \nabla \mathrm{div}\,\varphi \cdot (\nabla\varphi) q\, dx = \int_\Omega \frac{\partial^2\varphi_i}{\partial x_i \partial x_j}\frac{\partial\varphi_j}{\partial x_k} q_k\, dx =$$
$$= -\int_\Omega \frac{\partial\varphi_i}{\partial x_i}\frac{\partial^2\varphi_j}{\partial x_j\partial x_k} q_k\, dx - \int_\Omega \frac{\partial\varphi_i}{\partial x_i}\frac{\partial\varphi_j}{\partial x_k}\frac{\partial q_k}{\partial x_j}\, dx + \int_\Gamma \frac{\partial\varphi_i}{\partial x_i}\frac{\partial\varphi_j}{\partial x_k}\, q_k\nu_j\, d\sigma,$$

but

$$-\int_\Omega \frac{\partial\varphi_i}{\partial x_i}\frac{\partial^2\varphi_j}{\partial x_j\partial x_k} q_k\, dx = \frac{1}{2}\int_\Omega \frac{\partial\varphi_i}{\partial x_i}\frac{\partial\varphi_j}{\partial x_j}\frac{\partial q_k}{\partial x_k}\, dx - \frac{1}{2}\int_\Gamma \frac{\partial\varphi_i}{\partial x_i}\frac{\partial\varphi_j}{\partial x_j}\, q_k\nu_k\, d\sigma$$

and, taking into account Proposition 3.1,

$$\int_\Gamma \frac{\partial\varphi_i}{\partial x_i}\frac{\partial\varphi_j}{\partial x_k}\, q_k\nu_j\, d\sigma = \int_\Gamma \frac{\partial\varphi_i}{\partial x_i}\frac{\partial\varphi_j}{\partial x_j}\, q_k\nu_k\, d\sigma,$$

hence it follows that

$$\int_\Omega \nabla \mathrm{div}\,\varphi\cdot(\nabla\varphi)q\, dx = \frac{1}{2}\int_\Omega |\mathrm{div}\,\varphi|^2\, \mathrm{div}\, q\, dx - \int_\Omega \mathrm{div}\,\varphi\, \nabla\varphi : \nabla^t q\, dx + \frac{1}{2}\int_\Gamma |\mathrm{div}\,\varphi|^2\, q\cdot\nu\, d\sigma.$$

The other term gives

$$\int_\Omega \Delta\varphi \cdot (\nabla\varphi) q\, dx = \int_\Omega \frac{\partial^2\varphi_i}{\partial x_j \partial x_j}\frac{\partial\varphi_i}{\partial x_k} q_k\, dx =$$
$$= -\int_\Omega \frac{\partial\varphi_i}{\partial x_j}\frac{\partial^2\varphi_i}{\partial x_j\partial x_k} q_k\, dx - \int_\Omega \frac{\partial\varphi_i}{\partial x_j}\frac{\partial\varphi_i}{\partial x_k}\frac{\partial q_k}{\partial x_j}\, dx + \int_\Gamma \frac{\partial\varphi_i}{\partial x_j}\frac{\partial\varphi_i}{\partial x_k}\, q_k\nu_j\, d\sigma,$$

but

$$-\int_\Omega \frac{\partial\varphi_i}{\partial x_j}\frac{\partial^2\varphi_i}{\partial x_j\partial x_k} q_k\, dx = \frac{1}{2}\int_\Omega \frac{\partial\varphi_i}{\partial x_j}\frac{\partial\varphi_i}{\partial x_j}\frac{\partial q_k}{\partial x_k}\, dx - \frac{1}{2}\int_\Gamma \frac{\partial\varphi_i}{\partial x_j}\frac{\partial\varphi_i}{\partial x_j}\, q_k\nu_k\, d\sigma$$

and from Proposition 3.1,

$$\int_\Gamma \frac{\partial\varphi_i}{\partial x_j}\frac{\partial\varphi_i}{\partial x_k}\, q_k\nu_j\, d\sigma = \int_\Gamma \frac{\partial\varphi_i}{\partial x_j}\frac{\partial\varphi_i}{\partial x_j}\, q_k\nu_k\, d\sigma,$$

then

$$\int_\Omega \Delta\varphi\cdot(\nabla\varphi)q\,dx=\frac12\int_\Omega|\nabla\varphi|^2\operatorname{div}q\,dx-\int_\Omega\nabla\varphi:\nabla\varphi\nabla q\,dx+\frac12\int_\Gamma|\nabla\varphi|^2\,q\cdot\nu\,d\sigma.$$

Always with the multiplier $(\nabla\varphi)q$ the last term in (3.3a) gives in a classical manner

$$\int_Q\varphi''\cdot(\nabla\varphi)q\,dxdt=\frac12\int_Q|\varphi'|^2\operatorname{div}q\,dxdt-\frac12\int_\Sigma|\varphi'|^2\,q\cdot\nu\,d\sigma dt+(\varphi'\,,\,(\nabla\varphi)q)_{0,\Omega}\Big|_{t=0}^{T}.$$

Using the other multiplier $(\nabla^t q)\varphi$ for each term of the left hand side in (3.3a) it follows that

$$\int_\Omega\nabla\operatorname{div}\varphi\cdot(\nabla^t q)\varphi\,dx=\int_\Omega\frac{\partial^2\varphi_i}{\partial x_i\partial x_j}\frac{\partial q_k}{\partial x_j}\varphi_k\,dx=$$

$$=-\int_\Omega\frac{\partial\varphi_i}{\partial x_i}\frac{\partial^2 q_k}{\partial x_j\partial x_j}\varphi_k\,dx-\int_\Omega\frac{\partial\varphi_i}{\partial x_i}\frac{\partial q_k}{\partial x_j}\frac{\partial\varphi_k}{\partial x_j}\,dx+\int_\Omega\frac{\partial\varphi_i}{\partial x_i}\frac{\partial q_k}{\partial x_j}\,\varphi_k\nu_j\,dx$$

$$=-\int_\Omega\operatorname{div}\varphi\Delta q\cdot\varphi\,dx-\int_\Omega\operatorname{div}\varphi\nabla\varphi:\nabla q\,dx,$$

since $\varphi=0$ on Γ. For the second term

$$\int_\Omega\Delta\varphi\cdot(\nabla^t q)\varphi\,dx=\int_\Omega\frac{\partial^2\varphi_i}{\partial x_j\partial x_j}\frac{\partial q_k}{\partial x_i}\varphi_k\,dx=$$

$$=-\int_\Omega\frac{\partial\varphi_i}{\partial x_j}\frac{\partial^2 q_k}{\partial x_i\partial x_j}\varphi_k\,dx-\int_\Omega\frac{\partial\varphi_i}{\partial x_j}\frac{\partial q_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}\,dx+\int_\Gamma\frac{\partial\varphi_i}{\partial x_j}\frac{\partial q_k}{\partial x_i}\,\varphi_k\nu_j\,d\sigma$$

$$=-\int_\Omega\frac{\partial\varphi_i}{\partial x_j}\frac{\partial^2 q_k}{\partial x_i\partial x_j}\varphi_k\,dx-\int_\Omega\nabla\varphi:(\nabla^t q)\nabla\varphi\,dx.$$

For the last term, always with the second multiplier$(\nabla^t q)\varphi$, integration by parts in Q gives

$$\int_Q\varphi''\cdot(\nabla^t q)\varphi\,dxdt=-\int_Q(\nabla q)\varphi'\cdot\varphi'\,dxdt+\big(\varphi'\,,\,(\nabla^t q)\varphi\big)_{0,\Omega}\Big|_{t=0}^{T}.$$

Combining the identities above, a multiplier formula appears for each multiplier $(\nabla\varphi)q$ and $(\nabla^t q)\varphi$.

LEMMA 3.2. *Let φ be the solution of* (3.3). *For all $q\in W^{2,\infty}(\overline{\Omega})^N$ we have*

$$\begin{aligned}(3.11)\quad &(\varphi'\,,\,(\nabla\varphi)q)_{0,\Omega}\Big|_{t=0}^{T}+\frac12\int_Q\operatorname{div}q(|\varphi'|^2-\mu\,|\nabla\varphi|^2-(\lambda{+}\mu)\,|\operatorname{div}\varphi|^2)\,dxdt\;+\\&+\mu\int_Q\nabla\varphi:\nabla\varphi\nabla q\,dxdt+(\lambda{+}\mu)\int_Q\operatorname{div}\varphi\,\nabla\varphi:\nabla^t q\;dxdt=\\&=\frac12\int_\Sigma q\cdot\nu(\mu\,|\nabla\varphi|^2+(\lambda{+}\mu)\,|\operatorname{div}\varphi|^2)\,d\sigma dt,\end{aligned}$$

$$\begin{aligned}(3.12)\quad &\big(\varphi'\,,\,(\nabla^t q)\varphi\big)_{0,\Omega}\Big|_{t=0}^{T}=\int_Q(\nabla q)\varphi'\cdot\varphi'\,dxdt-\mu\int_Q\nabla\varphi:\nabla^t q\nabla\varphi\,dxdt\;+\\&-(\lambda{+}\mu)\int_Q\operatorname{div}\varphi\,\nabla\varphi:\nabla q\,dxdt-\mu\int_Q\frac{\partial\varphi_i}{\partial x_j}\frac{\partial^2 q_k}{\partial x_i\partial x_j}\varphi_k\,dxdt\;+\\&-(\lambda{+}\mu)\int_Q\operatorname{div}\varphi\,\Delta q\cdot\varphi\,dxdt.\end{aligned}$$

3.3. Choice of the rotated direction. The classical choice $q = x - x^0$, $x^0 \in \mathbb{R}^N$, in (3.11) and (3.12) gives (see [24])

$$(3.13)\quad \left(\varphi', \nabla\varphi(x-x^0)\right)_{0,\Omega}\Big|_{t=0}^{T} + \frac{N}{2}\int_Q (|\varphi'|^2 - \mu\,|\nabla\varphi|^2 - (\lambda+\mu)\,|\operatorname{div}\varphi|^2)\,dxdt +$$
$$+\int_Q (\mu\,|\nabla\varphi|^2 + (\lambda+\mu)\,|\operatorname{div}\varphi|^2)\,dxdt = \frac{1}{2}\int_\Sigma (\mu\,|\nabla\varphi|^2 + (\lambda+\mu)\,|\operatorname{div}\varphi|^2)(x-x^0)\cdot\nu\,d\sigma dt.$$

and

$$(3.14)\qquad (\varphi', \varphi)_{0,\Omega}\Big|_{t=0}^{T} = \int_Q (|\varphi'|^2 - \mu\,|\nabla\varphi|^2 - (\lambda+\mu)\,|\operatorname{div}\varphi|^2)\,dxdt.$$

Now, a rotated direction $q = A(x-x^0)$, $A = -A^t$ (skew-symmetric) in (3.11) and (3.12) gives the following new identities:

$$(3.15)\left(\varphi', \nabla\varphi A(x-x^0)\right)_{0,\Omega}\Big|_{t=0}^{T} - (\lambda+\mu)\int_Q \operatorname{div}\varphi\nabla\varphi : A\,dxdt =$$
$$= \frac{1}{2}\int_\Sigma (\mu\,|\nabla\varphi|^2 + (\lambda+\mu)\,|\operatorname{div}\varphi|^2)A(x-x^0)\cdot\nu\,d\sigma dt.$$

$$(3.16)\qquad (\varphi', A\varphi)_{0,\Omega}\Big|_{t=0}^{T} = (\lambda+\mu)\int_Q \operatorname{div}\varphi\,\nabla\varphi : A\;dxdt.$$

Remark 4 . – For $N = 2$ or $N = 3$, the tensorial product $\nabla\varphi : A$ in (3.15) and (3.16) can be written in terms of the curl and **curl** operators respectively. Indeed, let $\mathbf{e}_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_j\,\mathbf{e}_i^t$, where $\{\mathbf{e}_i\}_{i=1}^N$ is the canonical basis in $\mathbb{R}^N$. If $N = 2$ and $A = \alpha(\mathbf{e}_{21} - \mathbf{e}_{12})$, where $\alpha \in \mathbb{R}$, then

$$\nabla\varphi : A = \alpha\left(\frac{\partial\varphi_2}{\partial x_1} - \frac{\partial\varphi_1}{\partial x_2}\right) = \alpha\operatorname{curl}\varphi.$$

In $N = 3$ and $A = \alpha_1(\mathbf{e}_{32}-\mathbf{e}_{23})+\alpha_2(\mathbf{e}_{13}-\mathbf{e}_{31})+\alpha_3(\mathbf{e}_{21}-\mathbf{e}_{12})$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, then

$$\nabla\varphi : A = \alpha_1\left(\frac{\partial\varphi_3}{\partial x_2} - \frac{\partial\varphi_2}{\partial x_3}\right) + \alpha_2\left(\frac{\partial\varphi_1}{\partial x_3} - \frac{\partial\varphi_3}{\partial x_1}\right) + \alpha_3\left(\frac{\partial\varphi_2}{\partial x_1} - \frac{\partial\varphi_1}{\partial x_2}\right) = \alpha\cdot\mathbf{curl}\,\varphi.$$

One can compare with the analogous properties of Corollary 4.12 in Section 4.

3.4. Inverse inequality and exact controllability. Given the same notations as in Section 2, if we introduce the subset $\Gamma(x^0, d, A)$ of Γ as in (2.5), and the quantities $r(x^0, d, A)$ and $R(x^0)$ as in (2.7), we obtain the following observability inequality and exact controllability result.

THEOREM 3.3 (inverse inequality). *Given $x^0 \in \mathbb{R}^N$, $d > 0$ and a skew-symmetric matrix A normalized as in* (2.6)*, if $\lambda_0^2 = \inf\{||\nabla\varphi||_{0,\Omega}^2 \,/\, ||\varphi||_{0,\Omega}^2 ; \varphi \in H_0^1(\Omega)^N\}$ and if we define*

$$(3.17)\qquad T(x^0, d, A) = \frac{2}{\sqrt{\mu}}\left(R(x^0) + \frac{||A||_2}{\lambda_0}\right),$$

then for each $T > d^{-1}T(x^0, d, A)$ and for each weak solution φ of (3.3) *the following inequality holds*

$$E_0 \le \frac{r(x^0, d, A)}{2(dT - T(x^0, d, A))} \int_0^T \int_{\Gamma(x^0,d,A)} (\mu\,|\nabla\varphi|^2 + (\lambda{+}\mu)\,|\operatorname{div}\varphi|^2)\, d\sigma dt, \tag{3.18}$$

where the initial energy E_0 was defined in (3.4).

THEOREM 3.4 (exact controllability). *Suppose that there exist $x^0 \in I\!R^N$, $d > 0$ and a skew-symmetric matrix A normalized as in* (2.6) *such that $\Gamma(x^0, d, A)$ is not empty and $\Gamma(x^0, d, A) \subset \Gamma_0$, then for each $T > d^{-1}T(x^0, d, A)$ there exists a control $v \in L^2(\Sigma_0)^N$ such that the corresponding solution of* (3.1) *satisfies the final time condition* (3.2).

Proof. Adding up the classical formulas (3.13) multiplied by b with (3.14) multiplied by $d(N-1)/2$ let us to obtain

$$\left(\varphi'\,,\,\nabla\varphi(x-x^0) + \frac{d(N-1)}{2}\varphi\right)_{0,\Omega}\Bigg|_{t=0}^{T} + \frac{d}{2}\int_Q (|\varphi'|^2 - \mu\,|\nabla\varphi|^2 - (\lambda{+}\mu)\,|\operatorname{div}\varphi|^2)\, dxdt +$$
$$+\, d\int_Q (\mu\,|\nabla\varphi|^2 + (\lambda{+}\mu)\,|\operatorname{div}\varphi|^2)\, dxdt = \frac{d}{2}\int_\Sigma (\mu\,|\nabla\varphi|^2 + (\lambda{+}\mu)\,|\operatorname{div}\varphi|^2)(x-x^0)\cdot\nu\, d\sigma dt.$$

Now, the new formula (3.16) replaced into the new identity (3.15) gives

$$(\varphi'\,,\,\nabla\varphi A(x-x^0) - A\varphi)_{0,\Omega}\Big|_{t=0}^{T} = \frac{1}{2}\int_\Sigma (\mu\,|\nabla\varphi|^2 + (\lambda{+}\mu)\,|\operatorname{div}\varphi|^2) A(x-x^0)\cdot\nu\, d\sigma dt.$$

By subtracting the last two identities we establish that

$$\begin{aligned} &(X_1(t) + X_2(t))|_{t=0}^{T} + dTE_0 = \\ &= \frac{1}{2}\int_\Sigma (\mu\,|\nabla\varphi|^2 + (\lambda{+}\mu)\,|\operatorname{div}\varphi|^2)(x - x^0)\cdot(dI + A)\nu\, d\sigma dt, \end{aligned} \tag{3.19}$$

where we have defined the quantities:

$$X_1(t) = (\varphi'\,,\,A\varphi)_{0,\Omega} \quad\text{and}\quad X_2(t) = \left(\varphi'\,,\,\nabla\varphi(dI - A)(x-x^0) + d\frac{N-1}{2}\varphi\right)_{0,\Omega}.$$

We will prove the inverse inequality (3.18). On one hand, we deduce from the Cauchy-Schwartz inequality, the inequality $ab \le a^2/(4\varepsilon) + \varepsilon b^2$ with $\varepsilon = \lambda_0\sqrt{\mu}/(2\,||A||_2)$ and from the definition of λ_0 and E_0 that

$$\left|X_1(t)|_{t=0}^{T}\right| \le 2\frac{||A||_2}{2\lambda_0\sqrt{\mu}}\,||\varphi'||^2_{0,\Omega} + 2\frac{\lambda_0\sqrt{\mu}}{2\,||A||_2}\,||A||_2^2\,\frac{1}{\lambda_0^2}\,||\nabla\varphi||^2_{0,\Omega} \le 2\frac{||A||_2}{\lambda_0\sqrt{\mu}}E_0. \tag{3.20}$$

Note that if $||A||_2 = 0$ then $X_1 = 0$. On the other hand, using the Cauchy-Schwartz inequality and the same inequality as before with $\varepsilon = \sqrt{\mu}/(2R(x^0))$ we obtain

$$\begin{aligned} |X_2(t)| \;\le\; &\frac{R(x^0)}{2\sqrt{\mu}}\,||\varphi'||^2_{0,\Omega} + \frac{\mu}{2R(x^0)\sqrt{\mu}}\Big(\left\|\nabla\varphi(dI - A)(x-x^0)\right\|^2_{0,\Omega} + \\ &+ d^2\frac{(N-1)^2}{4}\,||\varphi||^2_{0,\Omega} + d(N-1)\left(\nabla\varphi(dI - A)(x-x^0)\,,\,\varphi\right)_{0,\Omega}\Big), \end{aligned} \tag{3.21}$$

where $R(x^0) = \max_{x\in\overline{\Omega}} |x - x^0| > 0$ was introduced in (2.7). We note that

$$(\nabla\varphi(dI - A)(x-x^0)\,,\,\varphi)_{0,\Omega} = -\frac{Nd}{2}\,\|\varphi\|^2_{0,\Omega}\,,$$

hence the last two terms in (3.21) have a negative sum $-d^2(N^2-1)\,\|\varphi\|^2_{0,\Omega}/4$. One also remarks that

$$\left\|\nabla\varphi(dI - A)(x-x^0)\right\|^2_{0,\Omega} \leq (d^2 + \|A\|_2^2)R(x^0)^2\,\|\nabla\varphi\|^2_{0,\Omega} \leq R(x^0)^2\,\|\nabla\varphi\|^2_{0,\Omega}\,,$$

since it has been assumed that $d^2 + \|A\|_2^2 = 1$. Finally we obtain

$$\left|X_2(t)|_{t=0}^{T}\right| \leq 2\frac{R(x^0)}{2\sqrt{\mu}}\,\|\varphi'\|^2_{0,\Omega} + 2\frac{1}{2R(x^0)\sqrt{\mu}}R(x^0)^2\mu\,\|\nabla\varphi\|^2_{0,\Omega} \leq 2\frac{R(x^0)}{\sqrt{\mu}}E_0. \tag{3.22}$$

The inequality (3.18) follows from (3.19), (3.20), (3.22) and definitions (2.5) and (2.7).

Theorem 3.4 follows immediately from Theorem 3.3 applying the HUM method (see [24], Chapter IV). □

4. Stokes system.

4.1. Control problem. Let Ω be an open bounded subset of $\mathbb{R}^N$ ($N = 2$ or 3) with boundary Γ of class C^2. Let ν be the unit exterior normal to Ω. If $N = 2$ we refer to the the tangent vector on Γ as $\tau = (-\nu_2, \nu_1)$.

DEFINITION 4.1. *For a vector field* $\mathbf{z}$ *defined on the boundary* Γ*, we introduce the operators* γ_n *and* γ_τ *defined by*

$$\gamma_n\mathbf{z} = \mathbf{z}\cdot\nu \tag{4.1}$$

$$\gamma_\tau\mathbf{z} = \begin{cases} \mathbf{z}\cdot\tau & \text{if } N = 2\\ \nu\times\mathbf{z} & \text{if } N = 3.\end{cases} \tag{4.2}$$

Remark 5 . – It should be noticed that on Γ

$$\begin{aligned} \mathbf{z} &= \gamma_n(\mathbf{z})\nu + \gamma_\tau(\mathbf{z})\tau \quad \text{if} \quad N = 2\\ \mathbf{z} &= \gamma_n(\mathbf{z})\nu + \gamma_\tau(\mathbf{z})\times\nu \quad \text{if} \quad N = 3.\end{aligned}$$

Therefore γ_n corresponds to the normal trace for $N = 2$ and $N = 3$ and γ_τ corresponds to the tangential trace for $N = 2$. In case $N = 3$, $\gamma_\tau \times \nu$ corresponds to the tangential components.

Now, let introduce the following classical functional spaces (see [43]) with their usual topologies

$$H = \{\mathbf{v}\in L^2(\Omega)^N \mid \operatorname{div}\mathbf{v} = 0\,, \gamma_n\mathbf{v} = 0 \text{ on } \Gamma\},\ V = \{\mathbf{v}\in H_0^1(\Omega)^N \mid \operatorname{div}\mathbf{v} = 0\},$$

in the standard embedding scheme $V' \subset H' \equiv H \subset V$.

The following control problem is considered. Let $T > 0$ and let Γ_0 be a subset of Γ with positive measure. Given $\mathbf{y}_0 \in H$ and $\mathbf{f} \in L^2(0,T;V')$, for each control v scalar field if $N = 2$ or $\mathbf{v}$ vector field in $\mathbb{R}^3$ if $N = 3$, we consider (formally at the moment) the solution $(\mathbf{y}, p)$ of the following evolution Stokes problem

$$\mathbf{y}' - \Delta\mathbf{y} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega\times(0,T) \tag{4.4a}$$

$$\operatorname{div}\mathbf{y} = 0 \quad \text{in} \quad \Omega\times(0,T) \tag{4.4b}$$

(4.4c) $$\gamma_n \mathbf{y} = 0 \quad \text{on} \quad \Gamma_0 \times (0,T)$$

(4.4d) $$\gamma_\tau \mathbf{y} = \begin{cases} v & \text{if } N=2 \\ \mathbf{v} & \text{if } N=3 \end{cases} \quad \text{on} \quad \Gamma_0 \times (0,T)$$

(4.4e) $$\mathbf{y} = 0 \quad \text{on} \quad (\Gamma \setminus \Gamma_0) \times (0,T)$$

(4.4f) $$\mathbf{y}(0) = \mathbf{y}_0 \quad \text{in} \quad \Omega.$$

Here $\mathbf{y}$ is the vector velocity field in $I\!R^N$ and p is the pressure, defined up to an additive constant. The symbol $'$ (prime) means derivation with respect to time and we use the usual notation $(\Delta \mathbf{y})_i = \partial^2 y_i / \partial x_j \partial x_j$ here (where repeated index means sum). In case $N = 3$, the control $\mathbf{v}$ must satisfy the following compatibility condition

(4.5) $$\gamma_n \mathbf{v} = 0 \quad \text{on} \quad \Gamma_0 \quad \text{if} \quad N = 3.$$

Our main goal is to find geometric conditions over Γ_0 in such a way that the space $\{\mathbf{y}(T)\}$ is dense in a suitable space when the control function v or $\mathbf{v}$ varies in a space also to be determined. In other words, we seek for conditions to have the tangential boundary approximate controllability.

Without loss of generality, for the approximate control problem, the initial datum $\mathbf{y}_0$ and $\mathbf{f}$ may be taken to be zero.

4.2. Tangential boundary approximate controllability. In Theorem 4.3 we obtain approximate controllability by using a rotated direction multiplier technique, with a geometric condition similar to that required in Sections 2 and 3. In Theorem 4.4 we state the result for analytic boundaries and, in this particular case, the geometric condition of Theorem 4.3 is not necessary.

In order to describe the condition appearing in Theorem 4.3, we set down an analogous to Definition 2.1 in two and three dimensions including the case $d = 0$ and other signs of the inner product appearing in the main condition.

DEFINITION 4.2. *Let x^0 be a vector in $I\!R^N$, $d \geq 0$ and $\alpha \in I\!R$ when $N = 2$ or $\alpha \in I\!R^3$ when $N = 3$. We define the following subset of Γ*

(4.6) $$\Gamma^+(x^0,d,\alpha) = \begin{cases} \{x \in \Gamma \text{ such that } (x - x^0)\cdot(d\nu + \alpha\tau) > 0\} & \text{if } N=2 \\ \{x \in \Gamma \text{ such that } (x - x^0)\cdot(d\nu + \alpha \times \nu) > 0\} & \text{if } N=3 \end{cases}$$

and analogously we define $\Gamma^-(x^0,d,\alpha)$ and $\Gamma^0(x^0,d,\alpha)$ if the sign of the inner product in (4.6) *is negative or zero respectively.*

In order to shorten notations let us define

$$\Sigma_0 = \Gamma_0 \times (0,T).$$

THEOREM 4.3 (approximate controllability). *We suppose that there exist $x^0 \in I\!R^N$, $d \geq 0$ and $\alpha \in I\!R$ when $N = 2$ or $\alpha \in I\!R^3$ when $N = 3$ such that Γ_0 satisfies the conditions*

(4.7a) $$\Gamma_0 \supseteq \Gamma^+(x^0,d,\alpha) \quad \text{if} \quad d > 0$$

(4.7b) $$\Gamma_0 \supseteq \Gamma^+(x^0,0,\alpha) \cup \Gamma^0(x^0,0,\alpha) \quad \text{if} \quad d = 0.$$

For each $T > 0$ and for each $v \in L^2(\Sigma_0)$ when $N = 2$ or $\mathbf{v} \in L^2(\Sigma_0)^3$ satisfying (4.5) *when $N = 3$, let us consider a solution $(\mathbf{y}, \mathbf{y}^T)$ of* (4.4) *in the weak sense of Definition* 4.6. *Then for all $T > 0$ the following sets are dense in V'*

(4.8) $$\begin{array}{ll} \{\mathbf{y}^T \text{ such that } v \in L^2(\Sigma_0)\} & \text{if} \quad N = 2 \\ \{\mathbf{y}^T \text{ such that } \mathbf{v} \in L^2(\Sigma_0)^3 \text{ satisfying (4.5)}\} & \text{if} \quad N = 3. \end{array}$$

Remark 6 . – With a boundary control in $L^2(\Sigma_0)$, we will give a sense to $\mathbf{y}(T)$ only in V'. More precisely, if $(\mathbf{y}, \mathbf{y}^T)$ is a weak solution of (4.4) in the sense of Definition 4.6, we will show that $\mathbf{y} \in C^0([0,T]; V')$.

Remark 7 . – If Γ_0 verifies $\Gamma_0 \supseteq \Gamma^+(x^0, 0, \alpha)$ for a choice of the parameters x^0, $d = 0$ and α, but the condition (4.7b) is not fulfilled, then it is showed that Γ_0 can be replaced by $\Gamma_0 \cup \Gamma^-(x^0, 0, \alpha)$ and Theorem 4.3 can be applied again with a new choice $\overline{x}^0$, $\overline{d}$, $\overline{\alpha}$ of the parameters (see the example provided by Figure 4.1).

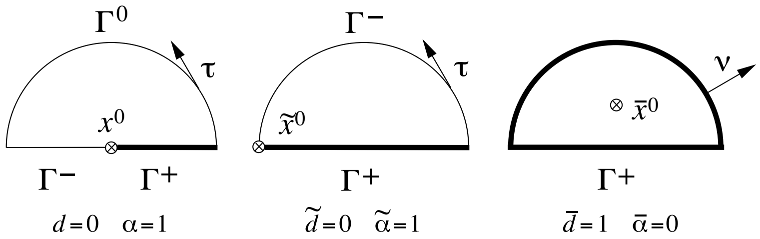

FIG. 4.1. *Example for Remark 7 in a half circle.* Left. *Take* $\Gamma_0 = \Gamma^+(x^0, 0, 1)$ *(bold line).* Γ_0 *can be extended to the whole diameter* $\Gamma_0 = \Gamma^+(x^0, 0, 1) \cup \Gamma^-(x^0, 0, 1)$. Center. *For a new* $\tilde{x}^0$, Γ_0 *can be extended to the whole boundary* $\Gamma_0 = \Gamma^+(\tilde{x}^0, 0, 1) \cup \Gamma^-(\tilde{x}^0, 0, 1)$. Right. *Another choice of parameters e.g.* $\overline{x}^0$, $\overline{d} = 1$, $\overline{\alpha} = 0$ *leads to apply Theorem 4.3 to conclude the approximate controllability.*

The following result shows that condition (4.7) of Theorem 4.3 is not necessary for analytic boundaries if $N = 2$.

THEOREM 4.4 (approximate controllability for analytic boundaries). *Assume that* $N = 2$ *and* Ω *connected. Let* $\{\Gamma^i\}_{i=1}^K$ *be the connected components of* Γ *and define* $\Gamma_0^i = \Gamma_0 \cap \Gamma^i$. *If for each* $i = 1, \ldots, K$ *we have* $\Gamma_0^i = \Gamma^i$ *or* Γ^i *analytic and* Γ_0^i *an arbitrary non empty open subset of* Γ^i, *then for each* $T > 0$ *the sets* (4.8) *are dense in* V'.

Remark 8 . – Condition (4.7) is not necessary in a generic sense: for all N and for a regular boundary Γ, if Γ_0 is an arbitrary non empty subset of Γ, we could always slightly modify Γ_0 in order to have the result of Theorem 4.3 (see [36]).

Remark 9 . – There is no counterexamples in order to decide if condition (4.7) is too restrictive, for instance, in the case of non analytic boundaries for $N = 2$.

4.3. A trace property. Here, we recall the definitions:

$$(\nabla \mathbf{z})_{ij} = \frac{\partial z_i}{\partial x_j} \quad \text{and} \quad \operatorname{curl} \mathbf{z} = \frac{\partial z_2}{\partial x_1} - \frac{\partial z_1}{\partial x_2} \quad \text{if} \quad N = 2,$$
$$\mathbf{curl}\, \mathbf{z} = \left(\frac{\partial z_3}{\partial x_2} - \frac{\partial z_2}{\partial x_3}, \frac{\partial z_1}{\partial x_3} - \frac{\partial z_3}{\partial x_1}, \frac{\partial z_2}{\partial x_1} - \frac{\partial z_1}{\partial x_2} \right) \quad \text{if} \quad N = 3.$$

PROPOSITION 4.5. *Let* $\mathbf{z} \in H^2(\Omega)^N$ *with* $\mathbf{z} = \mathbf{c}$ *on* Γ_0, *where* $\mathbf{c}$ *is a constant vector of* $I\!R^N$ *and* Γ_0 *is a subset of* Γ *of positive measure. Then, on* Γ_0, *we have*

$$\gamma_n((\nabla \mathbf{z})\nu) = \operatorname{div} \mathbf{z} \tag{4.10a}$$

$$\gamma_\tau((\nabla \mathbf{z})\nu) = \begin{cases} \operatorname{curl} \mathbf{z} & \textit{if } N = 2 \\ \mathbf{curl}\, \mathbf{z} & \textit{if } N = 3. \end{cases} \tag{4.10b}$$

Proof. We will prove the result for $\mathbf{z} \in C^1(\overline{\Omega})$. The condition $\mathbf{z} = \mathbf{c}$ on Γ_0 allow us to use Proposition 3.1, that is to say that the derivative and normal indexes can be permuted on Γ_0. Thanks to this property we easily see that

$$(\nabla \mathbf{z})\nu \cdot \nu = \frac{\partial z_i}{\partial x_j}\nu_j \nu_i = \frac{\partial z_i}{\partial x_i}\nu_j \nu_j = \operatorname{div} \mathbf{z}.$$

We recall the condensed formulas:

$$\operatorname{curl} \mathbf{z} = \varepsilon_{ij}\frac{\partial z_j}{\partial x_i} \quad \text{if} \quad N = 2 \quad \text{and} \quad (\mathbf{curl}\, \mathbf{z})_i = \varepsilon_{ijk}\frac{\partial z_k}{\partial x_j} \quad \text{if} \quad N = 3,$$

where ε_{ij} and ε_{ijk} are the signs of index permutations, i.e., if $\mathbf{e}_i$ is the i-th canonical base vector of $I\!R^N$, we have $\varepsilon_{ij} = \det(\mathbf{e}_i\, \mathbf{e}_j)$ and $\varepsilon_{ijk} = \det(\mathbf{e}_i\, \mathbf{e}_j\, \mathbf{e}_k)$. In case $N = 2$, if we remember that $\tau_i = -\varepsilon_{ik}\nu_k$ and $\varepsilon_{ik} = -\varepsilon_{ki}$, then

$$(\nabla \mathbf{z})\nu \cdot \tau = -\frac{\partial z_i}{\partial x_j}\nu_j \varepsilon_{ik}\nu_k = \varepsilon_{ki}\frac{\partial z_i}{\partial x_k}\nu_j \nu_j = \operatorname{curl} \mathbf{z}.$$

For $N = 3$, if we use the fact that $(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j\, b_k$, we obtain

$$(\nu \times (\nabla \mathbf{z})\nu)_i = \varepsilon_{ijk}\nu_j \frac{\partial z_k}{\partial x_l}\nu_l = \varepsilon_{ijk}\frac{\partial z_k}{\partial x_j}\nu_l \nu_l = (\mathbf{curl}\, \mathbf{z})_i.$$

The general case $\mathbf{z} \in H^2(\Omega)$ can be deduced by a density argument from the regular case. □

4.4. Weak formulation of the non smooth data problem (4.4). Without loss of generality, we consider the case $\mathbf{f} = \mathbf{0}$ and $\mathbf{y}_0 = \mathbf{0}$. We can always do this by choosing new variables $\mathbf{y} - \overline{\mathbf{y}}$ and $p - \overline{p}$ where $(\overline{\mathbf{y}}, \overline{p})$ is the solution of (4.4) for $v = 0$ (or $\mathbf{v} = \mathbf{0}$). We recall that ([43], Theorem 1.1, p. 254)

$$\begin{gathered}\overline{\mathbf{y}} \in L^2(0,T;V) \cap C([0,T];H),\ \overline{\mathbf{y}}' \in L^2(0,T;V') \\ \overline{p} = \overline{P}' \text{ with } \overline{P} \in C([0,T];L^2(\Omega)).\end{gathered}$$

DEFINITION 4.6. *For each $v \in L^2(\Sigma_0)$ if $N = 2$ or $\mathbf{v} \in L^2(\Sigma_0)^3$ verifying (4.5) if $N = 3$, we say that $(\mathbf{y}, \mathbf{y}^T)$ is a weak solution to Problem (4.4) if $\mathbf{y} \in L^2(0,T;H)$, $\mathbf{y}^T \in V'$ and*

$$\int_Q \mathbf{y} \cdot \mathbf{h}\, dxdt + \langle \mathbf{y}^T, \mathbf{z}^T \rangle_{V',V} = \begin{cases} -\int_{\Sigma_0} v \operatorname{curl} \mathbf{z}\, d\sigma dt & \text{if } N = 2 \\ -\int_{\Sigma_0} \mathbf{v} \cdot \mathbf{curl}\, \mathbf{z}\, d\sigma dt & \text{if } N = 3 \end{cases} \tag{4.11}$$

for each $\mathbf{h} \in L^2(0,T;H)$ and $\mathbf{z}^T \in V$, where $\mathbf{z}$ is the solution of

$$-\mathbf{z}' - \Delta \mathbf{z} + \nabla q = \mathbf{h} \quad in \quad \Omega \times (0,T) \tag{4.12a}$$

$$\operatorname{div} \mathbf{z} = 0 \quad in \quad \Omega \times (0,T) \tag{4.12b}$$

$$\mathbf{z} = \mathbf{0} \quad on \quad \Gamma \times (0,T) \tag{4.12c}$$

$$\mathbf{z}(T) = \mathbf{z}^T \quad on \quad \Omega. \tag{4.12d}$$

We know ([43], Proposition 1.2, p. 267) that Problem (4.12) has a unique solution

$$\mathbf{z} \in L^2(0,T;H^2(\Omega)^N \cap V) \text{ and } q \in L^2(0,T;H^1(\Omega)/\mathbb{R}),$$

with continuous dependence with respect to the data. From (4.12a) we also see that $\mathbf{z}' \in L^2(0,T;H)$ hence $\mathbf{z} \in C([0,T];V)$ (see [43], Chapter 1, Proposition 2.1 or [26], Chapter 3, Lemma 1.1), and then (4.12d) is meaningful.

LEMMA 4.7. *For each $v \in L^2(\Sigma_0)$ when $N = 2$ or $\mathbf{v} \in L^2(\Sigma_0)^3$ verifying* (4.5) *when $N = 3$, there exists a unique solution $(\mathbf{y}, \mathbf{y}^T)$ of* (4.11) – (4.12) *in $L^2(0,T;H) \times V'$. Moreover $\mathbf{y} \in C([0,T];V')$ and $\mathbf{y}(T) = \mathbf{y}^T$ in V'.*

Proof. The hardest part is to show that $\mathbf{y} \in C([0,T];V')$. Let $\mathcal{A}$ be the Stokes operator and let $D(\mathcal{A})$ be its domain,

$$D(\mathcal{A}) = \{\mathbf{v} \in V \text{ such that } \mathcal{A}\mathbf{v} \in H\} = H^2(\Omega)^N \cap V.$$

If $\mathbf{h} \in L^2(0,T;D(\mathcal{A}))$ and $\mathbf{z}^T = \mathbf{0}$, it can be shown that the solution of (4.12) satisfies $\mathbf{z}' \in L^2(0,T;D(\mathcal{A}))$ with a continuous dependence. Note that for each $\mathbf{h} \in \mathcal{D}((0,T);D(\mathcal{A}))$ we have in the sense of distributions

$$\langle \mathbf{y}', \mathbf{h} \rangle = -\langle \mathbf{y}, \mathbf{h}' \rangle = -\int_Q \mathbf{y} \cdot \mathbf{h}' \, dxdt$$

and then using Problem (4.11)-(4.12) with $\mathbf{z}^T = \mathbf{0}$, it follows that

$$|\langle \mathbf{y}', \mathbf{h} \rangle| = \left| \int_Q \mathbf{y} \cdot \mathbf{h}' \, dxdt \right| \leq ||v||_{L^2(\Sigma_0)} \, ||\mathrm{curl}\, \mathbf{z}'||_{L^2(\Sigma_0)} \leq C \, ||\mathbf{h}||_{L^2(0,T;D(\mathcal{A}))}$$

and then by density $\mathbf{y}' \in L^2(0,T;D(\mathcal{A})')$. Since $\mathbf{y} \in L^2(0,T;H)$ we have $\mathbf{y} \in C([0,T];X)$ (see [26]) with

$$X = [H, D(\mathcal{A})']_{\frac{1}{2}} = [D(\mathcal{A}), H]'_{\frac{1}{2}} = V',$$

where $[X_1, X_2]_{\frac{1}{2}}$ denotes the interpolated space between X_1 and X_2 (see [26], Chapter 1, Proposition 2.1). Once we know that $\mathbf{y} \in C([0,T];V')$ a density argument can be used to prove $\mathbf{y}(T) = \mathbf{y}^T$. Indeed, by taking $\mathbf{z} \in D((0,T);\mathcal{V})$, where $\mathcal{V} = \{\phi \in D(\Omega) \mid \mathrm{div}\, \phi = 0 \text{ in } \Omega\}$ in (4.11)-(4.12) we obtain the weak solution $\mathbf{y}$ satisfies (4.4a) in the sense of distributions. Then taking a test function $\mathbf{z} \in L^2(0,T;\mathcal{V})$ in this equation and observing that the Green formula

$$\int_0^T \langle \mathbf{y}', \mathbf{z} \rangle_{D(\mathcal{A})', D(\mathcal{A})} \, dt = -\int_0^T \langle \mathbf{y}, \mathbf{z}' \rangle_H \, dt + \langle \mathbf{y}(T), \mathbf{y}^T \rangle_{V',V}$$

is valid, after comparing with (4.11) we obtain that

$$\langle \mathbf{y}(T) - \mathbf{y}^T, \mathbf{z}^T \rangle_{V',V} = 0 \quad \text{for all} \quad \mathbf{z}^T \in \mathcal{V},$$

that is for all $\mathbf{z}^T \in V$ and this implies that $\mathbf{y}(T) = \mathbf{y}^T$ in V'. □

LEMMA 4.8. *If v (or $\mathbf{v}$) is a regular function, then Problem* (4.4) *is equivalent to Problem* (4.11)-(4.12).

Proof. If we multiply (4.4) by the solution of (4.12) and if we integrate by parts, we obtain

$$\int_Q \mathbf{y} \cdot \mathbf{h} \, dxdt + \int_\Omega \mathbf{y}(T) \cdot \mathbf{z}^T \, dx = -\int_{\Sigma_0} (\nabla \mathbf{z})\nu \cdot \mathbf{y} \, d\sigma dt.$$

But we know that $\mathbf{z}$ satisfies (4.12c), thus it is constant on Γ_0. By using Proposition 4.5, conditions (4.12b), (4.4c) and (4.4d), we infer that on Σ_0

$$(\nabla\mathbf{z})\nu\cdot\mathbf{y} = \begin{cases} \gamma_n((\nabla\mathbf{z})\nu)\gamma_n\mathbf{y} + \gamma_\tau((\nabla\mathbf{z})\nu)\gamma_\tau\mathbf{y} = v\,\mathrm{curl}\,\mathbf{z} & \text{if } N=2 \\ \gamma_n((\nabla\mathbf{z})\nu)\gamma_n\mathbf{y} + \gamma_\tau((\nabla\mathbf{z})\nu)\cdot\gamma_\tau\mathbf{y} = \mathbf{v}\cdot\mathbf{curl}\,\mathbf{z} & \text{if } N=3, \end{cases}$$

Conversely, in the previous Lemma we have proved that the solution of Problem (4.11)-(4.12) satisfies (4.4) except for (4.4d). But this can be easily shown in the regular case by comparison. ◻

4.5. Spectral decomposition. We want to prove that, under the hypothesis of Theorem 4.3 or Theorem 4.4, the set $\{\mathbf{y}^T\}$ of solutions to Problem (4.11)-(4.12) is dense in V' as v (or $\mathbf{v}$) varies in the control set. Given $\mathbf{z}^T \in V$, we suppose that for each $v\in L^2(\Sigma_0)$ if $N=2$ or $\mathbf{v}\in L^2(\Sigma_0)^3$ verifying (4.5) if $N=3$, we have

$$\langle \mathbf{y}^T, \mathbf{z}^T\rangle_{V',V} = 0.$$

We will show that $\mathbf{z}^T = 0$. We take $\mathbf{h}=0$ in Problem (4.11)-(4.12), therefore we have

$$\begin{cases} \int_{\Sigma_0} v\,\mathrm{curl}\,\mathbf{z}\,d\sigma dt = 0 & \text{if } N=2 \\ \int_{\Sigma_0} \mathbf{v}\cdot\mathbf{curl}\,\mathbf{z}\,d\sigma dt = 0 & \text{if } N=3 \end{cases}$$

for each $v\in L^2(\Sigma_0)$ when $N=2$ or $\mathbf{v}\in L^2(\Sigma_0)^3$ verifying (4.5) when $N=3$. Thus we have $\mathrm{curl}\,\mathbf{z}=0$ on Σ_0 when $N=2$. In case $N=3$, Proposition 4.5 implies that $\mathbf{curl}\,\mathbf{z}\cdot\nu = 0$ on Σ_0 and then we also have $\mathbf{curl}\,\mathbf{z}=\mathbf{0}$ on Σ_0.

After reversing time and changing notations by $\mathbf{z}^0 = \mathbf{z}^T$, we see that in order to prove Theorem 4.3 or Theorem 4.4 we need to show the following unique continuation property: Let $(\mathbf{z}, q)$ be a solution of

$$\mathbf{z}' - \Delta\mathbf{z} + \nabla q = \mathbf{0} \quad \text{in} \quad \Omega\times(0,T) \tag{4.13a}$$
$$\mathrm{div}\,\mathbf{z} = 0 \quad \text{in} \quad \Omega\times(0,T) \tag{4.13b}$$
$$\mathbf{z} = \mathbf{0} \quad \text{on} \quad \Gamma\times(0,T) \tag{4.13c}$$
$$\mathbf{z}(0) = \mathbf{z}^0 \quad \text{in} \quad \Omega. \tag{4.13d}$$

under the condition

$$\begin{cases} \mathrm{curl}\,\mathbf{z} = 0 & \text{if } N=2 \\ \mathbf{curl}\,\mathbf{z} = \mathbf{0} & \text{if } N=3 \end{cases} \quad \text{on} \quad \Sigma_0, \tag{4.14a}$$

then necessarily

$$\mathbf{z}=\mathbf{0} \;\text{ and }\; q = ct. \quad \text{in} \quad \Omega\times(0,T).$$

In order to study this property, we use an spectral decomposition method as in [27]. In this way, we first extend the solutions of (4.13) analytically for $t>0$ and we introduce the spectrum of the Stokes operator ordered as

$$0<\lambda_1<\lambda_2<\ldots\to\infty.$$

For each eigenvalue λ_i, $i\geq 1$, with multiplicity l_i, the associated eigenfunctions are designated by (φ_i^j, π_i^j), $j=1,\ldots,l_i$, and they form an orthonormal basis. Thus we have

$$-\Delta\varphi_i^j + \nabla\pi_i^j = \lambda_i\varphi_i^j \quad \text{in} \quad \Omega \tag{4.15a}$$
$$\mathrm{div}\,\varphi_i^j = 0 \quad \text{in} \quad \Omega \tag{4.15b}$$
$$\varphi_i^j = \mathbf{0} \quad \text{on} \quad \Gamma. \tag{4.15c}$$

If we decompose $\mathbf{z}^0 = \sum_{i\geq 1}\sum_{j=1,\ldots,l_i} a_i^j\varphi_i^j$, the solution of (4.13) can be written as follows, for every $t>0$

$$\mathbf{z} = \sum_{i\geq 1}\sum_{j=1}^{l_i} a_i^j \exp(-\lambda_i t)\varphi_i^j$$

and then, from condition (4.14), we have on Γ_0 for each $t>0$

$$\operatorname{curl}\mathbf{z} = 0 = \sum_{i\geq 1}\sum_{j=1}^{l_i} a_i^j \exp(-\lambda_i t)\operatorname{curl}\varphi_i^j.$$

Finally, instead of condition (4.14), from the strictly increasing ordering of the eigenvalues, we have the following condition for every $i\geq 1$:

$$\sum_{j=1}^{l_i} a_i^j \operatorname{curl}\varphi_i^j = 0 \quad \text{on} \quad \Sigma_0.$$

If now, for a fixed $i\geq 1$, we define

$$\varphi = \sum_{j=1}^{l_i} a_i^j\varphi_i^j \quad \text{and} \quad \nabla\pi = \sum_{j=1}^{l_i} a_i^j\nabla\pi_i^j,$$

we deduce that, for proving Theorem 4.3 or Theorem 4.4, we only have to show that the following unique continuation property on each frequency holds.

LEMMA 4.9. *Under the hypothesis of Theorem 4.3 or Theorem 4.4. If (φ,π,λ) is the solution of*

$$-\Delta\varphi + \nabla\pi = \lambda\varphi \quad \textit{in} \quad \Omega \tag{4.16a}$$

$$\operatorname{div}\varphi = 0 \quad \textit{in} \quad \Omega \tag{4.16b}$$

$$\varphi = \mathbf{0} \quad \textit{on} \quad \Gamma, \tag{4.16c}$$

with the additional condition that

$$\left\{\begin{array}{ll} \operatorname{curl}\varphi = 0 & \textit{if } N=2 \\ \mathbf{curl}\,\varphi = \mathbf{0} & \textit{if } N=3 \end{array}\right. \quad \textit{on} \quad \Gamma_0, \tag{4.17}$$

then

$$\varphi = \mathbf{0} \quad \textit{and} \quad \pi = ct. \quad \textit{in} \quad \Omega. \tag{4.18}$$

4.6. Proof of Lemma 4.9 under the hypothesis of Theorem 4.4. We take $\varphi = \mathbf{curl}\,w$. Then $-\Delta^2 w = \lambda\Delta w$ in Ω, $w=0$ on Γ^1, w constant on Γ^i, $i=2,\ldots,K$ and $\frac{\partial w}{\partial\nu} = 0$ on $\Gamma = \Gamma^1\cup\ldots\cup\Gamma^K$. From (4.16a) we also see that on Γ

$$\frac{\partial\pi}{\partial\nu} = \frac{\partial(\Delta w + \lambda w)}{\partial\tau} \quad \text{and} \quad \frac{\partial\pi}{\partial\tau} = -\frac{\partial(\Delta w + \lambda w)}{\partial\nu}. \tag{4.19}$$

Thanks to the assumed hypothesis we have $\Delta w = 0$ on Γ, then (4.19) implies that $\frac{\partial\pi}{\partial\nu} = 0$ on Γ. Since $\Delta\pi = 0$ in Ω then $\pi = ct.$ in Ω. Using again (4.19) we obtain that $\frac{\partial\Delta w}{\partial\nu} = 0$ on Γ and then

$$w = \frac{\partial w}{\partial\nu} = \Delta w = \frac{\partial\Delta w}{\partial\nu} = 0 \quad \text{on} \quad \Gamma^1,$$

and this implies that $w=0$, then $\varphi = \mathbf{0}$ by Holmgren's uniqueness property. This concludes the proof of Theorem 4.4.

4.7. Proof of Lemma 4.9 under the hypothesis of Theorem 4.3.

4.7.1. Step 1. Two multiplier formulas. In order to prove Lemma 4.9 under the hypothesis of Theorem 4.3, we have to introduce two new multiplier identities for the eigenvalue problem (4.16). We state the results in a more general manner than is required in this paper, since the results have an independent interest. We recall the notation $e : f = e_{ij} f_{ij}$ for e, f tensorial fields.

LEMMA 4.10. *Let* (φ, π) *and* (ϕ, ρ) *be solutions to the eigenvalue problem* (4.16) *for the same* λ. *Then* $\forall m \in W^{1,\infty}(\overline{\Omega})^N$ *we have the following two formulas*

$$\begin{aligned}(4.20)\quad \int_\Gamma (\nabla\varphi)\nu \cdot (\nabla\phi)\nu\ (m\cdot\nu)\, d\sigma &= \lambda \int_\Omega \varphi\cdot\phi\ \operatorname{div} m\, dx - \int_\Omega \nabla\varphi : \nabla\phi\ \operatorname{div} m\, dx\ + \\ &+ \int_\Omega \nabla\varphi : \nabla\phi(\nabla m + \nabla m^t)\, dx - \int_\Omega \pi\,\nabla\phi : \nabla m^t\, dx - \int_\Omega \rho\,\nabla\varphi : \nabla m^t\, dx,\end{aligned}$$

$$\begin{aligned}(4.21)\quad \int_\Omega \nabla\pi\cdot(\nabla m)^t\phi\, dx + \int_\Omega \nabla\rho\cdot(\nabla m)^t\varphi\, dx &= \lambda\int_\Omega \varphi\cdot(\nabla m + \nabla m^t)\phi\, dx\ + \\ &- \int_\Omega \nabla\varphi : (\nabla m + \nabla m^t)\nabla\phi\, dx + \int_\Omega \nabla\varphi : (\nabla\phi^t\nabla m + \nabla m\nabla\phi^t)\, dx.\end{aligned}$$

Proof. Briefly, we deduce the identities by using the multipliers $(\nabla\phi)m$ and $(\nabla\varphi)m$ in (4.16) in order to obtain (4.20) and the multipliers $(\nabla m)^t\phi$ and $(\nabla m)^t\varphi$ again in (4.16) to deduce (4.21). We now give the details. We multiply the pressure term in (4.16a) by $(\nabla\phi)m$ and we integrate the result by parts in Ω to obtain

$$\begin{aligned}(4.22)\quad \int_\Omega \nabla\pi\cdot(\nabla\phi)m\, dx &= \int_\Omega \frac{\partial\pi}{\partial x_i}\frac{\partial\phi_i}{\partial x_j} m_j\, dx \\ &= -\int_\Omega \pi\frac{\partial^2\phi_i}{\partial x_i\partial x_j} m_j\, dx - \int_\Omega \pi\frac{\partial\phi_i}{\partial x_j}\frac{\partial m_j}{\partial x_i}\, dx + \int_\Gamma \pi\frac{\partial\phi_i}{\partial x_j} m_j\,\nu_i\, d\sigma \\ &= -\int_\Omega \pi\nabla\phi : \nabla m^t\, dx + \int_\Gamma \pi\nu\cdot(\nabla\phi)m\, d\sigma = -\int_\Omega \pi\nabla\phi : \nabla m^t\, dx,\end{aligned}$$

since $\operatorname{div}\phi = 0$ and thanks to index change property (3.1)

$$(4.23)\qquad (\nabla\phi)m = \frac{\partial\phi_i}{\partial x_j} m_j = \frac{\partial\phi_i}{\partial x_j}\,\nu_k\,\nu_k\, m_j = \frac{\partial\phi_i}{\partial x_k}\,\nu_j\,\nu_k\, m_j = (\nabla\phi)\nu\,(m\cdot\nu)$$

and then from Proposition 4.5

$$(\nabla\phi)m\cdot\nu = (\nabla\phi)\nu\cdot\nu\,(m\cdot\nu) = \operatorname{div}\phi\,(m\cdot\nu) = 0.$$

Now, if we multiply the diffusion term in (4.16a) by $(\nabla\phi)m$ we have

$$\begin{aligned}(4.24)\quad \int_\Omega \Delta\varphi\cdot(\nabla\phi)m\, dx &= \int_\Omega \frac{\partial^2\varphi_i}{\partial x_j\partial x_j}\frac{\partial\phi_i}{\partial x_k} m_k\, dx \\ &= -\int_\Omega \frac{\partial\varphi_i}{\partial x_j}\frac{\partial^2\phi_i}{\partial x_k\partial x_j} m_k\, dx - \int_\Omega \frac{\partial\varphi_i}{\partial x_j}\frac{\partial\phi_i}{\partial x_k}\frac{\partial m_k}{\partial x_j}\, dx + \int_\Gamma \frac{\partial\varphi_i}{\partial x_j}\frac{\partial\phi_i}{\partial x_k} m_k\,\nu_j\, d\sigma \\ &= -\int_\Omega \nabla\varphi : (m\cdot\nabla)\nabla\phi\, dx - \int_\Omega \nabla\varphi : \nabla\phi\nabla m\, dx + \int_\Gamma (\nabla\varphi)\nu\cdot(\nabla\phi)m\, d\sigma,\end{aligned}$$

with the notation $(m\cdot\nabla)\nabla\phi \equiv m_k \frac{\partial}{\partial x_k}\nabla\phi$. To sum up, (4.16a) multiplied by $(\nabla\phi)m$ gives

$$\int_\Omega \nabla\varphi : (m\cdot\nabla)\nabla\phi\, dx + \int_\Omega \nabla\varphi : \nabla\phi\nabla m\, dx \;+ \\ - \int_\Gamma (\nabla\varphi)\nu\cdot(\nabla\phi)\nu\,(m\cdot\nu)\, d\sigma - \int_\Omega \pi\nabla\phi : \nabla m^t\, dx = \lambda\int_\Omega \varphi\cdot(\nabla\phi)m\, dx. \tag{4.25}$$

If we add up the identity (4.25) to the same one obtained by interchanging the roles of φ and ϕ we obtain the identity (4.20). We have only to remark that

$$\nabla\phi : \nabla\varphi\nabla m = \nabla\varphi : \nabla\phi\nabla m^t$$

to observe that

$$\begin{aligned} &\int_\Omega \varphi\cdot(\nabla\phi)m\, dx + \int_\Omega (\nabla\varphi)m\cdot\phi\, dx = -\int_\Omega \varphi\cdot\phi\ \operatorname{div} m\, dx \\ &\int_\Omega \nabla\varphi : (m\cdot\nabla)\nabla\phi\, dx + \int_\Omega (m\cdot\nabla)\nabla\varphi : \nabla\phi\, dx = \\ &\qquad = -\int_\Omega \nabla\varphi : \nabla\phi\ \operatorname{div} m\, dx + \int_\Gamma \nabla\varphi : \nabla\phi\,(m\cdot\nu)\, d\sigma \end{aligned}$$

and to transform the last boundary integral by proving the following relation on Γ

$$\nabla\varphi : \nabla\phi = \frac{\partial\varphi_i}{\partial x_j}\frac{\partial\phi_i}{\partial x_j} = \frac{\partial\varphi_i}{\partial x_j}\nu_k\,\nu_k\frac{\partial\phi_i}{\partial x_j} = \frac{\partial\varphi_i}{\partial x_k}\nu_j\,\nu_k\frac{\partial\phi_i}{\partial x_j} = (\nabla\varphi)\nu\cdot(\nabla\phi)\nu,$$

valid thanks to the Proposition 3.1.

To prove (4.21), we notice that if we multiply the diffusion term in (4.16a) by $(\nabla m)^t\phi$ and if we integrate by parts, then we obtain

$$\begin{aligned} \int_\Omega \Delta\varphi\cdot(\nabla m)^t\phi\, dx &= \int_\Omega \frac{\partial^2\varphi_i}{\partial x_k\partial x_k}\frac{\partial m_j}{\partial x_i}\phi_j\, dx \\ &= -\int_\Omega \frac{\partial\varphi_i}{\partial x_k}\frac{\partial^2 m_j}{\partial x_k\partial x_i}\phi_j\, dx - \int_\Omega \frac{\partial\varphi_i}{\partial x_k}\frac{\partial m_j}{\partial x_i}\frac{\partial\phi_j}{\partial x_k}\, dx \\ &= \int_\Omega \frac{\partial^2\varphi_i}{\partial x_k\partial x_i}\frac{\partial m_j}{\partial x_k}\phi_j\, dx + \int_\Omega \frac{\partial\varphi_i}{\partial x_k}\frac{\partial m_j}{\partial x_k}\frac{\partial\phi_j}{\partial x_i}\, dx - \int_\Omega \frac{\partial\varphi_i}{\partial x_k}\frac{\partial m_j}{\partial x_i}\frac{\partial\phi_j}{\partial x_k}\, dx \\ &= \int_\Omega \nabla\varphi : \nabla\phi^t\nabla m\, dx - \int_\Omega \nabla\varphi : \nabla m^t\nabla\phi\, dx. \end{aligned} \tag{4.26}$$

Therefore, the multiplier $(\nabla m)^t\phi$ in (4.16) gives

$$\begin{aligned} &-\int_\Omega \nabla\varphi : \nabla\phi^t\nabla m\, dx + \int_\Omega \nabla\varphi : \nabla m^t\nabla\phi\, dx\; + \\ &+\int_\Omega \nabla\pi\cdot(\nabla m)^t\phi\, dx = \lambda\int_\Omega \varphi\cdot(\nabla m)^t\phi\, dx. \end{aligned} \tag{4.27}$$

By interchanging the roles of φ and ϕ we obtain an analogous formula to (4.27). If we add up these two formulas, by using that (note the easy rules $F:GH = H:G^tF = G:FH^t$ and that $F:G = F^t:G^t$)

$$\begin{aligned} \nabla\varphi : \nabla\phi^t\nabla m &= \nabla\phi : \nabla m\nabla\varphi^t \\ \nabla\phi : \nabla m^t\nabla\varphi &= \nabla\varphi : \nabla m\nabla\phi, \end{aligned}$$

we obtain the identity (4.21). $\square$

4.7.2. Step 2. A particular case in the formulas. A particular case of formulas (4.20) and (4.21) in which we are interested is the choice $m = B(x - x^0)$, where B is a constant matrix in $I\!R^{N\times N}$ and x^0 is a constant vector in $I\!R^N$.

COROLLARY 4.11. *Let (φ, π) and (ϕ, ρ) be solutions to the eigenvalue problem (4.16) for the same λ. For each matrix $B \in I\!R^{N\times N}$ and for each $x^0 \in I\!R^N$ we have*

$$\int_\Gamma (\nabla\varphi)\nu \cdot (\nabla\phi)\nu \, ((x - x^0) \cdot B\nu)\, d\sigma = \int_\Omega \nabla\varphi : \nabla\phi(B + B^t)\, dx + \\ + \lambda \int_\Omega \varphi \cdot (B + B^t)\phi\, dx - \int_\Omega \nabla\varphi : (B + B^t)\nabla\phi\, dx. \tag{4.28}$$

$$\int_\Omega \pi\nabla\phi : B\, dx + \int_\Omega \rho\nabla\varphi : B\, dx = \\ = -\lambda \int_\Omega \varphi \cdot (B + B^t)\phi\, dx + \int_\Omega \nabla\varphi : (B + B^t)\nabla\phi\, dx. \tag{4.29}$$

Proof. We take $m = B(x - x^0)$ in (4.21). Since $\nabla m = B$ and

$$\int_\Omega \nabla\varphi : \nabla\phi^t B\, dx = \int_\Omega \frac{\partial\varphi_i}{\partial x_j}\frac{\partial\phi_k}{\partial x_i} b_{kj}\, dx = -\int_\Omega \frac{\partial\varphi_i}{\partial x_j}\phi_k \frac{\partial b_{kj}}{\partial x_i}\, dx = 0$$
$$\int_\Omega \nabla\varphi : B\nabla\phi^t\, dx = \int_\Omega \frac{\partial\varphi_i}{\partial x_j} b_{ik}\frac{\partial\phi_j}{\partial x_k}\, dx = -\int_\Omega \varphi_i \frac{\partial b_{ik}}{\partial x_j}\frac{\partial\phi_j}{\partial x_k}\, dx = 0,$$

we directly obtain (4.29), after noticing that

$$\int_\Omega \nabla\pi \cdot B^t\phi\, dx + \int_\Omega \nabla\rho \cdot B^t\varphi\, dx = -\int_\Omega \pi\nabla\phi : B\, dx - \int_\Omega \rho\nabla\varphi : B\, dx. \tag{4.30}$$

On the other hand, if we multiply (4.16a) by ϕ and if we integrate by parts, we obtain

$$\int_\Omega \nabla\varphi : \nabla\phi\, dx = \lambda \int_\Omega \varphi \cdot \phi\, dx. \tag{4.31}$$

By taking $m = B^t(x - x^0)$ in (4.20) and by using (4.30) and (4.31) we deduce (4.28). □

Remark 10 . – There are two cases where formulas (4.28) and (4.29) are simpler. The case $B = B^t$ (skew-symmetric) and the case $B = dI$, $d \in I\!R$, I the identity matrix in $I\!R^{N\times N}$. The case $B = I$ in (4.28) was also treated in [36] to study the generic simplicity of the Stokes' spectrum.

Now, we will restrict ourselves to the two and three dimensional cases. First we choose $B = A$ where A is a skew-symmetric matrix of the form

$$A = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} \text{ if } N = 2 \quad \text{and} \quad A = \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix} \text{ if } N = 3, \tag{4.32}$$

where $\alpha \in I\!R$ if $N = 2$ or $\alpha \in I\!R^3$ if $N = 3$.

From (4.29), we obtain the following orthogonality property.

COROLLARY 4.12. *Let* (φ,π) *and* (ϕ,ρ) *be solutions to the eigenvalue problem* (4.16) *for the same* λ. *Then*

$$\int_\Omega \pi \,\mathrm{curl}\,\phi\, dx + \int_\Omega \rho\,\mathrm{curl}\,\varphi\, dx = 0 \quad \textit{if} \quad N=2 \tag{4.33a}$$

$$\int_\Omega \pi \,\mathbf{curl}\,\phi\, dx + \int_\Omega \rho\,\mathbf{curl}\,\varphi\, dx = \mathbf{0} \quad \textit{if} \quad N=3. \tag{4.33b}$$

Now, we choose $B = dI + A$ in (4.28) with $d \geq 0$, I the identity matrix in $I\!R^{N\times N}$ and A always a skew-symmetric matrix in the form (4.32). The following result is obtained.

COROLLARY 4.13. *Let* (φ,π) *and* (ϕ,ρ) *solutions of* (4.16) *for the same* λ. *Then for each* $x^0 \in I\!R^N$, $d \geq 0$ *and* $\alpha \in I\!R$ *if* $N=2$ *or* $\alpha \in I\!R^3$ *if* $N=3$ *we have*

$$\int_\Gamma \mathrm{curl}\,\varphi\,\mathrm{curl}\,\phi\,(x-x^0)\cdot(d\nu+\alpha\tau)\,d\sigma = 2d\int_\Omega \nabla\varphi:\nabla\phi\,dx \ \textit{if}\ N=2 \tag{4.34a}$$

$$\int_\Gamma \mathbf{curl}\,\varphi\cdot\mathbf{curl}\,\phi\,(x-x^0)\cdot(d\nu+\alpha\times\nu)\,d\sigma = 2d\int_\Omega \nabla\varphi:\nabla\phi\,dx \ \textit{if}\ N=3. \tag{4.34b}$$

4.7.3. Step 3. Splitting the boundary. By taking $\varphi = \phi$ in (4.34), we obtain

$$\int_\Gamma |\mathrm{curl}\,\varphi|^2\ (x-x^0)\cdot(d\nu+\alpha\tau)\,d\sigma = 2d\int_\Omega |\nabla\varphi|^2\ dx \quad \text{if} \quad N=2 \tag{4.35a}$$

$$\int_\Gamma |\mathbf{curl}\,\varphi|^2\ (x-x^0)\cdot(d\nu+\alpha\times\nu)\,d\sigma = 2d\int_\Omega |\nabla\varphi|^2\ dx \quad \text{if} \quad N=3. \tag{4.35b}$$

Now, we split Γ as follows

$$\Gamma = \Gamma^+(x^0,d,\alpha)\cup\Gamma^0(x^0,d,\alpha)\cup\Gamma^-(x^0,d,\alpha). \tag{4.36}$$

Since Γ_0 satisfies the geometric condition (4.7a) of Theorem 4.3, then

$$\mathrm{curl}\,\mathbf{z} = 0 \ \text{ or } \ \mathbf{curl}\,\mathbf{z} = \mathbf{0} \quad \text{on} \quad \Gamma^+(x^0,d,\alpha). \tag{4.37}$$

If we consider decomposition (4.36) and condition (4.37) in identity (4.35), it follows that $d\int_\Omega |\nabla\varphi|^2\ dx \leq 0$ and therefore we directly obtain $\varphi = 0$ if $d>0$.

The case $d=0$ is more complicated. Using decomposition (4.36) and condition (4.37) in (4.35) with $d=0$ let us only deduce that

$$\mathrm{curl}\,\mathbf{z} = 0 \ \text{ or } \ \mathbf{curl}\,\mathbf{z} = \mathbf{0} \quad \text{on} \quad \Gamma^-(x^0,d,\alpha). \tag{4.38}$$

Nevertheless, thanks to the geometric condition (4.7b) of Theorem 4.3, we obtain that $\mathrm{curl}\,\mathbf{z} = 0$ or $\mathbf{curl}\,\mathbf{z} = \mathbf{0}$ on the whole Γ. Since $\Gamma \supseteq \Gamma^+(\overline{x}^0,1,0)$ for some $\overline{x}^0$ we deduce from the previous case ($d>0$) that $\varphi = 0$ in all Ω. This concludes the proof of Theorem 4.3. □

Remark 11 . – Condition (4.37) implies (4.38) if $d=0$. This justifies Remark 7.

Acknowledgments. The author wishes to thank Prof. E. Zuazua for fruitful discussions and suggestions.

REFERENCES

[1] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Contr. Optim., 30 (1992), pp. 1024–1465.

[2] C. Bardos, T. Masrour, and F. Tatout, *Condition nécessaire et suffisante pour la contrôlabilité exacte et la stabilisation du problème de l'élastodynamique. (Necessary and sufficient condition for the exact controllability and stabilisation of the elastodynamic problem).*, C. R. Acad. Sci., Paris, Ser. I, 320 (1995), pp. 1279–1281.

[3] N. Burq, *Contrôlabilité exacte des ondes dans des ouverts peu réguliers. (Exact controlability of waves in somewhat regular domains).*, Asymptotic Anal., 14 (1997), pp. 157–191.

[4] N. Burq and P. Gérard, *Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes*, C. R. Acad. Sci., Paris, Ser. I, 325 (1997), pp. 749–752.

[5] G. Chen, *Control and stabilization for the wave equation in a bounded domain.*, SIAM J. Control Optimization, 17 (1979), pp. 66–81.

[6] ———, *Control and stabilization for the wave equation in a bounded domain. II.*, SIAM J. Control Optimization, 19 (1981), pp. 114–122.

[7] P. Grisvard, *Contrôlabilité exacte avec conditions mêlées. (Exact controllability with mixed boundary conditions).*, C. R. Acad. Sci., Paris, Ser. I, 305 (1987), pp. 363–366.

[8] ———, *Contrôlabilité exacte dans les polygones et polyèdres. (Exact controllability in polygons and polyhedra).*, C. R. Acad. Sci., Paris, Sér. I, 304 (1987), pp. 367–370.

[9] ———, *Boundary control of cracked domains.*, in Evolution equations, control theory, and biomathematics. 3rd international workshop-conference, held at the Han-sur-Lesse Conference Center of the Belgian Ministry of Education. Proceedings. Basel: Dekker, (ISBN 0-8247-8885-0). Lect. Notes Pure Appl. Math., P. e. e. a. Clement, ed., vol. 155, 1993, pp. 235–240.

[10] L. Ho, *Observabilité frontière de l'équation des ondes. (Boundary observability of the wave equation).*, C. R. Acad. Sci., Paris, Ser. I, 302 (1986), pp. 443–446.

[11] L. Hormander, *Uniqueness theorems and estimates for normally hyperbolic partial differential equations of the second ordre*, Compt. Rend. 12 Congr. Math. Scandinaves, Lund, (1953), pp. 105–115.

[12] J. Kim, *Exact semi-internal control of an Euler-Bernoulli equation.*, SIAM J. Control Optimization, 30 (1992), pp. 1001–1023.

[13] ———, *On the energy decay of a linear thermoelastic bar and plate.*, SIAM J. Math. Anal., 23 (1992), pp. 889–899.

[14] K. Kime, *Control of matter waves in adjacent potential wells.*, Math. Methods Appl. Sci., 20 (1997), pp. 369–381.

[15] V. Komornik, *Exact controllability and stabilization. The multiplier method.*, Research in Applied Mathematics. 36. Chichester: Wiley, Paris: Masson, 1994.

[16] ———, *Stabilisation frontière des équations de Maxwell. (Boundary stabilization of Maxwell's equations).*, C. R. Acad. Sci., Paris, Ser. I, 318 (1994), pp. 535–540.

[17] J. Lagnese, *Boundary stabilization of linear elastodynamic systems.*, SIAM J. Control Optimization, 21 (1983), pp. 968–984.

[18] ———, *Exact boundary controllability of Maxwell's equations in a general region.*, SIAM J. Control Optimization, 27 (1989), pp. 374–388.

[19] ———, *Uniform asymptotic energy estimates for solutions of the equations of dynamic plane elasticity with nonlinear dissipation at the boundary.*, Nonlinear Anal., Theory Methods Appl., 16 (1991), pp. 35–54.

[20] ———, *Boundary controllability in problems of transmission for a class of second order hyperbolic systems*, ESAIM: Control, Optimization and Calculus of Variations, 2 (1997), pp. 343–357.

[21] I. Lasiecka and R. Triggiani, *Uniform exponential energy decay of wave equations in a bounded region with $L_2(0,\infty;L_2(\Gamma))$-feedback control in the Dirichlet boundary conditions.*, J. Differential Equations, 66 (1987), pp. 340–390.

[22] G. Lebeau, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl., IX. Ser. 71 (1992), pp. 267–291.

[23] G. Leugering, *On boundary feedback stabilisability of a viscoelastic beam.*, Proc. R. Soc. Edinb., Sect. A, 114 (1990), pp. 57–69.

[24] J.-L. Lions, *Contrôlabilité exacte, stabilisation et perturbation des systèmes distribués*, 1, coll. RMA, Masson, Paris, 1988.

[25] ———, *Exact controllability, stabilization and perturbations for distributed systems*, SIAM, Boston, 1986, SIAM Rev., 30 (1988), pp. 1–68.

[26] J.-L. Lions and E. Magenes, *Problèmes aux Limites non Homogènes et Applications*, vol. 1, Dunod, 1968.

[27] J.-L. Lions and E. Zuazua, *Approximate controllability of a hydro-elastic coupled system*, ESAIM: Control, Optimization and Calculus of Variations, 1 (1995), pp. 1–15.

[28] ———, *A generic uniqueness result for the Stokes system and its control theoretical consequences.*, in Partial differential equations and applications. Collected papers in honor of Carlo Pucci on the occasion of his 70th birthday. P. Marcellini et al. editors, Marcel Dekker, Lect. Notes Pure Appl. Math., vol. 177, New-York, 1996, pp. 221–235.

[29] E. Machtyngier, *Exact controllability for the Schroedinger equation.*, SIAM J. Control Optimization, 32 (1994), pp. 24–34.

[30] C. Morawetz, J. Ralston, and W. Strauss, *Decay of solutions of the wave equation outside nontrapping obstacles.*, Commun. Pure Appl. Math., 30 (1977), pp. 447–508.

[31] ———, *A correction to: Decay of solutions of the wave equation outside nontrapping obstacles.*, Commun. Pure Appl. Math., 31 (1978), p. 795.

[32] J. Nečas, *Les Méthodes Directes En Théorie Des Équations Elliptiques*, Masson, Paris, 1967.

[33] M. Niane and O. Seck, *Contrôlabilité exacte frontière de l'équation des ondes en présence de fissures par adjonction de contrôles internes au voisinage des sommets des fissures. (Exact boundary controllability of the wave equation in a domain with cracks by addition of internal controls near the crack tips).*, C. R. Acad. Sci., Paris, Ser. I, 316 (1993), pp. 695–700.

[34] S. Nicaise, *Boundary exact controllability of interface problems with singularities. I: Addition of the coefficients of singularities.*, SIAM J. Control Optimization, 34 (1996), pp. 1512–1532.

[35] J. Ortega, *Comportamiento asintótico, control y estabilización de algunos sistemas parabólicos y de placas*, PhD thesis, Universidad Complutense de Madrid, Espagne, 1997.

[36] J. Ortega and E. Zuazua, *Generic simplicity of the eigenvalues of the stokes system in two space dimensions.* Submitted to Journal of Differential and Integral Equations.

[37] A. Osses, *Une nouvelle famille des multiplicateurs et ses applications à la contrôlabilité exacte de l'équation d'ondes*, C. R. Acad. Sci., Paris, Ser. I, 326 (1998), pp. 1099–1104.

[38] A. Osses and J.-P. Puel, *Approximate controllability of a linear model in solid-fluid interaction*, ESAIM: Control, Optimization and Calculus of Variations, 4 (1999), pp. 497–513.

[39] K. Phung, *Contrôlabilité exacte et stabilization interne des équations de Maxwell. (Exact controllability and stabilization of Maxwell's equations).*, C. R. Acad. Sci., Paris, Ser. I, 323 (1996), pp. 169–174.

[40] B. Rao, *Contrôlabilité exacte frontière d'un système hybride en élasticité. (Exact boundary controllability of a hybrid system of elasticity by the HUM method).*, C. R. Acad. Sci., Paris, Ser. I, 324 (1997), pp. 889–894.

[41] F. Rellich, *Darstellung der eigenwerte von $\delta u + \lambda u$ durch ein randintegral*, Math. Z., 46 (1940), pp. 635–646.

[42] L. Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain.*, ESAIM: Control, Optimization and Calculus of Variations, 2 (1997), pp. 32–55.

[43] R. Temam, *Navier–Stokes Equations*, North–Holland, 1977.

[44] E. Zuazua, *Exponential decay for the semilinear wave equation with localized damping in unbounded domains.*, J. Math. Pures Appl., IX. Ser. 70 (1991), pp. 513–529.