Boundary Controllability of a Stationary Stokes System with Linear Convection Observed on an Interior Curve

A. Osses [1 2] and J.-P. Puel [3]

Abstract. We study the approximate controllability of the stationary Stokes system with linearized convection in a bounded domain of $\mathbb{R}^N$. The control acts on a part of the boundary and the velocity field is observed on an interior curve ($N = 2$) or surface ($N = 3$). We establish the L^2-approximate controllability under certain compatibility conditions and suitable geometrical assumptions on the curve (or surface). We build controls of minimal L^2-norm by duality. To compute the control, we propose a numerical method based on duality techniques which consists in minimizing a non quadratic functional coupled to a Stokes system. It is tested in several situations obtaining interesting numerical results.

Key Words. Stokes systems, inverse problems, approximate controllability, duality, computational methods.

---

[1] The first author gratefully acknowledges the support of the French Scientific Committee ECOS.

[2] Associate Professor, Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Santiago, Chile and Centre de Mathématiques Appliquées, Ecole Polytechnique, Palaiseau, France.

[3] Professor, Université de Versailles Saint-Quentin, Versailles, France and Centre de Mathématiques Appliquées, Ecole Polytechnique, Palaiseau, France.

1. Introduction and Main Results

1.1. Introduction. We consider a viscous incompressible fluid in a bounded cavity Ω of $\mathbb{R}^N$ ($N = 2$ or 3) with boundary Γ. The fluid is modeled by the Stokes equations with a linear convection term. We are interested in the movement of the fluid on an internal curve S ($N = 2$) or surface ($N = 3$) of the cavity Ω.

Viewed as an inverse problem, the main problem is the following one: given a velocity field $\vec{y}_1$ on S, we look for boundary conditions $\vec{v}$ on a part Γ_0 of Γ in order to retrieve a velocity field $\vec{y} = \vec{y}(\vec{v})$ in the cavity with a trace $\vec{y}(\vec{v})_{/S}$ on S close to the given field $\vec{y}_1$.

A control strategy is used in this paper. In control terminology, $\vec{v}$ is a boundary control and the goal is to obtain a small observed error $\vec{y}(\vec{v})_{/S} - \vec{y}_1$ on S.

This problem leads to studying the attainable set R of all the velocity traces $\vec{y}(\vec{v})_{/S}$ on S which can be achieved when the control $\vec{v}$ varies. For a viscous fluid, generally the attainable set can not be the whole space from which we take $\vec{y}_1$. More realistically, R is only a dense subset and then $\vec{y}_1$ can be arbitrarily approximated by $\vec{y}(\vec{v})_{/S}$.

This density property is called approximate controllability and it depends on the norm of the approximation. In this article we study the L^2-approximate controllability, which refers to approximations in L^2-norm.

Approximate controllability is equivalent to an unique continuation property. A general result of unique continuation for the generalized Stokes system has recently been proved by Fabre and Lebeau (Ref. 1). We use this result to prove the L^2-approximate controllability of our problem. In order to obtain the result, we have to impose some geometrical properties on S and compatibility conditions on $\vec{y}_1$.

This proof of L^2-approximate controllability does not provide an explicit method to find the control $\vec{v}$. In fact, $\vec{v}$ is not unique and it is necessary to impose additional criteria to characterize it.

There is an approach which was first presented by Lions (Refs. 2–4). It consists in finding a control of minimal norm, but a control that a priori ensures an error α in the approximation, i.e.

$$\min\{\ \frac{1}{2}\,\|\vec{v}\|^2_{0,\Gamma_0} \ \text{ for } \vec{v} \in L^2(\Gamma_0)^N \ \text{ s.t. } \ \left\|\vec{y}(\vec{v})_{/S} - \vec{y}_1\right\|_{0,S} \le \alpha\ \},$$

where $\|\ \|_{0,S}$ and $\|\ \|_{0,\Gamma_0}$ respectively are the L^2 norms on S and on the boundary Γ_0. The duality theory of Fenchel and Rockafellar (Refs. 5–6) shows that this problem is equivalent to minimizing another unconstrained non quadratic functional which will be precised later on.

We use the above method in section 2 to give a constructive proof of the L^2-approximate controllability and to obtain controls of minimal norm. Following Lions

(Ref. 4), we call this approach the duality method. However, one may note that this method can be extended to more general cases in which duality is not evident or simply does not exist (see Refs. 7–9).

There is also a classical approach: the optimal control point of view. The idea is to minimize over $L^2(\Gamma_0)^N$ the functional

$$H(\vec{v}) = \frac{1}{2}\,\|\vec{v}\|^2_{0,\Gamma_0} + \frac{1}{2\varepsilon}\,\left\|\vec{y}(\vec{v})_{/S} - \vec{y}_1\right\|^2_{0,S} \quad \text{with} \quad \varepsilon > 0.$$

It can be proved that this minimization problem has a unique solution characterized by two coupled Stokes systems. This is the well known optimality system.

In fact, we will not use the optimal control method in this article except for comparisons. Indeed, to clarify the theoretical framework, the relationship between the duality method and the classic optimal control theory for our problem is discussed in section 3.

Following the duality method, in section 4 we propose a discretization which leads to a numerical method to compute the control $\vec{v}$. Among many possibilities of discretization, we have chosen one which is simple, general and which allows a fast updating of the control $\vec{v}$ when the parameters α or $\vec{y}_1$ change. At the end of the paper we show several numerical examples to illustrate the characteristics of this numerical approach. We would like to mention that this method has shown interesting properties for the simpler case of the Laplace equation (see Ref. 10).

The following sections introduce the problem, explain the principal hypothesis and summarize the main theoretical results.

1.2. Control Problem. Let Ω be a bounded regular open set of $I\!R^N$. Without loss of generality, we will suppose that Ω is connected. Otherwise, the results of this article are valid for each component. We denote by Γ the boundary of Ω, and by ν the unit outward normal at a point of Γ. We suppose that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, where Γ_0 and Γ_1 are non-empty and relatively open. We represent by Γ_0 the boundary where the control $\vec{v}$ acts.

We are now given an internal regular surface $S \subset \Omega$ of dimension $N-1$. To fix the ideas, S represents the curve (N=2) or surface (N=3) where we can observe the solution (see Fig. 1).

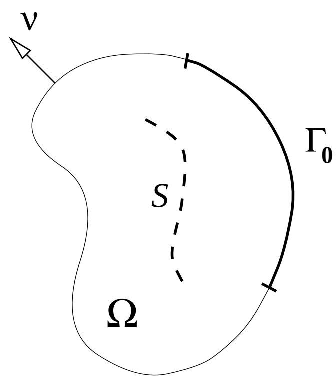

FIG 1. Domain and main notations.

We also introduce the closed subspace of $L^2(\Gamma_0)^N$

$$L^2(\Gamma_0)^N_* = \{\vec{v} \in L^2(\Gamma_0)^N \text{ such that} \int_{\Gamma_0} \vec{v}\cdot\nu \, d\sigma = 0\}. \tag{1}$$

Our aim is to study the following problem. In a sense which we will be made precise, for any $\vec{v} \in L^2(\Gamma_0)^N_*$ there exists a unique solution $\vec{y} = (y_i)$, $i = 1, \ldots, N$ –velocity– and p –pressure defined up to an additive constant– of the system

$$-\Delta y_i + \frac{\partial}{\partial x_j}(a_j\, y_i) = f_i - \frac{\partial p}{\partial x_i} \quad \text{in} \quad \Omega,\ i = 1, \ldots, N \tag{2a}$$

$$\operatorname{div} \vec{y} = 0 \quad \text{in} \quad \Omega \tag{2b}$$

$$\vec{y} = \vec{0} \quad \text{on} \quad \Gamma \setminus \Gamma_0 \tag{2c}$$

$$\vec{y} = \vec{v} \quad \text{on} \quad \Gamma_0, \tag{2d}$$

where $\vec{f} = (f_j) \in L^2(\Omega)^N$ and $\vec{a} = (a_j)$, $j = 1, \ldots, N$ are given functions. We suppose that

$$\vec{a} \in L^\infty(\Omega)^N,\ \operatorname{div}\vec{a} \in L^\infty(\Omega) \ \text{ and } \ \exists \beta < \lambda_1 \ \text{ s.t. } \ \operatorname{div}\vec{a} \geq -2\beta \ \text{ a.e. in } \Omega, \tag{3}$$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions.

Given $\vec{y}_1 \in L^2(S)^N$ and $\alpha > 0$, our problem is now to look for $\vec{v} \in L^2(\Gamma_0)^N_*$ such that

$$\left\| \vec{y}(\vec{v})_{/S} - \vec{y}_1 \right\|_{0,S} \leq \alpha, \tag{4}$$

where $\| \ \|_{0,S}$ denotes the standard L^2-norm on S. That is to say, our problem consists in studying the boundary L^2-approximate controllability of the system (2) observed on S.

1.3. General Hypothesis. Solution of (2) with $\vec{v} \in L^2(\Gamma_0)_*^N$ is not obvious. We will use a very weak formulation of problem (2) using a transposition method (Refs. 11–12) and we will obtain $\vec{y} \in L^2(\Omega)^N$, but we need to give a meaning to the trace $\vec{y}_{/S}$. We can show that if $\mathcal{U}$ is a neighborhood of Γ_0 then $\vec{y}$ is regular in $\Omega \setminus \mathcal{U}$. Thus the trace $\vec{y}_{/S}$ makes sense provided that S and Γ_0 are strictly separated. To avoid contact difficulties, we will suppose that S and the whole boundary Γ are strictly separated. Our first geometrical hypothesis on S is then the following one:

$$\overline{S} \cap \Gamma = \emptyset. \tag{5}$$

At this point a little remark can be made. Of course $\alpha = 0$ would be ideal in (4) –exact controllability–, but this case is in general impossible for regularity reasons. For example, take $\vec{f} = 0$, then (5) implies that $\vec{y}(\vec{v})$ is regular on S and $\vec{y}_1$ is only taken in $L^2(S)^N$.

We need an additional geometrical condition which relates Γ_0 and S. It should ensure that the effect of the control on Γ_0 can reach S. Let us define the set of all the arcs which connect S to Γ_0 without crossing $\overline{S}$, that is

$$\mathcal{A}(S;\Gamma_0) = \{\, a \in C([0,1];\overline{\Omega});\ a(t) \in \Omega \setminus \overline{S}\ \forall t \in]0,1[,\ a(0) \in S,\ a(1) \in \Gamma_0 \,\}.$$

The second geometrical conditions on S can be stated as follows:

$$\forall x \in S,\ \exists a_x \in \mathcal{A} \quad \text{s.t. } a_x(0) = x\ , \tag{6}$$

that is, each point on S can be connected to Γ_0 by an arc in Ω which does not cross $\overline{S}$.

Under the geometrical conditions (5) and (6), we can prove the L^2-approximate controllability property, provided that S has no closed components. Otherwise, we have to impose an extra compatibility condition on $\vec{y}_1$. With this objective, let us define the exterior and interior sets of S relative to Γ_0 as

$$S_{ext} = \{\, a(t);\ a \in \mathcal{A}(S;\Gamma_0),\ t \in]0,1[\, \} \tag{7}$$

$$S_{int} = \Omega \setminus \overline{S}_{ext}. \tag{8}$$

It is easy to see that S has closed components if and only if the set S_{int} is not empty. Let $\{S^i_{int}\}_{i=1}^K$ be the connected components of S_{int}. We will suppose that $\vec{y}_1$ satisfies the following compatibility hypothesis:

$$\int_{\partial S^i_{int}} \vec{y}_1 \cdot \nu^{(i)}\, ds = 0 \quad \text{for all} \quad i = 1, \dots, K, \tag{9}$$

where $\nu^{(i)}$ is the unit interior normal to S^i_{int}. As we will see later, if the hypothesis (9) is not satisfied for some index i, there is no physical solution satisfying (4) for α small, since there is no extension of $\vec{y}_1$ to S^i_{int} satisfying $\operatorname{div} \vec{y}_1 = 0$.

In brief, the geometrical conditions (5) and (6) on S and the compatibility condition (9) on $\vec{y}_1$ –in the case that S has closed components– will ensure the L^2-approximate controllability property for our problem.

1.4. Main Results. To present our principal theoretical results, we introduce the following closed subspace of $L^2(S)^N$

$$L^2(S)^N_* = \{\vec{y}_1 \in L^2(S)^N \text{such that (9) holds}\}. \tag{10}$$

We remark that if S has no closed components all the results of this article are valid replacing $L^2(S)^N_*$ by $L^2(S)^N$.

Theorem 1.1 Under the geometrical conditions (5) and (6), if $\vec{y}(\vec{v})$ denotes the solution of (2) corresponding to $\vec{v} \in L^2(\Gamma_0)^N_*$, then the attainable subspace

$$R_0 = \{\, \vec{y}(\vec{v})_{/S} \;\text{ s.t. }\; \vec{v} \in L^2(\Gamma_0)^N_*,\; \vec{f} = \vec{0}\,\} \tag{11}$$

is dense in $L^2(S)^N_*$.

If we look for controls of minimal norm, a more constructive L^2-approximate controllability study can be made. We define the following non quadratic functional over $L^2(S)^N_*$

$$J(\vec{\varphi}_0) = \frac{1}{2}\int_{\Gamma_0}\left|-\frac{\partial\vec{\varphi}}{\partial\nu} + q\nu\right|^2 d\sigma + \alpha\,\|\vec{\varphi}_0\|_{0,S} - \int_S (\vec{y}_1 - \overline{\vec{y}}_{/S})\cdot\vec{\varphi}_0\, ds, \tag{12}$$

where for each $\vec{\varphi}_0 \in L^2(S)^N_*$, the pair $(\vec{\varphi}, q)$ is the unique solution of

$$-\Delta\vec{\varphi}_i - a_j\frac{\partial\vec{\varphi}_i}{\partial x_j} = \vec{\varphi}_{0_i}\,\delta_{(S)} - \frac{\partial q}{\partial x_i} \quad \text{in} \quad \Omega \tag{13a}$$

$$\operatorname{div}\vec{\varphi} = 0 \quad \text{in} \quad \Omega \tag{13b}$$

$$\vec{\varphi} = \vec{0} \quad \text{on} \quad \Gamma, \tag{13c}$$

where $\delta_{(S)}$ denotes the Dirac distribution on S and with the additional condition

$$\int_{\Gamma_0} q\, d\sigma = 0. \tag{13d}$$

We will show that the traces of q and $\frac{\partial\vec{\varphi}}{\partial\nu}$ over Γ_0 make sense. Thus (12) and (13d) have meaning. The pair $(\overline{\vec{y}}, \overline{p}) \in H^1(\Omega)^N \times L^2(\Omega)/I\!R$ is the solution of

$$-\Delta\overline{y}_i + \frac{\partial}{\partial x_j}(a_j\,\overline{y}_i) = f_i - \frac{\partial\overline{p}}{\partial x_i} \quad \text{in} \quad \Omega,\; i = 1,\ldots,N \tag{14a}$$

$$\operatorname{div}\overline{\vec{y}} = 0 \quad \text{in} \quad \Omega \tag{14b}$$

$$\overline{\vec{y}} = \vec{0} \quad \text{on} \quad \Gamma. \tag{14c}$$

Theorem 1.2 Under the geometrical conditions (5) and (6), for each $\vec{y}_1 \in L^2(S)_*^N$ and $\alpha > 0$, there exists a control $\vec{v} \in L^2(\Gamma_0)_*^N$ such that the solution $\vec{y}(\vec{v})$ of problem (2) satisfies (4). Moreover the minimization problem

$$\min\left\{\frac{1}{2}\int_{\Gamma_0} |\vec{v}|^2 \, d\sigma \text{ for } \vec{v} \in L^2(\Gamma_0)_*^N \text{such that (4) holds}\right\} \tag{15}$$

has a unique solution $\widehat{\vec{v}} \in L^2(S)_*^N$. On the other hand, the problem of minimizing J in $L^2(S)_*^N$ also has a unique solution $\widehat{\vec{\varphi}}_0$. We have

$$J(\widehat{\vec{\varphi}}_0) = \min_{\vec{\varphi}_0 \in L^2(S)_*^N} J(\vec{\varphi}_0) = -\frac{1}{2}\int_{\Gamma_0} \left|\widehat{\vec{v}}\right|^2 d\sigma \tag{16}$$

and the minimal norm control $\widehat{v}$ is given explicitly as a function of $\widehat{\vec{\varphi}}_0$ by

$$\widehat{\vec{v}} = -\frac{\partial \widehat{\vec{\varphi}}}{\partial \nu} + \widehat{q}\nu \quad \text{on} \quad \Gamma_0, \tag{17}$$

where $(\widehat{\vec{\varphi}}, \widehat{q})$ is the solution of (13) associated to $\widehat{\vec{\varphi}}_0$. For the error we have a.e. for $x \in S$:

$$(\vec{y}(\widehat{\vec{v}})_{/S} - \vec{y}_1)(x) = \begin{cases} -\dfrac{\alpha}{\left\|\widehat{\vec{\varphi}}_0\right\|_{0,S}} \widehat{\vec{\varphi}}_0(x) & \text{if } \ \alpha \le \|\vec{y}_1\|_{0,S} \\ -\vec{y}_1(x) & \text{otherwise.} \end{cases} \tag{18}$$

We can also show some variants of these theorems to find controls of minimal L^∞-norm, which are in fact quasi bang-bang controls, but we will not follow this direction here (see Refs. 7 and 8). It is also possible to extend these results to some non-linear systems with $\vec{a} = \vec{a}(\vec{y})$ by using a fixed point argument (Ref. 9), but we need $\vec{a}$ to be bounded in $L^\infty(\Omega)^N$ and to satisfy for example $\operatorname{div} \vec{a} = 0$. As a result it seems to be difficult to obtain relevant non-linearities. For the time dependent problem, there are no conditions over the divergence of $\vec{a}$ and this results can be extended for a wider class of non-linearities (Ref. 13). We remark that an interesting theorem of approximate controllability for the Navier-Stokes equations with non-standard Navier boundary conditions has appeared recently in Ref. 14, but the case of Dirichlet or Neumann boundary conditions is still open.

2. L^2-Approximate Controllability.

After introducing some standard definitions and notations in section 2.1, we define in section 2.2 the very weak solutions of problem (2) and we prove some auxiliary results. In section 2.3 we state the unique continuation property of Proposition 2.3 and we use it to prove Theorem 1.1 and Theorem 1.2 of approximate controllability. We remark that in both approaches the unique continuation property is essential.

For the sake of simplicity and without a loss of generality, in all this section we suppose that $\vec{f} = \vec{0}$. Otherwise, we first make a translation using problem (14).

2.1. Some Classic Spaces and Notations. We recall the classic Sobolev spaces related to the Stokes system:

$$\mathcal{V} = \{\phi \in \mathcal{D}(\Omega)^N \ ; \ \operatorname{div} \phi = 0\}$$
$$H = \{\vec{v} \in L^2(\Omega)^N \ ; \ \operatorname{div} \vec{v} = 0 \ , \vec{v} \cdot \nu|_\Gamma = 0\}$$
$$V = \{\vec{v} \in H_0^1(\Omega)^N \ ; \ \operatorname{div} \vec{v} = 0\}.$$

We recall that $\mathcal{V}$ is dense in H and in V (see for example Ref. 15).

We introduce some notations of the tensorial calculus. Let $\mathcal{H}$ be any functional space over $I\!R$, $\vec{a}, \vec{b} \in \mathcal{H}^N$, $e, f \in \mathcal{H}^{N\times N}$. We denote by a_i the i-th component of the vector $\vec{a}$ and by e_{ij} the element of the row i and the column j of the tensor e. We use the following notations, where the repeated indexes mean summation:

$$(\Delta\vec{a})_i = \frac{\partial^2 a_i}{\partial x_j \partial x_j}, \quad (\nabla\vec{a})_{ij} = \frac{\partial a_i}{\partial x_j}, \quad (\operatorname{div} e)_i = \frac{\partial e_{ij}}{\partial x_j}, \quad e : f = e_{ij} f_{ij}.$$

We will also use the classic notations in fluid dynamics

$$((\vec{a}\cdot\nabla)\vec{b})_i = \frac{\partial b_i}{\partial x_j} a_j \quad \text{and} \quad (\vec{a}\otimes\vec{b})_{ij} = a_i b_j.$$

2.2. Very Weak Solutions. In this section we define a very weak solution of the non-smooth problem (2). We also obtain some a priori estimates for these solutions and we justify the use of some traces on S. The following definition can be essentially found in Ref. 12. We only make some modifications to include the convection term. The definition is based in the fact that the dual space of

$$H_\Omega = \{g \in H_0^1(\Omega) \ ; \ \int_\Omega g \, dx = 0\} \tag{19}$$

can be identified with $H^{-1}(\Omega)/I\!R$. We denote by $\langle \ , \ \rangle_{-1,1}$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Definition 2.1 Given $\vec{v} \in L^2(\Gamma_0)_*^N$, a pair $(\vec{y}, p) \in L^2(\Omega)^N \times H^{-1}(\Omega)/I\!R$ is said a very weak solution of (2) (with $\vec{f} = \vec{0}$) if and only if

$$\int_\Omega \vec{y}\cdot\vec{h}\, dx - \langle p, g\rangle_{-1,1} = \int_{\Gamma_0} \vec{v}\cdot(-\frac{\partial\vec{\varphi}}{\partial\nu} + q\nu)\, d\sigma \quad \forall\vec{h} \in L^2(\Omega)^N, \forall g \in H_\Omega, \tag{20}$$

where $\vec{\varphi} \in H^1(\Omega)^N$ and $q \in L^2(\Omega)$ are the solutions of the Stokes problem

$$-\Delta\vec{\varphi} - (\vec{a}\cdot\nabla)\vec{\varphi} = \vec{h} - \nabla q \quad \text{in} \quad \Omega \tag{21a}$$
$$\operatorname{div}\vec{\varphi} = g \quad \text{in} \quad \Omega \tag{21b}$$
$$\vec{\varphi} = \vec{0} \quad \text{on} \quad \Gamma. \tag{21c}$$

Formally, if we suppose that we have a solution $\vec{y} \in H^1(\Omega)^N$ and $p \in L^2(\Omega)$ of (2) (with $\vec{f} = \vec{0}$), multiplying (2a) by $\vec{\varphi}$ solution of (21) and integrating by parts, we can see that

$$-\int_\Omega \vec{y} \cdot \Delta\vec{\varphi}\, dx - \int_\Omega \vec{y} \cdot (\vec{a} \cdot \nabla)\vec{\varphi}\, dx + \int_{\Gamma_0} \vec{v} \cdot \frac{\partial \vec{\varphi}}{\partial \nu}\, d\sigma = \int_\Omega p \operatorname{div} \vec{\varphi}\, dx, \qquad (22)$$

from which we obtain (20) and (21).

In a more precise sense, we have the following results (proofs as in Ref. 12).

Lemma 2.1 For all $\vec{v} \in L^2(\Gamma_0)_*^N$ a unique solution $(\vec{y}, p) \in L^2(\Omega)^N \times H^{-1}(\Omega)/I\!R$ of (20) exists. Moreover, there exists a constant $C = C(\Omega)$ such that

$$\|\vec{y}\|_{0,\Omega} + \|p\|_{H^{-1}(\Omega)/I\!R} \leq C \, \|\vec{v}\|_{0,\Gamma_0}. \qquad (23)$$

Lemma 2.2 Let $(\vec{y}, p) \in L^2(\Omega)^N \times H^{-1}(\Omega)/I\!R$ be a solution of (20). Then $\vec{y}$ and p satisfy (2a) and (2b) in the sense of $H^{-1}(\Omega)$. Moreover, the velocity field $\vec{y}$ satisfies (2c) and (2d) for the normal component in the sense of $H^{-1/2}(\Gamma)$.

Remark 2.1: Let $(\vec{y}, p) \in L^2(\Omega)^N \times H^{-1}(\Omega)/I\!R$ be a solution of (20). From (2a), we know that in a distribution sense

$$\Delta p = \operatorname{div} \Delta \vec{y} - \operatorname{div}\operatorname{div}(\vec{y} \otimes \vec{a}) = -\operatorname{div}\operatorname{div}(\vec{y} \otimes \vec{a}) \quad \in H^{-2}(\Omega)$$

and a classic elliptic regularity results (Ref. 11, theorem 3.2, p. 138) implies that $p \in L^2_{loc}(\Omega)$. Using this result again with

$$\Delta \vec{y} = \nabla p + \operatorname{div}(\vec{y} \otimes \vec{a}) \quad \in H^{-1}_{loc}(\Omega)^N,$$

we obtain that $\vec{y} \in H^1_{loc}(\Omega)^N$. Thanks to the geometrical hypothesis (5), we have that $\vec{y}_{/S} \in H^{\frac{1}{2}}(S)^N$. Therefore, the trace of the very weak solution $\vec{y}$ of problem (2) on S makes sense as a function of $H^{\frac{1}{2}}(S)^N$.

Let us define

$$Z = \{\psi \in H^{\frac{1}{2}}(\Gamma)^N \text{ such that } \psi \cdot \nu = 0\}.$$

Definition 2.2 Given the solution $(\vec{y}, p) \in L^2(\Omega)^N \times H^{-1}(\Omega)/I\!R$ of (20), we can define the generalized tangential trace of $\vec{y}$ on Γ as the following functional over Z

$$\vec{y}_t(\psi) = \int_\Omega \vec{y} \cdot (\Delta\phi + (\vec{a} \cdot \nabla)\phi)\, dx + \langle p, \operatorname{div} \phi \rangle_{-1,1}, \qquad (24)$$

where $\phi = \phi(\psi) \in H^2(\Omega)^N \cap H^1_0(\Omega)$ is the unique solution of the problem

$$\Delta^2 \phi = \vec{0} \quad \text{in} \quad \Omega \qquad (25a)$$

$$\phi = \vec{0} \quad \text{on} \quad \Gamma \qquad (25b)$$

$$\frac{\partial \phi}{\partial \nu} = \psi \quad \text{on} \quad \Gamma. \qquad (25c)$$

Lemma 2.3 Let $(\vec{y}, p) \in L^2(\Omega)^N \times H^{-1}(\Omega)/I\!R$ a solution of (20). Then the velocity field $\vec{y}$ satisfies (2c) and (2d) for the tangential component in the sense of Z^*, i.e.

$$\vec{y}_t(\psi) = \int_{\Gamma_0} (\vec{v} - (\vec{v} \cdot \nu)\nu) \cdot \psi \, d\sigma \qquad \forall \psi \in Z. \tag{26}$$

The proofs of Lemmas 2.1, 2.2 and 2.3 make use of the following Proposition.

Proposition 2.1 If $\vec{h} \in L^2(\Omega)^N_*$, there exists a unique solution $(\vec{\varphi}, q)$ of (21) in $H^1_0(\Omega)^N \cap H^2(\Omega)^N \times H^1(\Omega)$. Moreover, there exists a constant $C > 0$ such that

$$\|\vec{\varphi}\|_{2,\Omega} + \|q\|_{1,\Omega} \leq C(\left\|\vec{h}\right\|_{0,\Omega} + \|g\|_{1,\Omega} \cdot \tag{27}$$

Proof: The proof is classical. See Refs. 15 and also Theorem 7.3 and Remark 2, pp. 668-669 in Ref. 16 for the convection term.

■

The following Proposition gives meaning to condition (25c) with (24), i.e. the fact that $\operatorname{div} \phi \in H^1_0(\Omega)$. It will be also useful in the next section.

Proposition 2.2 If $\vec{\varphi} \in H^2(\Omega)^N$ with $\vec{\varphi} = \vec{cte}$ on Γ then $\frac{\partial \vec{\varphi}}{\partial \nu} \cdot \nu = \operatorname{div} \vec{\varphi}$ on Γ.

Proof: The condition $\vec{\varphi} = \vec{cte}$ on Γ can be characterized as $\nabla \vec{\varphi}_i \wedge \nu = 0$ on Γ for all $i = 1, \ldots, N$. Then

$$\frac{\partial \vec{\varphi}_i}{\partial x_j} \nu_k = \frac{\partial \vec{\varphi}_i}{\partial x_k} \nu_j \qquad \forall i, j, k = 1, \ldots, N. \tag{28}$$

Using (28) we have

$$((\nabla \vec{\varphi})^t \nu)_i = \frac{\partial \vec{\varphi}_j}{\partial x_i} \nu_j = \frac{\partial \vec{\varphi}_j}{\partial x_j} \nu_i = \operatorname{div} \vec{\varphi} \, \nu_i,$$

and $(\nabla \vec{\varphi})^t \nu \cdot \nu = (\nabla \vec{\varphi}) \nu \cdot \nu$, from which we deduce the Proposition.

■

Remark 2.2: This proof holds if Γ is regular, but there is another proof –a longer proof that we do not present here– which shows that this condition is not essential.

2.3. Unique Continuation Property. In this section we establish the unique continuation problem of Proposition 2.3 and we use it to prove Theorems 1.1 and 1.2.

Lemma 2.4 Under the geometrical hypothesis (5), for each $\vec{\varphi}_0 \in L^2(S)_*^N$, there exists a unique solution $(\vec{\varphi}, q)$ in $H^1(\Omega)^N \times L^2(\Omega)$ of the problem

$$-\Delta\vec{\varphi} - (\vec{a}\cdot\nabla)\vec{\varphi} = \vec{\varphi}_0\delta_{(S)} - \nabla q \quad \text{in} \quad \Omega \tag{29a}$$

$$\operatorname{div}\vec{\varphi} = 0 \quad \text{in} \quad \Omega \tag{29b}$$

$$\vec{\varphi} = \vec{0} \quad \text{on} \quad \Gamma \tag{29c}$$

$$\int_{\Gamma_0} q\, d\sigma = 0. \tag{29d}$$

Moreover, $\frac{\partial\vec{\varphi}}{\partial\nu}$ and $q\nu$ are well defined on Γ_0 as elements of $H^{\frac{1}{2}}(\Gamma_0)^N$ and we also have the estimates

$$\left\| -\frac{\partial\vec{\varphi}}{\partial\nu} + q\nu \right\|_{\frac{1}{2},\Gamma_0} \leq C(\|\vec{\varphi}\|_{1,\Omega} + \|q\|_{0,\Omega}) \leq C'\, \|\vec{\varphi}_0\|_{0,S}\,. \tag{30}$$

Proof: The first part is trivial as $\vec{\varphi}_0\delta_{(S)} \in H^{-1}(\Omega)$ (see Refs. 15 or 17, Theorem 5.1, p. 80). The condition (29d) uniquely defines q and this condition has meaning as we will see now.

Let $\theta \in \mathcal{C}^\infty(\overline{\Omega})$ be such that $\theta = 1$ in an ε-neighbourhood of Γ_0 and $\theta = 0$ in an ε-neighbourhood of S with $\varepsilon > 0$. This function exists thanks to the geometrical hypothesis (5). Let us remark that

$$\begin{aligned} -\Delta(\theta\vec{\varphi}) &= -\vec{\varphi}\Delta\theta - 2(\nabla\theta\cdot\nabla)\vec{\varphi} - \theta\Delta\vec{\varphi} \\ &= -\vec{\varphi}\Delta\theta - 2(\nabla\theta\cdot\nabla)\vec{\varphi} + ((\vec{a}\cdot\nabla)\vec{\varphi})\theta - \theta\nabla q, \end{aligned}$$

thus we obtain

$$-\Delta(\theta\vec{\varphi}) + \nabla(\theta q) = \vec{h} \quad \text{in} \quad \Omega \tag{31a}$$

$$\operatorname{div}(\theta\vec{\varphi}) = g \quad \text{in} \quad \Omega \tag{31b}$$

$$\theta\vec{\varphi} = \vec{0} \quad \text{on} \quad \Gamma, \tag{31c}$$

where

$$\vec{h} = -\vec{\varphi}\Delta\theta - 2(\nabla\theta\cdot\nabla)\vec{\varphi} + ((\vec{a}\cdot\nabla)\vec{\varphi})\theta + q\nabla\theta \ \in L^2(\Omega)^N \tag{32a}$$

$$g = \vec{\varphi}\cdot\nabla\theta \ \in H_0^1(\Omega). \tag{32b}$$

Since the compatibility condition

$$\int_\Omega g\, dx = \int_\Omega \vec{\varphi}\cdot\nabla\theta\, dx = \int_\Gamma \vec{\varphi}\cdot\theta\nu\, d\sigma = 0$$

is satisfied, problem (31) has a unique solution (see Ref. 15)

$$(\theta\vec{\varphi}, \theta q) \in H^2(\Omega)^N \cap H_0^1(\Omega)^N \times H^1(\Omega). \tag{33}$$

We can see that on Γ_0

$$\frac{\partial(\theta\vec{\varphi})}{\partial\nu} = \frac{\partial\vec{\varphi}}{\partial\nu} \in H^{\frac{1}{2}}(\Gamma_0)^N \quad \text{and} \quad \theta q\nu = q\nu \in H^{\frac{1}{2}}(\Gamma_0)^N.$$

Then, using trace theorems, estimates given by the regularity results for the solution of (31) (Ref. 15) and the expressions (32), we can deduce that

$$\begin{aligned}\left\| -\frac{\partial\vec{\varphi}}{\partial\nu} + q\nu \right\|_{\frac{1}{2},\Gamma_0} &\leq C(\|\theta\vec{\varphi}\|_{2,\Omega} + \|\theta q\|_{1,\Omega}) \\ &\leq C(\left\|\vec{h}\right\|_{0,\Omega} + \|g\|_{1,\Omega}) \\ &\leq C(\|\vec{\varphi}\|_{1,\Omega} + \|q\|_{0,\Omega}).\end{aligned}$$

The last inequality in (30) is standard and follows from $\left\|\vec{\varphi}_0\,\delta_{(S)}\right\|_{-1,\Omega} \leq \|\vec{\varphi}_0\|_{0,S}$. ■

We can easily prove the following geometrical properties about the partition of Ω in the sets S_{ext} and S_{int} defined in (7) and (8).

Lemma 2.5 Under the geometrical conditions (5) and (6) we have the following properties:

(i) S_{ext} is a non empty open set and we have : $\overline{S} \subset \partial S_{ext}$, $\Gamma_0 \subset \partial S_{ext}$. Moreover, if S_{ext} has several connected components, the boundary of each of them contains a non-empty open subset of Γ_0.

(ii) S_{int} may be empty but if it is not, $\partial S_{int} \subset \overline{S} \cup \Gamma$.

Proposition 2.3 Under the geometrical hypothesis (5) and (6), if $\vec{\varphi}_0$ belongs to $L^2(S)^N_*$ and $(\vec{\varphi}, q)$ is the solution of (29), then

$$-\frac{\partial\vec{\varphi}}{\partial\nu} + q\nu = 0 \text{ on } \Gamma_0 \quad \text{implies} \quad \vec{\varphi} = 0 \text{ and } q = 0 \quad \text{in all} \quad \Omega. \tag{34}$$

Proof: The solution $(\vec{\varphi}, q)$ of (29) is regular near Γ_0 as we have seen in Lemma 2.4. Applying Proposition 2.2 we can see that the conditions

$$\frac{\partial\vec{\varphi}}{\partial\nu} + q\nu = \vec{0} \text{ on } \Gamma_0, \; \vec{\varphi} = \vec{0} \text{ on } \Gamma_0 \text{ and } \operatorname{div}\vec{\varphi} = 0 \text{ in } \Omega$$

imply

$$\vec{\varphi} = \frac{\partial\vec{\varphi}}{\partial\nu} = \vec{0} \text{ on } \Gamma_0 \text{ and } q = 0 \text{ on } \Gamma_0. \tag{35}$$

Moreover, if $\{S^i_{ext}\}_{i=1}^L$ are the connected components of S_{ext}, the solution of (29) satisfies for each $i = 1, \dots, L$

$$\vec{\varphi} \in H^2(S^i_{ext})^N \text{ and } q \in H^1(S^i_{ext}). \tag{36}$$

From Lemma 2.5(i) we also see that

$$\forall i = 1 \dots, L \quad S^i_{ext} \text{ contains a non-empty open subset of } \Gamma_0. \tag{37}$$

Conditions (35), (36) and (37) let us to use the results of unique continuation by Fabre and Lebeau (see Ref. 1) in each connected component of S_{ext} to deduce that

$$\forall i = 1, \dots, L \quad \vec{\varphi} = \vec{0} \text{ and } q = 0 \text{ in } S^i_{ext}.$$

In particular, Lemma 2.5(i) implies $\vec{\varphi} = \vec{0}$ on S and then from Lemma 2.5(ii) we deduce that $\vec{\varphi} = 0$ on ∂S_{int}. Let $\{S^j_{int}\}_{j=1}^K$ be the connected components of S_{int}. Then we have for each $j = 1, \dots, K$ an homogeneous Dirichlet Stokes problem in S^j_{int}. Then

$$\forall j = 1 \dots, K \quad \vec{\varphi} = \vec{0} \text{ and } q = c_i \text{ in } S^j_{int}, \tag{38}$$

where c_j is a constant on S^j_{int}. But $\vec{\varphi} = \vec{0}$ in all Ω and then

$$\nabla q = \vec{\varphi}_0 \delta_{(S)} \text{ in } \Omega. \tag{39}$$

Multiplying (39) by a function $\phi \in H^1_0(\Omega)^N$ and integrating by parts in Ω we can deduce that

$$-\sum_{i=1}^K c_i \int_{S^j_{int}} \operatorname{div} \phi \, ds = \int_S \vec{\varphi}_0 \cdot \phi \, ds$$

and taking ϕ such that $\phi_{/\partial S^k_{int}} = \delta_{kj} \nu^{(j)}$, where $\nu^{(j)}$ is the unit interior normal to S^j_{int}, after integration by parts we obtain

$$c_j = \frac{1}{\left| S^j_{int} \right|} \int_{\partial S^j_{int}} \vec{\varphi}_0 \cdot \nu^{(j)} \, ds = 0. \tag{40}$$

We have used the fact that $\vec{\varphi}_0$ belongs to $L^2(S)^N_*$. By definition $\Omega = S_{ext} \cup S_{int} \cup S$ and then Proposition 2.3 is proved.

■

Proof of Theorem 1.1. Classic Approach. Let $\vec{\varphi}_0 \in L^2(S)_*^N$ and $\vec{v} \in L^2(\Gamma_0)_*^N$ be given and let $\vec{y}(\vec{v})$ be the very weak solution of (2) in the sense of Definition 2.1. Let us suppose that

$$\int_S \vec{y}(\vec{v}) \cdot \vec{\varphi}_0 \, ds = 0 \quad \forall \vec{v} \in L^2(\Gamma_0)_*^N.$$

We will show that this implies $\vec{\varphi}_0 = \vec{0}$ on S. Let $(\vec{\varphi}, q)$ be the solution of (29) associated to $\vec{\varphi}_0$. Multiplying (2a) by $\vec{\varphi}$ and integrating by parts we obtain

$$\int_S \vec{y}(\vec{v}) \cdot \vec{\varphi}_0 \, ds = \int_{\Gamma_0} \vec{v} \cdot (-\frac{\partial \vec{\varphi}}{\partial \nu} + q\nu) \, d\sigma = 0 \quad \forall \vec{v} \in L^2(\Gamma_0)_*^N,$$

and then there exists a constant c_0 such that

$$(-\frac{\partial \vec{\varphi}}{\partial \nu} + q\nu) = c_0 \nu \quad \text{sur} \quad \Gamma_0.$$

Using Proposition 2.2 can see that $q = c_0$ on Γ_0. But from (29d)

$$c_0 = \frac{1}{|\Gamma_0|} \int_{\Gamma_0} q \, d\sigma = 0. \tag{41}$$

Then $-\frac{\partial \vec{\varphi}}{\partial \nu} + q\nu = \vec{0}$ on Γ_0 and Proposition 2.3 implies that $\vec{\varphi} = \vec{0}$ and $q = 0$ in Ω. Finally, from (29a) we conclude $\vec{\varphi}_0 = \vec{0}$ on S.

■

Proof of Theorem 1.2. Constructive Approach. There exists a more constructive approach to the L^2-approximate controllability which is in fact an explicit method to find controls of minimal norm. The method was introduced by Lions (Refs. 2-4) and uses the duality theory of Fenchel and Rockafellar. We will use here the following adapted duality result (see Ref. 6 and the bibliography therein).

Proposition 2.4 Let V be a Banach space and Y a separable topological vector space. Let $F : V \to I\!R$ and $G : Y \to I\!R$ be two convex functions. Let $L : V \to Y$ be a linear continuous operator. We suppose that there exists $v_0 \in V$ such that $F(v_0) < +\infty$, $G(Lv_0) < +\infty$ and G is continuous at Lv_0. Then, if the infimum is finite, we have:

$$\inf_{v \in V} (F(v) + G(Lv)) = \sup_{w^* \in Y^*} (-F^*(L^* w^*) - G^*(-w^*)). \tag{42}$$

Moreover, the supremum in (42) is attained at least in one point $\widehat{w}^* \in Y^*$. If the infimum is attained at $\widehat{v} \in V$ (for example if V is reflexive and $F(v) + G(Lv) \to +\infty$ as $\|v\| \to \infty$), the following extremal relations are satisfied:

$$F(\widehat{v}) + F^*(L^* \widehat{w}^*) - \langle L^* \widehat{w}^*, \widehat{v} \rangle = 0 \tag{43a}$$

$$G(L\widehat{v}) + G^*(-\widehat{w}^*) + \langle \widehat{w}^*, L\widehat{v} \rangle = 0. \tag{43b}$$

The respective duality products between V and Y and their topological dual spaces V^* and Y^* are denoted equally by $\langle \cdot , \cdot \rangle$. The respective norms are $\| \cdot \|$ and $\| \cdot \|_*$. We recall that F^* is the conjugate function of F in V, i.e.

$$F^*(v^*) = \sup_{v \in V} (\langle v^*, v \rangle - F(v)) \quad \text{for each} \quad v^* \in V^*$$

and similarly for G^* defined in Y^*. Also L^* denotes the adjoint operator of L. The symbol $I_{[-\alpha,\alpha]}$ denotes for $\alpha > 0$ the convex function equal to 0 in the interval $[-\alpha, \alpha]$ and to $+\infty$ in its complement.

Continuation of proof of Theorem 1.2. We apply Proposition 2.4 with $V = L^2(\Gamma_0)_*^N$, $Y = L^2(S)^N$. We also take

$$F(\vec{v}) = \frac{1}{2} \|\vec{v}\|_{0,\Gamma_0}^2 \quad , \quad G(\vec{w}) = I_{[-\alpha,\alpha]}(\|\vec{w} - \vec{y}_1\|_{0,S}) \tag{44}$$

and

$$L(\vec{v}) = \vec{y}(\vec{v})_{/S}. \tag{45}$$

We can write (15) in the form of the infimum in (42). In this case, the infimum is attained at a unique point. To see this, remark first that the set $\mathcal{U}_{ad}$ is not empty, since the existence of a control function $\vec{v}_0 \in L^2(S)_*^N$ that satisfies (4) is ensured in virtue of Theorem 1.1. Moreover, it is easy to see that $\mathcal{U}_{ad}$ is a convex set, then, since $\| \cdot \|_{0,S}^2$ is a strictly convex and coercive functional, there exists a unique $\widehat{\vec{v}} \in L^2(S)_*^N$ which minimizes (15).

Clearly, F and G are convex functions. By Theorem 1.1 we can choose $\vec{v}_0$ such that

$$\left\| \vec{y}(\vec{v}_0)_{/S} - \vec{y}_1 \right\|_{0,S} < \alpha$$

then $F(\vec{v}_0) < +\infty$ and $G(\vec{y}(\vec{v}_0)_{/S}) = 0$ and since the dependence of $\vec{y}(\vec{v})$ on $\vec{v}$ is continuous (see (23)), this is also true in a neighborhood of $\vec{v}_0$. Thus G is continuous at $\vec{y}(\vec{v}_0)_{/S}$. Therefore, the identity (16) is valid in this case. We denote by $\widehat{\vec{\varphi}}_0$ a function where the supremum is attained.

We now deduce formula (16) for J. It is easy to verify that

$$F^*(\vec{v}^*) = \frac{1}{2} \|\vec{v}^*\|_{0,\Gamma_0}^2 \quad \text{and} \quad G^*(\vec{w}^*) = \alpha \|\vec{w}^*\|_{0,S} + \int_S \vec{y}_1 \cdot \vec{w}^* \, ds,$$

and then

$$F^*(L^*\vec{w}^*) + G(-\vec{w}^*) = \frac{1}{2} \|L^*\vec{w}^*\|_{0,\Gamma_0}^2 + \alpha \|\vec{w}^*\|_{0,S} - \int_S \vec{y}_1 \cdot \vec{w}^* \, ds. \tag{46}$$

To identify L^*, let us consider $(\vec{\varphi}, q)$ solution of (35) associated to $\vec{\varphi}_0$. Multiplying equation (2a) by $\vec{\varphi}$ and integrating by parts we obtain

$$\int_S \vec{y}(\vec{v}) \cdot \vec{\varphi}_0 \, ds = \int_{\Gamma_0} \vec{v} \cdot (-\frac{\partial \vec{\varphi}}{\partial \nu} + q\nu) \, d\sigma,$$

thus

$$L^* \vec{\varphi}_0 = -\frac{\partial \vec{\varphi}}{\partial \nu} + q\nu. \tag{47}$$

We obtain from (46) and (47) the expression (12) for J.

The extremal relation (43a) can be written as:

$$\frac{1}{2} \left\| \widehat{\vec{v}} \right\|_{0,\Gamma_0}^2 + \frac{1}{2} \left\| L^* \widehat{\vec{\varphi}}_0 \right\|_{0,\Gamma_0}^2 - \int_{\Gamma_0} \widehat{\vec{v}} \cdot L^* \widehat{\vec{\varphi}}_0 \, d\sigma = 0, \tag{48}$$

then

$$\int_{\Gamma_0} (\frac{1}{2} \left| \widehat{\vec{v}}(x) \right|^2 + \frac{1}{2} \left| L^* \widehat{\vec{\varphi}}_0(x) \right|^2 - \widehat{\vec{v}}(x) \cdot L^* \widehat{\vec{\varphi}}_0(x)) \, d\sigma = 0,$$

from where we deduce (17). To see that $\widehat{\vec{\varphi}}_0$ is unique, we use the fact that $\widehat{\vec{v}}$ is unique. Indeed, if $\widehat{\vec{\varphi}}_0^1$ and $\widehat{\vec{\varphi}}_0^2$ are two functions where the minimum of J is attained, then from (17)

$$L^* (\widehat{\vec{\varphi}}_0^1 - \widehat{\vec{\varphi}}_0^2) = 0.$$

Then Proposition 2.3 implies $\widehat{\vec{\varphi}}_0^1 = \widehat{\vec{\varphi}}_0^2$.

On the other hand, since we know that $\left\| L\widehat{\vec{v}} - \vec{y}_1 \right\|_{0,S} \leq \alpha$, the extremal relation (43b) gives

$$\alpha \left\| \widehat{\vec{\varphi}}_0 \right\|_{0,S} - \int_S \vec{y}_1 \cdot \widehat{\vec{\varphi}}_0 \, ds + \int_S \widehat{\vec{\varphi}}_0 \cdot L\widehat{\vec{v}} \, ds = 0. \tag{49}$$

Thus

$$\alpha \left\| \widehat{\vec{\varphi}}_0 \right\|_{0,S} = -\int_S (L\widehat{\vec{v}} - \vec{y}_1) \widehat{\vec{\varphi}}_0 \, ds \leq \left\| L\widehat{\vec{v}} - \vec{y}_1 \right\|_{0,S} \left\| \widehat{\vec{\varphi}}_0 \right\|_{0,S},$$

from where, if $\widehat{\vec{\varphi}}_0 \neq \vec{0}$

$$\left\| L\widehat{\vec{v}} - \vec{y}_1 \right\|_{0,S} = \alpha. \tag{50}$$

But, using (49) and (50) we can see that

$$\begin{aligned} \left\| \alpha \frac{\widehat{\vec{\varphi}}_0}{\left\| \widehat{\vec{\varphi}}_0 \right\|_{0,S}} + (L\widehat{\vec{v}} - \vec{y}_1) \right\|_{0,S}^2 &= \alpha^2 + \alpha^2 + 2\alpha \frac{1}{\left\| \widehat{\vec{\varphi}}_0 \right\|_{0,S}} \int_S \widehat{\vec{\varphi}}_0 \cdot (L\widehat{\vec{v}} - \vec{y}_1) \, ds \\ &\leq 2\alpha^2 - 2\alpha^2 = 0. \end{aligned}$$

If $\widehat{\vec{\varphi}}_0 = \vec{0}$, (48) implies that $\widehat{\vec{v}} = \vec{0}$ and then $\left\| L\widehat{\vec{v}} - \vec{y}_1 \right\|_{0,S} = \|\vec{y}_1\|_{0,S} \leq \alpha$. Reciprocally, if $\|\vec{y}_1\|_{0,S} \leq \alpha$ then $J(\vec{\varphi}_0) \geq \vec{0}$ for all $\vec{\varphi}_0 \in L^2(S)^N$ and $J(\vec{0}) = 0$ hence by uniqueness $\widehat{\vec{\varphi}}_0 = \vec{0}$. This shows the error identity (18).

■

Remark 2.2: From the above proof it follows that, under the hyphotesis of Theorem 1.2:

$$\widehat{\vec{\varphi}}_0 \neq \vec{0} \Longleftrightarrow \alpha < \|\vec{y}_1\|_{0,S}\,.$$

3. Relationship between Duality Method and Optimal Control Theory.

We make here an analysis of perturbations such as it has been made in Ref. 18 for the heat equation and in Ref. 19 in a more general framework. In order to compare the duality method and the optimal control approach applied to our control problem, we write both as perturbations of a same linear problem.

In all this section we will denote

$$(\vec{u}, \vec{v})_{0,S} = \int_S \vec{u} \cdot \vec{v}\, ds. \tag{51}$$

3.1. Main Operator.

Let us define the operator Λ from $L^2(S)_*^N$ onto itself by

$$\Lambda \vec{\varphi}_0 = \vec{y}_{/S}, \tag{52}$$

where $\vec{y}$ is the solution of the system (53)-(55)

$$-\Delta \vec{y} + \operatorname{div}(\vec{y} \otimes \vec{a}) = -\nabla p \quad \text{in} \quad \Omega \tag{53a}$$

$$\operatorname{div} \vec{y} = 0 \quad \text{in} \quad \Omega \tag{53b}$$

$$\vec{y} = \vec{0} \quad \text{on} \quad \Gamma \setminus \Gamma_0 \tag{53c}$$

$$\vec{y} = \vec{v} \quad \text{on} \quad \Gamma_0 \tag{53d}$$

$$-\Delta \vec{\varphi} - (\vec{a} \cdot \nabla)\vec{\varphi} = -\nabla q + \vec{\varphi}_0 \delta_{(S)} \quad \text{in} \quad \Omega \tag{54a}$$

$$\operatorname{div} \vec{\varphi} = 0 \quad \text{in} \quad \Omega \tag{54b}$$

$$\vec{\varphi} = \vec{0} \quad \text{on} \quad \Gamma. \tag{54c}$$

$$\int_{\Gamma_0} q\, d\sigma = 0 \tag{54d}$$

$$\vec{v} = -\frac{\partial \vec{\varphi}}{\partial \nu} + q\nu \quad \text{on} \quad \Gamma_0. \tag{55}$$

It is clear that

$$\Lambda = LL^* \tag{56}$$

where the linear operator L^* from $L^2(S)_*^N$ onto $L^2(\Gamma_0)_*^N$ is defined by

$$L^*\vec{\varphi}_0 = -\frac{\partial \vec{\varphi}}{\partial \nu} + q\nu \quad \text{on} \quad \Gamma_0 \tag{57}$$

with $(\vec{\varphi}, q)$ the solution of (54) corresponding to $\vec{\varphi}_0$ and L is defined from $L^2(\Gamma_0)_*^N$ onto $L^2(S)_*^N$ by

$$L\vec{v} = \vec{y} \quad \text{on} \quad S \tag{58}$$

with $\vec{y}$ the solution of (53) corresponding to $\vec{v}$.

Proposition 3.1 The operator Λ is symmetric and positive definite in $L^2(S)_*^N$. Moreover $\forall \vec{\varphi}_0,\ \widetilde{\vec{\varphi}}_0 \in L^2(S)_*^N$ we have

$$(\Lambda\vec{\varphi}_0, \widetilde{\vec{\varphi}}_0)_{0,S} = \int_{\Gamma_0} \left(-\frac{\partial \vec{\varphi}}{\partial \nu} + q\nu\right) \cdot \left(-\frac{\partial \widetilde{\vec{\varphi}}}{\partial \nu} + \widetilde{q}\nu\right) d\sigma, \tag{59}$$

where $(\vec{\varphi}, q)$ and $(\widetilde{\vec{\varphi}}, \widetilde{q})$ are the solutions of (54) for $\vec{\varphi}_0$ and $\widetilde{\vec{\varphi}}_0$ respectively.

Proof: The fact that $\Lambda \geq 0$ and the identity (59) follow directly from (56). The operator Λ is injective if L^* is inyective and this follows from the unique continuation property of Proposition 2.3.

■

Remark 3.1: The operator Λ is not coercive in $L^2(S)_*^N$.

Remark 3.2: In terms of Λ, the exact controllability problem (see Ref. 2) can be written as follows: find $\widehat{\vec{\varphi}}_0 \in L^2(S)_*^N$ such that

$$\Lambda\widehat{\vec{\varphi}}_0 = \vec{y}_1, \tag{60}$$

and generally this problem has no solution if $\vec{y}_1$ is not regular enough.

3.2. Duality Method. In the previous section we have seen that the problem of finding $\widehat{\vec{v}}$ which satisfies

$$\min_{L^2(\Gamma_0)_*^N} \frac{1}{2} \|\vec{v}\|_{0,\Gamma_0}^2 \quad \text{under the condition} \quad \|L\vec{v} - \vec{y}_1\|_{0,S} \leq \alpha, \tag{61}$$

is equivalent to finding $\widehat{\vec{\varphi}}_0$ which minimizes over $L^2(S)_*^N$ the functional

$$J(\vec{\varphi}_0) = \frac{1}{2}(\Lambda\vec{\varphi}_0, \vec{\varphi}_0)_{0,S} + \alpha \|\vec{\varphi}_0\|_{0,S} - (\vec{y}_1, \vec{\varphi}_0)_{0,S}, \tag{62}$$

and then $\widehat{\vec{v}} = L^*\widehat{\vec{\varphi}}_0$. For $\alpha < \|\vec{y}_1\|_{0,S}$ (otherwise $\widehat{\vec{\varphi}}_0 = \vec{0}$), writing the Euler equation, problem (56) becomes equivalent to finding $\widehat{\vec{\varphi}}_0$ in $L^2(S)_*^N$ such that

$$\frac{\alpha}{\left\|\widehat{\vec{\varphi}}_0\right\|_{0,S}} \widehat{\vec{\varphi}}_0 + \Lambda\widehat{\vec{\varphi}}_0 = \vec{y}_1. \tag{63}$$

Thus, we obtain a non linear perturbation of the exact controllability problem (60).

3.3. Optimal Control Approach. In this case we look for $\widetilde{\vec{v}}$ which minimizes over $L^2(\Gamma_0)_*^N$ the functional

$$H_\varepsilon(\vec{v}) = \frac{1}{2}\|\vec{v}\|_{0,\Gamma_0}^2 + \frac{1}{2\varepsilon}\|L\vec{v} - \vec{y}_1\|_{0,S}^2 . \tag{64}$$

Writing also the Euler equation in this case, it is clear that the optimal control $\widetilde{\vec{v}}$ is given by the following optimality system

$$\widetilde{\vec{y}} = L\widetilde{\vec{v}} \tag{65a}$$

$$\widetilde{\vec{v}} = -\frac{1}{\varepsilon}L^*(\widetilde{\vec{y}} - \vec{y}_1). \tag{65b}$$

Equivalently, $\widetilde{\vec{v}}$ is the solution of

$$\widetilde{\vec{v}} + \frac{1}{\varepsilon}L^*(L\widetilde{\vec{v}} - \vec{y}_1) = 0. \tag{66}$$

If we define $\widetilde{\vec{\varphi}}_0 = -1/\varepsilon(L\widetilde{\vec{v}} - \vec{y}_1)$ then equation (66) with $\widetilde{\vec{v}} = L^*\widetilde{\vec{\varphi}}_0$ can be written as

$$\varepsilon\widetilde{\vec{\varphi}}_0 + \Lambda\widetilde{\vec{\varphi}}_0 = \vec{y}_1, \tag{67}$$

and it is a linear perturbation of the exact controllability problem (60).

3.4. Concluding Remarks. The analysis of the previous sections let us easily compare the duality approach and the optimal control approach when $\alpha < \|\vec{y}_1\|_{0,S}$. The former represents a non linear perturbation (63) of the exact controllability (60) and the latter can be written as a linear perturbation (67) of the exact controllability (60). We have exactly the same solution with both methods if and only if

$$\varepsilon = \frac{\alpha}{\left\|\widehat{\vec{\varphi}}_0\right\|_{0,S}}. \tag{68}$$

In fact, this equivalence and relation (68) above are verified for general operators L and L^* (Refs. 19 and 20).

In our particular problem, and in view of numerical approximation, the direct Stokes system associated to operator L has nonsmooth data and its real sense is given by the transposition method as we have seen in section 2. Thus the implementation of a method which must solve the direct problem turns out to be difficult (Ref. 21). But we remark that in all the cases we can avoid the direct solution of operator L. Indeed, if we consider relation (59), the duality and the optimal control equations (63) and (67) can be solved respectively by

$$\min_{L^2(S)_*^N} \frac{1}{2}(L^*\vec{\varphi}_0, L^*\vec{\varphi}_0)_{0,S} + \alpha\|\vec{\varphi}_0\|_{0,S} - (\vec{y}_1, \vec{\varphi}_0)_{0,S} \tag{69}$$

$$\min_{L^2(S)_*^N} \frac{1}{2}(L^*\vec{\varphi}_0, L^*\vec{\varphi}_0)_{0,S} + \varepsilon\|\vec{\varphi}_0\|_{0,S}^2 - (\vec{y}_1, \vec{\varphi}_0)_{0,S}. \tag{70}$$

Indeed, (70) is easier to solve, but we do not know the error $\|L\vec{v} - \vec{y}_1\|_{0,S}$ a priori. Recently, a method based on the relation (68) has been proposed (Ref. 19). It consists in approximating the solution of (69) by using a sequence of solutions of (70) and relation (68). We do not use this method here. In this article we will solve directly (69). This functional has a non linear variable metric and this fact motivates the use of a quasi Newton method. Moreover, we take advantage of another particularity of our problem: in a discretization, the number of degrees of freedom associated to the surface or curve is small respect to the number of degrees of freedom of all the cavity. Then, we can solve the operator L^* only for a finite dimensional basis of its domain and then construct a discrete version of (69). With this method we do not need to solve L^* again if the parameters α or $\vec{y}_1$ are changed. We will present such a method in the next section and we will give some successful computations.

4. Numerical Method

The discrete method is essentially based on the duality method of Theorem 1.2 studied in the previous sections. Accordingly, we take the previous notations and for the sake of simplicity we also suppose that $\vec{f} = \vec{0}$. We assume that S verifies the geometrical hypothesis (5) and (6) and that the parameters $\vec{y}_1 \in L^2(S)_*^N$ and $\alpha > 0$ are given.

4.1. Discrete Method. Let us consider a triangulation τ_h of Ω where h is the discretization parameter. In the sake of simplicity, we suppose that $\overline{\Omega}$ is the reunion of triangles (N=2) or tetrahedrons (N=3) of τ_h, i.e.

$$\overline{\Omega} = \bigcup_{T \in \tau_h} T.$$

We also suppose that Γ_0 and S consist of segments (N=2) or faces (N=3) of elements in τ_h.

We use a particular discretization of the continuous method which enables us to calculate easily the control function when the parameters α or $\vec{y}_1$ change. We take advantage of the fact that it is possible to uncouple the Stokes system associated to (29) and the minimization problem (16) by solving (29) only for a suitable basis associated to S.

We denote by L_h^* a finite dimensional discretization of $L^2(S)_*^N$, endowed with the inner product and norm of $L^2(S)_*^N$. If M is the total number of vertices of τ_h in S, the dimension of L_h^* will be in general a function of M denoted here $D(M)$. The fact that L_h^* is generated by a basis $\{\vec{\phi}_k\}$, $k = 1, \ldots, D(M)$, will be written as

$$L_h^* = < \vec{\phi}_1, \ldots, \vec{\phi}_{D(M)} > . \tag{71}$$

We define

$$T : \vec{\varphi}_0 \in L^2(S)_*^N \to (\vec{\varphi}, q) \in H_0^1(\Omega)^N \times L^2(\Omega) \tag{72}$$

the linear operator which associates the unique solution $(\vec{\varphi}, q)$ of (29a,b,c) for each $\vec{\varphi}_0$ (of course q is defined up to a constant), i.e., the solution of the variational problem:

$$\text{Find } (\vec{\varphi}, q) \in H_0^1(\Omega)^N \times L^2(\Omega) \text{ such that} \tag{73a}$$

$$B(\vec{\varphi}, q; \vec{w}, g) = \int_S \vec{\varphi}_0 \cdot \vec{w}\, ds \qquad \forall (\vec{w}, g) \in H_0^1(\Omega)^N \times L^2(\Omega), \tag{73b}$$

where the bilinear form B is defined by

$$B(\vec{\varphi}, q; \vec{w}, g) = \nu \int_\Omega \nabla \vec{\varphi} : \nabla \vec{w}\, dx - \int_\Omega (\vec{a}\cdot\nabla)\vec{\varphi}\cdot\vec{w}\, dx - \int_\Omega q \operatorname{div} \vec{w}\, dx - \int_\Omega g \operatorname{div} \vec{\varphi}\, dx. \tag{74}$$

We introduce the spaces V_h and W_h as finite dimensional approximations of $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively. Now, let us define the following discrete approximation of T:

$$T_h : \vec{\varphi}_{0h} \in L_h^* \to (\vec{\varphi}_h, q_h) \in V_h^N \times W_h, \tag{75}$$

where $T_h \vec{\varphi}_{0h} = (\vec{\varphi}_h, q_h)$ is the solution of

$$\text{Find } (\vec{\varphi}_h, q_h) \in V_h^N \times W_h \text{ such that} \tag{76a}$$

$$B(\vec{\varphi}_h, q_h; \vec{w}, g) = \int_S \vec{\varphi}_{0h} \cdot \vec{w}\ ds \qquad \forall (\vec{w}, g) \in V_h^N \times W_h. \tag{76b}$$

Here q_h is also defined up to an additive constant.

With the previous discretizations, let us define the matrices G, C as

$$G_{kl} = \int_{\Gamma_0} \partial T_h \vec{\phi}_k \cdot \partial T_h \vec{\phi}_l\ d\sigma \tag{77a}$$

$$C_{kl} = \int_S \vec{\phi}_k \cdot \vec{\phi}_l\ ds, \tag{77b}$$

where we take the notation

$$\partial(\vec{\varphi}, q) = -\frac{\partial \vec{\varphi}}{\partial \nu} + q\nu. \tag{78}$$

Each $\vec{\phi} \in L_h^*$ has the decomposition

$$\vec{\phi} = \sum_{k=1}^{D(M)} \beta_k \vec{\phi}_k \tag{79}$$

and then the following quantities can be expressed in terms of $\beta = (\beta_1, \ldots, \beta_{D(M)})^t$:

$$\frac{1}{2}\int_{\Gamma_0} \left|\partial T_h \vec{\phi}\right|^2 d\sigma = \frac{1}{2}\int_{\Gamma_0} \partial T_h \vec{\phi} \cdot \partial T_h \vec{\phi}\; d\sigma$$
$$= \frac{1}{2}\sum_k \sum_l \beta_k \beta_l \int_{\Gamma_0} \partial T_h \vec{\phi}_k \cdot \partial T_h \vec{\phi}_l\; d\sigma = \frac{1}{2}\,\beta^t\, G\beta$$

$$\left\|\vec{\phi}\right\|_{L_h}^2 = \int_S \left(\sum_k \beta_k \vec{\phi}_k\right)\left(\sum_l \beta_l \vec{\phi}_l\right)\; ds$$
$$= \sum_k \sum_l \beta_k \beta_l \int_S \vec{\phi}_k \vec{\phi}_l\; ds = \beta^t\, C\beta.$$

Also if $\vec{y}_1 = \sum_{k=1}^{D(M)} f_{1k}\vec{\phi}_k$ we have

$$\int_S \vec{y}_1 \vec{\phi}\; ds = \sum_k \sum_l \beta_k f_{1l} \int_S \vec{\phi}_k \cdot \vec{\phi}_l\; ds = \beta^t\, C f_1.$$

Therefore, we find the following approximation of the functional J defined in (12) and its derivative J' (defined for $\beta \neq 0$):

$$J_h(\beta) = \frac{1}{2}\,\beta^t\, G\beta + \alpha\,(\beta^t\, C\beta)^{1/2} - \beta^t\, C f_1 \tag{80}$$
$$J_h{}'(\beta) = G\beta + \alpha\,(\beta^t\, C\beta)^{-1/2} C\beta - C f_1. \tag{81}$$

Therefore, if $\widehat{\beta} = (\widehat{\beta}_1, \ldots, \widehat{\beta}_{D(M)})$ is the solution of the unconstrained minimizing problem

$$J_h(\widehat{\beta}) = \min_{\beta \in I\!R^{D(M)}} J_h(\beta) \tag{82}$$

and if $\widehat{\vec{\varphi}}_{0h} = \sum_k \widehat{\beta}_k \vec{\phi}_k$, the discrete control $\vec{v}_h$ on Γ_0 is given by

$$\vec{v}_h = -\partial T_h \widehat{\vec{\varphi}}_{0h} = -\sum_{k=1}^{D(M)} \widehat{\beta}_k\, \partial T_h \vec{\phi}_k. \tag{83}$$

Operator T_h and Stokes System. Condition (29d) is essential to proof the unique continuation property of Proposition 2.10 and to conclude Theorems 1.1 and 1.2. In order to be consistent with this condition we have to set the constant which uniquely defines q_h in (73) such that

$$\frac{1}{|\Gamma_0|}\int_{\Gamma_0} q_h\, d\sigma = 0. \tag{84}$$

We introduce the following finite element spaces

$$V_h = \{w \in C^0(\overline{\Omega}) \mid w|_T \in P_2(T),\ \forall T \in \tau_h\} \cap H_0^1(\Omega). \tag{85a}$$

$$W_h = \{g \in C^0(\overline{\Omega}) \mid g|_T \in P_1(T),\ \forall T \in \tau_h\,, \int_{\Gamma_0} g\, d\sigma = 0\} \tag{85b}$$

In the above definitions $P_1(T)$ are polynomials of degree less than or equal to 1 and the associated degrees of freedom are the evaluations at the vertices of T. $P_2(T)$ are polynomials of degree less than or equal to 2 and the degrees of freedom are the evaluations at the vertices and at the middle points of each side of T. The dimension of W_h is then the total number NV of vertices in $\overline{\Omega}$ minus one. The dimension of V_h is the total number NI of interior vertices and middle points (i.e. the total number of interior nodes) in Ω. Then

$$\dim V_h^N = NI \times N \tag{86a}$$

$$\dim W_h = NV - 1. \tag{86b}$$

We introduce a canonical finite element basis for V_h^N and W_h

$$V_h^N =< \vec{w}_1, \ldots, \vec{w}_{NI\times N} >$$
$$W_h =< g_1, \ldots, g_{NV-1} >$$

and we define for $i, j = 1, \ldots, NI \times N$ and $k = 1, \ldots, NV - 1$ the quantities

$$M_{ij} = \nu \int_\Omega \nabla \vec{w}_i : \nabla \vec{w}_j \, dx$$
$$D_{ij} = \int_\Omega (\nabla \vec{w}_i)\vec{a} \cdot \vec{w}_j \, dx$$
$$N_{ik} = -\int_\Omega g_k \operatorname{div} \vec{w}_i \, dx$$
$$b_j = \int_S \vec{\varphi}_{0h} \cdot \vec{w}_j \, ds.$$

With this definitions, it is clear that the solution $T_h\vec{\varphi}_{0h} = (\vec{\varphi}_h, q_h)$ of (75)-(76) can be written thus as

$$\vec{\varphi}_h = \sum_{i=1}^{NI\times N} \eta_i\, \vec{w}_i \quad \text{and} \quad q_h = \sum_{k=1}^{NV-1} \xi_k\, g_k,$$

where (η, ξ) is the solution of the linear system:

$$(M + D)\vec{\eta} + N\vec{\xi} = \vec{b} \tag{87a}$$

$$N^t\vec{\eta} = \vec{0}. \tag{87b}$$

The total size of this discrete Stokes system is

$$NT = NI \times N + NV - 1.$$

In fact, in our method it is only necessary to solve the Stokes system (87) for the $D(M)$ right hand sides:

$$b_j(\vec{\phi}_k) = \int_S \vec{\phi}_k \cdot \vec{w}_j \; ds, \tag{88}$$

where the set $\{\vec{\phi}_k\}_{k=1}^{D(M)}$ corresponds to the basis (71) of L_h^*.

Normal Derivatives. We explain how we compute the ∂ operator defined in (78) to obtain the G matrix (77a). We only consider the case N=2, the case N=3 is similar. We have supposed that the boundary Γ_0 consists of $K-1$ segments $[x_{2i-1}, x_{2i+1}]$, $i = 1, \dots, K-1$, $i = 1, \dots, K-1$ of length l_i, where $\{x_j\}_{j=1}^{2K-1}$ are the nodes of τ_h which belongs to Γ_0. We introduce $\{\psi_k\}_{k=1}^{2K-1}$ the canonical finite element basis in V_h associated to these nodes, that is to say

$$\psi_j \in V_h \quad \psi_j(n_k) = \delta_{jk} \text{ for } j,k = 1, \dots, 2K-1.$$

Multiplying (29a) by $\psi\vec{e}_m$, where $\{\vec{e}_m\}_{m=1,2}$ is the canonical basis in $I\!R^2$, and integrating by parts, we obtain for each $k = 1, \dots .2K-1$ and $m = 1, 2$ the identity

$$\int_{\Gamma_0} \partial(\vec{\varphi}, q) \cdot \psi_k \vec{e}_m \, d\sigma = -\int_\Omega \nabla\vec{\varphi} : \nabla(\psi_k \vec{e}_m)\, dx + \\ + \int_\Omega (\vec{a}\cdot\nabla)\vec{\varphi} \cdot \psi_k \vec{e}_m \, dx + \int_\Omega q \,\mathrm{div}\,(\psi_k \vec{e}_m)\, dx. \tag{89}$$

We can notice that the integrands in (89) have small support: the union of all the elements in τ_h which have a side or a vertex on Γ_0.

We write the left hand side in (89) as a sum of integrals on I_i and we use a quadrature approximation to compute each integral on I_i

$$\int_{I_i} \partial(\vec{\varphi}, q) \cdot \psi_k \vec{e}_m = l_i \sum_j \lambda_j \, \psi_k(z_j) \partial(\vec{\varphi}, q)(z_j) \cdot \vec{e}_m, \tag{90}$$

where λ_j are specific weights and z_j are suitable quadrature points. A Simpson quadrature formula (exact for P_3) with $j \in \{2i-1, \; 2i, \; 2i+1\}$ and

$$\lambda_{2i-1} = \frac{1}{6}, \; \lambda_{2i} = \frac{2}{3}, \; \lambda_{2i+1} = \frac{1}{6},$$

transforms (89) in a diagonal system with a matrix of the form

$$\mathrm{diag}\left(\dots, \; \frac{1}{6}(l_{i-1} + l_i), \; \frac{2}{3} l_i, \; \frac{1}{6}(l_i + l_{i+1}), \dots\right),$$

but this is not an exact quadrature since the integrand in the left hand side of (90) is a P_4 polynomial. Then we have to use the Boole-Villarceau formula (exact for P_5) with $j \in \{2i-1,\ 2i-\frac{1}{2},\ 2i,\ 2i+\frac{1}{2},\ 2i+1\}$ and

$$\lambda_{2i-1} = \lambda_{2i+1} = \frac{7}{90},\ \lambda_{2i-\frac{1}{2}} = \lambda_{2i+\frac{1}{2}} = \frac{16}{45},\ \lambda_{2i} = \frac{2}{15}.$$

After some calculations in order to eliminate the points indexed $2i-\frac{1}{2}$ and $2i+\frac{1}{2}$, identity (90) becomes equivalent to the following sparse system

$$\begin{pmatrix} \ddots & & & & \\ & \frac{2}{15}(l_{i-1}+l_i) & \frac{1}{15}l_i & -\frac{1}{30}l_i & \\ & \frac{1}{15}l_i & \frac{8}{15}l_i & \frac{1}{15}l_i & \\ & -\frac{1}{30}l_i & \frac{1}{15}l_i & \frac{2}{15}(l_i+l_{i+1}) & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \partial^m_{2i-1} \\ \partial^m_{2i} \\ \partial^m_{2i+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ h^m_{2i-1} \\ h^m_{2i} \\ h^m_{2i+1} \\ \vdots \end{pmatrix},$$

where $\partial^m_j = \partial(\vec{\varphi}, q)(x_j) \cdot \vec{e}_m$ and h^m_j denotes the right hand side of (89). From this system we easily obtain δ and by applying again the Boole-Villarceau formula we can compute the G matrix.

For a discussion of other possible methods to compute the normal derivatives see for example Refs. 22 and 23.

Choice of Basis in L^*_h . We also discuss the case N=2. We suppose that S has been discretized in $M-1$ segments $I_i = [x_i, x_{i+1}]$ of length l_i which are sides of triangles of τ_h. We take as a discretization of $L^2(S)^N_*$, the piecewise constant functions on each segment:

$$L^*_h = \{\vec{\phi} \in L^2(S)^N_* \mid \vec{\phi}|_{I_i} \in P_0(I_i)^N, i = 1, \ldots, M-1\}.$$

We have $\dim L^*_h = D(M) = N(M-1)$. In the case S_{int} is empty we have $L^2(S)^N = L^2(S)^N_*$ and we can choose a orthogonal basis, for example, with the following functions:

$$\phi_i = \begin{cases} 1 & x \in I_i \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$L^*_h =< \phi_i\, \vec{e}_k >_{i=1,\ldots,M-1;\ k=1,2} .$$

We see that the respective products are equal to

$$\int_S \phi_i\, \vec{e}_k \cdot \phi_j\, \vec{e}_l\ ds = \delta_{kl} \int_{I_i \cap I_j} ds = l_i\ \delta_{ijkl}$$

and then the C matrix defined in (77b) is diagonal

$$C = \delta_{ijkl} l_i = \mathrm{diag}\,(l_1, \ldots, l_{M-1}, l_1, \ldots, l_{M-1}).$$

If S_{int} is not empty, C is in general more complex. In this case, we have to modify the basis functions $\phi_i \vec{e}_k$ in order to assure the compatibility condition (9). We define a new basis $\{\widetilde{\vec{\phi}}_{ik}\}$. To simplify, let us suppose that S_{int} has only one connected component since the general case is analogous. We define

$$\widetilde{\vec{\phi}}_{ik} = \phi_i \vec{e}_k - c_{ik}\nu_{int} \tag{91}$$

where

$$c_{ik} = \frac{1}{|\partial S_{int}|} \int_{\partial S_{int}} \phi_i \vec{e}_k \cdot \nu_{int} \, ds, \tag{92}$$

and ν_{int} is the unit interior normal to S_{int}. In this case

$$\widetilde{C}_{ijkl} = \delta_{ijkl} l_i - |\partial S_{int}| \, c_{ik} c_{jl}$$

and since $\vec{e}_k \cdot \nu_{int}$ is a constant denoted ν_{ki} in the interval I_i, we obtain

$$\widetilde{C}_{ijkl} = \delta_{ijkl} l_i - \frac{l_i l_j}{|\partial S_{int}|} \nu_{ki} \nu_{lj}.$$

If $l = \max l_i$, $\widetilde{C}$ is only a perturbation of order $\mathcal{O}(l/M)$ of C and then $\widetilde{C}$ is non-singular.

Minimization Problem The minimizing problem is solved using the variant Broyden–Fletcher–Goldfarb–Shanno (BFGS) of the classic Davison–Fletcher–Powell (DFP) algorithm (Ref. 24). The stop tests are a "small" gradient or a limit of the precision machine.

Whatever the iterative method used to minimize (82), we can say something about its initial guess. Using notations of the previous section, we remark that

$$0 < (\Lambda \widehat{\vec{\varphi}}_0, \widehat{\vec{\varphi}}_0)_{0,S} = (\vec{y}_1, \widehat{\vec{\varphi}}_0)_{0,S} - \alpha \left\| \widehat{\vec{\varphi}}_0 \right\|_{0,S},$$

then

$$\frac{(\vec{y}_1, \widehat{\vec{\varphi}}_0)_{0,S}}{\|\vec{y}_1\|_{0,S} \left\| \widehat{\vec{\varphi}}_0 \right\|_{0,S}} > \frac{\alpha}{\|\vec{y}_1\|} > 0.$$

If we define

$$\cos\theta_0 = \frac{\alpha}{\|\vec{y}_1\|_{0,S}} \quad \text{and} \quad \cos\widehat{\theta} = \frac{(\vec{y}_1, \widehat{\vec{\varphi}}_0)_{0,S}}{\|\vec{y}_1\|_{0,S} \left\| \widehat{\vec{\varphi}}_0 \right\|_{0,S}},$$

since $0 < \cos\theta_0 < 1$ and $\cos\widehat{\theta} > \cos\theta_0$, then

$$0 < \widehat{\theta} < \theta_0 < \frac{\pi}{2}.$$

Thus, the angle between $\widehat{\vec{\varphi}}_0$ and $\vec{y}_1$ is acute. This fact leads us to choose as a first approximation of $\widehat{\vec{\varphi}}_0$, a function $\vec{\varphi}_0^0$ parallel to $\vec{y}_1$, i.e.

$$J(\vec{\varphi}_0^0) = \min_{\lambda \in I\!R^+} J(\lambda \vec{y}_1).$$

From $\frac{\partial J}{\partial \lambda} = 0$ we can easily deduce that

$$\vec{\varphi}_0^0 = \begin{cases} \|\vec{y}_1\|_{0,S} \frac{(\|\vec{y}_1\|_{0,S} - \alpha)}{(\Lambda \vec{y}_1, \vec{y}_1)_{0,S}} \vec{y}_1 & \text{if} \quad \alpha < \|\vec{y}_1\|_{0,S} \\ 0 & \text{otherwise.} \end{cases} \tag{93}$$

We then have the BFGS algorithm with the initial guess

$$\beta_0 = (f_1^t C f_1)^{\frac{1}{2}} \frac{((f_1^t C f_1)^{\frac{1}{2}} - \alpha)}{f_1^t G f_1} \; f_1.$$

Remark 4.1: In practice, G is ill conditioned, since it is a discrete version of the operator Λ of section 3. To overcome this problem, it is necessary to work with double precision variables, to choose the basis $\{\vec{\phi}_k\}_{k=1}^{D(M)}$ so that it does not have superfluous elements and to use, for example, the following preconditioning:

$$\beta = \text{diag}\,(G_{ii}^{-1/2})\beta'.$$

We can also do a restart of the algorithm if necessary. We make some modifications of the BFGS method following this Remark.

4.2. Test Examples. To test the behavior of the algorithm, we take some representative examples in two dimentions and in all of them we use the same geometry (see Fig. 2). We choose Ω the square $]0,1[\times]0,1[$. We take a coordinate system with an orthogonal basis $\vec{e}_1$, $\vec{e}_2$ parallel to the sides of Ω with its origin $(0,0)$ at the bottom left corner. The curve S is a circumference of radius 0.25 centered in Ω. We call θ the angle with vertex at the center of the circumference S measured in anti clockwise direction from $\vec{e}_1$. In all the examples we take viscosity equals to 1.0 and a constant function $\vec{a} = (1.0, 1.0)$.

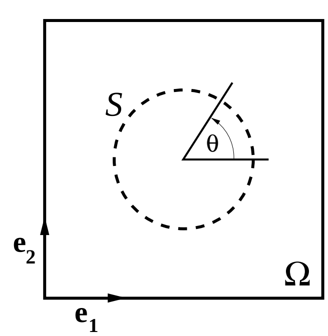

FIG. 2. Geometry in all the Examples.

We take a finite element mesh $\mathcal{T}_h$ as described in Section 4.2. Table 1 summarizes the main characteristics of this discretization.

TABLE 1. Main characteristics of test problem.

Vertices in $\mathcal{T}_h$ = NV	510
Interior nodes in $\mathcal{T}_h$ = NI	1797
Degrees of freedom pressure = NV-1	509
Degrees of freedom velocity = 2×NI	3594
Size of Stokes System (87) =NT×NT	4103×4103
Vertices in S = M	31
Number of right hand sides = 2(M-1)	60
Size of minimization problem (82) = 2(M-1)×2(M-1))	60×60

The computation of the elementary matrices is done by using a specific C++ numeric code to obtain sparse matrices. The discrete Stokes system (87) with the right sides (88) is solved by a LU factorization. In order to obtain the matrices G and C involved in the minimization problem, we follow the methods explained in Section 4.2.

The four different examples are obtained from two different choices of $\vec{y}_1$ and two control boundaries Γ_0. In Examples 4.1 and 4.2, we take $\vec{y}_1 \cdot \vec{e}_1 = -\sin\theta$ and $\vec{y}_1 \cdot \vec{e}_2 = \cos\theta$. In Example 4.1, Γ_0 is the whole boundary while, in the Example 4.2, Γ_0 is the whole boundary except for the bottom side. Both examples consist in finding a minimal norm control $\vec{v}_h$ such that the velocity field $\vec{y}(\vec{v}_h)$ of the controlled Stokes system be almost tangent to the circumference S, with unit norm and with an error in $L^2(S)^N$ lower than or equal to α.

For each value of the parameter α, the precision is studied comparing the theoretical error

$$\|\vec{y} - \vec{y}_1\|_{0,S} = \min(\alpha, \|\vec{y}_1\|_{0,S})$$

and the numerical error

$$\|\vec{y}_h - \vec{y}_1\|_{0,S}\,,$$

where $\vec{y}_h$ is a discrete solution of (2) for the computed control $\vec{v}_h$ and for the same mesh $\mathcal{T}_h$. We remark that the error due to this extra calculation is added to the numerical errors presented here, but it is not an intrinsic error of the method. To make comparisons easier, we also introduce the corresponding relative errors, that is to say, the errors divided by $\|\vec{y}_1\|_{0,S}$. Fig. 3 shows the comparison between the numerical and theoretical relative errors for Example 4.1.

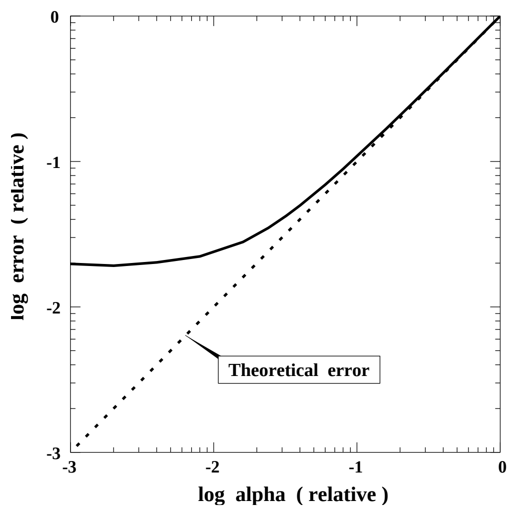

FIG. 3. Comparison between the theoretical (dashed diagonal line) and numerical relative errors for Example 4.1 (solid line) varying α.

We remark that the numerical error does not decrease as expected theoretically when $\alpha \to 0$. We have tested a 65% more refined mesh and the results are similar. We define α^* as the minimal value of α corresponding to a minimal numerical error e^*.

In Fig. 4 we show the two examples corresponding to the case $\alpha = \alpha^*$. We mention in each case the numerical error e^* and the numerical relative error between parenthesis. Fig. 4 shows the velocity field of the controlled Stokes system for each example. Fig 5. shows the comparison between the velocity field $\vec{y}_h(\vec{v}_h) \cdot \vec{e}_1$ (solid line) and $\vec{y}_1 \cdot \vec{e}_1$ (dashed line) on S for the horizontal component as a function of the angle θ.

We take the Example 4.1 with α^* to give an idea of the numerical cost at each stage of the algorithm. Let us see the total computation CPU system time in a work station HP–9000 with a UNIX environment: mesh and elementary matrices (10 minutes), factorization of the $NT \times NT$ Stokes system (10 minutes), solution of the $2(M-1)$ right hand sides associated to the base on S (3 minutes), computation of the $2(M-1)$ normal derivatives on Γ_0 (63 seconds), minimization of the $2(M-1) \times 2(M-1)$ problem and computation of the control on Γ_0 (4 seconds). We see that, if α or $\vec{y}_1$ are modified, the time required to calculate again the control is only 4 seconds. The situation is analogous in the other examples.

The memory stockage to solve the factorization of the Stokes system is of order $\mathcal{O}(NT^2)$, but in practice the matrices are sparse and a profile stockage is used. The total stockage for the other stages involves saving in memory $\partial(T_h\vec{\phi}_k)$ on Γ_0, and G, f_1, C, which is only of order $\mathcal{O}(M^2)$.

4.3. Conclusions. To sum up, the presented numerical method is of fast updating and works with a small memory stockage in a fixed geometry. The reason is that in this case we can uncouple the Stokes system and the minimization problem.

As a consequence the number of degrees of freedom of the problem is only related to the total number of nodes on the curve, and this fact reduces notoriously the numerical computation time.

We have seen that in all the examples the numerical method does not improve the precision under a minimal value of alpha as expected theoretically. It is not only a discretization problem: we see that the more the mesh is refined, the worse is the conditioning of the minimization problem. Here, we find a tipical situation in numerical control problems.

Finally, we would like to emphasize the fact that the discretization method and its numerical implementation that we have presented, can be easily adapted to other similar linear control problems.

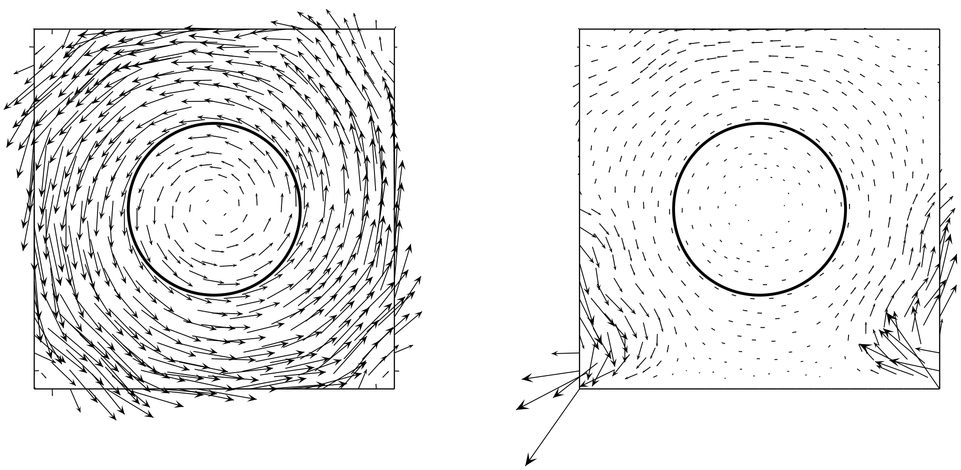

FIG. 4. Left: Example 4.1 with $\alpha^* = 0.0025$, $e^* = 0.024$ (1.9%).
Right: Example 4.2 with $\alpha^* = 0.075$, $e^* = 0.09$ (7.4%).

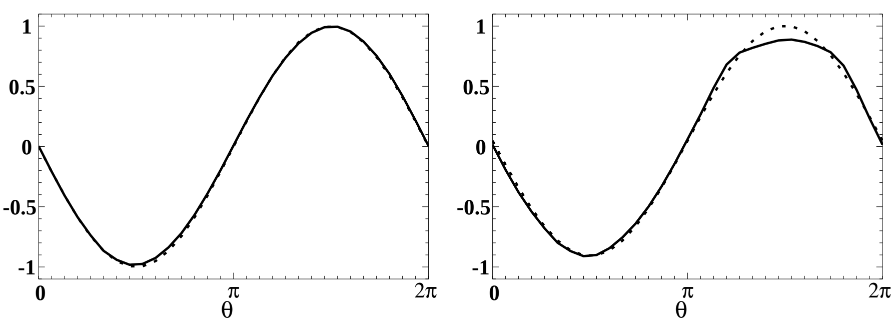

FIG. 5. Comparison between the controlled velocity field (solid line) and $\vec{y}_1$ (dashed line) on S for the horizontal component as a function of θ. Left: Example 4.1. Right: Example 4.2.

References

1. FABRE, C., and LEBEAU, G., *Prolongement Unique Des Solutions de l'Equation de Stokes*, Communications in Partial Differential Equations, Vol. 21, pp. 573–596, 1996.

2. LIONS, J.-L., *Exact Controllability, Stabilization and Perturbations for Distributed Systems*, SIAM Review, Vol. 30, pp. 1–68, 1988.

3. LIONS, J.-L., *Remarques sur la Contrôlabilité Approchée*, Proceedings of Jornadas Hispano-Francesas sobre Control de Sistemas Distribuidos, University of Malaga, Malaga, Spain, 1990.

4. LIONS, J.-L., *Exact Controllability for Distributed Systems: Some Trends and Some Problems*, Applied and Industrial Mathematics, Edited by R. Spigler, Kluwer, Dordretcht, Holland, pp. 59–84, 1991.

5. ROCKAFELLAR, E. T., *Convex Functions and Duality in Optimization Problems and Dynamics*, Lecture Notes in Operational Research and Mathematical Education, Springer, Berlin, Germany, Vol. 2, 1969.

6. EKELAND, Y., and TEMAM, R., *Analyse Convexe et Problèmes Variationnels*, Dunod, Gauthier-Villars, Paris, France, 1974.

7. FABRE, C., PUEL, J.-P., and ZUAZUA, E., *Contrôlabilité Approchée de l'Equation de la Chaleur Semi-Linéaire*, Comptes Rendus de l'Academie des Sciences de Paris, Série I, Vol.315, pp. 807–812, 1992.

8. FABRE, C., PUEL, J.-P., and ZUAZUA, E., *Contrôlabilité Approchée de l'Equation de la Chaleur Linéaire avec des Contrôles de Norme* L^∞ *Minimale*, Comptes Rendus de l'Academie des Sciences de Paris, Série I, Vol. 316, pp. 679–684, 1993.

9. FABRE, C., PUEL, J.-P., and ZUAZUA, E., *Approximate Controllability of the Semilinear Heat Equation*, Proceedings of the Royal Society of Edinburgh, Vol. 125A, pp. 31–61, 1995.

10. OSSES, A., and PUEL, J.-P., *Some Extensions of Approximate Controllability Results to Inverse Problems*, ESAIM Proceedings: Elaticité, Viscoelasticité et Contrôle Optimal, Huitièmes Entretiens du Centre Jacques Cartier, Vol. 1, pp. 157–168, 1996.

11. LIONS, J.-L., and MAGENES, E., *Problèmes aux Limites Non Homogènes et Applications*, Vol. 1, Dunod, Paris, France, 1968.

12. CONCA, C., *Stokes Equations with Nonsmooth Data*, Revista de Matemáticas Aplicadas de la Universidad de Chile, Vol. 10, pp. 115–122, 1989.

13. FABRE, C., *Uniqueness Results for a General Stokes Equations and Their Consequences in Control Problems for Linear and Nonlinear Equations*, ESAIM: Control, Optimization and Calculus of Variations. Vol. 1, pp. 267–302, 1996.

14. Coron, J.-M., *On the Controllability of the 2-D Incompressible Navier–Stokes Equations with the Navier Slip Boundary Conditions*, ESAIM: Control, Optimization and Calculus of Variations, Vol. 1, pp. 35–75, 1996.

15. Temam, R., *Navier-Stokes Equations*, North-Holland, Amsterdam, Holland, 1977.

16. Agmon, S., Douglis, A., and Nirenberg, L., *Estimates Near The Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions, I*, Communications on Pure and Applied Mathematics, Vol. 12, pp. 623–727, 1959.

17. Girault, V., and Raviart, P., *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*, Springer Series in Computational Mathematics, Springer Verlag, Berlin, Germany, Vol. 5, 1986.

18. Carthel, C., Glowinski, R., and Lions, J.-L., *On Exact and Approximate Boundary Controllabilities for the Heat Equation: A Numerical Approach*, Journal on Optimization Theory and its Applications, Vol. 82, pp. 429–484, 1994.

19. Demeestère, P., *Mèthode des Sentinelles : Etude Comparative et Application à l'Identification d'une Frontière. Contrôle du Temps d'Explosion d'une Equation de Combustion. Annexe A: Régularisation de Fenchel et Régularisation de Tychonov*, PhD Thesis, Université de Technologie de Compiègne, Compiègne, France, 1997.

20. Osses, A., and Puel, J.-P., *On the Controllability of the Semilinear Laplace Equation Observed on an Interior Curve*, Revista de Matemáticas Aplicadas de la Universidad Complutense de Madrid, to appear in Vol. 11, no. 2, October 1998.

21. Lions, J.-L., *A Remark on the approximation of nonhomogeneus hyperbolic boundary value problems*, Vistas in Applied Mathematics, Edited by A. V. Balakrishnan, A. A. Dorodnitsyn, J.-L. Lions., Optimization Software, Inc., New York, pp. 114–126, 1986.

22. Glowinski, R., Li, C.H., and Lions, J.-L., *A Numerical Approach to the Exact Boundary Controllability of the Wave Equation. Dirichlet Controls, Description of Numerical Methods*, Report, Department of Mathematics, University of Houston, 1988.

23. Glowinski, R., Lions, J.-L., and Trémolières, R., *Analyse Numérique des Inéquations Variationnelles*, Dunod, Paris, France, 1976.

24. Press, W., et al, *Numerical Recipes in C, the Art of Scientific Computing*, Cambridge University Press, Cambridge, England, 1992.