Added Mass and Damping in Fluid-Structure Interaction[+]

C. Conca
Departamento de Ingeniería Matemática
Universidad de Chile, Casilla 170/3 – Correo 3
Santiago (Chile)

A. Osses
Centre de Mathématiques Appliquées
Ecole Polytechnique, 91128 Palaiseau Cedex
Paris (France)

J. Planchard
Electricité de France, Etudes et Recherches
1 avenue du Général de Gaulle
92141 Clamart (France)

Abstract

This paper is concerned with the added mass matrix for a mechanical structure vibrating in an incompressible liquid. It is shown in particular that this matrix does not depend on viscosity and, from this fact, can be calculated as if the fluid is perfect. The viscous effect on the mechanical system can then be represented by a damping term of type time-convolution. The presence of a flowing fluid around the structure leads to additional damping terms proportional to the fluid density.

Subject indices: 01, 07, 18, 37

[+] Submitted to the Journal Computer Methods in Applied Mechanics and Engineering.

1. Introduction

An important notion in fluid-structure interaction studies is that of added mass of mechanical systems vibrating in a liquid. The added mass is generally described as a matrix which models the interaction of the elements of the mechanical structure via the fluid pressure fluctuations. Its interest is to allow investigating of the dynamical behaviour of the structure without determining the fluid motion and consequently to reduce the number of freedom degrees and then to save computer times. Numerous applications of this idea may be found in the problems of vibration of heat exchanger tube bundle, fuel assemblies of nuclear reactors (we refer to the Chen's book [1] and the papers by Paidoussis and his coworkers [2], [3], [4], [5], etc.), space engineering (see Morand and Ohayon [6]), etc.

The added mass is generally calculated assuming an ideal perfect motionless fluid because the computation may be easily done. The viscosity of the real fluid is then modeled by means of a damping term introduced in the dynamical equation of the structure and the damping coefficient is often obtained from measurements. The aim of this paper is to justify this above assumption, even if the fluid is not perfect. However, it will be shown that viscosity leads to a damping term which is of time-convolution type. The case of a mechanical structure placed in a cross-flows will be also investigated and it is seen that another damping terms result from the linearized convection operator in the Navier-Stokes equations.

In order to simplify the presentation, we consider a simple harmonic oscillator, for instance a rigid tube elastically supported by a spring system, a piano string for instance, and we shall sketch how the added mass can be defined for a general elastic structure.

2. Some reminders about added mass for a perfect fluid at rest.

2.1 The case of harmonic oscillator.

The fluid, of specific density ρ, is supposed to be perfect, incompressible, and initially at rest, it occupies a bounded region Ω and, to fix the ideas, Ω is two-dimensional (but of course $3D$-problems can be considered). Γ denotes the wall of the cavity Ω containing the fluid. The mechanical structure is a single tube of wall γ (γ and Γ are, in fact, the cross-sections of the different walls, see Fig. 1).

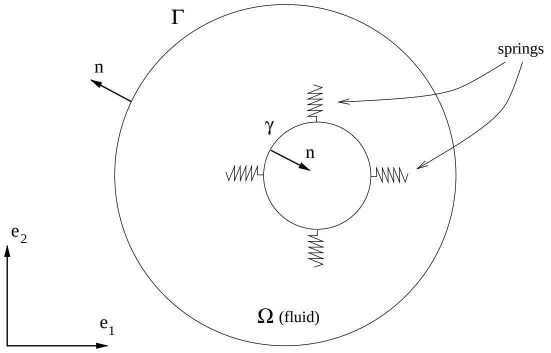

Fig. 1 A mechanical harmonic oscillator.

The tube, of mass m (per length unit), is supported by a spring system of stiffness k, allowing only transversal motions. The displacement $\vec{s}(t)$ of the tube at time t, is assumed to be small enough so that the geometrical variations of the domain Ω, due to the motion of the oscillator, may be neglected.

If $\vec{u}(x,t)$ denotes the speed of fluid particles at x, we have in Ω (p is the pressure):

$$\rho\frac{\partial\vec{u}}{\partial t} - \nabla p = 0 \ , \tag{2.1}$$

$$div\ \vec{u} = 0 \ , \tag{2.2}$$

with the boundary conditions (sliding condition):

$$\vec{u}\cdot\vec{n} = 0 \quad \text{on} \ \ \Gamma, \tag{2.3}$$

$$\vec{u}\cdot\vec{n} = \frac{d\vec{s}}{dt}\cdot\vec{n} \quad \text{on} \ \gamma, \tag{2.4}$$

in which $\vec{n}$ is the unit-normal, oriented outside the fluid (Γ is supposed to be fixed).

The tube dynamical equation is:

$$(m\frac{d^2}{dt^2} + k)\vec{s}(t) = \int\limits_\gamma p(x,t)\vec{n}d\gamma + \vec{f}(t), \tag{2.5}$$

where $\vec{f}$ is a specified external force applied to the cylinder.

The liquid being initially at rest, we have curl $\vec{u}(x,t) = 0$ for $t = 0$ and, from (2.1), for any positive t. Hence $\vec{u}$ derives from a potential $\phi(x,t) : \vec{u} = \nabla\phi$ and again from (2.1), we have

$$p(x,t) = -\rho\frac{\partial\phi}{\partial t}(x,t) + C(t)$$

in which $C(t)$ is a certain constant depending only on time. (2.2), (2.3) and (2.4) lead to the relations:

$$\Delta\phi(x,t) = 0 \quad \text{in } \Omega, \tag{2.6}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \Gamma, \tag{2.7}$$

$$\frac{\partial\phi}{\partial n} = \frac{d\vec{s}}{dt}\cdot\vec{n} \quad \text{on } \gamma. \tag{2.8}$$

The pressure force in (2.5) can be replaced by $-\rho\int\limits_{\gamma}\frac{\partial\phi}{\partial t}\vec{n}d\gamma$ (where the constant $C(t)$ disappears).

The potential ϕ clearly may be written as

$$\phi(x,t) = \sum_{j=1}^{2}\chi_j(x)\frac{ds_j}{dt}(t),$$

where the s_j are the components of $\vec{s}$, with respect to an orthonormal reference basis $(\vec{e}_j)$, and the functions $\chi_j(x)$ satisfy the system:

$$\begin{cases} \Delta\chi_j = 0 & \text{in } \Omega, \quad \gamma = 1,2, \\ \frac{\partial\chi_j}{\partial n} = 0 & \text{on } \Gamma, \\ \frac{\partial\chi_j}{\partial n} = n_j & \text{on } \gamma, \end{cases} \tag{2.9}$$

where n_j is the j^{th} direction-cosine of $\vec{n}$ (with respect to the basis $\vec{e}_j$). χ_j is uniquely determined if we impose, for example, the condition:

$$\int\limits_{\Omega}\chi_j dx = 0. \tag{2.10}$$

(Note that $\int\limits_{\gamma} n_j d\gamma = 0$, so that (2.9) is a well-posed system).

Injecting the expansion of ϕ into (2.5), one obtains:

$$(m\frac{d^2}{dt^2} + k)\vec{s}(t) = -\rho H\frac{d^2\vec{s}}{dt^2}(t) + \vec{f}(t), \tag{2.11}$$

in which H is the 2×2 matrix of entries

$$h_{ij} = \int\limits_{\gamma}\chi_i(x)n_j(x)d\gamma.$$

ρH is the so-called added mass matrix. H has the following property:

Proposition 1. [7], [8]

H is symmetric and positive definite.

Sketch of the proof.

It is not difficult to check that the χ_j are linearly independent.

By using the Green identity, one has

$$h_{ij} = \int\limits_{\gamma} \chi_i n_j d\gamma = \int\limits_{\Omega} \nabla \chi_i \cdot \nabla \chi_j dx = h_{ji},$$

whence the symmetry.

For any vector $\vec{\xi} = \binom{\xi_1}{\xi_2} \neq 0$, we set

$$\varphi(x) = \sum_{j=1}^{2} \xi_j \chi_j(x).$$

Then

$$\begin{aligned} H\vec{\xi} \cdot \vec{\xi} = \sum_{i,j} h_{ij} \xi_i \xi_j &= \sum_{i,j} \xi_i \xi_j \int\limits_{\Omega} \nabla \chi_i \cdot \nabla \chi_j dx \\ &= \int\limits_{\Omega} |\nabla \varphi|^2 dx > 0. \end{aligned}$$

(Note that $\nabla \varphi = 0$ would imply $\varphi \equiv 0$, which is not possible since the χ_j are linearly independent). ■

Now, we are interested by the sinusoidal solutions of (2.11) with $\vec{f} \equiv 0$, of the form $\vec{s}(t) = e^{i\omega t} \vec{z}$. Then ω and $\vec{z}$ satisfy the matrix eigenproblem

$$(m + \rho H)\vec{z} = \frac{k}{\omega^2} \vec{z} \tag{2.12}$$

in which ω^2 is an eigenvalue of the matrix $k(m+\rho H)^{-1}$. ω^2 is obviously a positive number (H is positive definite). Clearly, from the properties of H, the two eigenfrequencies ω of (2.12) are strictly smaller than $\omega_0 = (k/m)^{1/2}$ which is the eigenfrequency of the tube when placed in vacuum.

In case of several parallel tubes $\gamma_1, \gamma_2, ..., \gamma_N$, the added mass matrix ρH is defined by means of the harmonic functions $\chi_{\ell j}(x), \ell = 1, 2, ..., N, j = 1$, with the conditions:

$$\int_\Omega \chi_{\ell j} dx = 0, \quad \frac{\partial}{\partial n}\chi_{\ell j} = 0 \text{ on } \Gamma, \quad \frac{\partial}{\partial n}\chi_{\ell j} = n_j \delta_{\ell k} \quad \text{on each } \gamma_k$$

($\delta_{\ell k}$ is the Kronecker symbol).

H is made up with the integrals $\int\limits_{\gamma_k} \chi_{\ell i} n_j d\gamma_k, \quad \ell, k = 1, 2, ..., N, \quad i, j = 1, 2$ (H is of order $2N$). H has the same properties than in Proposition 1.

Let us turn back to equation (2.11), and denote by $\vec{z}_j$ with $j = 1, 2$, the eigenvectors of (2.12). The eigenvectors can be chosen to be orthonormalized:

$$\vec{z}_i \cdot \vec{z}_j = \delta_{ij}.$$

Then, the response $\vec{s}(t)$ to the force $\vec{f}(t)$ may the expressed as $\vec{s}(t) = \sum\limits_j \alpha_j(t)\vec{z}_j$ where the components $\alpha_j(t)$ satisfy the equations:

$$\frac{d^2}{dt^2}\alpha_j(t) + \omega_j^2 \alpha_j(t) = \omega_j^2 f_j(t), j = 1, 2, \tag{2.13}$$

(where $f_j(t) = \vec{f}(t) \cdot \vec{z}_j$) with adequate initial conditions and whose the solution is quite evident. Thus, the solutions of equations (2.1) to (2.5) may be easily expressed by means of the eigenvectors of the added mass matrix.

2.2. Some Remarks

2.2.1. When the fluid is compressible, ϕ must satisfy a wave hyperbolic equation. It is possible to define an added mass matrix H depending on time, so that the term $H\frac{d^2\vec{s}}{dt^2}$ must be replaced by a time-convolution one $H * \frac{d^2\vec{s}}{dt^2}$; that means that the action of a tube on itself or another one is not instantaneous, due to the fact that the pressure propagates with a finite velocity (see [9], [10]).

2.2.2. In this section, the geometrical variations of Ω due to the tube motion, have been neglected; these variations may be taken into a account, leading to additional terms into (2.11), of the form $D\frac{d\vec{s}}{dt}$ of damping type (see [11]).

2.2.3. In the case of large tube bundle in which the tubes are placed at the tops of a regular rectangular network, the homogenization technique can be used to liken the fluid-bundle system as an homogeneous material. In this situation, the added mass matrix is replaced by an integro-differential operator (in fact a pseudo-differential operator of order zero). We refer to [10], [12], [13], for a general theory and the applications.

2.3. The case of elastic shell

The shell γ can be deformed under the action of the fluid and $W(x,t)$ denotes its small deformation along the normal $\vec{n}$. In this situation, the fluid potential satisfies:

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega, \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \Gamma, \\ \frac{\partial\phi}{\partial n} = \frac{\partial W}{\partial t} & \text{on } \gamma. \end{cases} \tag{2.14}$$

The dynamical equation for γ is

$$\rho_s \frac{\partial^2 W}{\partial t^2} + \mathcal{E}W = p \equiv -\rho\frac{\partial\phi}{\partial t}, \tag{2.15}$$

where ρ_s is the specific density of γ and $\mathcal{E}$ is the elastic stiffness operator; $\mathcal{E}$ is a certain differential operator with respect to the space tangential variables of γ (we do not exhibit it).

We consider the eigenvalues of γ:

$$\mathcal{E}W_j = \rho_s\omega_j^2 W_j, \quad j = 1, 2, \text{ etc.} \tag{2.16}$$

If γ presents certain symmetries, there is an infinite set of eigenmodes such that:

$$\int_\gamma W_j d\gamma = 0, \tag{2.17}$$

and it will be assumed that it is the case.

Because the fluid is incompressible, the third equation (2.14) implies that $\int_\gamma W d\gamma = 0$. Consequently, W can be expressed as

$$W(x,t) = \sum_j{}^* \alpha'_j(t) W_j(x), \tag{2.18}$$

where $\sum^*$ means the summation is done on the indices j for which (2.17) is valid. It inmediately results that ϕ can be written as:

$$\phi(x,t) = -\rho\sum_j{}^* \alpha'_j(t)\chi_j(x), \tag{2.19}$$

where

$$\begin{cases} \Delta \chi_j = 0 \qquad \text{in } \Omega, \\ \frac{\partial \chi_j}{\partial n} = 0 \quad \text{on } \Gamma, \quad \frac{\partial \chi_j}{\partial n} = W_j \quad \text{on } \gamma, \\ \int\limits_{\Omega} \chi_j dx = 0. \end{cases} \tag{2.20}$$

The prime denotes the time-derivative.

The function χ_j are linearly independent. Taking the scalar product of (2.15) with W_j, we have

$$\rho_s(\frac{\partial^2 W}{\partial t^2}, W_j)_\gamma + (\mathcal{E}W, W_j)_\gamma = -\rho(\frac{\partial \phi}{\partial t}, W_j)_\gamma$$

(where $(\varphi, \psi)_\gamma = \int\limits_\gamma \varphi \bar{\psi} d\gamma$) and, thanks to (2.18) and (2.19):

$$\rho_s \alpha_j''(t) + \rho_s \omega_j^2 \alpha_j(t) = -\rho \sum_i {}^* \alpha_i''(t) (\chi_i, W_j)_\gamma$$

in which it is assumed the orthonormalization condition $(W_i, W_j)_\gamma = \delta_{ij}$. The above equation is rewritten as

$$(\rho_s + \rho \mathcal{H}) \alpha''(t) + \mathcal{K} \alpha(t) = 0 \tag{2.21}$$

where $\mathcal{H}$ and $\mathcal{K}$ are infinite matrices

$$\mathcal{H} = ((\chi_i, W_j)_\gamma), \mathcal{K} = \rho_s \ \ diag(\omega_i^2),$$

$\alpha(t)$ is the vector of infinite length with components $\alpha_j(t)$. It is obviously understood that the indices i and j are such that (2.17) is true. $\rho\mathcal{H}$, so defined, is symmetric and positive definite: it represents the generalized added mass.

3. The case of a quiescent viscous fluid

3.1. The equations

Now our objective is to extend the notion of added mass to a viscous fluid. This one is supposed to be at rest so that $\vec{u}$ satisfies the Stokes equations:

$$\rho \frac{\partial \vec{u}}{\partial t} - \nu \Delta \vec{u} + \nabla p = 0 \tag{3.1}$$

$$div \vec{u} = 0, \tag{3.2}$$

where ν is the viscosity.

The boundary condition for $\vec{u}$ is of type adherence, i.e.

$$\vec{u} = 0 \text{ on } \Gamma, \quad \vec{u} = \frac{d\vec{s}}{dt} \quad \text{on } \gamma. \tag{3.3}$$

The dynamical equation for γ is :

$$(m\frac{d^2}{dt^2} + k)\vec{s}(t) = -\int_\gamma \sigma(\vec{u})\vec{n}d\gamma + \vec{f}(t), \tag{3.4}$$

where $\sigma(\vec{u}) = -pI + 2\nu e(\vec{u})$ (stress tensor),

$$e(\vec{u}) = (e_{ij}(\vec{u})), e_{ij}(\vec{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}).$$

It is noted that the field $\vec{u}$ cannot be easily eliminated from the above equation, because the presence of $\frac{\partial \vec{u}}{\partial t}$ in (3.1).

3.2. Some related eigenvalue problems

It is useful to consider the following associated eigenproblem. Let us set

$$\vec{u} = \vec{u}(x)e^{\mu t}, \quad p = p(x)e^{\mu t}, \quad \vec{s}(t) = \vec{s}e^{\mu t},$$

one gets for $\vec{f} \equiv 0$:

$$\begin{cases} \mu\rho\vec{u} - \nu\Delta\vec{u} + \nabla p = 0, \quad div\vec{u} = 0, \\ \vec{u} = 0 \text{ on } \Gamma, \quad \vec{u} = \mu\vec{s} \text{ on } \gamma, \\ (m\mu^2 + k)\vec{s} = -\int_\gamma \sigma(\vec{u})\ \vec{n}d\gamma. \end{cases} \tag{3.5}$$

(3.5) forms a quadratic eigenvalue problem in which μ is the eigenvalue. We have the not obvious result:

Proposition 2.

The system (3.5) has a countable infinite set of eigenvalues $\mu_m, m = 1, 2$, etc. Moreover

$$Re\mu_m < 0 \quad \text{for any } m,$$

(Re denotes the real part), and there are at most 4 complex eigenvalues and these last ones lie inside the circle of radius $(k/m)^{1/2}$ and centered at the origin of the complex plane. The real eigenvalues μ_m tend to $-\infty$ as m increases.

The proof of this proposition is too long to be presented here and may be found in ref. [10], [13], [14].

There is no complex eigenvalue if the viscosity ν is high enough and it can be shown that the real eigenvalues converge to zero as $\nu \to 0$ (see [10]).

Let us denote by $\{\vec{u}_m(x), \vec{s}_m\}$ the eigenvector associated at each eigenvalue μ_m. We have then the important following result:

Proposition 3. [15]

The eigenvectors of (3.5), accompanied with the possible generalized eigenvectors, form a complete basis for the fields $\vec{v}(x)$ with $div\vec{v} = 0$.

The proof of Proposition 3 is based on a theorem of Dunford & Schwartz [16] and cannot be reproduced here. The above results means that any field $\vec{v}(x)$ with $\int\limits_{\Omega} |\vec{v}|^2 dx < +\infty$, $div\vec{v} = 0$ may be expressed as $\vec{v} = \sum\limits_n \alpha_n \vec{u}_n(x)$.

An important particular case of (3.5) is the following self-adjoint eigenproblem:

$$\begin{cases} -\lambda\rho\vec{v} - \nu\Delta\vec{v} + \nabla p = 0, \quad div\vec{v} = 0 \\ \vec{v} = 0 \ \text{ on } \ \partial\Omega = \Gamma \cup \gamma. \end{cases} \tag{3.6}$$

i.e. when the cylinder γ is fixed (be careful to the sign of λ in (3.6)). In this situation we have:

Proposition 4. (Temam [17]).

The problem (3.6) has a countable infinite set of positive eigenvalues λ_m and the eigenvectors $\vec{u}_m$ form a complete set for the vectors whose divergence is zero.

Note that the λ_m are proportional to ν:

$$\lambda_m = \nu\lambda_m^0 \tag{3.7}$$

in which λ_m^0 is the m^{th} eigenvalue of (3.6) for $\nu = 1$.

We observe that a direct consequence of Propositions 2 and 3 is: $\vec{u}(x,t)$ and $\vec{s}(t)$ tend to zero as $t \to \infty$. More precisely:

$$\vec{u}(x,t) = \mathcal{O}(e^{\mu_1 t}) \text{ for large } t \text{ (idem for } \vec{s}(t)).$$

where μ_1 is the eigenvalue of (3.5) of greatest real part (in algebraic value). In other words, the tube oscillates with damping with a frequency smaller than $\omega_0 = (k/m)^{1/2}$.

Let us come back to equations (3.1), (3.2), (3.3) and suppose that the fluid is at rest at $t = 0 : \vec{u}(x, 0) \equiv 0$. From the parabolic character of this system, it results that $\vec{u}$ can be expressed as

$$\vec{u}(x,t) = \int_0^t \int_\gamma g_{st}(x, \xi; t-\tau) \frac{d\vec{s}}{d\tau}(\tau) d\gamma_\xi d\tau \tag{3.8}$$

or more symbolically

$$\vec{u}(x,t) = G_{st} * \frac{d\vec{s}}{dt};$$

g_{st} is the Green function associated to the Stokes equation and then it defines a semi-group. If is interesting to expand g_{st} with the eigenfunctions $\vec{v}_m(x)$ of (3.6).

Setting $A\vec{u} \equiv -\nu\Delta\vec{u} + \nabla p$, we have to solve the problem:

$$\begin{cases} \rho\frac{\partial \vec{u}}{\partial t} + A\vec{u} = 0, \quad div\vec{u} = 0, \vec{u} = 0 \text{ for } t = 0, \\ \vec{u} = 0 \text{ on } \Gamma, \quad \vec{u} = \vec{g} \text{ on } \gamma, \end{cases} \tag{3.9}$$

where $\vec{g}$ is a given function.

For any fields $\vec{\varphi}(x)$ and $\vec{\psi}(x)$, we introduce the standard $L^2(\Omega)$ scalar product

$$(\vec{\varphi}, \vec{\psi}) = \sum_i \int_\Omega \varphi_i(x)\overline{\psi}_i(x) dx, |\vec{\varphi}| = (\vec{\varphi}, \vec{\varphi})^{1/2},$$

and

$$a(\vec{\varphi}, \vec{\psi}) = \nu \sum_{i,j} \int_\Omega \frac{\partial \varphi_i}{\partial x_j} \frac{\partial \overline{\psi}_i}{\partial x_j} dx.$$

Now, the solution of (3.9) is expressed as

$$\vec{u}(x,t) = \sum_{m=1}^{\infty} \alpha_m(t) \vec{v}_m(x),$$

in which the eigenvectors $\vec{v}_m$ of (3.6) are orthonormalized: $(\vec{v}_m, \vec{v}_n) = \delta_{mn}$.

Multiplying (3.9) by $\vec{v}_m$, we have

$$\rho(\frac{\partial \vec{u}}{\partial t}, v_m) + (A\vec{u}, \vec{v}_m) = 0, \tag{3.10}$$

and by the Green identity:

$$(A\vec{u}, \vec{v}_m) = a(\vec{u}, \vec{v}_m) - \int\limits_{\partial\Omega} \sigma(\vec{u})\vec{n} \cdot \vec{v}_m ds.$$

The surface integral cancels since $\vec{v}_m = 0$ on $\partial\Omega$. On the other side:

$$a(\vec{u}, \vec{v}_m) = \rho\lambda_m(\vec{u}, v_m) = (\vec{u}, A\vec{v}_m)$$

and after integrating by parts:

$$\rho\lambda_m(\vec{u}, \vec{v}_m) = a(\vec{u}, \vec{v}_m) - \int\limits_{\gamma} \sigma(\vec{v}_m)\vec{n} \cdot \vec{g} d\gamma;$$

we obtain then, from (3.10):

$$\rho[\frac{d}{dt}(u, \vec{v}_m) + \lambda_m(\vec{u}, \vec{v}_m)] = -\int\limits_{\gamma} \sigma(\vec{v}_m)\vec{n} \cdot \vec{g} d\gamma$$

or

$$\alpha'_m(t) + \lambda_m \alpha_m(t) = g_m(t)$$

where

$$g_m(t) = -\frac{1}{\rho}\int\limits_{\gamma} \sigma(\vec{v}_m)\vec{n} \cdot \vec{g} d\gamma,$$

from which it results that:

$$\alpha_m(t) = \int\limits_{0}^{t} e^{-\lambda_m(t-\tau)} g_m(\tau) d\tau.$$

Whence

$$\vec{u}(x,t) = -\frac{1}{\rho}\sum_{m=1}^{\infty}\int\limits_{0}^{t} e^{-\lambda_m(t-\tau)}\int\limits_{\gamma} \sigma(\vec{v}_m)\vec{n} \cdot \vec{g}(\xi,\tau) d\gamma_\xi d\tau \vec{v}_m(x). \tag{3.11}$$

That explicites the Green kernel for the Stokes equation (3.1). It is observed that $g_{st}(x, \xi; t)$ behaves as $e^{-\lambda_1 t}$ as $t \to +\infty$, where λ_1 is the smallest positive eigenvalue of (3.6).

Now, we consider the function $\vec{g}(x,t)$ constant along the contour γ. For such a function, we can write

$$\vec{g}(t) = \sum_{i=1}^{2} g_i(t)\vec{e}_i$$

and the corresponding $\vec{u}$ is $\vec{u} = \vec{u}_1 + \vec{u}_2$ where

$$\vec{u}_i(x,t) = -\frac{1}{\rho}\sum_m \int_0^t e^{-\lambda_m(t-\tau)} g_i(\tau)d\tau(\vec{k}_m \cdot \vec{e}_i)\vec{v}_m(x) \tag{3.12}$$

in which

$$\vec{k}_m = \int_\gamma \sigma(\vec{v}_m)\vec{n}d\gamma.$$

Let us set

$$\vec{\phi}_i(x,t) = -\frac{1}{\rho}\sum_m (\vec{k}_m \cdot \vec{e}_i)\vec{v}_m(x)e^{-\lambda_m t}. \tag{3.13}$$

Clearly

$$\vec{u}_i(x,t) = \vec{\phi}_i * g_i$$

and $\vec{\phi}_i$ is the solution of

$$\begin{cases} \rho\frac{\partial\vec{\phi}_i}{\partial t} - \nu\Delta\vec{\phi}_i + \nabla p_i = 0, \quad div\vec{\phi}_i = 0 \quad \text{in } \Omega \\ \phi_i = 0 \text{ on } \Gamma \text{ and } \vec{\phi}_i = \delta(t)\vec{e}_i \text{ on } \gamma, \\ \text{with zero initial condition inside } \Omega, \end{cases} \tag{3.16}$$

where $\delta(t)$ is the Dirac time-distribution at $t = 0$ ($\vec{\phi}_i$ is then the impulsional response). Replacing $\vec{g}$ by $\frac{d\vec{s}}{dt}$, we have

$$\vec{u}(x,t) = \sum_{i=1}^{2} \phi_i(x,t) * \frac{ds_i}{dt}(t). \tag{3.17}$$

Remark: The decreasing of $\vec{\phi}_i$ with respect to time corresponds to a loss of kinetic energy caused by viscosity; it results that the vectors $\vec{\phi}_i$ are, in norm, uniformly bounded by a constant independent of ν. ■

3.3. The added mass and viscous damping

Let us turn out to the first equation (3.5) and taking its divergence, we obtain:

$$\Delta p = 0. \tag{3.18}$$

In order to get boundary conditions for the pressure, we multiply (3.5) by $\vec{n}$ on $\partial\Omega$. Thus

$$\frac{\partial p}{\partial n} = \begin{cases} -\rho\frac{d^2\vec{s}}{dt^2} \cdot \vec{n} + \nu\Delta\vec{u} \cdot \vec{n} & \text{on } \gamma, \\ \nu\Delta\vec{u} \cdot \vec{n} & \text{on } \Gamma \end{cases}. \tag{3.19}$$

Because (3.18) and (3.19) are direct consequences of (3.5), the following compatibility condition necessarily holds

$$\int\limits_{\partial\Omega} \Delta\vec{u}\cdot\vec{n}ds = 0 \tag{3.20}$$

(to obtain the equality, integrate (3.18) on Ω and use the Green identity).

It is easily seen that the pressure may be decomposed as $p = p_0 + q$ in which the first term results from the tube motion and the second from the viscosity effects, i.e.,

$$\begin{cases} \Delta p_0 = 0 & \text{in } \Omega, \\ \frac{\partial p_0}{\partial n} = 0 & \text{on } \Gamma, \quad \frac{\partial p_0}{\partial n} = -\rho\frac{d^2\vec{s}}{dt^2}\cdot\vec{n} \text{ on } \gamma, \end{cases} \tag{3.21}$$

$$\begin{cases} \Delta q = 0 & \text{in } \Omega, \\ \frac{\partial q}{\partial n} = \nu\Delta\vec{u}\cdot\vec{n} & \text{on } \partial\Omega. \end{cases} \tag{3.22}$$

Clearly

$$-\int\limits_{\gamma} p_0\vec{n}d\gamma = \rho H\frac{d^2\vec{s}}{dt^2}, \tag{3.23}$$

where ρH is the added mass matrix defined in Section 2 for a perfect fluid.

The second term q linearly depends on $\nu\Delta\vec{u}\cdot\vec{n}$ and then on $\frac{d\vec{s}}{dt}$ via the relation (3.17). If $N(x,\xi)$ denotes the Neumann function associated with the Laplacian operator and Neumann boundary condition on $\partial\Omega$ (see Appendix), we have:

$$q(x,t) = \nu\int\limits_{\partial\Omega} N(x,\xi)(\Delta\vec{u}(\xi,t)\cdot\vec{n}(\xi))ds_\xi$$

and then, from (3.17):

$$q(x,t) = \nu\sum_{i=1}^{2}\int\limits_{\partial\Omega} N(x,\xi)\int\limits_0^t \Delta\vec{\phi}_i(\xi,t-\tau)\frac{ds_i}{d\tau}(\tau)\cdot\vec{n}(\xi)ds_\xi. \tag{3.24}$$

It is important to note that the function $\vec{u}(x,t)$ (i.e. the $\vec{\phi}_j$) must be sufficiently regular in order that the boundary condition for (3.22) has a sense ($\Delta\vec{u}\cdot\vec{n}$ must be a distribution belonging to the Sobolev space $H^{-1/2}(\gamma)$, see Lions-Magenes [18]). It is possible if γ is smooth what we assume.

The resultant of the viscous forces acting on γ is

$$2\nu\int\limits_{\gamma} e(\vec{u})\vec{n}d\gamma = 2\nu\sum_i\int\limits_{\gamma} e(\vec{\phi}_i)\vec{n}d\gamma * \frac{ds_i}{dt}, \tag{3.25}$$

(e is the strain tensor). Collecting (3.23), (3.24) and (3.25), the dynamical equation becomes:

$$(m+\rho H)\frac{d^2\vec{s}}{dt^2}(t) + \nu D * \frac{d\vec{s}}{dt}(t) + k\vec{s}(t) = \vec{f}(t) \tag{3.26}$$

in which D is a matrix, of order two, depending on time, whose the i^{th} column is:

$$-2\int\limits_{\gamma} e(\vec{\phi}_i(x,t))\vec{n}d\gamma + \int\limits_{\gamma} \vec{n}(x) \int\limits_{\partial\Omega} N(x,\xi)(\Delta\vec{\phi}_i(\xi,t)\cdot\vec{n}(\xi))ds_\xi d\gamma_x.$$

The matrix D depends also on viscosity through the functions $\vec{\phi}_i$ but it is uniformly bounded with respect to ν, from the remark of subsection 3.2. νD is the *damping matrix*, which is $\mathcal{O}(\nu)$ as $\nu \to 0$. It is observed that the viscosity effect occurs in (3.26) via a time-convolution operation. In the absence of external force $\vec{f}$, the displacement vector $\vec{s}(t)$ obviously tends to zero as t increases to infinity.

Remark: The damping term in (3.26) may be written in another form. We have indeed:

$$\int\limits_0^t D(t-\tau)\frac{d\vec{s}}{dt}(\tau)d\tau = -\int\limits_0^t D'(t-\tau)\vec{s}(\tau)d\tau + D(t)\vec{s}(t) - D(0)\vec{s}(0).$$

Thus, the viscous force can be modelled by an added stiffness matrix and an external force depending on the initial condition. The two formulations obviously are equivalent; it is only a question of terminology. However, for low viscosity, the convolution term with D' may be neglected, being of order of ν (see relations (3.7) and (3.11)); it is maybe the reason for which certain authors prefer to use the added stiffness rather than damping to simulate the viscous effects. ■

3.4. Dynamical behaviour of the structure when $\nu \to 0$.

Let $\vec{s}_\nu$ and $\vec{s}_0$ be the displacement of the tube for $\nu \neq 0$ and for $\nu = 0$. We have then, in absence of external force:

$$\begin{cases} (m+\rho H)\vec{s}_\nu{}'' + \nu D * \vec{s}_\nu{}' + k\vec{s}_\nu = 0, \\ (m+\rho H)\vec{s}_0{}'' + k\vec{s}_0 = 0, \end{cases}$$

with the same initial conditions (the prime denotes the time derivative). Subtracting the two above equations and after multiplying by $\vec{s}_\nu' - \vec{s}_0'$ and integrating from 0 to $t < T$, where T is fixed and finite, we get:

$$\begin{aligned} &\frac{1}{2}[(m+\rho H)(\vec{s}_\nu(t) - \vec{s}_0(t))\cdot(\vec{s}_\nu{}'(t) - \vec{s}_0{}'(t)) \\ &+ k|\vec{s}_\nu(t) - \vec{s}_0(t)|^2] = -\nu\int\limits_0^t (D * \vec{s}_\nu{}')\cdot(\vec{s}_\nu{}' - \vec{s}_0{}')dt. \end{aligned} \tag{3.27}$$

But $\vec{s}_0{}', \vec{s}_\nu, \vec{s}_\nu{}', D$ are bounded with respect to time, so that we have, since the matrix $m + \rho H$ is positive definite:

$$|\vec{s}_\nu(t) - \vec{s}_0(t)| = \mathcal{O}(\nu^{1/2}), \tag{3.28}$$

$$|\vec{s}_\nu{}'(t) - \vec{s}_0{}'(t)| = \mathcal{O}(\nu^{1/2}), \tag{3.29}$$

where these estimates are valid on any bounded interval $[0, T]$.

Thus $s_\nu(t) \to s_0(t)$ uniformly on $[0, T]$ as $\nu \to 0$. The convergence obviously is not uniform for infinite T since $\vec{s}_\nu(\infty) = 0$ while $\vec{s}_0(t)$ is oscillatory for any time. Note that the estimates (3.28) and (3.29) can be improved by injecting it into the right hand side of equation (3.27). Thus we have proved the following result.

Proposition 5.

When the viscosity tends to zero, the displacement vector $\vec{s}_\nu$ converges to the corresponding one for $\nu = 0$, over any bounded time interval.

Let us come back to the eigenproblem (3.5). In the proof of existence of the real eigenvalues μ_m (see [19]), it was shown there exists an eigenvalue $\lambda_{m'}$ of (3.6) such that:

$$0 > \mu_m > -\lambda_{m'} = -\nu \lambda^0_{m'}.$$

This inequality implies that the real μ_m tend to zero as the viscosity becomes smaller and smaller.

It has been seen that $\vec{s}_\nu(t)$ may be expressed with the eigenvectors of (3.5) (Propositions 2 and 3) while $\vec{s}_0(t)$ is also expressed with the eigenvectors $\vec{z}_j$ of the matrix H. Because $\vec{s}_\nu(t)$ converges to $\vec{s}_0(t)$ (which is oscillatory), that means that the complex eigenvalues of (3.5) tend, as $\nu \to 0$, to the numbers $\pm i\omega_j$, where ω_j are the eigenfrequencies of the tube when placed in a perfect fluid (otherwise $\vec{s}_\nu$ would not converge to $\vec{s}_0$!).

As a corollary of Proposition 5, we have:

Proposition 6.

The complex eigenvalues of the system (3.5) corresponding to a viscous fluid, converge as $\nu \to 0$ to $\pm i\omega_j$ where ω_j are the eigenfrequencies of the mechanical structure immersed in a perfect fluid, and the real eigenvalues tend to zero.

A more direct proof of this result will be presented in a forthcoming paper [20].

Remark. All these results are extended to the case of several parallel tubes. The case of an elastic shell can be investigated in a similar way; but a difficulty lies in the fact that there is no theorem like Proposition 2 allowing localization of the eigenvalues μ.

4. The case of a flowing fluid.

4.1 Linearized Navier-Stokes equations

In this section, we consider a fluid flowing in a channel whose the domain is denoted Ω of boundary Γ:

$$\Gamma = \Gamma_{lat} \cup \Gamma_{in} \cup \Gamma_{out};$$

Γ_{lat} is the (physical) lateral wall while Γ_{in} and Γ_{out} are respectively the inlet and outlet (the fluid enters by Γ_{in} and goes out by Γ_{out}, Fig. 2).

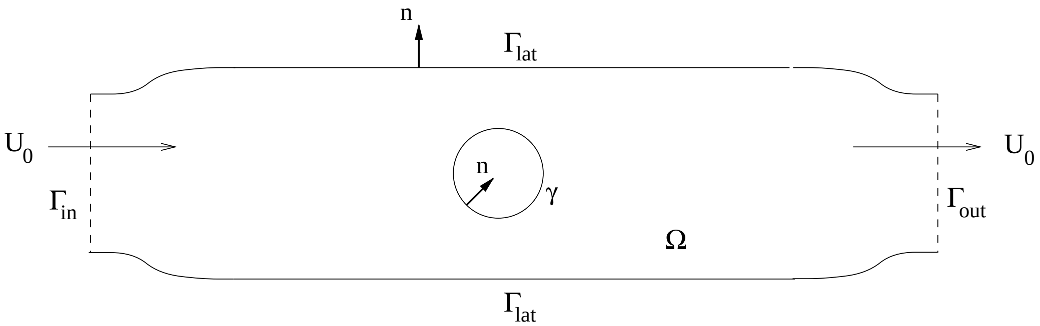

Fig. 2 Typical configuration.

A mechanical harmonic oscillator (of wall γ) is placed inside Ω. A steady state of the coupled system is now considered. For this equilibrium situation, the steady flow $\vec{u}_0(x)$ obeys to the Navier-Stokes equations in Ω:

$$\begin{cases} \rho(\vec{u}_0 \cdot \nabla)\vec{u}_0 - \nu\Delta\vec{u}_0 + \nabla p_0 = 0, \\ div\vec{u}_0 = 0, \\ \vec{u}_0 = 0 \text{ on } \gamma \text{ and } \Gamma_{lat}, \\ \vec{u}_0(x) = \vec{U}_0(x) \text{ on } \Gamma_0 \equiv \Gamma_{in} \cup \Gamma_{out}, \end{cases} \tag{4.1}$$

where $\vec{U}_0(x)$ is the prescribed flow at the inlet and the outlet (one could also consider prescribed values of the pressure at Γ_{in} and Γ_{out}). That obviously suppose that

$$\int_{\Gamma_0} \vec{U}_0 \cdot \vec{n} d\Gamma_0 = 0,$$

what is the total mass conservation law.

Under the action of fluid, the solid body γ is moved with a translation vector $\vec{s}_0$ given by

$$k\vec{s}_0 = -\int\limits_{\gamma} \sigma(\vec{u}_0)\vec{n}d\gamma.$$

We suppose that k is sufficiently high so that the geometrical variations of Ω are negligible.

The unsteady state of the systems is governed by:

$$\begin{cases} \rho(\frac{\partial\vec{\mathcal{U}}}{\partial t} + (\vec{\mathcal{U}}\cdot\nabla)\vec{\mathcal{U}}) - \nu\Delta\vec{\mathcal{U}} + \nabla\mathcal{P} = 0, \\ div\vec{\mathcal{U}} = 0, \\ \vec{\mathcal{U}} = \frac{d\vec{\xi}}{dt} \text{ on } \gamma, \quad \vec{\mathcal{U}} = \vec{U}_0 \text{ on } \Gamma_0, \\ \vec{\mathcal{U}} = 0 \text{ on } \Gamma_{lat}, \\ (m\frac{d^2}{dt^2} + k)\vec{\xi}(t) = -\int\limits_{\gamma} \sigma(\vec{\mathcal{U}})\vec{n}d\gamma, \end{cases} \tag{4.2}$$

in which $\vec{\xi}(t)$ denotes the displacement of γ.

In writing the boundary conditions for $\vec{\mathcal{U}}$, it was implicitly assumed that the inlet and outlet are far enough from γ so that the values of $\vec{U}_0$ on Γ_{in} and Γ_{out} are not perturbed by the motion of the moving body.

We want to study the behaviour of the system around the steady state. Putting

$$\vec{\mathcal{U}} = \vec{u}_0 + \vec{u}, \quad \mathcal{P} = p_0 + p, \quad \vec{\xi} = \vec{s}_0 + \vec{s},$$

where the disturbances $\vec{u}, p, \vec{s}$ are supposed to be small; we have then, neglecting the second order term for the fluid flow:

$$\begin{cases} \rho[\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} + (\vec{u}\cdot\nabla)\vec{u}_0] - \nu\Delta\vec{u} + \nabla p = 0, \\ div\vec{u} = 0, \\ \vec{u} = 0 \text{ on } \gamma, \quad \vec{u} = \frac{d\vec{s}}{dt} \text{ on } \gamma, \\ (m\frac{d^2}{dt^2} + k)\vec{s}(t) = -\int\limits_{\gamma} \sigma(u)\vec{n}d\gamma. \end{cases} \tag{4.4}$$

In order to investigate the stability of the steady state (4.1), it is useful to consider the eigenvalues μ of the linearized equation (4.3) and (4.4) (replace the time-derivative by the factor μ). The infinite set of eigenvalues is located inside the region of the complex plane defined by the following inequalities (see Fig. 3):

$$\begin{cases} |z| \leq \frac{B(\vec{u}_0) + \sqrt{B(u_0)^2 + 4\omega_0^2|sin\theta|^2}}{2|sin\theta|}, \\ (\theta = \text{ argument of } z \text{ }), \\ Rez \leq -\xi_1, \end{cases} \tag{4.5}$$

in which $\omega_0^2 = k/m, B(\vec{u}_0) = \rho \max\limits_{x \in \Omega} \|r(\vec{u}_0(x))\|,^{(*)}$ $r(\vec{u}_0)$ is the skew-symmetric part of $\nabla \vec{u}_0$, and ξ_1 is the first eigenvalue of

$$\begin{cases} -\nu \Delta \vec{v} + \rho e(\vec{u}_0)\vec{v} + \nabla \varphi = \xi \rho \vec{v}, \\ div \vec{v} = 0, \\ \vec{v} = 0 \text{ on } \Gamma_o \cup \Gamma_{lat}, \\ \sigma(\vec{v})\vec{n} = \xi \frac{m\vec{v}}{\ell(\gamma)} \text{ on } , \ell(\gamma) = \int\limits_\gamma d\gamma. \end{cases} \tag{4.6}$$

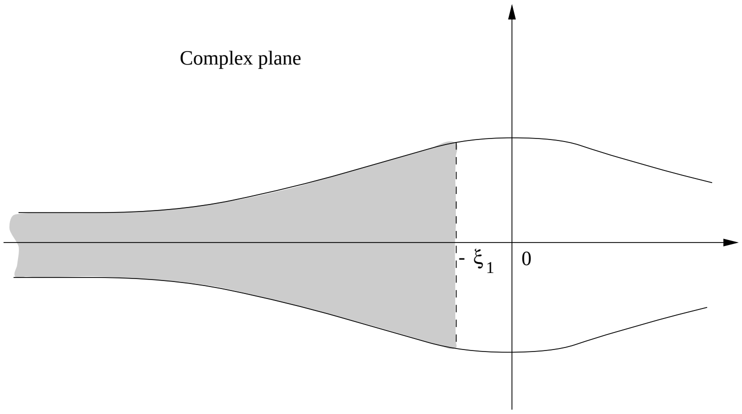

Fig. 3 Zone of location of the eigenvalues μ_m (dashed region) when the fluid-structure system is stable.

Moreover, $Re\mu_n \to -\infty$ as $n \to \infty$, and there are at most a finite number of eigenvalues with positive real part. The proof of these results is given in [19], [21] (see also [10]). Note that (4.6) is a self-adjoint eigenvalue problems so that ξ_1 (and the other eigenvalues) is a real number. This localization of the μ_m requires that the flow $\vec{u}_0$ is smooth in order that the quantity $B(\vec{u}_0)$ can be defined$^{(**)}$. Obviously when $\vec{u}_0 \equiv 0$, we are in the situation of Proposition 2. Observe that $-\xi_1$, for $\vec{u}_0 \equiv 0$, provides an upper bound for the real parts of the eigenvalues of (3.5).

The coupled system is stable when all eigenvalues μ_n have strictly negative real parts and unstability occurs as at least one eigenvalue has a positive real part. The system is obviously stable when $\vec{U}_0 \equiv 0$. As $\vec{U}_0$ increases, one eigenvalue can cross the imaginary axis and the system becomes unstable. But it is shown in [21] that none μ crosses the axis at the origin of complex plane; in other words, only pairs of conjugate complex eigenvalues can cross this axis and, as a consequence, the unstability appears via the

---

(*) $\|\cdot\|$ is the usual norm for matrices.

(**) It is sufficient that $\vec{u}_0$ is piecewise C^1.

occurrence of time-periodic fluctuations whose the frequency is related to the imaginary part of the crossing pair. Such a phenomenon is known as Hopf bifurcation (see [22], [23]).

Let us come back to the linearized Navier-Stokes equations (4.3). As the previous section, $\vec{u}(x,t)$ may be written as

$$\vec{u}(x,t) = \int_0^t G(x, t-\tau)\frac{d\vec{s}}{d\tau}(\tau)d\tau \tag{4.7}$$

where the semi-group G takes account of the linearized convection terms. G may be explicited by means of the eigenvectors associated with the linearized Navier-Stokes problem$^{(*)}$

$$\begin{cases} -\nu\Delta\vec{v} + \rho[(\vec{u}_0\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\vec{u}_0] + \nabla\varphi = \lambda\rho\vec{v} \\ div\vec{v} = 0, \quad \vec{v} = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.8}$$

(4.8) has an infinite set of eigenvalues located inside a half-band of the complex plane axed on the real axis. We have indeed:

$$a(\vec{v},\vec{v}) + \rho(\vec{u}_0\cdot\nabla\vec{v},\vec{v}) + \rho(\vec{v}\cdot\nabla\vec{u}_0,\vec{v}) = \lambda\rho|v|^2$$

in which the second term disappears since $div\vec{v} = 0$, and taking the real and imaginary parts of the above equality:

$$\begin{cases} a(\vec{v},\vec{v}) + \rho(e(\vec{u}_0)\vec{v},\vec{v}) = \rho Re\lambda|\vec{v}|^2, \\ Im(r(\vec{u}_0)\vec{v},\vec{v}) = Im\lambda|\vec{v}|^2, \end{cases}$$

from which it results that:

$$\begin{cases} Re\lambda \geq \theta_1, \\ |Im\lambda| \leq \max\limits_{x\in\Omega}\|r(\vec{u}_0(x))\|, \end{cases}$$

in which θ_1 is the smallest eigenvalue of the self-adjoint equations:

$$\begin{cases} -\nu\Delta\vec{\psi} + e(\vec{u}_0)\psi + \nabla q = \theta\rho\vec{\psi}, \\ div\vec{\psi} = 0, \quad \vec{\psi} = 0 \text{ on } \partial\Omega. \end{cases}$$

Clearly, because $Re\lambda$ is bounded below by a finite number, (4.8) has only a finite number (possibly equal to zero) of eigenvalues with negative real parts.

(*) And those of the adjoint problem. Adapting the proof of reference [15] in an adequate manner, it can be proved that the eigenvectors form a complete basis.

Thus, the behaviour of the semi-group G, for large times, is controlled by the eigenvalue of (4.8) of smallest real part. In particular $G(t)$ tends to zero when θ_1 is positive. It is also noted that $\vec{u}$ may be written:

$$\vec{u} = \sum_i \vec{\phi}_i * \frac{ds_i}{dt},$$

whose the $\vec{\phi}_i$ are the impulsional response of (3.16) written with the additional linearized convection terms.

4.2. Added mass and damping.

We presently take the divergence of the first equation (4.3), obtaining:

$$\rho div[(\vec{u}_0 \cdot \nabla)\vec{u} + (\vec{u} \cdot \nabla)\vec{u}_0] + \Delta p = 0.$$

But

$$\begin{aligned} div(\vec{u}_0 \cdot \nabla)\vec{u} &= \sum_i \frac{\partial}{\partial x_i}(\sum_j u_{0j}\frac{\partial}{\partial x_j})u_i \\ &= \sum_{i,j} \frac{\partial u_{0j}}{\partial x_i}\frac{\partial u_i}{\partial x_j} \text{ because } div\vec{u}_0 = 0. \end{aligned}$$

In a similar way, $div(\vec{u} \cdot \nabla)\vec{u}_0$ gives the same expression. Finally, we have

$$-\Delta p = 2\rho \sum_{i,j} \frac{\partial u_{0j}}{\partial x_i}\frac{\partial u_i}{\partial x_j}. \tag{4.9}$$

The boundary condition for the pressure is obtained by multiplying (4.3) by $\vec{n}$:

$$\frac{\partial p}{\partial n} = -\rho \frac{d^2\vec{s}}{dt^2} \cdot \vec{n}\chi_\gamma(x) + \nu\Delta\vec{u} \cdot \vec{n} - \rho\vec{z}(x,t) \cdot \vec{n} \tag{4.10}$$

where χ_γ is equal to 1 for $x \in \gamma$ and zero elsewhere, and

$$\vec{z}(x,t) = (\vec{u}_0 \cdot \nabla)\vec{u} + (\vec{u} \cdot \nabla)\vec{u}_0.$$

It is easy to see that:

$$\begin{cases} \vec{z} = 0 & \text{on } \Gamma_{lat}, \\ \vec{z} = (\frac{d\vec{s}}{dt} \cdot \nabla)\vec{u} & \text{on } \gamma, \\ \vec{z} = (\vec{U}_0 \cdot \nabla)\vec{u} & \text{on } \Gamma_0 \equiv \Gamma_{in} \cup \Gamma_{out}. \end{cases} \tag{4.11}$$

The pressure p is decomposed as follows:

$$p = q_0 + q_1 + q_2 + q_3, \tag{4.12}$$

where the $q_j's$ are the solutions of the cascade of equations:

$$\begin{cases} \Delta q_0 = 0 & \text{in } \Omega, \\ \frac{\partial q_0}{\partial n} = -\rho \frac{d^2 \vec{s}}{dt^2} \cdot \vec{n} & \text{on } \gamma, \\ \frac{\partial q_0}{\partial n} = 0 & \text{on } \Gamma_0 \cup \Gamma_{lat}, \end{cases} \tag{4.13}$$

$$\begin{cases} -\Delta q_1 = \rho \sum\limits_{ij} \frac{\partial u_{0j}}{\partial x_i} \frac{\partial u_i}{\partial x_j} \equiv \rho div(\vec{u} \cdot \nabla)\vec{u}_0 \quad \text{in} \quad \Omega \\ \frac{\partial q_1}{\partial n} = -\rho(\vec{u} \cdot \nabla)\vec{u}_0 \cdot \vec{n} \equiv -\rho((\frac{d\vec{s}}{dt} \cdot \nabla)\vec{u}_0) \cdot \vec{n} \quad \text{on} \quad \gamma, \\ \frac{\partial q_1}{\partial n} = 0 \quad \text{on} \quad \Gamma_0 \cup \Gamma_{lat}, \end{cases} \tag{4.14}$$

$$\begin{cases} -\Delta q_2 = \rho div(\vec{u}_0 \cdot \nabla)\vec{u} \quad \text{in} \quad \Omega \\ \frac{\partial q_2}{\partial n} = -\rho(\vec{u}_0 \cdot \nabla)\vec{u} \cdot \vec{n} \equiv -\rho(\vec{U}_0 \cdot \nabla)\vec{u} \cdot \vec{n} \quad \text{on} \quad \Gamma_0 \\ \frac{\partial q_2}{\partial n} = 0 \quad \text{on} \quad \gamma \cup \Gamma_{lat}, \end{cases} \tag{4.15}$$

$$\begin{cases} -\Delta q_3 = 0 \quad \text{in} \quad \Omega, \\ \frac{\partial q_3}{\partial n} = \nu \Delta \vec{u} \cdot \vec{n} \quad \text{on entire} \quad \partial\Omega. \end{cases} \tag{4.16}$$

We already know that (4.13) has a solution q_0 which linearly depends on $\frac{d^2\vec{s}}{dt^2}$. It is necessary to check that (4.14) and (4.15) are solvable. To do that, it suffices to observe that the two compatibility conditions hold:

$$\begin{aligned} \int\limits_{\Omega} div(\vec{u} \cdot \nabla)\vec{u}_0 dx &= \int\limits_{\partial\Omega} (\vec{u} \cdot \nabla)\vec{u}_0 \cdot \vec{n} ds \\ &= \int\limits_{\gamma} (\frac{d\vec{s}}{dt} \cdot \nabla)\vec{u}_0 \cdot \vec{n} d\gamma, \end{aligned}$$

$$\begin{aligned} \int\limits_{\Omega} div(\vec{u}_0 \cdot \nabla)\vec{u} dx &= \int\limits_{\partial\Omega} (\vec{u}_0 \cdot \nabla)\vec{u} \cdot \vec{n} ds \\ &= \int\limits_{\Gamma_0} (\vec{U}_0 \cdot \nabla)\vec{u}_0 \cdot \vec{n} d\Gamma_0 \ ; \end{aligned}$$

that results from Green identity and (4.11). Then (4.15) and (4.16) are well-posed and that implies immediately that, for (4.16), the condition $\int\limits_{\partial\Omega} \Delta\vec{u} \cdot \vec{n} ds = 0$ is automatically fulfilled (remind that (4.9) and (4.10) are direct consequences of (4.3)).

Looking for the equations (4.12) to (4.16), we note, from (4.7), that q_1, q_2, q_3 may be written in the form:

$$\begin{cases} q_1(x,t) = \rho Z_1(x,t) * \frac{d\vec{s}}{dt} + \rho \hat{Z}_1 \frac{d\vec{s}}{dt}, \\ q_2(x,t) = \rho Z_2(x,t) * \frac{d\vec{s}}{dt}, \\ q_3(x,t) = \rho Z_3(x,t) * \frac{d\vec{s}}{dt}, \end{cases} \tag{4.17}$$

in which the Z_i and $\hat{Z}_1$ are linear operators obviously depending on $\vec{u}_0$ (via the Neumann function of the Laplacian operator on Ω); $\hat{Z}_1$ is time-independent. The partial pressure p_0, so defined, leads in the dynamical equation, to the classical added mass matrix ρH.

Finally using (4.17), the displacement vector of γ satisfies a relation of the form:

$$(m+\rho H)\frac{d^2\vec{s}}{dt^2} + \rho(D_1 * \frac{d\vec{s}}{dt} + D_2\frac{d\vec{s}}{dt}) + \nu D_3 * \frac{d\vec{s}}{dt} + k\vec{s} = 0, \tag{4.18}$$

in which the matrices D_1, D_2 depend linearly on $\vec{u}_0$ (dependence of $\vec{u}_0$ is nonlinear for D_3).

Thus the linearized convection leads to additional damping terms in the dynamical equation, while the standard added mass remains unchanged. As it was already remarked in Subsection 3.3., $\rho D_1 * \frac{d\vec{s}}{dt}$ may be transformed into an added stiffness term $\rho(-D_1' * \vec{s} + D_1\vec{s})$.

It is important to observe that this added mass is always valid even if the quadratic term $(\vec{u}\cdot\nabla)\vec{u}$ is kept in equation (4.3), but $\vec{u}$ nonlinearly depends on $\frac{d\vec{s}}{dt}$ and, from this fact, cannot obviously be explicited.

5. Conclusion.

In this paper, it was shown that the added mass for a moving body immersed in a fluid moving in an incompressible liquid does not depend on the viscosity ν and the flow around the structure, and it can be calculated as if the fluid is perfect and quiescent. The viscous effects can be then modelled by a time-convolution damping term of order ν. In the case of small perturbations near an equilibrium state when the fluid flows around the mechanical structure, the linearized convection introduces supplementary damping terms, one classical and one of convolution type and both proportional to the fluid density.

References

[1] S.S. Chen, Flow-Induced Vibration of circular cylindrical structures, *Hemisphere Publishing Corporation* (1987).

[2] D. Jagannath, M.P. Paidoussis, "Solid-fluid interaction in the vibration of nuclear fuel bundles" *Proc. BNES Int. Conf. Vibrat. Nucl. Plants, Keswick, U.K.* (1978).

[3] M.P. Paidoussis, "Fluidelastic vibration of cylinder arrays in axial and cross-flow: state of the art", J. Sound Vibrat, **76**(3), (1981), pp. 329-360.

[4] M.P. Paidoussis, "A review of flow-induced vibrations in reactors and reactor components", Nucl. Eng. Design, **74**, (1982), pp. 31-60.

[5] M.P. Paidoussis, D. Mavriplis, S.J. Price, "A potential-flow theory for the dynamics of cylinders arrays in a cross-flow" *J. Fluids Mech.*, **146**, (1984), pp. 27-252.

[6] H. Morand, R. Ohayon, "Interactions Fluides-Structures, Masson, Collection RMA, Paris (1992).

[7] M. Ilnou-Zahir, J. Planchard, "Natural frequencies of tube bundle in an incompressible fluid", *Comp. Meth. Appl. Mech. Eng.*, **41**, (1983), pp. 47-68.

[8] C. Conca, J. Planchard, M. Vanninathan, "Existence and location of eigenvalues for fluid-solid structures", *Comp. Meth. Appl. Mech. Eng.*, **77**, (1989), pp. 253-291.

[9] J. Planchard, "Comportement dynamique des faisceaux de tubes immergés dans un fluide", in "Aspects Théoriques et Numériques de la Dynamique des Structures" (Y. Bamberger, J. Donéa, H. Laval, J. Planchard, R.P. Shaw), *Eyrolles, Collection des Etudes et Recherches d'EDF, Paris*, (1988).

[10] C. Conca, J. Planchard, B. Thomas, M. Vanninathan, "Problèmes Mathématiques en Couplage Fluide-Structure" *Eyrolles, Collection des Etudes et Recherches d'EDF*, Paris, (1994).

[11] J. Planchard, B. Thomas, "On the added mass, damping diffness matrices for an elastic structure placed in a potential cross-flow", Proceedings "Computational Methods for Fluid-Structure Interaction" (Edits. J. Crolet, R. Ohayon), Longman, (1994), pp. 1-16.

[12] J. Planchard, "Global behaviour of large elastic tube bundle immersed in a fluid", *Comput. Mech.*, **2**(55), (1987), pp. 105-118.

[13] C. Conca, J. Planchard, M. Vanninathan, "Fluids and Periodic Structures", *Masson, Collection RMA, Paris*, (1995).

[14] C. Conca, M. Duran, J. Planchard, "A quadratic eigenvalue problem involving Stokes equations" *Comp. Meth. Appl. Mech. Eng.*, **100**, (1992), pp. 295-313.

[15] J. Planchard, “Sur les vecteurs propes d’un problème d’interaction fluide-stucture”, *Electricité de France, DER, Report No. 94NJ00093*, (1993).

[16] N. Dunford, J.T. Schwartz, “Linear Operators”, *Interscience, New York*, (1962).

[17] R. Témam, “Navier-Stokes Equations”, *North Holland, Amsterdam* (1977).

[18] J. L. Lions, E. Magenes, “Problèmes aux Limites non Homogènes et Applications”, *Dunod, Paris* (1968).

[19] J. Planchard, B. Thomas, “On the dynamical stability of tube-bundles in cross-flow”, *J. Fluids and Structures*, **7**, (1993), pp. 321-339.

[20] C. Conca, A. Osses, J. Planchard, “Asymptotic analysis relating spectral models in fluid-solid vibrations”, (paper in preparation).

[21] J. Planchard, M. Thomas, “Sur la localisation des valeurs propres d’un problème d’interaction fluide-structure”, *Bulletin des Etudes et Recherches d’Electricité de France*, Série C, No. 4, (1991), pp. 81-85.

[22] D.D. Joseph, “Stability of Fluid Motions”, Springer-Verlag, Berlin, (1976).

[23] D.H. Sattinger, “Topics in Stability and Bifurcation Theory”, Springer-Verlag, Berlin, (1973).

Appendix

The Neumann function

One has to solve

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = h & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain. In order that (1) has a solution, it is necessary that f and h satisfy the compatibility condition:

$$\int_{\Omega} f dx + \int_{\partial\Omega} h ds = 0, \tag{2}$$

and u is uniquely determined if one imposes the condition:

$$\int_{\Omega} u dx + \int_{\partial\Omega} u ds = 0. \tag{3}$$

A way to express u in an integral form with respect to f and h, is to use the Steklov eigenfunctions of the Laplacian operator on Ω. They are defined by

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega \\ \frac{\partial v}{\partial n} = \lambda v & \text{on } \partial\Omega, \quad \frac{\partial}{\partial n} = \text{outward normal derivative} \end{cases} \cdot \tag{4}$$

(4) has an infinite set of eigenvalues λ_m:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \ldots$$

and the corresponding eigenvectors form a complete set. The first eigenfunction is $v_0 \equiv 1$ and all the other ones are orthogonal to it in the sense:

$$\int_{\Omega} v_m dx + \int_{\partial\Omega} v_m ds = 0. \tag{5}$$

The v_m are orthonormalized such that

$$\int_{\Omega} v_m v_n dx + \int_{\partial\Omega} v_m v_n ds = \delta_{mn}. \tag{6}$$

From the condition (3), we express u as

$$u(x) = \sum_{n=1}^{\infty} \alpha_n v_n(x)$$

so that (3) in automatically satisfied from (5).

From the following equality:

$$\int\limits_{\Omega} \nabla u \cdot \nabla v_m dx = \int\limits_{\Omega} f v_m dx + \int\limits_{\partial\Omega} h v_m ds$$

for any m, we deduce, thanks to (6):

$$\alpha_m = \frac{1}{\lambda_m}(\int\limits_{\Omega} f v_m dx + \int\limits_{\partial\Omega} h v_m ds),$$

and then

$$u(x) = \int\limits_{\Omega} N(x,\xi) f(\xi) d\xi + \int\limits_{\partial\Omega} N(x,\xi) h(\xi) ds_{\xi}, \tag{7}$$

in which:

$$N(x,\xi) = \sum_{m=1}^{\infty} \frac{v_m(x) v_m(\xi)}{\lambda_m}.$$

$N(x,\xi)$ so defined, is the Neumann function of the Laplacian with Neumann boundary condition.

Note that (7) has a sense even f and h do not satisfy the condition (2) but the function so obtained (say u_0) is not the solution of (1). Indeed, suppose that $\int\limits_{\Omega} f dx + \int\limits_{\partial\Omega} h ds = c \neq 0$, then the corresponding u_0 satisfies:

$$\begin{cases} -\Delta u_0 = f - c_0, \\ \frac{\partial u_0}{\partial n} = h - c_0, \quad c_0 = \frac{1}{2} \frac{c}{\int\limits_{\Omega} dx + \int\limits_{\partial\Omega} ds}. \end{cases}$$