Extremal graph theory:
degree and substructure in finite and infinite graphs

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der Fakultät für Mathematik, Informatik
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der Universität Hamburg
genehmigte
Habilitationsschrift

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Tag der Habilitation: 22.11.2010

Hamburg, Jahr der Drucklegeung: 2010

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Chapter 1

Introduction

Extremal graph theory is a branch of graph theory that seeks to explore the properties of graphs that are in some way extreme. *The* classical extremal graph theoretic theorem and a good example is Turán's theorem. This theorem reveals not only the edge-density but also the structure of those graphs that are 'extremal' without a complete subgraph of some fixed size, where extremal means that upon the addition of any edge the forbidden subgraph will appear. This is the type of question we study in extremal graph theory, for finite as well as for infinite graphs. In general one asks whether some invariant – which instead of the edge-density might be the minimum degree, or the chromatic number, etc – has an influence on the appearance of substructures, or on another graph invariant.

A typical example of modern extremal graph theory is the Loebl–Komlós–Sós conjecture (LKS-conjecture for short) from 1992. This conjecture is about whether the existence of subtrees in a graph can be forced by assuming a large median degree. More precisely, the LKS-conjecture states that every graph G that has at least $|G|/2$ vertices of degree at least some $k \in \mathbb{N}$, contains as subgraphs all trees with k edges. We shall discuss the LKS-conjecture and steps towards a solution, as well as an application of the conjecture in Ramsey theory, in Chapters 2, 3, 4 and 5.

Our main contribution to this field is an approximate version of the LKS-conjecture, which will be presented in Chapter 3. The proof of this version builds on work of Ajtai, Komlós and Szemerédi [1] and features an application of Szemerédi's regularity lemma. This chapter is based on work from [74].

In Chapter 4, which is based on work from [75], we shall solve the LKS-conjecture for special classes of trees. In Chapter 5 we discuss the impact of the conjecture in Ramsey theory.

A theory belonging to extremal graph theory in a broader sense will be the topic of Chapters 6 and 7. The much studied perfect graphs (those for which the chromatic number of every induced subgraph H equals the clique number of H) were introduced by Berge in the early 1960's. One can characterise perfect graphs in terms of their stable set polytope (SSP for short), and in this context Chvátal [25] proposed the concept of t-perfect graphs. These are defined via properties of their SSP, in fact, by a slight modification of the properties which the SSP of a perfect graph must have, involving a second polytope, namely TSTAB.

We will be concerned with colourings and characterisations of t-perfect graphs by forbidden t-minors. So, if the relation of the SSP and the TSTAB of a graph (both to be formally defined in Chapter 6) is viewed as one of the graph's invariants, then what we are studying is this invariant's impact on the chromatic number, and if we should succeed in a characterisation, we would indeed force substructure via an invariant. We discuss t-perfect and the closely related strongly t-perfect graphs in Chapters 6 and 7, which are based on [13, 15].

In general, extremal graph theory has been a very active area of graph theory during the last decades. However, until recently, an extremal branch of infinite graph theory did practically not exist. The reason for this is that in general, the behaviour of infinite graphs is not as well understood as the behaviour of finite graphs. Often it is not clear how certain invariants translate 'correctly' to infinite graphs. For example, how does a condition on the edge-density translate to an infinite graph? How should one define the average degree of an infinite graph?

Even for parameters that appear to have an obvious counterpart in infinite graph theory, it may happen that they lose the power they have in finite graphs. One example is the minimum degree. In finite graphs, a high minimum degree can imply the existence of large complete subgraphs (this is a corollary of Turán's theorem mentioned above). But in infinite graphs, the assumption of a high minimum degree loses its strength, as any minimum degree condition can be met by an infinite tree which is not 'dense' enough to contain an interesting substructure.

A solution to this dilemma are the *end degrees*, to be defined in Chapter 8 (and a variant in Chapter 10), which, if large enough, provide a certain denseness 'at infinity' and thus make it possible to force substructure in infinite graphs. The most interesting of these substructures are doubtlessly large complete minors. With weaker assumptions, we can still force highly connected subgraphs and grid minors. These and related topics will be the subject of Chapters 8, 9 and 10, which are based on work from [83, 86].

We shall encounter another application of the end degrees in infinite extremal graph theory in Chapter 11. The topic of this chapter, which is based on work from [82], are minimally k-connected graphs. For finite graphs, mainly two types of minimality have been investigated: minimality with respect to edge-deletion, which we shall call *edge-minimality*, and with respect to vertex-deletion, which we shall call *vertex-minimality* (In the literature, one often encounters the terms *minimality* for the former and *criticality* for the latter type). In finite such graphs bounds on the minimum degree and on the number of vertices which attain it have been much studied. We give an overview of the results known for finite graphs and show that basically all of the results carry over to infinite graphs if we consider ends of small degree as well as vertices.

Chapter 12 is on duality of infinite graphs. This chapter is based on work from [14]. The main interest will be the end space of a graph in relation to the ends space of its dual. (Duals of certain classes of infinite graphs have been introduced in [11].) We shall see in Chapter 12 that a duality also exists between the sets of ends of two dual graphs, in form of a homeomorphism between the two end spaces. Moreover, the degrees of the ends are preserved under this homeomorphism.

Chapter 2

The Loebl–Komlós–Sós conjecture

2.1 History of the conjecture

A typical question in extremal graph theory is one of the following type: Making certain assumptions on some global parameters of a graph, can we force certain substructures. The Loebl–Komlós–Sós conjecture is good example for such an extremal question: It asks for the appearance of all trees of a given size as subgraphs, imposing a minimal degree condition on part of the vertex set of the graph.

The conjecture was formulated by Komlós and Sós in 1992; the background which led to its formulation was the study of the discrepency of trees [37]. Earlier, Loebl had conjectured the following preliminary form, sometimes called the $n/2$–$n/2$–$n/2$ conjecture.

Conjecture 2.1.1 (Loebl conjecture [37])**.** *Let $n \in \mathbb{N}$, and let G be a graph of order n so that at least $n/2$ vertices of G have degree at least $n/2$. Then every tree with at most $n/2$ edges is a subgraph of G.*

The conjecture was then generalised by Komlós and Sós, and in this new form became known as the Loebl–Komlós–Sós conjecture (or short LKS-conjecture).

Conjecture 2.1.2 (Loebl–Komlós–Sós conjecture [37])**.** *Let $k, n \in \mathbb{N}$, and let G be a graph of order n so that at least $n/2$ vertices of G have degree at least k. Then every tree with at most k edges is a subgraph of G.*

We discuss the bounds of Conjecture 2.1.2 in Section 2.4.

A solution to Conjecture 2.1.1 has been given by Zhao [96] for large graphs, see Section 2.3.

2.2 Special cases of the Loebl–Komlós–Sós conjecture

Observe that for stars, that is, trees of diameter 2, Conjecture 2.1.2 is trivial. Furthermore, it is not difficult to see that the LKS-conjecture holds for trees of diameter 3.

In fact, trees of diameter 3 are exactly those that consist of two stars with adjacent centres. So, it is enough to realise that the set $L \subseteq V(G)$ of vertices of degree at least k cannot be independent. But, if this is not the case, then one easily reaches a contradiction by double-counting the number of edges between L and the set $S := V(G) \setminus L$.

Barr and Johansson [3], and independently Sun [87], proved Conjecture 2.1.2 for all trees of diameter 4. In Chapter 4, we shall show Conjecture 2.1.2 for all trees of diameter at most 5.

On the other extreme of the spectrum of the trees (as opposed to stars) are paths. Paths and path-like trees constitute another class of trees for which Conjecture 2.1.2 has been solved. Bazgan, Li, and Woźniak [4] proved the conjecture for paths and for all trees that can be obtained from a path and a star by identifying one of the vertices of the path with the centre of the star. In Chapter 4, we shall extend their result to a larger class of trees, allowing for two stars instead of one, under certain restrictions.

2.3 The regularity approach

A completely different approach towards a solution of Conjectures 2.1.1 and 2.1.2 has first been proposed by Ajtai, Komlós and Szemerédi [1]. Their approach makes use of the regularity method, together with a Gallai-Edmonds decomposition of the cluster graph. This allowed Ajtai, Komlós and Szemerédi to prove an approximate version of Conjecture 2.1.1.

Theorem 2.3.1 (Ajtai, Komlós and Szemerédi [1]). *For every η there is an $n_0 \in \mathbb{N}$ such that for every graph G on $n \geq n_0$ vertices the following is true.*

If at least $(1+\eta)n/2$ vertices of G have degree at least $(1+\eta)n/2$, then G contains all trees with at most $n/2$ edges.

Zhao [96] extended this approach adding some stability arguments, and could thus verify the exact form of Conjecture 2.1.1 for large graphs.

Theorem 2.3.2 (Zhao [96])**.** *There is an $n_0 \in \mathbb{N}$ such that each graph G of order at least $n \geq n_0$ with at least $n/2$ vertices of degree at least $n/2$ contains every tree with at most $n/2$ edges as a subgraph.*

In [1], it is conjectured that an extension of Theorem 2.3.1, namely the approximate dense version of the Loebl–Komlós–Sós conjecture, also holds. We prove this approximate version in Chapter 3, which is based on work from [74].

Theorem 2.3.3. [74] *For every $\eta, q > 0$ there is an $n_0 \in \mathbb{N}$ such that for every graph G on $n \geq n_0$ vertices and every $k \geq qn$ the following is true.*

If at least $n/2$ vertices of G have degree at least $(1+\eta)k$, then G contains all trees with at most k edges.

Observe that we do not need the approximation factor $(1 + \eta)$ for the number of vertices of large degree. This is due to a not overly complicated reduction, for details see Chapter 3.

Combining the stability arguments also used by Zhao, and our methods exposed in Chapter 3, a sharp version of Conjecture 2.1.2 for $n = O(k)$ has been proved very recently by Hladký and Piguet [57], and independently by Cooley [27].

The sparse case (i.e. the case when k is not linear in n) of the Loebl–Kómlos–Sós conjecture remains open. It is not surprising that this should be the most difficult case to solve, as in fact it implies the dense case.

Let us give a short sketch of this folklore observation. Assume that there is a counterexample to Conjecture 2.1.2 for the dense case, i. e., there exists a graph G of order n with half of its vertices of degree k, where $n = O(k)$, that does not contain some tree of order $k + 1$. By taking many copies of G, we could then construct a counterexample to Conjecture 2.1.2 for the sparse case.

2.4 Discussion of the bounds

In this chapter, we shall discuss the bounds from Conjecture 1. As T could be a star, it is clear that we need that G has a vertex of degree at least k.

On the other hand, we also need a certain amount of vertices of large degree. In fact, the amount $\frac{n}{2}$ we require cannot be lowered by a factor of

$\frac{k-1}{k+1}$. We shall show now that if we require only $\frac{k-1}{k+1}\frac{n}{2} = \frac{n}{2} - \frac{n}{k+1}$ vertices to have degree at least k, the conjecture becomes false whenever $k+1$ is even and divides n.

To see this, construct a graph G on n vertices as follows. Divide $V(G)$ into $\frac{2n}{k+1}$ sets A_i, B_i, so that $|A_i| = \frac{k-1}{2}$, and $|B_i| = \frac{k+3}{2}$, for $i = 1, \ldots, \frac{n}{k+1}$. Insert all edges inside each A_i, and insert all edges between each pair A_i, B_i. Now, consider the tree T we obtain from a star with $\frac{k+1}{2}$ edges by subdividing each edge but one. Clearly, T is not a subgraph of G.

A similar construction shows that we need more than $\frac{n}{2} - \frac{2n}{k+1}$ vertices of large degree, when $k+1$ is odd and divides n. By adding some isolated vertices, our example can be modified for arbitrary k. This shows that at least $\frac{n}{2} - 2\lfloor\frac{n}{k+1}\rfloor - (n \mod (k+1))$ vertices of large degree are needed, for each k. Hence, when $\max\{\frac{n}{k}, n \mod k\} \in o(n)$, the bound $\frac{n}{2}$ is asymptotically best possible.

Chapter 3

An approximate version of the LKS conjecture

3.1 An approximate version and an extension

In this chapter, which is based on work from [74], we shall prove an approximate version of the Loebl–Komlós–Sós conjecture for large, dense graphs, which has been conjectured in [1].

Our proof of Theorem 2.3.3 is inspired by the proof of the approximate version of the Loebl conjecture by Ajtai, Komlós and Szemerédi [1]. We use the regularity lemma followed by a Gallai-Edmonds decomposition of the reduced cluster graph. This enables us to find a certain substructure in the cluster graph, which contains a large matching, and captures the degree condition on G. The tree is then embedded mainly into the matching edges.

We shall see that in the case that $k \geq n/2$, it is not difficult to obtain the same structure as in [1]. Our proof then follows [1], providing all details.

In the case that $k < n/2$, however, the situation is more complex. We will have to content ourselves with a less favourable structure in the cluster graph, which complicates the embedding of the tree. For a brief outline of the crucial ideas we then employ, see Section 3.3.1. The full proof is given in the remainder of Section 3.3.

Using similar ideas, we extend Theorem 2.3.3 in a different direction. We pursue the question which other subgraphs are contained in our graph G from Theorem 2.3.3.

Our second result of this chapter asserts that we can replace the trees with bipartite graphs that may have a few more edges than trees. It will be proved in Section 3.4

Theorem 3.1.1. [**74**] *For every $\eta, q > 0$ and for every $c \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ so that for each graph G on $n \geq n_0$ vertices and each $k \geq qn$ the following is true.*

If at least $n/2$ vertices of G have degree at least $(1+\eta)k$, then each connected bipartite graph Q on $k+1$ vertices with at most $k+c$ edges is a subgraph of G.

In particular, the condition of Theorem 2.3.3 allows for embedding even cycles in G:

Corollary 3.1.2. [**74**] *For every $\eta, q > 0$ there is an $n_0 \in \mathbb{N}$ so that for all graphs G on $n \geq n_0$ vertices and each $k \geq qn$ the following is true.*

If at least $n/2$ vertices of G have degree at least $(1+\eta)k$, then G contains all even cycles of length at most $k+1$.

Observe that a sharp version of Theorem 3.1.1 does not hold, as is witnessed by the following example. Take the complete graph on k vertices and the empty graph on k vertices. Connect these two graphs with a matching of order k. The graph we obtain satisfies the condition of the sharp version of Theorem 3.1.1, but does not contains the cycle of length $k+1$.

Also, the condition that Q is bipartite is necessary. This can be seen by considering the complete bipartite graph $K_{(1+\eta)k,(1+\eta)k}$. This graph satisfies the condition of Theorem 3.1.1, but all its subgraphs are bipartite.

3.2 Preliminaries

The purpose of this section is to introduce the two main tools used in the proofs of Theorem 2.3.3 and Theorem 3.1.1. The first of these tools is the well-known regularity lemma. The second is Lemma 3.2.3, which will give structural information on our graph G from Theorem 2.3.3 (and Theorem 3.1.1). We derive it from the Gallai-Edmonds matching theorem.

3.2.1 Regularity

In this subsection, we introduce the notion of regularity, state Szemerédi's regularity lemma, and review a few useful properties of regularity. All of this is well-known, so the advanced reader is invited to skip this section. For an instructive survey on the regularity lemma and its applications, consult [58].

Let us first go through some necessary notation. For a graph $G = (V, E)$, with $W \subseteq E$ and $S \subseteq V$, we will write $G - W$ for the subgraph $(V, E \setminus W)$ of

G, and $G - S$ the subgraph of G which is obtained by deleting all vertices of S and all incident edges. For subsets X and Y of the vertex set $V(G)$, define $N_Y(X)$ as the set of all neighbours of X in $Y \setminus X$. If X and Y are disjoint, then let $e(X, Y)$ denote the number of edges between X and Y. The *density* of the pair (X, Y) is $d(X, Y) := \frac{e(X,Y)}{|X||Y|}$.

A bipartite graph G with partition classes C_1 and C_2 is called ε*-regular* if for all subsets $C_1' \subseteq C_1$, $C_2' \subseteq C_2$ with $|C_1'| \geq \varepsilon |C_1|$ and $|C_2'| \geq \varepsilon |C_2|$, it is true that $|d(C_1, C_2) - d(C_1', C_2')| < \varepsilon$.

A partition $C_0 \cup C_1 \cup \cdots \cup C_N$ of $V(G)$ is called (ε, N)*-regular*, if

- $|C_0| \leq \varepsilon n$ and $|C_i| = |C_j|$ for $i, j = 1, \ldots, N$,
- all but at most εN^2 pairs (C_i, C_j) with $i \neq j$ are ε-regular.

We are now ready to state Szemerédi's regularity lemma.

Theorem 3.2.1 (Regularity lemma, Szemerédi [88]). *For every $\varepsilon > 0$ and $m_0 \in \mathbb{N}$, there exist $M_0, N_0 \in \mathbb{N}$ so that every graph G of order $n \geq N_0$ admits an (ε, N)-regular partition of its vertex set $V(G)$ with $m_0 \leq N \leq M_0$.*

Call the partition classes C_i of G *clusters*. Now, for each graph G, for each (ε, N)-regular partition of $V(G)$, and for any density p define the *cluster graph* (sometimes called *reduced graph*) in the following standard way.

First, we construct an auxiliary graph G_p obtained from G by deleting all edges inside the clusters C_i, all edges that are incident with C_0, all edges between irregular pairs, and all edges between regular pairs (C_i, C_j) of density $d(C_i, C_j) < p$. Set $s := |C_i|$, and observe that

$$|E(G - G_p)| \leq N\frac{s^2}{2} + \varepsilon n^2 + \varepsilon N^2 s^2 + \frac{N^2}{2} p s^2 \leq (\frac{1}{2m} + 2\varepsilon + \frac{p}{2}) n^2. \quad (3.1)$$

Now, the *cluster graph* $H = H_p$ on the vertex set $\{C_i\}_{1 \leq i \leq N}$ has an edge C_iC_j for each pair (C_i, C_j) of clusters that has positive density in G_p. We shall prefer to work with the *weighted cluster graph* $\bar{H} = \bar{H}_p$ which we obtain from H by assigning weights

$$w(C_iC_j) := d(C_i, C_j)s$$

to the edges $C_iC_j \in E(H)$.

In the setting of weighted graphs, the (weighted) *degree* of a vertex v is defined as

$$\bar{\deg}(v) := \sum_{u \in N(v)} w(vu),$$

and the degree into a subset $U \subseteq V(\bar{H})$, where we only count the weights of v–U edges, is denoted by $\bar{\deg}_U(v)$. We shall adopt this notation for our weighted cluster graph $\bar{H}$. For a subset $X \subseteq C_j$, we write

$$\bar{\deg}_X(C_i) := \frac{e(X, C_i)}{s}.$$

For a set $\mathcal{Y}$ of subsets of distinct clusters from $G_p - C_i$, we shall write $\bar{\deg}_{\mathcal{Y}}(C_i)$ for $\sum_{Y\in\mathcal{Y}} \bar{\deg}_Y(C_i)$.

We shall often use edges of $\bar{H}$ to represent the respective subgraph of G_p, or its vertex set. For example, an edge $e = CD \in E(\bar{H})$, might refer to the subgraph of G_p induced by $C \cup D$, or to $C \cup D$ itself. And for a set $U \subseteq C \cup D$, we sometimes use the shorthand $e \cap U$ for $(C \cup D) \cap U$.

Let us review some basic properties of G_p and $\bar{H}$. Let $C, D \in V(\bar{H})$: We call a set $D' \subseteq D$ *significant*, if $|D'| \geq \varepsilon s$. A vertex $v \in C$ is called *typical* to a significant set D' if $\deg_{D'}(v) \geq \bar{\deg}_{D'}(C) - 2\varepsilon s$. Observe that

$$\text{At most } \varepsilon s \text{ vertices of } C \text{ are not typical to a given significant set } D'. \tag{3.2}$$

Similarly, we have that

$$\text{all but at most } \varepsilon s \text{ vertices } v \text{ of } C \text{ have degree } \deg_{G_p}(v) \leq \bar{\deg}(C) + 2\varepsilon s. \tag{3.3}$$

Also, almost all vertices of any cluster $C \in V(\bar{H})$ are *typical* to almost all significant sets, in the following sense.

If $\mathcal{Y}$ is a set of significant subsets of clusters in $V(\bar{H})$, then

$$|\{Y \in \mathcal{Y} : \deg_Y(v) \geq \bar{\deg}_Y(C) - 2\varepsilon s\}| \geq (1 - \sqrt{\varepsilon})|\mathcal{Y}|, \tag{3.4}$$

for all but at most $\sqrt{\varepsilon}s$ vertices $v \in C$.

To see this, assume that the set $C' \subseteq C$ of vertices not satisfying (3.4) is larger than $\sqrt{\varepsilon}s$. Then

$$\begin{aligned}\sum_{Y\in\mathcal{Y}} |\{v \in C : v \text{ is not typical to } Y\}| &\geq \sum_{v\in C'} |\{Y \in \mathcal{Y} : v \text{ is not typical to } Y\}| \\ &\geq |C'|\sqrt{\varepsilon}|\mathcal{Y}| \\ &> \varepsilon s|\mathcal{Y}|.\end{aligned}$$

Thus there is a $Y \in \mathcal{Y}$ such that more than $\varepsilon|C|$ vertices in C are not typical to Y, a contradiction to (3.2).

3.2.2 The matching

The main interest in this subsection is Lemma 3.2.3, which will give us important structural information on the cluster graph H that corresponds to the graph G from Theorem 2.3.3 (or later Theorem 3.1.1). A weaker variant of this lemma, Lemma 3.2.4 below, appeared in [1].

For the proof of Lemma 3.2.3, we need a simplified version of the Gallai-Edmonds matching theorem, a proof of which can be found for example in [31].

A 1-*factor*, or *perfect matching*, of a graph G is a 1-regular spanning subgraph of G. We call G *factor-critical*, if for each $v \in V(G)$, there exists a perfect matching of $G - v$.

Theorem 3.2.2 (Gallai, Edmonds). *Every graph G contains a set $S \subseteq V(G)$ so that each component of $G - S$ is factor-critical, and so that there is a matching in G that matches the vertices of S to vertices of different components of $G - S$.*

We are now ready for one of the key tools in the proof of Theorem 2.3.3. (Recall that we write $\bar{\deg}_{M \cup L}(v)$ for $\bar{\deg}_{V(M) \cup L}(v)$.)

Lemma 3.2.3.[74] *Let $\bar{H}$ be a weighted graph on N vertices, and let $K \in \mathbb{R}$. Let L be the set of those vertices $v \in V(\bar{H})$ with $\bar{\deg}(v) \geq K$. If $|L| > N/2$, then there are two adjacent vertices $v_A, v_B \in L$, and a matching M in $\bar{H}$ such that one of the following holds.*

(a) M covers $N(\{v_A, v_B\})$,

(b) M covers $N(v_A)$, and $\bar{\deg}_{M \cup L}(v_B) \geq K/2$. Moreover, each edge in M has at most one endvertex in $N(v_A)$.

Proof. Observe that we may assume that $Y := V(\bar{H}) \setminus L$ is independent. (In fact, otherwise we simply delete the edges in $E(Y)$, which will not affect the degree of the vertices in L.)

Theorem 3.2.2 applied to the unweighted version of $\bar{H}$ yields a set $S \subseteq V(\bar{H})$. Among all matchings M' satisfying the conclusion of Theorem 3.2.2, choose M' so that it contains a maximal number of vertices of Y. Extend M' to a maximal matching M of $\bar{H}$.

Set $L' := L \setminus S$. Clearly, if there is an edge $v_A v_B$ with endvertices $v_A, v_B \in L'$, then (a) holds. Therefore, we may assume that L' is independent.

Then, each edge of $\bar{H}$ that is not incident with S has one endvertex in L', and one in Y. Consider any component C of $\bar{H} - S$. Since C is factor-critical,

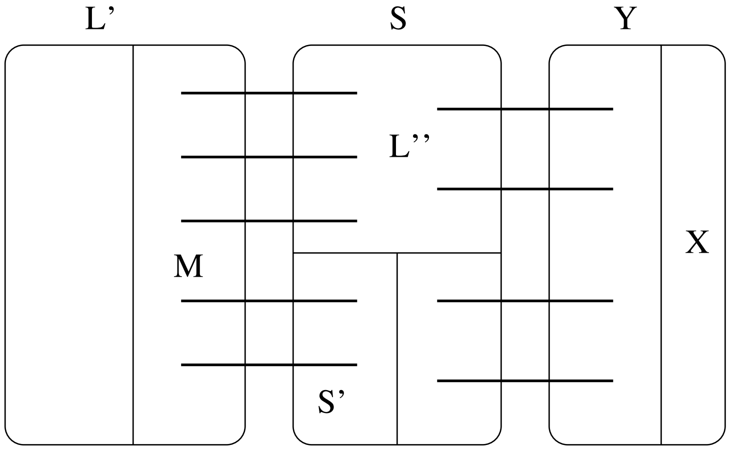


Figure 3.1: The graph $\bar{H}$ with the matching M, and sets L, S and Y.

we have that $|(C-u)\cap Y| = |(C-u)\cap L'|$, for every $u \in V(C)$. Hence, C consists of only one vertex, and so must every component of $\bar{H}-S$.

Denote by X the subset of Y that is not covered by M. Set $\tilde{L} := N(L')\cap L \subseteq S$ (see Figure 1). Now, if there is a vertex $v_B \in \tilde{L}$ whose weighted degree into $\bar{H}-X$ is at least $K/2$, then v_B, together with any of its neighbours v_A in L', satisfies (b). So, we may assume that for each $u \in \tilde{L}$,

$$\bar{\deg}_{\bar{H}-X}(u) < K/2, \tag{3.5}$$

and hence $\bar{\deg}_X(u) \geq K/2$.

On the other hand, $\bar{\deg}_{\tilde{L}}(w) < K$ for each $w \in X$. Thus, by double (weighted) edge-counting, it follows that

$$|X| \geq \frac{|\tilde{L}|}{2}. \tag{3.6}$$

Set $S' := S \cap Y$. By (3.5), the total weight of the edges in $E(\tilde{L}\cup S', L')$ is less than $|\tilde{L}|K/2 + |S'|K$, while each vertex of L' has weighted degree at least K into $\tilde{L}\cup S'$. Thus, again by double edge-counting, and by (3.6),

$$|X| + |S'| \geq \frac{|\tilde{L}|}{2} + |S'| > |L'|. \tag{3.7}$$

Furthermore, since Y is independent, M matches $S' \subseteq Y$ to L'. Thus $|L'| \geq |S'| + |L \setminus M|$, and so, by (3.7),

$$|X| > |L \setminus M|.$$

Since $|L| > \frac{N}{2}$, this implies that M contains an edge uv with both $u, v \in L$. We may assume that $v \in L'$ and $u \in \tilde{L}$. By (3.5), u has a neighbour w in X. Hence, the matching $M' \cup \{uw\} \setminus \{uv\}$ covers more vertices of Y than M' does, a contradiction to the choice of M'. □

Note that in the case $K \geq N/2$ the situation in Lemma 3.2.3 is less complicated. In that case, observe that clearly $|S| \leq |V(\bar{H} - S)|$. So, either $|S| = |V(\bar{H} - S)|$ (in which case conclusion (a) of Lemma 3.2.3 holds), or there is a component C of $\bar{H} - S$ that has more than one vertex. Thus, as C is factor-critical, there exists an L'–L' edge in C, and (a) holds again.

This proves the following lemma, which appeared in [1].

Lemma 3.2.4. *If $K \geq N/2$, then Lemma 3.2.3 always yields case (a).*

In the case $k \geq n/2$, this observation simplifies our proof of Theorem 2.3.3 considerably, as then only the simplest case needs to be treated. We shall not make use of Lemma 3.2.4 in our proof of Theorem 2.3.3.

3.3 Proof of Theorem 2.3.3

The organisation of this section is as follows. The first subsection is devoted to an outline of our proof, highlighting the main ideas, leaving out all details. In Subsection 3.3.2, assuming that we are given a host graph G and a tree T^* as in Theorem 2.3.3, we shall first apply the regularity lemma to G. We then use Lemma 3.2.3 to find a substructure of the corresponding weighted cluster graph $\bar{H}$, which will facilitate the embedding of T^*.

We shall prepare T^* for this by cutting it into small pieces in Subsections 3.3.3 and 3.3.4. Then, in Subsection 3.3.5, we partition the matching given by Lemma 3.2.3, according to the decomposition of the tree T^*. In Subsection 3.3.6, we expose tools that we need for our embedding. What remains is the actual embedding procedure, which we divide into the two cases given by Lemma 3.2.3, and treat separately in Subsections 3.3.7 and 3.3.8.

3.3.1 Overview

In this subsection, we shall give an outline of our proof of Theorem 2.3.3. So, assume that we are given $\eta > 0$ and $q > 0$. The regularity lemma applied to parameters depending on η and q yields an $n_0 \in \mathbb{N}$. Now, let $n \geq n_0$, let $k \geq qn$, let G be a graph of order n that satisfies the condition of

Theorem 2.3.3, and let T^* be a tree with k edges. We wish to find a subgraph of G that is isomorphic to T^*, i.e. we would like to *embed* T^* in G.

In order to do so, consider the weighted cluster graph $\bar{H}$ corresponding to G that is given by the regularity lemma. Denote by $L \subseteq V(\bar{H})$ the set of those clusters that have degree at least $(1+\pi')k$ in $\bar{H}$, where $\pi' = \pi'(\eta, q) > 0$. Apply Lemma 3.2.3 to $\bar{H}$ and $K := (1 + \pi')k$ which yields vertices $A, B \in V(\bar{H})$ and a matching M. The rest of our proof will be divided into two cases, corresponding to the two possible conclusions (a) and (b) of Lemma 3.2.3.

As the technical details for these two cases overlap, we will not completely separate them later on in the proof. In this outline, however, we think it is more instructive to present first the easier proof for case (a), and then turn our attention to case (b).

If the output of Lemma 3.2.3 is Case (a), then we shall decompose T^* into small subtrees (of order much below ηk) and a small set SD of vertices (of constant order in n), so that between any two of our subtrees lies a vertex from SD (the name SD stands for 'seeds'). In fact, SD is the disjoint union of two sets SD^A and SD^B, and each tree of $T^* - SD$ is adjacent to only one of these two sets. Denote the set of trees adjacent to SD^A by $\mathcal{T}_A$, and the set of trees adjacent to SD^B by $\mathcal{T}_B$. The formal definition of SD, $\mathcal{T}_A$ and $\mathcal{T}_B$ can be found in Section 3.3.3.

Next, in Section 3.3.5, we partition the matching M from Lemma 3.2.3 into M_A and M_B. This is done in a way so that $\bar{\deg}_{M_A}(A)$ is large enough so that $F_A := \bigcup \mathcal{T}_A$ fits into M_A, and $\bar{\deg}_{M_B}(B)$ is large enough so that $F_B := \bigcup \mathcal{T}_B$ fits into M_B.

Finally, in Section 3.3.7, we embed SD^A in A and SD^B in B and use the regularity of the edges in $\bar{H}$ to embed the small trees of $\mathcal{T}_A \cup \mathcal{T}_B$, one after the other, levelwise, into $M_A \cup M_B$. The order of this embedding procedure will be such that the already embedded part of T^* is always connected.

Moreover, the structure of our decomposition of T^*, and the fact that we embed the trees from $\mathcal{T}_A \cup \mathcal{T}_B$ in the matching edges, ensures that the predecessor of any vertex $r \in SD^A \cup SD^B$ is embedded in a cluster that is adjacent to A, respectively to B (in which we wish embed r). This enables us to embed all of SD in $A \cup B$, as planned.

An important detail of our embedding technique is that we shall always try to *balance* the embedding in the matching edges, in the sense that the used part of either side should have about the same size. We only allow for an unbalanced embedding if the degree of A resp. B into one of the endclusters of the concerned edge is already 'exhausted' (cf. Property ($\diamond$) in Section 3.3.6). In practice, this means that whenever we have the choice into which endcluster of an edge $e \in M$ we embed the root of some tree of $\mathcal{T}_A \cup \mathcal{T}_B$, we shall choose the side carefully.

In this manner, we can ensure that all of T^* will fit into M (or more precisely into the corresponding subgraph of G). This finishes the embedding of T^* in case (a) of Lemma 3.2.3.

In case (b) of Lemma 3.2.3, it is not possible to partition the matching M into M_A and M_B so that F_A fits into M_A and F_B fits into M_B, as in case (a). More precisely, for any partition of M into M_A and M_B, if $\bar{\deg}_{M_A}(A)$ allows for the embedding of a forest of order t, say, in M_A, then $\bar{\deg}_{M_B \cup L}(B)$ only guarantees for the embedding of a forest of order at most $(k-t)/2$ in the subgraph of G_p induced by M_B and the edges incident with L', where $L' := L \setminus M$. For more details on this, see Lemma 3.3.1.

We use a combination of two strategies to overcome this problem. Firstly, we shall embed T^* in two phases, leaving for the second phase some subtrees that are (each) adjacent to only one vertex from SD. Secondly, we shall embed some of the trees from $\mathcal{T}_B$ in part of the matching reserved for F_A. This means that we 'switch' some of our trees to $\mathcal{T}_A$.

Let us explain the two strategies in more detail. We modify our sets $\mathcal{T}_A \cup \mathcal{T}_B$, in the following way. Denote by $\bar{\mathcal{T}}_A$ the set of those trees from $\mathcal{T}_A$ that are adjacent to only one vertex from SD^A, and similarly define $\bar{\mathcal{T}}_B$. (Observe that thus the deletion of any tree in $\bar{\mathcal{T}}_A \cup \bar{\mathcal{T}}_B$ leaves T^* connected.)

We may assume that

$$|V(\bigcup \bar{\mathcal{T}}_A)| \geq |V(\bigcup \bar{\mathcal{T}}_B)|.$$

Finally, set $\mathcal{T}' := (\mathcal{T}_A \cup \mathcal{T}_B) \setminus (\bar{\mathcal{T}}_A \cup \bar{\mathcal{T}}_B)$. Our plan now is to first embed the trees from $\mathcal{T}' \cup \bar{\mathcal{T}}_B$ together with the vertices from SD and to postpone the embedding of $\bar{F}_A := \bigcup \bar{\mathcal{T}}_A$ to a later stage. As the part of the tree embedded in the first phase is connected, we avoid the difficulty of having to connect already embedded parts of T^* in the second phase.

Now, we shall partition M into M_F and $\bar{M}_B$ so that $\bar{\deg}_{M_F}(A)$ allows for the embedding of $\bigcup \mathcal{T}'$, and $\bar{\deg}_{\bar{M}_B \cup L}(B)$ allows for the embedding of $\bar{F}_B := \bigcup \bar{\mathcal{T}}_B$. This actually means that the place we reserved for the embedding of $F_B - V(\bar{F}_B)$ lies in M_F. Therefore, we shall 'switch' this forest to $\mathcal{T}_A$ (which is the second of our strategies).

Let us explain what we mean by *switching*. For each tree $T \in \mathcal{T}_B \setminus \bar{\mathcal{T}}_B$, delete all vertices from T that are adjacent to SD^B in T^* and add them to SD^A. Put the components of what remains of T into $\mathcal{T}_A$. Denote the thus enlarged SD^A by $\bar{SD}^A$ and set $\bar{SD} := \bar{SD}^A \cup SD^B$.

After switching all trees $T \in \mathcal{T}_B \setminus \bar{\mathcal{T}}_B$, denote by $\mathcal{T}_F$ the (enlarged) set $\mathcal{T}_A \setminus \bar{\mathcal{T}}_A$. That is, $\mathcal{T}_F$ consists of all trees from the original $\mathcal{T}_A \setminus \bar{\mathcal{T}}_A$, together with all trees we generated by switching. It will be easy to verify that the switching procedure does not increase too much the number of seeds.

Also, each tree from $\mathcal{T}_F$ and $\bar{\mathcal{T}}_A$ is adjacent only to the enlarged $\bar{SD}^A$, and each tree from $\bar{\mathcal{T}}_B$ is still adjacent only to SD^B. For details on the switching procedure, consult Section 3.3.4.

It remains to embed T^* in G, which is done in Section 3.3.8. We first embed the vertices from $\bar{SD}^A \cup SD^B$ in $A \cup B$, embed $F_F := \bigcup \mathcal{T}_F$ in M_F, and embed part of $\bar{\mathcal{T}}_B$ in $\bar{M}_B$, in the same way as in case (a). In a second phase, we embed the remaining trees from $\bar{\mathcal{T}}_B$ into edges of H that are incident with L'. For each tree, we are able to find a free space in a suitable edge because of the high degree of the clusters from L'.

In the remaining third phase we wish to embed $\bar{F}_A$. We shall now use all of M, forgetting about the partition into M_F and $\bar{M}_B$. The neighbours of the trees from $\bar{\mathcal{T}}_A$ in $\bar{SD}^A$ have already been embedded in the first phase. Having chosen their images carefully then, ensures that now they have still large enough degree into what is not yet used of M. Hence, there is enough place for $\bar{F}_A$ in M.

Also, it is essential here that each edge of M meets $N(A)$ in at most one cluster. The reason is that parts of these clusters might have been used in the first and second phases of the embedding. So, some of the edges involved might be unbalanced, in the sense above, because e. g. the degree of B was such that we were not able to choose the endcluster in which we embedded the roots of the trees from $\bar{\mathcal{T}}_B$. However, as each edge of M has at most one endcluster in $N(A)$, it is irrelevant whether the embedding is balanced or not in these edges.

The embedding itself of $\bar{F}_A$ is done as before. This finishes the sketch of our proof in case (b).

3.3.2 Preparations

We shall now prove Theorem 2.3.3. First of all, we fix a few constants depending on η and q. Set

$$\pi := \min\{\eta, q\}, \quad \varepsilon := \frac{\pi^7 q}{25 \cdot 10^7} \quad \text{and} \quad m_0 := \frac{500}{q\pi^3}.$$

The regularity lemma (Theorem 3.2.1) applied to ε, and m_0 yields natural numbers M_0 and N_0.

Fix

$$\beta := \frac{\varepsilon}{M_0}, \quad p := \frac{\pi^3 q}{250} \quad \text{and} \quad n_0 := \max\left\{N_0, \frac{8M_0}{\beta}\cdot\frac{8}{p}\right\}.$$

Thus our constants satisfy the following relations

$$\frac{1}{n_0} \ll \beta \ll \varepsilon \ll \frac{1}{m_0} < p \ll \pi \leq q,$$

where $a \ll b$ stands for the fact that $a < \frac{\pi}{100}b$.

In particular, p satisfies

$$4\varepsilon + \frac{1}{m_0} < p = \frac{\pi^3 q}{250}. \tag{3.8}$$

Let $n \geq n_0$, let $k \geq qn$, and let G be a graph of order n which has at least $\frac{n}{2}$ vertices of degree at least $(1+\eta)k$. Suppose T^* is a tree of order $k+1$. Our aim is to find an embedding $\varphi : V(T^*) \to V(G)$ that preserves adjacency.

Now, by Theorem 3.2.1 there exists an (ε, N)-regular partition of $V(G)$, with $m_0 \leq N \leq M_0$. As in Section 3.2.1, let G_p be the subgraph of G that preserves exactly the edges between regular pairs of density at least p.

By (3.1) and by (3.8),

$$|E(G - G_p)| < pn^2 \leq \frac{\pi^3}{250}kn.$$

Thus, for all but at most $\frac{\pi^2}{50}n$ vertices v, we have that $\deg_{G_p}(v) \geq \deg_G(v) - \frac{\pi}{5}k$. Hence,

$$G_p \textit{ has at least } (1 - \frac{\pi^2}{25})\frac{n}{2} \textit{ vertices of degree at least } (1 + \frac{4\pi}{5})k.$$

Let $\bar{H} = \bar{H}_p$ be the weighted cluster graph corresponding to G_p. Denote by L the set of those clusters in $V(\bar{H})$ that contain more than εs vertices of degree at least $(1 + \frac{4\pi}{5})k$ in G_p. A simple calculation shows that $|L| > (1 - \frac{\pi^2}{5})\frac{N}{2}$.

Now, delete $\min\{\pi^2 N/5, |V(\bar{H}) \setminus L|\}$ clusters in $V(\bar{H}) \setminus L$ to obtain a subgraph of the cluster graph $\bar{H}$. As this subgraph is very similar (or identical) to $\bar{H}$, in the rest of the text we shall denote it as well by $\bar{H}$. So from now on, by $\bar{H}$, we shall always refer to this subgraph. Each cluster in L drops its degree by at most $\frac{\pi^2}{5}Ns \leq \frac{\pi k}{5}$. Thus, by (3.3), each cluster X in L has degree

$$\bar{\deg}_{\bar{H}}(X) > (1 + \frac{3\pi}{5})k - 2\varepsilon n > (1 + \frac{\pi}{5})k. \tag{3.9}$$

Then Lemma 3.2.3 applied to $\bar{H}$ and $K := (1+\frac{\pi}{5})k$ yields an edge $AB \in E(\bar{H})$ with $A, B \in L$, together with a matching M' of $\bar{H}$, which satisfy (a) or (b) of Lemma 3.2.3. Obtain M from M' by deleting all edges that are incident with A or with B. In case (a) of Lemma 3.2.3, we calculate that

$$\begin{aligned} \min\{\bar{\deg}_M(A), \bar{\deg}_M(B)\} &\geq (1+\frac{\pi}{5})k - \frac{3n}{N} \\ &\geq (1+\frac{\pi}{5} - \frac{3}{qm_0})k \\ &\geq (1+\frac{\pi}{10})k. \end{aligned} \tag{3.10}$$

Similarly, in case (b) it follows that

$$\bar{\deg}_M(A) \geq (1+\frac{\pi}{10})k \quad \text{and} \quad \bar{\deg}_{M\cup L}(B) \geq (1+\frac{\pi}{10})\frac{k}{2}. \tag{3.11}$$

Thus, for the remainder of our proof of Theorem 2.3.3 we shall work with the assumption that there is a matching M of $\bar{H}$ and vertices $A, B \notin V(M)$ so that

1. $\bar{\deg}_M(A), \bar{\deg}_M(B) \geq (1+\frac{\pi}{10})k$, or
2. $\bar{\deg}_M(A) \geq (1+\frac{\pi}{10})k$, $\bar{\deg}_{M\cup L}(B) \geq (1+\frac{\pi}{10})\frac{k}{2}$, and each cluster in $N(A)$ meets a different edge of M.

We shall refer to these two cases as 'Case 1' and 'Case 2', respectively. We will embed the tree T^* in the subgraph $G'_p \subseteq G_p$ corresponding to $\bar{H}$, using two different strategies in Case 1 and in Case 2.

3.3.3 Partitioning the tree

In this section, we shall cut our tree into small pieces. More precisely, we shall define a set $SD \subseteq V(T^*)$, and sets $\mathcal{T}_A$ and $\mathcal{T}_B$ of disjoint small subtrees of T^* which are connected through the vertices from SD. Moreover, SD together with the union of all trees from $\mathcal{T}_A \cup \mathcal{T}_B$ will span T^*.

Fix a root R of T^*, and regard T^* as a poset having R as the minimal element. For a vertex x of a subtree $T \subseteq T^*$, denote by $T(x)$ the subtree of T induced by x and all vertices y greater than x in the tree-order of T^*. (That is, $T(x)$ contains all vertices y such that the path between the root R and y contains the vertex x.) If $R \notin V(T)$, then define the *seed* $sd(T)$ of T as the maximal vertex of T^* which is smaller than every vertex of T.

Our sets $SD = SD^A \cup SD^B$, $\mathcal{T}_A$ and $\mathcal{T}_B$ will satisfy:

(I) $SD^A \cap SD^B = \emptyset$,

(II) $r \in SD$ lies at even distance to R if and only if $r \in SD^A$, and $R \in SD^A$,

(III) $\mathcal{T}_A \cup \mathcal{T}_B$ consists of the components of $T^* - SD$,

(IV) $|V(T)| \leq \beta k$, and $sd(T) \in SD$, for each $T \in \mathcal{T}_A \cup \mathcal{T}_B$,

(V) $\max\{|SD^A|, |SD^B|\} \leq \frac{2}{\beta}$, and

(VI) $e_{T^*}(V(F_A), SD^B) = 0$, and $e_{T^*}(V(F_B), SD^A) = 0$,

where $F_A := \bigcup_{T \in \mathcal{T}_A} T$ and $F_B := \bigcup_{T \in \mathcal{T}_B} T$ are the forests spanned by $\mathcal{T}_A$ and $\mathcal{T}_B$.

Let us first define SD. To this end, we shall inductively find vertices x_i, and define auxiliary trees $T^i \subseteq T^*$. Set $T^0 := T^*$.

In step $i \geq 1$, let $x_i \in V(T^*)$ be the maximal vertex in the tree-order of $V(T^{i-1})$ with

$$|V(T^{i-1}(x_i))| > \beta k, \tag{3.12}$$

as illustrated in Figure 3.2(a), and define

$$T^i := T^{i-1} - (T^{i-1}(x_i) - x_i).$$

If there is no vertex satisfying (3.12), then set $x_i := R$, and stop the definition process.

Say our process stops in some step j. Let A' be the set of all x_i, $i \leq j$, with even distance to the root R, and let B' be the set of all other x_i.

Note that, because of (3.12), at each step $i \leq j-1$,

$$|V(T_i)| \leq |V(T_{i-1})| - (\beta k - 1),$$

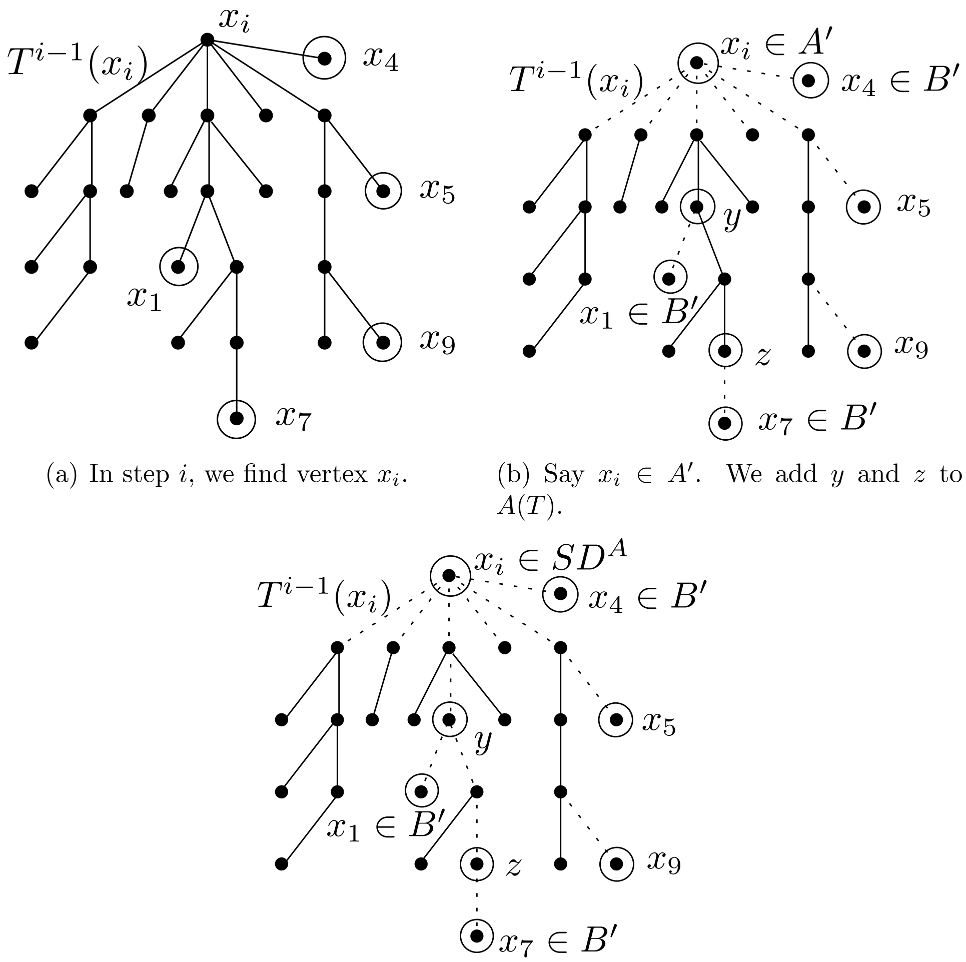
and thus, by the definition of n_0,

$$j - 1 \leq \frac{|V(T^*)|}{\beta k - 1} \leq \frac{k+1}{\beta k - 1} \leq \frac{3}{2\beta}.$$

Hence,

$$|A' \cup B'| \leq \frac{2}{\beta}. \tag{3.13}$$

For the sake of condition (VI), we shall now add a few more vertices to our sets A' and B', which will result in the desired SD.



(a) In step i, we find vertex x_i.

(b) Say $x_i \in A'$. We add y and z to $A(T)$.

(c) $T^{i-1}(x_i) - SD \subseteq \mathcal{T}_A$.

Figure 3.2: Phases of the partition of T^*.

Let $\mathcal{C}$ be the set of the components of $T^* - (A' \cup B')$. For each $T \in \mathcal{C}$ with $sd(T) \in A'$, denote by $A(T)$ the set of vertices adjacent to B'. Similarly, if $sd(T) \in B'$, then denote by $B(T)$ the set of vertices adjacent to A' (cf. Figure 3.2(b)). Set

$$SD^A := A' \cup \bigcup_{T \in \mathcal{C}} A(T), \quad \text{and} \quad SD^B := B' \cup \bigcup_{T \in \mathcal{C}} B(T)$$

and set $SD := SD^A \cup SD^B$.

Since each vertex in B' has at most one neighbour in the union of the

$A(T)$, it follows that

$$|SD^A \setminus A'| \leq |B'|,$$

and analogously,

$$|SD^B \setminus B'| \leq |A'|.$$

Thus,

$$\max\{|SD^A|, |SD^B|\} \leq |A' \cup B'|. \tag{3.14}$$

Finally, we shall define $\mathcal{T}_A$ and $\mathcal{T}_B$. Let $\mathcal{C}'$ be the set of the components of $T^* - SD$. Set

$$\mathcal{T}_A := \{T \in \mathcal{C}' : \ sd(T) \in SD^A\} \quad \text{and} \quad \mathcal{T}_B := \{T \in \mathcal{C}' : \ sd(T) \in SD^B\},$$

as shown in Figure 3.2(c), and define the forests

$$F_A := \bigcup_{T \in \mathcal{T}_A} T \quad \text{and} \quad F_B := \bigcup_{T \in \mathcal{T}_A} T.$$

Observe that Conditions (I)–(IV) and (VI) are clearly met and that (V) holds because of (3.13) and (3.14).

This finishes our manipulation of the tree T^* in Case 1.

3.3.4 The switching

In Case 2 from Section 3.3.2, we shall not only cut our tree to small pieces (cf. Section 3.3.3), but also switch some of our small subtrees from one of the two sets $\mathcal{T}_A$, $\mathcal{T}_B$ to the other. We achieve this by adding some more vertices to SD, thus naturally refining our partition of T^*.

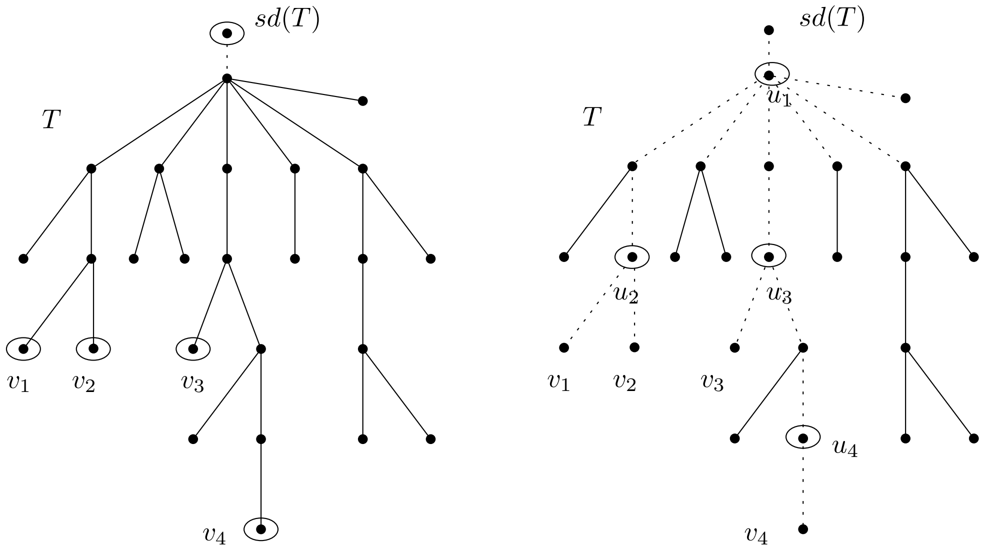
Set

$$\bar{\mathcal{T}}_A := \{T \in \mathcal{T}_A : e(V(T), SD - sd(T)) = \emptyset\}, \quad \text{and}$$
$$\bar{\mathcal{T}}_B := \{T \in \mathcal{T}_B : e(V(T), SD - sd(T)) = \emptyset\}.$$

We may assume that

$$| \bigcup_{T \in \bar{\mathcal{T}}_A} V(T) \ | \geq | \bigcup_{T \in \bar{\mathcal{T}}_B} V(T) \ |. \tag{3.15}$$

Now, consider a tree $T \in \mathcal{T}_B \setminus \bar{\mathcal{T}}_B$ as in Figure 3.3(a). Denote by $S(T)$ the set of all vertices in $V(T)$ that in T^* are adjacent to some vertex of SD^B. For illustration see Figure 3.3(b).



(a) A tree $T \in \mathcal{T}_B \setminus \bar{\mathcal{T}}_B$, with $sd(T), x_1, x_2, x_3, x_4 \in SD^B$.

(b) The set $S(T) = \{y_1, \dots, y_4\}$, and the subtrees of T generated by the switching.

Figure 3.3: The switching procedure.

Set

$$\bar{SD}^A := SD^A \cup \bigcup_{T \in \mathcal{T}_B \setminus \bar{\mathcal{T}}_B} S(T) \quad \text{and} \quad \bar{SD} := \bar{SD}^A \cup SD^B.$$

Finally, define

$$\mathcal{T}'_A := \bigcup_{T \in \mathcal{T}_B \setminus \bar{\mathcal{T}}_B} \{C : C \text{ is a component of } T - S(T)\}$$

and

$$\mathcal{T}_F := (\mathcal{T}_A \setminus \bar{\mathcal{T}}_A) \cup \mathcal{T}'_A.$$

(The F in $\mathcal{T}_F$ stands for 'first', as this part of the tree is to be embedded first.) Finally, set

$$F_F := \bigcup_{T \in \mathcal{T}_F} T,$$

$$\bar{F}_A := \bigcup_{T \in \bar{\mathcal{T}}_A} T \quad \text{and} \quad \bar{F}_B := \bigcup_{T \in \bar{\mathcal{T}}_B} T.$$

Observe that our sets $\bar{SD} = \bar{SD}^A \cup SD^B$, $\mathcal{T}_F \cup \bar{\mathcal{T}}_A$, and $\bar{\mathcal{T}}_B$ still satisfy conditions (I)-(IV) and (VI) from Section 3.3.3 (with SD, SD^A, $\mathcal{T}_A$, $\mathcal{T}_B$, F_A, and F_B replaced by $\bar{SD}$, $\bar{SD}^A$, $\mathcal{T}_F \cup \bar{\mathcal{T}}_A$, $\bar{\mathcal{T}}_B$, $\bar{F}_A$, and $\bar{F}_B$, respectively). Instead of (V), we now have the similar

(V)' $|\bar{SD}| \leq \frac{8}{\beta}$,

since by the definition of $\bar{SD}^A$ we know that for each vertex x of SD^B, we have created at most 2 vertices of $\bar{SD}^A \setminus SD^A$ (between x and the next vertex of SD^B in direction of the root R). Thus,

$$|\bar{SD}^A| \leq |SD^A| + 2|SD^B| \leq \frac{6}{\beta},$$

as needed for (V)'.

3.3.5 Partitioning the matching

In this subsection, we shall divide the matching M into two parts, into which we will later embed the two forests F_A, F_B, respectively F_F and $\bar{F}_B$, of T^* that we defined in Subsection 3.3.3, resp. in Subsection 3.3.4. (The forest $\bar{F}_A$ will be embedded later).

For this, we will need the following number-theoretic lemma, which appeared also in [1]. We give a short proof.

Lemma 3.3.1. *Let I be a finite set, and let $a, b, \Delta > 0$. For $i \in I$, let $a_i, b_i \in (0, \Delta]$. Suppose that*

$$\frac{a}{\sum_{i\in I} a_i} + \frac{b}{\sum_{i\in I} b_i} \leq 1. \tag{3.16}$$

Then there is a partition of I into I_a and I_b such that $\sum_{i\in I_a} a_i > a - \Delta$ and $\sum_{i\in I_b} b_i \geq b$.

Proof. Define a total order $\preceq$ on I in a way that $i \preceq j$ implies $\frac{a_i}{b_i} \leq \frac{a_j}{b_j}$ for all $i, j \in I$. Let $\ell \in I$ be minimal in this order with $a \geq \sum_{i \succ \ell} a_i$.

Set $I_a := \{i \in I : \; i \succ \ell\}$ and set $I_b := I \setminus I_a$. It is clear that $\sum_{i\in I_a} a_i > a - \Delta$, by the definition of ℓ and as $a_\ell \leq \Delta$. So, all we have to show is that $\sum_{i\in I_b} b_i \geq b$.

Indeed, suppose otherwise. Then by (3.16), and by the definition of ℓ, we have that

$$\begin{aligned}\frac{\sum_{i\in I_b} b_i}{\sum_{i\in I} b_i} &< \frac{b}{\sum_{i\in I} b_i}\\ &\leq \frac{a-\sum_{i\in I_a} a_i}{\sum_{i\in I} a_i}+\frac{b}{\sum_{i\in I} b_i}\\ &\leq 1-\frac{\sum_{i\in I_a} a_i}{\sum_{i\in I} a_i}\\ &= \frac{\sum_{i\in I_b} a_i}{\sum_{i\in I} a_i}.\end{aligned}$$

Multiply the two sides of this inequality with $\sum_{i\in I} a_i \cdot \sum_{i\in I} b_i$, subtract the term $\sum_{i\in I_b} a_i \cdot \sum_{i\in I_b} b_i$, and divide by $\sum_{i\in I_a} b_i \sum_{i\in I_b} b_i$ to obtain

$$\frac{a_\ell}{b_\ell} \leq \frac{\sum_{i\in I_a} a_i}{\sum_{i\in I_a} b_i} < \frac{\sum_{i\in I_b} a_i}{\sum_{i\in I_b} b_i} \leq \frac{a_\ell}{b_\ell},$$

(where the first and last inequality follow from the definition of $\preceq$). This yields the desired contradiction. □

We shall now apply Lemma 3.3.1 to partition our matching $M = \{e_i\}_{i\leq|M|}$. We do this separately for the two cases from Section 3.3.2.

In Case 1, we set

$$a := |V(F_A)| + \frac{\pi k}{20}, \quad b := |V(F_B)| + \frac{\pi k}{20}, \quad \text{and} \quad \Delta := 2s.$$

For $i \leq |M|$, set $a_i := \bar{\deg}_{e_i}(A) \leq \Delta$, and $b_i := \bar{\deg}_{e_i}(B) \leq \Delta$. Now, (3.10) implies that

$$\frac{a}{\sum_{i=1}^{|M|} a_i} + \frac{b}{\sum_{i=1}^{|M|} b_i} \leq \frac{|V(F_A)| + |V(F_B)| + \frac{\pi k}{10}}{(1+\frac{\pi}{10})k} \leq 1.$$

Hence, Lemma 3.3.1 yields a partition of M into M_A and M_B such that

$$\bar{\deg}_{M_A}(A) > |V(F_A)| + \frac{\pi k}{40} \text{ and } \bar{\deg}_{M_B}(B) > |V(F_B)| + \frac{\pi k}{40}. \tag{3.17}$$

In Case 2, set

$$a := |V(F_F)| + \frac{\pi k}{20}, \quad b := |V(\bar{F}_B)| + \frac{\pi k}{40}, \quad \text{and} \quad \Delta := 2s.$$

For $i = 1, \dots, |M|$, again set $a_i := \bar{\deg}_{e_i}(A)$, and $b_i := \bar{\deg}_{e_i}(B)$. Set $L' := L \setminus M$. For $i = |M|+1, \dots, |M|+|L'|$, set $a_i := 0$, and set $b_i := \bar{\deg}_{C_i}(B)$, where C_i is the ith cluster in L'.

Observe that by (3.15),

$$|V(\bar{F}_B)| \leq \frac{k - |V(F_F)|}{2}. \tag{3.18}$$

Now, let us check that the conditions of Lemma 3.3.1 hold. Clearly, $a_i, b_i \leq \Delta$ for all $i \leq |M| + |L'|$.

Moreover, Condition (3.16) holds since (3.11) and (3.18) imply that

$$\begin{aligned} \frac{a}{\sum_{i=1}^{|M|+|L'|} a_i} + \frac{b}{\sum_{i=1}^{|M|+|L'|} b_i} &\leq \frac{|V(F_F)| + \frac{\pi k}{20}}{(1+\frac{\pi}{10})k} + \frac{|V(\bar{F}_B)| + \frac{\pi}{40}}{(1+\frac{\pi}{10})\frac{k}{2}} \\ &\leq \frac{|V(F_F)| + 2|V(\bar{F}_B)| + \frac{\pi k}{10}}{(1+\frac{\pi}{10})k} \\ &\leq 1. \end{aligned}$$

We thus obtain a partition of M into M_F and $\bar{M}_B$ such that

$$\bar{\deg}_{M_F}(A) > |V(F_F)| + \frac{\pi k}{40} \quad \text{and} \quad \bar{\deg}_{\bar{M}_B \cup L'}(B) \geq |V(\bar{F}_B)| + \frac{\pi k}{40}. \tag{3.19}$$

Let $\mathcal{T}_B^M \subseteq \bar{\mathcal{T}}_B$ be maximal with

$$\bar{\deg}_{\bar{M}_B}(B) \geq |\bigcup_{T \in \mathcal{T}_B^M} V(T)| + \frac{\pi k}{40N}|\bar{M}_B|. \tag{3.20}$$

Set $\mathcal{T}_B^L := \bar{\mathcal{T}}_B \setminus \mathcal{T}_B^M$. Let $F_B^M := \bigcup_{T \in \mathcal{T}_B^M} T$ and let $F_B^L := \bar{F}_B - V(F_B^M)$. Observe that if $\mathcal{T}_B^M \neq \bar{\mathcal{T}}_B$, then the maximality of $\mathcal{T}_B^M$ ensures that

$$\bar{\deg}_{\bar{M}_B}(B) < |V(F_B^M)| + \frac{\pi k}{40N}|\bar{M}_B| + \beta k.$$

Hence, by (3.19), either $\mathcal{T}_B^L = \emptyset$, or

$$\bar{\deg}_{L'}(B) \geq |V(F_B^L)| + \frac{\pi k}{80N}|L'|. \tag{3.21}$$

3.3.6 Embedding lemmas for trees

In this section, we shall prove some preparatory lemmas on embedding trees in regular pairs of $\bar{H}$.

Let $C, D \in V(\bar{H})$, and let $U, N \subseteq C \cup D$. We say that U has Property $(\star)$ in CD for N if it satisfies the following.

$(\star)$ If $||C \cap U| - |D \cap U|| > \beta k + \varepsilon s$, then $\min\{|N \cap C|, |N \cap D|\} \leq \min\{|C \cap U|, |D \cap U|\} + 2\varepsilon s + \beta k$.

Lemma 3.3.2. *Let (T, r) be a rooted tree of order at most βk. Let $CD \in E(\bar{H})$. Suppose that $U, N \subseteq C \cup D$ are such that*

$$\min\{|N \cap C \setminus U|, |D \setminus U|\} > \frac{2}{p}(\varepsilon s + \beta k).$$

Then there is an embedding φ of T in $(C \cup D) \setminus U$ such that $\varphi(r) \in N$ and such that the following holds.

$(\star\star)$ *If U has Property $(\star)$ in CD for N, then also $U_\varphi := U \cup \varphi(V(T))$ has Property $(\star)$ in CD for N.*

Proof. Write $V(T) = r \cup L_1 \cup L_2 \cup \ldots$, where L_ℓ is the ℓth level of T (i. e. the set of vertices at distance ℓ to r).

First, suppose that $|N \cap D \setminus U| \leq \varepsilon s$. In this case, choose $\varphi(r) \in N \cap C \setminus U$ typical w. r. t. $D \setminus U$. (This is possible, as by (3.2), there are at most εs vertices that are not typical.)

Embed the rest of $V(T)$ levelwise, choosing for $\varphi(L_\ell)$ unused vertices of $D \setminus U$ that are typical with respect to $C \setminus U$, if ℓ is odd; and choosing vertices of $C \setminus U$ that are typical with respect $D \setminus U$, if ℓ is even.

Now, suppose that $|N \cap D \setminus U| > \varepsilon s$. In this case, we may alternatively wish to embed r in $N \cap D$. We do so in either of the following cases

1. $|\bigcup_{\ell \in \mathbb{N}} L_{2\ell-1}| > |\bigcup_{\ell \in \mathbb{N}} L_{2\ell}|$ and $|C \setminus U| \geq |D \setminus U|$, or
2. $|\bigcup_{\ell \in \mathbb{N}} L_{2\ell-1}| < |\bigcup_{\ell \in \mathbb{N}} L_{2\ell}|$ and $|C \setminus U| \leq |D \setminus U|$,

and otherwise embed r in $N \cap C$, as before. After having thus chosen a place for the root r, the rest of T is embedded analogously as above (possibly swapping the roles of C and D). We have thus completed the embedding of T.

It remains to prove $(\star\star)$. So assume that U_φ has Property $(\star)$ for N in some edge CD. Furthermore, assume that

$$||C \cap U_\varphi| - |D \cap U_\varphi|| > \beta k + \varepsilon s. \tag{3.22}$$

Now, if $||C \cap U| - |D \cap U|| > \beta k + \varepsilon s$, then Property $(\star)$ for U_φ follows from Property $(\star)$ for U. So suppose otherwise, that is

$$||C \cap U| - |D \cap U|| \leq \beta k + \varepsilon s. \tag{3.23}$$

By (3.22), this means that we could not choose into which of $N \cap C$ and $N \cap D$ we would embed the root of T. Hence,

$$\min_{Y=C,D}\{|N \cap Y \setminus U|\} \leq |N \cap D \setminus U| \leq \varepsilon s.$$

Using (3.23), this gives

$$\begin{aligned}
\min\{|N \cap C|, |N \cap D|\} \leq\ & \max\{|C \cap U|, |D \cap U|\} + \min_{Y=C,D}\{|N \cap Y \setminus U|\} \\
\leq\ & \max\{|C \cap U|, |D \cap U|\} + \varepsilon s \\
\leq\ & \min\{|C \cap U|, |D \cap U|\} + 2\varepsilon s + \beta k \\
\leq\ & \min\{|C \cap U_\varphi|, |D \cap U_\varphi|\} + 2\varepsilon s + \beta k,
\end{aligned}$$

as desired. □

Let $C, D, X \in V(\bar{H})$, let $X' \subseteq X$, let $\mathcal{Z} \subseteq V(\bar{H})$, let $U \subseteq \bigcup V(\bar{H})$, let $m \in \mathbb{N}$, and let (T, r) be a rooted tree.

We say that U has Property $(\diamond)$ in (C, D) with respect to X if it satisfies the following.

$(\diamond)$ If $||C \cap U| - |D \cap U|| > \beta k$, then
$\min\{\bar{\deg}_C(X), \bar{\deg}_D(X)\} \leq \min\{|C \cap U|, |D \cap U|\} + 5\varepsilon s + \beta k$.

An embedding φ of T is a *(v, X', U)-embedding* in $\mathcal{Z}$, if $\varphi(V(T) \setminus \{r\}) \subseteq \bigcup \mathcal{Z} \setminus U$, if $\varphi(r) = v$, and if each vertex at odd distance to the root r is mapped to a vertex that is typical to X'.

A vertex is *$\mathcal{Z}$-typical*, if it is typical to each cluster from $\mathcal{Z}$.

The set $\mathcal{Z}$ is said to be *(m, U)-large for X*, if

$$\bar{\deg}_{\mathcal{Z}}(X) > m + |U \cap \bigcup \mathcal{Z}| + \frac{\pi k}{100N}|\mathcal{Z}|.$$

Lemma 3.3.3. *Let $(T, r), X', X$ and U be as above with $|X'| \geq |X|/2$.*
A) *Suppose M_X is a matching in $\bar{H} - X$ so that $V(M_X)$ is $(|V(T)|, U)$-large for X, so that $v \in X$ is $V(M_X)$-typical, and so that U has Property $(\diamond)$ in (C, D) with respect to X, for each $CD \in M_X$.*
Then, there is a (v, X', U)-embedding φ of T in $V(M_X)$ such that $U \cup \varphi(V(T))$ has Property $(\diamond)$ with respect to X for every $CD \in M_X$.
B) *Let $L_X, N_{L_X} \subseteq V(\bar{H})$ be such that L_X is $(|V(T)|, U)$-large for X, and N_{L_X} is $(|V(T)|, U)$-large for each $Y \in L_X$. If $v \in X$ is L_X-typical, then there is a (v, X', U)-embedding φ of T in $L_X \cup N_{L_X}$.*

Proof. We map r to v and embed the trees from the forest $F := T - \{r\}$ inductively. In each step $j \geq 1$, we embed a tree T^j of the forest F. Denote by $V^{<j} \subseteq V(F)$ the set $\bigcup_{i<j} V(T^i)$ of vertices we have already embedded before step j. Let S be the set of vertices in $\bigcup V(\bar{H})$ that are not typical to X'. Set $U^{<j} := U \cup S \cup \varphi(V^{<j})$. In particular, $U^{<1} = U \cup S$.

For Part **A)**, we shall moreover use two properties of U during our embedding. Firstly, if $CD \in M_X$ satisfies $||C \cap U| - |D \cap U|| \leq \beta k$, then we require that in each step $j \geq 1$

(I) $U^{<j}$ has Property ($\star$) for $N(v) \cap (C \cup D)$.

This property holds for $j = 1$, as the condition of Property ($\star$) is void, and we shall check it for each later step.

Secondly, for the edges with $||C \cap U| - |D \cap U|| > \beta k$, observe that, as the sets $U^{<j}$ are growing, Property ($\diamond$) ensures that for all $j \geq 1$

(II) $\min_{Y\in\{C,D\}}\{\bar{\deg}_Y(X)\} \leq \min_{Y\in\{C,D\}}\{|Y \cap U^{<j}|\} + 5\varepsilon s + \beta k$.

So, assume now that we are in step $j \geq 1$, that is, $\varphi(x)$ has been defined for all $x \in V^{<j}$, and we are about to embed T^j.

Claim 3.3.4. *There is an edge CD, with $CD \in M_X$ for Part* **A)** *and with $C \in L$, for Part* **B)***, such that*

$$\min\{|(N(v) \cap C) \setminus U^{<j}|, |D \setminus U^{<j}|\} \geq \frac{2}{p}(\varepsilon s + \beta k).$$

Before proving Claim 3.3.4, we shall show how we complete our embedding of T^j under the assumption that the claim holds for the edge $e = CD$.

Set $N := N(v) \cap e$ and let $r^j := N(r) \cap V(T^j)$ be the root of T^j. Apply Lemma 3.3.2 to embed (T^j, r^j) in $e \setminus U^{<j}$, mapping r^j to $N(v)$. Lemma 3.3.2 together with (I) for j ensures (I) for $j + 1$. As our embedding avoids S, all vertices in $\varphi(T)$ are typical to X'. This terminates step j.

Say we terminate the embedding procedure in step ℓ. Then φ is a (v, X', U)-embedding. So, for Part B), we are done. For Part A), however, we still have to prove that $U^{<\ell} \setminus S$ has Property ($\diamond$) in each $CD \in M_X$.

To this end, assume that $||C \cap (U^{<\ell} \setminus S)| - |D \cap (U^{<\ell} \setminus S)|| > \beta k$. If $||C \cap U| - |D \cap U|| \leq \beta k$, then by (I), $U^{<\ell}$ has Property ($\star$) in CD for $N(v) \cap (C \cup D)$. Hence,

$$\begin{aligned}
\min_{Y=C,D}\{\bar{\deg}_Y(X)\} &\leq \min_{Y=C,D}\{\deg_Y(v)\} + 2\varepsilon s \\
&\leq \min_{Y=C,D}\{|Y \cap (U^{<\ell})|\} + 4\varepsilon s + \beta k \\
&\leq \min_{Y=C,D}\{|Y \cap (U^{<\ell} \setminus S)|\} + 5\varepsilon s + \beta k.
\end{aligned}$$

On the other hand, if $||C \cap U| - |D \cap U|| > \beta k$, then (II) ensures that

$$\min_{Y\in\{C,D\}}\{\bar{\deg}_Y(X)\} \le \min_{Y\in\{C,D\}}\{|Y \cap (U^{<\ell} \setminus S)|\} + 5\varepsilon s + \beta k.$$

This shows that $U^{<\ell} \setminus S$ has Property ($\diamond$) in each $CD \in M_X$ for Part **A)**. It only remains to prove Claim 3.3.4.

Proof of Claim 3.3.4: First, suppose we are in Case **A)**. Let us start by showing that there is an edge $e = CD \in M_X$ which satisfies

$$\bar{\deg}_e(X) - |e \cap U^{<j}| \ge \frac{8}{p}(\varepsilon s + \beta k) + 2\varepsilon s. \tag{3.24}$$

Indeed, suppose there is no such edge. Then, as $V(M_X)$ is $(V(T), U)$-large, we have that

$$\begin{aligned}
\frac{8}{p}(\varepsilon s + \beta k)|M_X| &> \sum_{e\in M_X}(\bar{\deg}_e(X) - |e \cap U^{<j}| - 2\varepsilon s)\\
&= \bar{\deg}_{M_X}(X) - |U \cap \bigcup M_X| - |U^{<j} \setminus U| - 2\varepsilon s|M_X|\\
&\ge \bar{\deg}_{M_X}(X) - |U \cap \bigcup M_X| - |V(T)| - |S \cap M_X| - 2\varepsilon s|M_X|\\
&\ge \frac{\pi k}{100N}|V(M_X)| - 4\varepsilon s|M_X|\\
&> \frac{\pi k}{100N}|M_X|,
\end{aligned}$$

which, as $\beta k \le \frac{\varepsilon}{M_0}n \le \varepsilon s$, implies that $16\varepsilon/p > \pi q/100$, a contradiction.

So, assume now that we have chosen an edge e for which (3.24) holds. Clearly, we can write $e = CD$ such that

$$\frac{4}{p}(\varepsilon s + \beta k) \le \bar{\deg}_C(X) - 2\varepsilon s - |C \cap U^{<j}| \le |N(v) \cap C \setminus U^{<j}|. \tag{3.25}$$

We claim that

$$|D \setminus U^{<j}| \ge \frac{2}{p}(2\varepsilon s + \beta k). \tag{3.26}$$

Indeed, suppose for contradiction (3.26) does not hold. Then (3.25) implies that

$$\begin{aligned}
|C \cap U^{<j}| \le\ & s - \frac{4}{p}(\varepsilon s + \beta k)\\
=\ & |D \cap U^{<j}| + |D \setminus U^{<j}| - \frac{2}{p}(2\varepsilon s + \beta k) - \frac{2}{p}\beta k\\
\le\ & |D \cap U^{<j}| - \frac{2}{p}\beta k.
\end{aligned}$$

Hence, either by (I), or by (II), it holds that

$$\min\{\bar{\deg}_C(X), \bar{\deg}_D(X)\} \leq |C \cap U^{<j}| + 5\varepsilon s + \beta k.$$

Thus, by (3.24),

$$\begin{aligned}
\frac{8}{p}(\varepsilon s + \beta k) + 2\varepsilon s &\leq \bar{\deg}_e(X) - |C \cap U^{<j}| - |D \cap U^{<j}| \\
&\leq \bar{\deg}_e(X) - \min\{\bar{\deg}_C(X), \bar{\deg}_D(X)\} + 5\varepsilon s + \beta k - |D \cap U^{<j}| \\
&\leq s + 5\varepsilon s + \beta k - |D \cap U^{<j}| \\
&< |D \setminus U^{<j}| + 5\varepsilon s + 2\beta k.
\end{aligned}$$

So, $|D \setminus U^{<j}| > (\frac{8}{p} - 3)(\varepsilon s + \beta k)$, a contradiction to our assumption that (3.26) does not hold. This proves (3.26), which together with (3.25) then implies Claim 3.3.4 for Case **A)**.

Now, assume that we are in Case **B)**. First we show that if some $\mathcal{Z} \subseteq V(\bar{H})$ is $(|V(T)|, U)$-large for some $Y \in V(\bar{H})$, then there is a $Z \in \mathcal{Z} \cap N(Y)$ such that

$$\bar{\deg}_Z(Y) - |Z \cap U^{<j}| \geq \frac{2}{p}(\varepsilon s + \beta k) + 2\varepsilon s.$$

Indeed, otherwise

$$\begin{aligned}
\frac{2}{p}(\varepsilon s + \beta k)|\mathcal{Z}| &> \sum_{Z \in \mathcal{Z}} (\bar{\deg}_Z(Y) - |Z \cap U^{<j}| - 2\varepsilon s) \\
&\geq \bar{\deg}_{\mathcal{Z}}(Y) - |U^{<j} \cap \bigcup \mathcal{Z}| - 2\varepsilon s|\mathcal{Z}| \\
&> (\frac{\pi k}{100N} - 4\varepsilon s)|\mathcal{Z}| \\
&\geq \frac{\pi k}{200N}|\mathcal{Z}|,
\end{aligned}$$

a contradiction, as above.

So there is a $C \in L_X$ and a $D \in N_{L_X} \cap N(C)$ such that

$$|N(v) \cap C \setminus U^{<j}| \geq \bar{\deg}_C(X) - |C \cap U^{<j}| - 2\varepsilon s \geq \frac{2}{p}(\varepsilon s + \beta k)$$

and

$$|D \setminus U^{<j}| \geq \bar{\deg}_D(C) - |D \cap U^{<j}| \geq \frac{2}{p}(\varepsilon s + \beta k),$$

as desired for Claim 3.3.4.

□

3.3.7 The embedding in Case 1

In this subsection, we shall complete the proof of Theorem 2.3.3 under the assumption that Case 1 of Section 3.3.2 holds. So, we assume that there are an edge $AB \in E(\bar{H})$ and a matching $M = M_A \cup M_B$ in $\bar{H} - \{A, B\}$ as in Section 3.3.5. These, together with the sets $SD = SD^A \cup SD^B$, $\mathcal{T}_A$ and $\mathcal{T}_B$ from Section 3.3.3, satisfy (3.17).

Our embedding φ will be defined in $|SD|$ steps. In each step $i \geq 1$, we choose a suitable vertex $r_i \in SD$ and embed it together with all trees from

$$\mathcal{T}_i := \{T \in \mathcal{T}_A \cup \mathcal{T}_B : \ sd(T) = r_i\}.$$

Set $V_0 := \emptyset$ and for $i \geq 1$, let

$$V_i := V_{i-1} \cup \{r_i\} \cup \bigcup_{T \in \mathcal{T}_i} V(T).$$

We start with $r_1 := R$, and in each step $i > 1$, we shall choose a vertex $r_i \in SD \setminus V_{i-1}$ that is adjacent to V_{i-1}. The seed r_i will be embedded in a vertex $v_i \in A \cup B$, while $\mathcal{T}_i$ will be mapped to edges from M (or more precisely, to the corresponding subgraph of G'_p). Set $U_0 := \emptyset$, and once φ is defined on V_i, set $U_i := \varphi(V_i)$.

For each $i \geq 0$, the following conditions will hold.

(i) $|(A \cup B) \cap U_i| \leq i$,

(ii) if $x \in V_i \cap N(SD^A)$, resp. $x \in V_i \cap N(SD^B)$, then $\varphi(x)$ has at least $\frac{p}{4}s$ neighbours in A, resp. in B,

(iii) for $CD \in M_A$, the set U_i has Property $(\diamond)$ in CD with respect to A.

(iv) for $CD \in M_B$, the set U_i has Property $(\diamond)$ in CD with respect to B.

Observe that properties (i)–(iv) trivially hold for $i = 0$.

So, suppose now that we are in some step $i \geq 1$ of our embedding process. Choose $r_i \in SD$ as detailed above. Let us assume that $r_i \in SD^A$, the case when $r_i \in SD^B$ is analogous.

We embed r_i in a vertex $v_i = \varphi(r_i) \in A$ that is typical with respect to B and typical w. r. t. all but at most $2\sqrt{\varepsilon}|M_A|$ clusters of M_A. Properties (i) and (ii) for $i-1$ ensure that if x is the predecessor of r_i in T^*, then $\varphi(x)$ has at least $\frac{ps}{4} - i$ neighbours in $A \setminus U_{i-1}$. By (3.2) and (3.4), at most $2\sqrt{\varepsilon}s$ of these vertices do not have the required properties. Hence, there are at least $(\frac{p}{4} - 2\sqrt{\varepsilon})s - i \geq 1$ suitable vertices we may choose $v_i := \varphi(r_i)$ from.

Let $M_A^i \subseteq M_A$ be a maximal submatching such that v_i is typical w. r. t. each of the end-clusters of each edge of M_A^i. Then by (3.4) and (3.17) we obtain

$$\begin{aligned}
\bar{\deg}_{M_A^i}(A) \geq &\ \bar{\deg}_{M_A}(A) - 4\sqrt{\varepsilon}|M_A|s \\
> &\ |V(F_A)| + \frac{\pi k}{40} - 4\sqrt{\varepsilon}|M_A|s \\
> &\ |V(F_A)| + \frac{\pi k}{80} \\
> &\ |V(\bigcup \mathcal{T}_i)| + |U_{i-1} \cap \bigcup V(M_A)| + \frac{\pi k}{80N}|V(M_A^i)|.
\end{aligned}$$

Now we use Lemma 3.3.3 Part **A)** letting (T, r) be the tree induced by r_i and the trees from $\mathcal{T}_i$, and setting $M_X := M_A^i$, $U := U_{i-1}$, $v := v_i$, and $X = X' = A$. It is easy to see that (i), (ii), and (iv) hold for i, as they hold for $i-1$, and by our choice of $\varphi(V_i \setminus V_{i-1})$. Lemma 3.3.3 Part **A)** ensures ($\diamond$) for all edges $CD \in M_A^i$. As we did not embed anything in the edges outside M_A^i, (iii) for $i-1$ implies (iii) for i, for all $CD \in M_A$.

This completes the embedding of the tree T^* in $G'_p \subseteq G$ in Case 1.

3.3.8 The embedding in Case 2

We shall now complete the proof of Theorem 2.3.3 under the assumption that Case 2 of Section 3.3.2 holds. That is, there are an edge $AB \in E(\bar{H})$ and a matching $M = M_F \cup \bar{M}_B$ in $\bar{H} - \{A, B\}$ together with sets $\bar{SD} = \bar{SD}^A \cup SD^B$, $\mathcal{T}_F$, $\bar{\mathcal{T}}_A$, $\mathcal{T}_B^M$ and $\mathcal{T}_B^L$ from Sections 3.3.3 and 3.3.4 satisfying (3.20) and (3.21) from Section 3.3.5.

Our embedding will be defined in three phases. In the first phase, we shall embed all vertices from $\bar{SD}$ in $A \cup B$, embed F_F in edges of M_F, and embed F_B^M in edges of $\bar{M}_B$. In the second phase, we shall embed F_B^L in edges incident with $L' \cap N(B)$, and in the third phase, we shall embed $\bar{F}_A$ in the remaining space inside edges from M.

Denote by A' the set of vertices in A that are typical to all but at most $2\sqrt{\varepsilon}|M|$ clusters of $V(M)$, and denote by B' the set of vertices in B that are typical to all but at most $\sqrt{\varepsilon}|L'|$ clusters of L'.

The first phase is done analogously as in Case 1, always considering A' and B' instead of A and B. In each step, Lemma 3.3.3 Part **A)** is used in the following setting.

The rooted tree (T, r) is the tree induced by r_i and the trees from

$$\mathcal{T}_i := \{T \in \mathcal{T}_F \cup \mathcal{T}_B^M : sd(T) = r_i\}.$$

We set either $(X', X) = (A', A)$ or $(X', X) = (B', B)$, and let $v = \varphi(r_i)$. The matching M_X is a maximal submatching either of M_F or of $\bar{M}_B$, so that $\varphi(r_i)$ is $V(M_X)$-typical. Finally, the set U is the set of the vertices used before step i.

For the second phase, assume that $V(F_B^L) \neq \emptyset$ (otherwise we shall skip the second phase). We define the second phase of our embedding process in $|SD^B|$ steps.

In each step $i \geq 1$, we embed the trees $\mathcal{T}^i := \{T \in \mathcal{T}_B^L : sd(T) = r_i\}$ in edges incident with L'. (Recall that $L' = L \setminus M$.) Suppose that we are at step i of this procedure, i. e. that we have already embedded the trees from $\mathcal{T}^1, \ldots, \mathcal{T}^{i-1}$. Denote by U_{i-1} the set of vertices used so far for the embedding. Let L'_i be the set of those clusters of L' to which $\varphi(r_i)$ is typical. As $\varphi(r_i) \in B'$, (3.4) and (3.21) imply that

$$\bar{\deg}_{L'_i}(B) \geq |V(\bigcup \mathcal{T}^i)| + |U_{i-1} \cap L'_i| + \frac{\pi k}{100N}|L'_i|.$$

Furthermore, by (3.9), for all $Y \in L'_i$ we have that

$$\bar{\deg}(Y) \geq |V(\bigcup \mathcal{T}^i)| + |U_{i-1}| + \frac{\pi k}{100}.$$

Use Lemma 3.3.3 Part **B)** to embed $\mathcal{T}_i$, letting the rooted tree be the tree induced by r_i and the trees from $\mathcal{T}^i$, and setting $X := B$, $X' := B'$, $v := \varphi(r_i)$, $L_X := L'_i$, $N_{L_X} := N(L'_i)$, and $U := U_{i-1}$.

The third phase of our embedding process takes place in $|\bar{SD}^A|$ steps, where in each step $i \geq 1$, we embed the trees from $\mathcal{T}^i := \{T \in \bar{\mathcal{T}}_A : sd(T) = r_i\}$. Suppose that we are at step i of this procedure, i. e. that we have already embedded the trees from $\mathcal{T}^1, \ldots, \mathcal{T}^{i-1}$. Denote by $\bar{U}_{i-1}$ the set of vertices used so far for the embedding. Let M_i be the maximal submatching of M such that $\varphi(r_i)$ is typical to all cluster of $V(M_i)$. As $\varphi(r_i) \in A'$, we have by (3.4) and (3.10) that

$$\bar{\deg}_{M_i}(A) \geq |V(\bigcup \mathcal{T}^i)| + |\bar{U}_i| + \frac{\pi k}{100}.$$

Observe that, as each edge $CD \in M$ meets $N(A)$ in at most one end-cluster, the set U_i trivially has Property $(\diamond)$ in CD with respect to A. We use Lemma 3.3.3 Part **A)** to embed $\mathcal{T}_i$, letting (T, r) be the tree induced by r_i together with the trees from $\mathcal{T}^i$, and setting $X := A$, $X' := A'$, $v := \varphi(r_i)$, $M_X := M_i$, and $U := \bar{U}_{i-1}$.

This terminated our embedding of T^*, and thus the proof of Theorem 2.3.3.

3.4 Proof of Theorem 3.1.1

Our proof of Theorem 3.1.1 follows closely the lines of the proof of Theorem 2.3.3. We embed a rooted spanning tree (T^*, R) of Q, and choosing φ carefully, we ensure the adjacencies for the edges from $E(Q) \setminus E(T^*)$.

Proof of Theorem 3.1.1. Set $\pi := \min\{\eta, q\}$ and set

$$\varepsilon' := \frac{\varepsilon^{c+1}}{c+4}, \quad \text{and} \quad m_0 := \frac{500}{\pi^2 q},$$

where ε is the constant from the proof of Theorem 2.3.3. As in the proof of Theorem 2.3.3, the regularity lemma applied to ε', and m_0, yields natural numbers N_0 and M_0'. Set $M_0 := \max\{M_0', c\}$, define β and p accordingly, and set

$$n_0 := \max\left\{N_0, \frac{9M_0}{\beta} \cdot \left(\frac{8}{p}\right)^{c+1}\right\}.$$

Now, let G be a graph on $n \geq n_0$ vertices which satisfies the condition of Theorem 3.1.1, let $k \geq qn$, and let Q be a connected bipartite graph of order $k+1$ with at most $k+c$ edges, with a spanning tree T^*. Fix a root R in T^*. Denote by M^* the subgraph of Q induced by the edges in $E(Q) \setminus E(T^*)$ and let N^* be the set of predecessors of $V(M^*)$ in the tree order of T^*.

We decompose T^* as in Section 3.3.3, with the difference that we now add the vertices from $V(M^*) \cup N^*$ to the sets A' and B' (from the definition of SD), depending on the parity of their distance in T^* to R. In this way, and since Q is bipartite, we obtain, after the switching, two independent sets $\bar{SD}^A$ and SD^B so that

$$|\bar{SD}^A| + |SD^B| \leq \frac{8}{\beta} + 8c < \frac{9}{\beta},$$

which is constant in n.

The definition of our the embedding φ is similar as in the proof of Theorem 2.3.3, except for some extra precautions we take for vertices from

$V(M^*) \cup N^*$. At step i, for each vertex $r \in \bar{SD}^A$, define

$$N_r^i := \bigcap_{\ell=1}^{j} N(\varphi(x_\ell)) \cap A,$$

where $x_1, \dots x_j$ are the already embedded neighbours of r in $\bar{SD}^A$. If none of the neighbours of r in $\bar{SD}^A$ has been embedded before step i, then set $N_r^i := A$. Analogously define N_r^i for $r \in SD^B$.

In each step i of our embedding process, we shall ensure the following.

(i) If $r \in V(M^*)$ is not yet embedded, then $|N_r^i| \geq \left(\frac{p}{4}\right)^j s$,

where $j = j(r, i)$ is the number of neighbours of r in $\bar{SD}^A$ resp. SD^B that have already been embedded before step i.

Observe that in step $i = 0$, either $N_r^0 = A$ or $N_r^0 = B$, and therefore condition (i) is met.

Suppose that at step $i \geq 1$ of our embedding process, we are about to embed a vertex $r = r_i \in V(M^*) \cup N^*$. Assume that $r \in \bar{SD}^A$ (the case when $r \in SD^B$ is analogous). Denote by $x_1, \dots, x_\ell$ the neighbours of r in $V(M^*)$ that have not been embedded yet.

Now, embed r in a vertex from N_r^{i-1} that satisfies the at most 3 conditions of typicality from the proof of Theorem 2.3.3, except the typicality w. r. t. B, which we replace with typicality w. r. t. each $N_{x_j}^{i-1}$, for $1 \leq j \leq \ell$. This is possible, since our embedding scheme and the condition on the number of edges of Q ensure that r has at most $c+1$ neighbours in Q that are already embedded. Thus, it follows from (i) for $i-1$ and for r, from (3.2), and from the choice of n_0 that there are at least

$$\left(\left(\frac{p}{4}\right)^{c+1} - (c+3)\varepsilon'\right) s - |\bar{SD}| + 1 \geq \frac{1}{2}\left(\frac{p}{4}\right)^{c+1} s - \frac{9}{\beta} + 1 \geq 1$$

unused typical vertices we can choose $\varphi(r)$ from.

Finally, observe that since we chose $\varphi(r)$ typical w. r. t. each $N_{x_j}^{i-1}$, we have ensured property (i) for i and for every $r' \in V(M^*)$ that is not yet embedded. This completes the proof of Theorem 3.1.1. □

Chapter 4

Solution of the LKS conjecture for special classes of trees

4.1 Trees of small diameter and caterpillars

In this section, which is based on work from [75] we will prove the Loebl–Kómlos–Sós conjecture for two classes of trees.

The first class of trees for which we shall prove Conjecture 2.1.2, is the class of all tress that have diameter at most 5. Our result implies the results of Barr and Johansson [3], and Sun [87].

Theorem 4.1.1. **[75]** *Let $k, n \in \mathbb{N}$, and let G be a graph of order n so that at least $n/2$ vertices of G have degree at least k.*
Then every tree of diameter at most 5 and with at most k edges is a subgraph of G.

The second of the classes for which we shall prove Conjecture 2.1.2 is a subclass of the caterpillars. This extends results of Bazgan, Li, and Woźniak [4].

Let $\mathcal{T}(k, \ell, c)$ be the class of all trees with k edges which can be obtained from a path P of length $k - \ell$, and two stars S_1 and S_2 by identifying the centres of the S_i with two vertices that lie at distance c from each other on$P$.

Theorem 4.1.2. **[75]** *Let $k, \ell, c, n \in \mathbb{N}$ such that $\ell \geq c$. Suppose that c is even, or that $\ell + c \geq \lfloor n/2 \rfloor + 1$. Let $T \in \mathcal{T}(k, \ell, c)$, and let G be a graph of order n so that at least $n/2$ vertices of G have degree at least k.*
Then T is a subgraph of G.

4.2 Proof of Theorem 4.1.1

We shall prove the Theorem 4.1.1 by contradiction. So, assume that there are $k, n \in \mathbb{N}$, and a graph G with $|V(G)| = n$, such that at least $n/2$ vertices of G have degree at least k. Furthermore, suppose that T is a tree of diameter at most 5 with $|E(T)| \leq k$ such that $T \not\subseteq G$.

We may assume that among all such counterexamples G for T, we have chosen G edge-minimal. In other words, we assume that the deletion of any edge of G results in a graph which has less than $n/2$ vertices of degree k.

Denote by L the set of those vertices of G that have degree at least k, and set $S := V(G) \setminus L$. Observe that, by our edge-minimal choice of G, we know that S is independent. Also, we may assume that S is not empty (otherwise $T \subseteq G$ trivially).

Clearly, our assumption that $T \not\subseteq G$ implies that for each set M of leaves of T it holds that

$$\text{there is no embedding } \varphi \text{ of } V(T) \setminus M \text{ in } V(G) \text{ so that } \varphi(N(M)) \subseteq L. \tag{4.1}$$

In what follows, we shall often use the fact that both the degree of a vertex and the cardinality of a set of vertices are integers. In particular, assume that $a, b \in \mathbb{N}$, and $x \in \mathbb{Q}$. Then the following implication holds.

$$\textit{If } a < x + 1 \textit{ and } b \geq x, \textit{ then } a \leq b. \tag{4.2}$$

Let us now define a useful partition of $V(G)$. Set

$$\begin{aligned} &A := \{v \in L : \deg_L(v) < \frac{k}{2}\}, \\ &B := L \setminus A, \\ &C := \{v \in S : \deg(v) = \deg_L(v) \geq \frac{k}{2}\}, \textit{ and} \\ &D := S \setminus C. \end{aligned}$$

Let $r_1 r_2 \in E(T)$ be such an edge that each vertex of T has distance at most 2 to at least one of r_1, r_2. Set

$$\begin{aligned} V_1 &:= N(r_1) \setminus \{r_2\}, & V_2 &:= N(r_2) \setminus \{r_1\}, \\ W_1 &:= N(V_1) \setminus \{r_1\}, & W_2 &:= N(V_2) \setminus \{r_2\}. \end{aligned}$$

Furthermore, set

$$V_1' := N(W_1) \quad \text{and} \quad V_2' := N(W_2).$$

Observe that $|V_1 \cup V_2 \cup W_1 \cup W_2| < k$. So, without loss of generality (since we can otherwise interchange the roles of r_1 and r_2), we may assume that

$$|V_2 \cup W_1| < \frac{k}{2}. \tag{4.3}$$

Since $|V_1'| \leq |W_1|$, this implies that

$$|V_1' \cup V_2| < \frac{k}{2}. \tag{4.4}$$

Now, assume that there is an edge $uv \in E(G)$ with $u, v \in B$. We shall conduct this assumption to a contradiction to (4.1) by proving that then we can define an embedding φ so that $\varphi(V_1' \cup V_2 \cup \{r_1, r_2\}) \subseteq L$. Define the embedding φ as follows. Set $\varphi(r_1) := u$, and set $\varphi(r_2) := v$. Map V_1' to a subset of $N(u) \cap L$, and V_2 to a subset of $N(v) \cap L$ that is disjoint from $\varphi(V_1')$. This is possible, as (4.2) and (4.4) imply that $|V_1' \cup V_2| + 1 \leq \deg_L(v)$.

We have thus reached the desired contradiction to (4.1). This proves that

$$B \textit{ is independent.} \tag{4.5}$$

Set

$$N := N(B) \cap L \subseteq A.$$

We claim that each vertex $v \in N$ has degree

$$\deg_L(v) < \frac{k}{4}. \tag{4.6}$$

Then, (4.5) and (4.6) together imply that

$$|B|\frac{k}{2} \leq e(N, B) \leq |N|\frac{k}{4},$$

and hence,

$$|N| \geq 2|B|. \tag{4.7}$$

In order to see (4.6), suppose otherwise, i. e., suppose that there is a vertex $v \in N$ with $\deg_B(v) \geq \frac{k}{4}$. Observe that by (4.4), $|V_1' \cup V_2'| < \frac{k}{2}$ and hence we may assume that at least one of $|V_1'|$, $|V_2'|$, say $|V_1'|$, is smaller than $\frac{k}{4}$. The embedding φ is defined as for the proof of (4.5), by embedding first $V_1' \cup \{r_2\}$ in $N(v)$ and then V_2' in a subset $N(\varphi(r_2)) \cap L$, that is disjoint from $\varphi(V_1')$. The case when $|V_2'| < \frac{k}{4}$ is done analogously. This yields the desired contradiction to (4.1), and thus proves (4.6).

Now, set

$$X := \{v \in L : \deg_{C\cup L}(v) \geq \frac{k}{2}\} \supseteq B.$$

We claim that the number of edges between X and C

$$e(X, C) = 0. \quad (4.8)$$

Observe that then

$$X = B, \quad (4.9)$$

and,

$$e(B, C) = 0. \quad (4.10)$$

In order to see (4.8), suppose for contradiction that there exists an edge uv of G with $u \in X$ and $v \in C$. We define an embedding φ of $V_1' \cup V_2 \cup W_1^C \cup \{r_1, r_2\}$ in $V(G)$, where W_1^C is a certain subset of W_1, as follows.

Set $\varphi(r_1) := u$, and set $\varphi(r_2) := v$. Embed a subset V_1^C of V_1' in $N(u) \cap C$, and a subset $V_1^L = V_1' \setminus V_1^C$ in $N(u) \cap L$. We can do so because of (4.2) and (4.4), which implies that $|V_1'| < \frac{k}{2}$.

Next, map $W_1^C := N(V_1^C) \cap W_1$ and V_2 to L, preserving all adjacencies. Indeed, observe that by the independence of S, each vertex in C has at least $\frac{k}{2}$ neighbours in L, while by (4.3), we have that

$$|V_1^L \cup W_1^C \cup V_2 \cup \{u\}| \leq |W_1 \cup V_2| + 1 < \frac{k}{2} + 1.$$

We have hence mapped V_1', V_2, W_1^C and the vertices r_1 and r_2 in a way so that the neighbours of $(V_1 \setminus V_1') \cup (W_1 \setminus W_1^C) \cup W_2$ are mapped to L. This yields the desired contradiction to (4.1). We have thus shown (4.8), and consequently, also (4.9) and (4.10).

Observe that $D \neq \emptyset$. Indeed, otherwise $C \neq \emptyset$ and thus by (4.8), we have that $A \neq \emptyset$. By (4.9), this implies that $D \neq \emptyset$, contradicting our assumption.

Next, we claim that there is a vertex $w \in N$ with

$$\deg_{C\cup L}(w) \geq \frac{k}{4}. \quad (4.11)$$

Indeed, suppose otherwise. By (4.9) and since D is non-empty, we obtain that[1]

$$|A \setminus N|\frac{k}{2} + |N|\frac{3k}{4} \leq e(A, D) < |D|\frac{k}{2}.$$

[1] $e(A, D)$ is defined as neither A nor D can be empty.

Dividing by $\frac{k}{4}$, it follows that

$$2|A| + |N| < 2|D|.$$

Together with (4.7), this yields

$$|D| > |A| + |B| \geq \frac{n}{2},$$

a contradiction, since by assumption $|D| \leq |S| \leq \frac{n}{2}$. This proves (4.11).

Using a similar argument as for (4.8), we can now show that

$$|V_1'| \geq \frac{k}{4}. \tag{4.12}$$

Indeed, otherwise by (4.11), we can map r_1 to w, r_2 to any $u \in N(w) \cap B$, and embed V_1' in $C \cup L$, and V_2 and W_1^C (defined as above) in L, preserving the adjacencies. This yields the desired contradiction to (4.1).

Observe that (4.12) implies that $\frac{k}{4} \leq |V_1'| \leq |W_1|$, and hence, by (4.3),

$$|V_2| < \frac{k}{4}. \tag{4.13}$$

We claim that moreover

$$|V_1' \cup W_2| \geq \frac{k}{2}. \tag{4.14}$$

Suppose for contradiction that this is not the case. We shall then define an embedding φ of $V_1' \cup V_2' \cup \{r_1, r_2\} \cup W_2^C$ in $V(G)$, for a certain $W_2^C \subseteq W_2$, as follows.

Set $\varphi(r_2) := w$, and choose for $\varphi(r_1)$ any vertex $u \in N(w) \cap B$. Map a subset V_2^C of V_2' to $N(w) \cap C$, and map $V_2^L := V_2' \setminus V_2^C$ to $N(w) \cap L$. This is possible, as by (4.2), by (4.11), and by (4.13), we have that $\deg_{C \cup L}(w) \geq |V_2'| + 1$.

Let $W_2^C := N(V_2^C) \cap W_2$. Then

$$|V_2^L| \leq |W_2 \setminus W_2^C|,$$

and by our assumption that $|V_1' \cup W_2| < \frac{k}{2}$, we obtain that

$$|V_1' \cup V_2^L \cup W_2^C \cup \{r_2\}| \leq |V_1' \cup W_2| + 1 < \frac{k}{2} + 1.$$

Thus, by (4.2), for each $v \in C$, we have that $\deg(v) \geq |V_2^L \cup W_2^C| + 1$. Observe that (4.10) implies that $u \notin N(C)$. So, we can map W_2^C to L, preserving all adjacencies, and V_1' to a subset of $N(u) \cap L$ which is disjoint from $\varphi(V_2^L \cup W_2^C \cup \{v\})$.

We have thus embedded all of $V(T)$ except $(V_1 \setminus V_1') \cup (W_2 \setminus W_2^C) \cup W_1$ whose neighbours have their image in L. This yields a contradiction to (4.1), and hence proves (4.14).

Now, by (4.14),

$$|W_2| \geq \frac{k}{2} - |V_1'|,$$

and since $|W_1| \geq |V_1'|$, and $|V(T) \setminus \{r_1, r_2\}| < k$,

$$\begin{aligned} |V_1 \cup V_2| &< k - |W_1| - (\frac{k}{2} - |V_1'|) \\ &\leq \frac{k}{2}. \end{aligned} \tag{4.15}$$

The now gained information on the structure of T enables us to show next that for each vertex v in $\tilde{N} := N(B \cup C) \cap L$ it holds that

$$\deg_L(v) < \frac{k}{4}. \tag{4.16}$$

Suppose for contradiction that this is not the case, i. e., that there exists a $v \in \tilde{N}$ with $\deg_L(v) \geq \frac{k}{4}$. We define an embedding φ of $V(T) \setminus (W_1 \cup W_2)$ in $V(G)$ so that $N(W_1 \cup W_2)$ is mapped to L.

Set $\varphi(r_2) := v$ and choose for $\varphi(r_1)$ any vertex $u \in N(v) \cap (B \cup C)$. By (4.13), and since we assume that (4.16) does not hold, we can map V_2 to $N(v) \cap L$. Moreover, since by (4.2) and (4.15) we have that

$$\deg_L(u) \geq |V_1 \cup V_2 \cup \{r_2\}|,$$

we can map V_1 to $N(u) \cap L$. We have hence mapped all of $V(T)$ but $W_1 \cup W_2$ to L, which yields the desired contradiction to (4.1) and thus establishes (4.16).

We shall finally bring (4.16) to a contradiction. We use (4.5), (4.9), (4.10) and (4.16) to obtain that

$$\begin{aligned} |D|\frac{k}{2} &\geq e(D, L) \\ &\geq |A \setminus \tilde{N}|\frac{k}{2} + |\tilde{N}|\frac{3k}{4} - e(C, \tilde{N}) + |B|k - e(B, \tilde{N}) \\ &\geq |A|\frac{k}{2} + |\tilde{N}|\frac{k}{4} + |B|k - e(B \cup C, \tilde{N}). \end{aligned}$$

Since $|S| \leq |L|$ by assumption, this inequality implies that

$$\begin{aligned} |B|\frac{k}{2} + |C|\frac{k}{2} + |\tilde{N}|\frac{k}{4} &\leq |B|\frac{k}{2} + (|A| + |B| - |D|)\frac{k}{2} + |\tilde{N}|\frac{k}{4} \\ &\leq e(B \cup C, \tilde{N}) \\ &\leq |\tilde{N}|\frac{k}{2}, \end{aligned}$$

where the last inequality follows from the fact that $\tilde{N} \subseteq A = L \setminus X$, by (4.9).

Using (4.16), a final double edge-counting now gives

$$\begin{aligned} (|A| + |B| + |C|)\frac{k}{2} &\leq |A|\frac{k}{2} + |\tilde{N}|\frac{k}{4} \\ &\leq e(A, S) \\ &< |D|\frac{k}{2} + |C|k \\ &= |S|\frac{k}{2} + |C|\frac{k}{2}, \end{aligned}$$

implying that $|L| < |S|$, a contradiction. This completes the proof of Theorem 4.1.1.

4.3 Proof of Theorem 4.1.2

In this section, we shall prove Theorem 4.1.2. We shall actually prove something stronger, namely Lemmas 4.3.2 and 4.3.3.

A *caterpillar* is a tree T where each vertex has distance at most 1 to some central path $P \subseteq T$. In this paper, we shall consider a special subclass of caterpillars, namely those that have at most two vertices of degree greater than 2. Observe that any such caterpillar T can be obtained from a path P by identifying two of its vertices, v_1 and v_2, with the centres of stars. We shall write $T = C(a, b, c, d, e)$, if P has length $a + c + e$, and v_1 and v_2 are the $(a + 1)$th and $(a + c + 1)$th vertex on P, and have b, resp. d, neighbours outside P. Therefore, if $a, e > 0$, then $C(a, b, c, d, e)$ has $b + d + 2$ leaves.

We call P the *body*, and v_1 and v_2 the *joints* of the caterpillar. For illustration, see Figure 4.1.

So $\mathcal{T}(k, \ell, c)$, as defined in the introduction, denotes the class of all caterpillars $C(a, b, c, d, e)$ with $b+d = \ell$, and $a+b+c+d+e = k$. We can thus

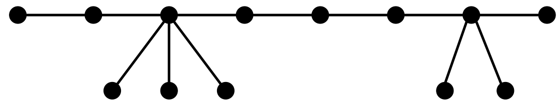


Figure 4.1: The caterpillar $C(2,3,4,2,1)$ or $C(2,3,4,3,0)$.

state the result of Bazgan, Li, and Woźniak mentioned in the introduction as follows.

Theorem 4.3.1 (Bazgan, Li, Woźniak [4]). *Let $k, \ell, c \in \mathbb{N}$, and let $T = C(a, 0, c, d, e)$ be a tree from $\mathcal{T}(k, \ell, c)$. Let G be a graph so that at least half of the vertices of G have degree at least k. Then T is a subgraph of G.*

Theorem 4.1.2 will follow from the following two lemmas. The first deals with even c, the second with odd c.

Lemma 4.3.2. *Let $k, \ell, c \in \mathbb{N}$ so that c is even and $\ell \geq c$. Let $T \in \mathcal{T}(k, \ell, c)$, and let G be a graph such that at least half of the vertices of G have degree at least k. Then T is a subgraph of G.*

Proof. Observe that we may assume that $\ell \geq 2$. Let v_1 and v_2 be the joints of T, and let P be its body. As above, denote by L the set of those vertices of G that have degree at least k and set $S := V(G) \setminus L$. We may assume that S is independent.

By Theorem 4.3.1, there is a path $P_k := \{x_0, x_1, \ldots, x_k\}$ of length k in G. Let φ be an embedding of $V(P)$ in $V(P_k)$ which maps the starting vertex of P to x_0.Now, if both $u_1 := \varphi(v_1)$ and $u_2 := \varphi(v_2)$ are in L, then we can easily extend φ to $V(T)$.

On the other hand, if both u_1 and u_2 lie in S, then let $\varphi'(v) = x_{i+1}$ whenever $\varphi(v) = x_i$. The embedding φ' maps both v_1 and v_2 to L, and can thus be extended to an embedding of $V(T)$. We call φ' a *shift* of $\varphi(V(P))$.

To conclude, assume that one of the two vertices u_1 and u_2 lies in L and the other lies in S. As c is even and S is independent, it follows that there are two consecutive vertices x_j and x_{j+1} on $u_1 P_k u_2$ which lie in L.

Similarly as above, shift $\varphi(V(P))$ repeatedly until u_1 is mapped to x_j. If the iterated shift φ' maps v_2 to L, we are done. Otherwise, we shift $\varphi'(V(P))$ once more. Then both v_1 and v_2 are mapped to L, and we are done.

Observe that in total, we have shifted $\varphi(V(P))$ at most c times. We could do so, since $|P_k| - |P| = \ell \geq c$ by assumption. □

Lemma 4.3.3. *Let $k, \ell, c, n \in \mathbb{N}$ be such that $\ell \geq c$. Let $T = C(a, b, c, d, e)$ be a tree in $\mathcal{T}(k, \ell, c)$, and let G be a graph of order n such that at least $n/2$ vertices of G have degree at least k. Suppose that*

(i) $k \geq \lfloor n/2 \rfloor + 2\min\{a, e\} + 1$, *if* $\max\{a, e\} \leq k/2$, *and*

(ii) $k \geq \lfloor n/4 \rfloor + a + e + 2$, *if* $\max\{a, e\} > k/2$.

Then T is a subgraph of G.

Observe that in case (ii) of Lemma 4.3.3 it follows that

$$k \geq \lfloor n/4 \rfloor + \min\{a, e\} + \max\{a, e\} + 1 > \lfloor n/4 \rfloor + \min\{a, e\} + k/2 + 1,$$

and hence, because $2\lfloor \frac{n}{4} \rfloor + 1 \geq \lfloor \frac{n}{2} \rfloor$, similar as in (i),

$$k \geq \lfloor n/2 \rfloor + 2\min\{a, e\} + 1.$$

Proof of Lemma 4.3.3. As before, set $L := \{v \in V(G) : \deg(v) \geq k\}$ and set $S := V(G) \setminus L$. We may assume that S is independent, and that that $a, e \neq 0$. Because of Theorem 4.3.1, we may moreover assume that $b, d > 0$ (and thus $\ell \geq 2$), and by Lemma 4.3.2, that c is odd. Assume that $a \leq e$ (the case when $a < e$ is similar).

Suppose that $T \not\subseteq G$. Using the same shifting arguments as in the proof of Lemma 4.3.2, we know that for any path in G of length m, we can shift its first $(a+c+e)$ vertices at least $m-(a+c+e)$ times. So we may assume that every path in G of length at least k zigzags between L and S, except possibly on its first a and its last e edges. In fact, as c is odd, we can even assume that every path in G of length at least $k-1$ zigzags between L and S, except possibly on its first a and its last e edges.

As paths are symmetric, we may actually assume that every path $Q = x_0 \ldots x_m$ in G of any length $m \geq k-1$ zigzags on its subpaths $x_a Q x_{m-e}$ and $x_e Q x_{m-a}$. Observe that these subpaths overlap exactly if $e \leq m/2$. Our aim is now to find a path that does not zigzag on the specified subpaths, which will yield a contradiction.

So, let $\mathcal{Q}$ be the set of those subpaths of G that have length at least $k-1$ and end in L. Observe that by Theorem 4.3.1, and since S is independent, $\mathcal{Q} \neq \emptyset$. Among all paths in $\mathcal{Q}$, choose $Q = x_0 \ldots x_m$ so that it has a maximal number of vertices in L.

This choice of Q guarantees that $N(x_m) \subseteq S \cup V(Q)$. Observe that the remark after the statement of Lemma 4.3.3 implies that in both cases (i) and (ii) ,

$$\begin{aligned}\deg(x_m) \geq k &\geq \lfloor n/2 \rfloor + 2a + 1\\ &\geq |S| + 2a + 1.\end{aligned}$$

Since $a > 0$, we thus obtain that x_m has a neighbour $x_s \in L \cap V(Q)$ with

$$s \in [a, m - a - 1].$$

Moreover, in the case that $e > m/2$, condition (ii) of Lemma 4.3.3 implies that

$$\begin{aligned}\deg(x_m) \geq k &\geq 2(\lfloor n/4 \rfloor + a + e + 2) - k\\ &\geq \lfloor n/2 \rfloor - 1 + 2a + 2e + 4 - (m + 1)\\ &= |S| + 2a + 2e + 2 - m.\end{aligned}$$

Hence, in this case we can guarantee that

$$s \in [a, m - e - 1] \cup [e, m - a - 1].$$

Consider the path Q^* we obtain from Q by joining the subpaths x_0Qx_s and $x_{s+1}Qx_m$ with the edge x_sx_m. Then Q^* is a path of length $m \geq k - 1$ which contains the $L - L$ edge x_sx_m. Note that x_sx_m is neither one of the first a nor of the last a edges on Q^*. Furthermore, in the case that $e > m/2$, we know that x_sx_m is none of the middle $2e-m$ edges on Q^*. This contradicts our assumption that every path of length at least $k - 1$ zigzags between L and S except possibly on these subpaths. □

Now it is easy to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. Assume we are given graphs G and $T \in \mathcal{T}(k, \ell, c)$ as in Theorem 4.1.2. If c is even, it follows from Lemma 4.3.2 that $T \subseteq G$. So assume that $\ell + c \geq \lfloor n/2 \rfloor$. We shall now use Lemma 4.3.3 to see that $T \subseteq G$. Suppose that $T = C(a, b, c, d, e)$. We have

$$k - 2\min\{a, e\} \geq k - a - e = \ell + c \geq \lfloor n/2 \rfloor + 1 \geq \lfloor n/4 \rfloor + 2,$$

since we may assume that $n \geq 4$, as otherwise Theorem 4.1.2 holds trivially. □

Chapter 5

An application of the LKS conjecture in Ramsey Theory

5.1 Ramsey numbers

Conjecture 2.1.2 has an important application in Ramsey theory. The Ramsey number $r(H, H')$ of two graphs, H and H', is defined as the minimum integer n such for every graph G of order at least n either H is a subgraph of G, or H' is a subgraph of the complement $\bar{G}$ of G. One can reformulate this definition using edge-colourings. Then $r(H, H')$ is the minimum integer n such for every edge-colouring of the complete graph K^n of order n with two colours, say red and blue, we either find a red copy of H or a blue copy of H' in K^n.

Originally, Ramsey numbers were introduced only for complete graphs H and H'. Even for this class, they are very difficult to compute. This is the reason why there are few other classes for graphs for which the Ramsey number has been studied. These classes include trees, mainly paths and stars, and cycles, also wheels.

Ramsey numbers of trees of small maximum degree have been studied for instance in [54]). For a more general overview see [47].

For our purposes here, let us extend the definition given above. We denote by $r(\mathcal{H}, \mathcal{H}')$ the Ramsey number of two classes of graphs, $\mathcal{H}$ and $\mathcal{H}'$, that is, $r(\mathcal{H}, \mathcal{H}')$ is the minimum integer n such for every graph G of order at least n either each graph $H \in \mathcal{H}$ is a subgraph of G, or each graph $H' \in \mathcal{H}'$ is a subgraph of the complement $\bar{G}$ of G. We write $r(\mathcal{H})$ as shorthand for $r(\mathcal{H}, \mathcal{H})$.

5.2 Ramsey numbers of trees

We are here interested in the Ramsey number classes of trees. For $i \in \mathbb{N}$, let $\mathcal{T}_i$ denote the class of all trees of order i.

Zhao's result [96] that Conjecture 2.1.1 holds for large graphs (see Chapter 2 implies almost at once that for large k, the Ramsey number $r(\mathcal{T}_{k+1})$ of the class of all graphs of order at most $k + 1$ is at most $2k$.

In fact, for any colouring of the edges of the complete graph K_{m+k} with two colours, either half of the vertices have degree k in the subgraph induced by the first colour, or half of the vertices have degree m in the subgraph induced by the second colour. Hence, if $m + k$ is large enough so that the Loebl conjecture holds, it follows that $r(\mathcal{T}_{k+1}) \leq k$.

In the same way we can deduce from Conjecture 2.1.2, if true, a bound on $r(\mathcal{T}_{k+1}, \mathcal{T}_{m+1})$. Indeed with the same argument as above we would get that $r(\mathcal{T}_{k+1}, \mathcal{T}_{m+1}) \leq k + m$. This upper bound has been conjectured in [37], and it is not difficult to see that the bound is best possible.

So, in particular, the bound $k + m$ holds for all classes of trees for which Conjecture 2.1.2 is known to hold. Our results for special cases, Theorem 4.1.1 and Theorem 4.1.2, thus have the following corollary.

Corollary 5.2.1. *Let T_1, T_2 be trees with k resp. m edges such that, for $i = 1, 2$, either T_i is as in Theorem 4.1.2 or has diameter at most 5 (or both). Then $r(T_1, T_2) \leq k + m$.*

Furthermore, we may use Theorem 2.3.3, to prove that the conjectured bound is asymptotically correct. Our proof is based on ideas from [37].

Proposition 5.2.2. *$r(\mathcal{T}_{k+1}, \mathcal{T}_{m+1}) \leq k + m + o(k + m)$, as $k + m \to \infty$.*

Let us remark that the argument of the proof of Proposition 5.2.2 would apply also for the graphs from Theorem 3.1.1, if we use Theorem 3.1.1 instead of Theorem 2.3.3. We thus obtain an upper bound of $k + m + o(k + m)$ for the Ramsey numbers of graphs Q_k, Q_m as in Theorem 3.1.1. However, the sharp bound does not hold. This can be seen considering the example given in Chapter 3.

Finally, we remark that the exact bound of $r(\mathcal{T}_{k+1}, \mathcal{T}_{m+1}) \leq k + m$ also follows from a positive answer to the Erdős–Sós conjecture. This well-known conjecture states that each graph with average degree greater than $k - 1$ contains all trees with at most k edges as subgraphs. For partial results on the Erdős–Sós conjecture, see e.g. [8, 78, 95]. Ajtai, Komlós, Simonovits and Szemerédi have announced a proof of the Erdős–Sós conjecture for large n.

5.3 Proof of Proposition 5.2.2

Given $0 < \varepsilon < 1/4$, we apply Theorem 2.3.3 to $\eta = q = \varepsilon/4$ to obtain an $n_0 \in \mathbb{N}$. Now, let $n \geq n_0$, and let G be a graph on $n' = (1+2\varepsilon)n+1$ vertices. Let k and m be such that $k+m=n$.

Clearly, either at least half of the vertices of G have degree at least $k+\varepsilon n$, or in the complement $\bar{G}$ of G, at least half of the vertices have degree at least $m+\varepsilon n$.

First, suppose that the former of these assertions is true. Then it is easy to calculate that

$$k + \varepsilon n \geq (1+\eta)(k+qn').$$

Thus, we may apply Theorem 2.3.3, which yields that each tree in $\mathcal{T}_{k+qn'+1}$ is a subgraph of G. Hence, also each tree in $\mathcal{T}_{k+1}$ is a subgraph of G.

Now, assume that the second assertion from above holds, that is, in the complement $\bar{G}$ of G, there are at least $\frac{1}{2}(1+\varepsilon)n$ vertices of degree at least $m + \frac{\varepsilon}{2}n$. We then find all trees from $\mathcal{T}_{m+1}$ as subgraphs in $\bar{G}$. This is done analogously.

We have thus shown that for every $\varepsilon > 0$ there is an n_0 so that for all k, m with $k+m \geq n_0$, we have that $r(\mathcal{T}_{k+1}, \mathcal{T}_{m+1}) \leq (1+2\varepsilon)(k+m)+1$. This proves Proposition 5.2.2.

Chapter 6

t-perfect graphs

6.1 An introduction to t-perfect graphs

Perfect graphs can be determined by the structure of their stable set polytope. The *stable set polytope*, or SSP for short, is the convex hull of the characteristic vectors of independent vertex sets, the stable sets. In the case of a perfect graph, the SSP is fully described by non-negativity and clique inequalities. Vice versa, if the SSP of some graph is given by these types of inequalities then the graph is perfect.

In analogy to the relationship between perfect graphs and the SSP, Chvátal [25] proposed to investigate a class of graphs now called t-perfect: the class of graphs whose SSP is determined by non-negativity, edge and odd-cycle inequalities. (For precise definitions see next section.) The class of t-perfect graphs includes the series-parallel graphs (Boulala and Uhry [7]) and the almost bipartite graphs, i.e. those graphs that become bipartite upon deletion of a single vertex (Fonlupt and Uhry [38]). Gerards and Shepherd [45] characterise the graphs with all subgraphs t-perfect. A prime example of graph that is not t-perfect is the complete graph on four vertices, the K_4. Indeed, this graph will play an important role in what follows.

In this chapter, which is based on work from [13], we prove two theorems for t-perfect graphs that are, in addition, claw-free. We show that these graphs can be 3-coloured and we characterise them in terms of forbidden substructures.

Standard polyhedral methods assert that the fractional chromatic number of a t-perfect graph is at most 3. Shepherd suggested that t-perfect graphs might always be k-colourable for some fixed small k. As Laurent and Seymour found a t-perfect graph with $\chi = 4$ (see [81, p. 1207]), this number

k has to be at least 4.

Conjecture 6.1.1. *Every t-perfect graph is 4-colourable.*

We prove that if the graphs are additionally claw-free then three colours suffice.

Theorem 6.1.2. [13] *Every claw-free t-perfect graph is 3-colourable.*

Moreover, such a 3-colouring can be computed in polynomial time (Corollary 6.6.1).

We remark that compared to a result of Chudnovsky and Ovetsky [22] our Theorem 6.1.2 yields an improvement of 1. Indeed, Chudnovsky and Ovetsky show that the chromatic number of a quasi-line graph G is bounded by $\frac{3}{2}\omega(G)$. As no t-perfect graph can contain a clique of at least four vertices and, furthermore, as a claw-free t-perfect graph is quasi-line, Chudnovsky and Ovetsky's bound is applicable and yields $\chi \leq 4$ for all claw-free t-perfect graphs.

The celebrated strong perfect graph theorem of Chudnovsky, Robertson, Seymour and Thomas [23] characterises perfect graphs in terms of forbidden induced subgraphs: a graph is perfect if and only if it does not contain odd holes or anti-holes. We prove an analogous, although much more modest, result for claw-free t-perfect graphs. While, in order to describe perfect graphs, induced subgraphs are suitable as forbidden substructures, for t-perfect graphs a more general type of substructure, called a t-minor, is more appropriate. Briefly, a t-minor is any graph obtained from the original graph by two kinds of operations, both of which preserve t-perfection: vertex deletions and simultaneous contraction of all the edges incident with a vertex whose neighbourhood forms an independent set. With this notion our second result is as follows.

Theorem 6.1.3. [13] *A claw-free graph is t-perfect if and only if it does not contain any of K_4, W_5, C_7^2 and C_{10}^2 as a t-minor.*

Here, K_4 denotes the complete graph on four vertices, W_5 is the 5-wheel, and for $n \in \mathbb{N}$ we denote by C_n^2 the square of the cycle C_n on n vertices, see Figure 6.1. (The square of a graph is obtained by adding edges between any two vertices of distance 2.)

The graphs from Theorem 6.1.3 already appear implicitly in Galluccio and Sassano [42]. They showed that every rank facet in a claw-free graph comes from a combination of inequalities describing cliques, line graphs of 2-connected factor-critical graphs, and circulant graphs $C_{\alpha\omega+1}^{\omega-1}$. However, as

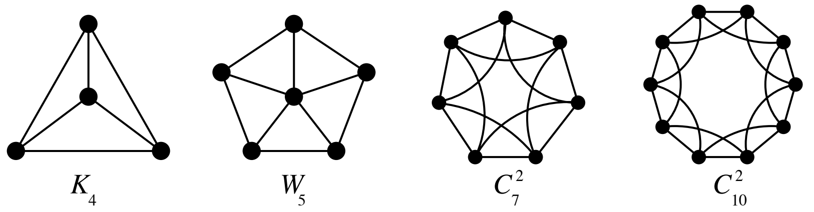


Figure 6.1: The forbidden t-minors.

a claw-free graph may have non-rank facets we will not be able to make use of these results.

Ben Rebea's conjecture describes the structure of the stable set polytope of quasi-line graphs. As the conjecture has been solved (see Eisenbrand et al [36] and Chudnovsky and Seymour [24]), and as claw-free t-perfect graphs are quasi-line, it seems conceivable to use Ben Rebea's conjecture to prove Theorem 6.1.3. We have not pursued this approach for three reasons. First, Theorem 6.1.3 does not appear to be a direct consequence of the conjecture. Second, the solution of the conjecture rests on Chudnovsky and Seymour's characterisation of claw-free graphs, which is far from trivial. Finally, our proof of Theorem 6.1.3 (with a little extra effort) yields a 3-colouring of claw-free t-perfect graphs.

6.2 The polytopes SSP and TSTAB

Let $G = (V, E)$ be a graph. The *stable set polytope* $\mathrm{SSP}(G) \subseteq \mathbb{R}^V$ of G is defined as the convex hull of the characteristic vectors of stable, i.e. independent, subsets of V. We define a second polytope $\mathrm{TSTAB}(G) \subseteq \mathbb{R}^V$ for G, given by

$$\begin{aligned} &x \geq 0, \\ &x_u + x_v \leq 1 \text{ for every edge } uv \in E, \\ &x(C) \leq \lfloor |C|/2 \rfloor \text{ for every induced odd cycle } C \text{ in } G. \end{aligned} \tag{6.1}$$

These inequalities are respectively known as non-negativity, edge and odd-cycle inequalities. Clearly, it holds that $\mathrm{SSP}(G) \subseteq \mathrm{TSTAB}(G)$.

We say that the graph G is *t-perfect* if $\mathrm{SSP}(G)$ and $\mathrm{TSTAB}(G)$ coincide. Equivalently, G is t-perfect if and only if $\mathrm{TSTAB}(G)$ is an integral polytope, i.e. if all its vertices are integral vectors.

Neither the complete graph on four vertices K_4 nor the 5-wheel W_5 are t-perfect. Indeed, for K_4 the vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ lies in TSTAB but not in the SSP of K_4 as the sum of over all entries is larger than $\alpha(K_4) = 1$. The vector that assigns a value of $\frac{2}{5}$ to each vertex on the rim and a value of $\frac{1}{5}$ to the centre shows that 5-wheel is t-imperfect. Again, the vector lies in TSTAB but the sum of all entries is larger than $\alpha(W_5) = 2$.

The following fact is well-known:

$$\textit{bipartite graphs are } t\textit{-perfect.} \tag{6.2}$$

In fact, the SSP of a bipartite graph is fully described by just non-negativity and edge inequalities.

It is easy to check that vertex deletion preserves t-perfection (edge deletion, however, does not). A second operation that maintains t-perfection is described in Gerards and Shepherd [45]:

for a vertex v for which $N(v)$ is a stable set contract all edges in $E(v)$. (∗)

We call this operation a *t-contraction at v*. Let us say that H is a *t-minor* of G if it is obtained from G by repeated vertex-deletion and t-contraction. Then, if G is t-perfect, so is H. We call a graph *minimally t-imperfect* if it is not t-perfect but every proper t-minor of it is t-perfect. Obviously, in order to characterise t-perfect graphs in terms of forbidden t-minors it suffices to find all minimally t-imperfect graphs.

The following simple lemma ensures that we stay within the class of claw-free graphs when taking t-minors. (For a proof, observe that a claw in a t-minor can only arise from an induced subdivided claw in the original graph.)

Lemma 6.2.1. *Every t-minor of a claw-free graph is claw-free.*

For more on t-perfect and claw-free graphs we refer the reader to Schrijver [81, Chapters 68 and 69].

6.3 t-perfect line graphs

We begin by proving our main results for line graphs (Lemma 6.3.3 and Lemma 6.3.4). Cao and Nemhauser [20], among other results, already characterise t-perfect line graphs in terms of forbidden subgraphs. Unfortunately, their characterisation appears erroneous. While we therefore cannot make use of their theorem, we will pursue an approach that is inspired by their work. In particular, we take advantage of Edmonds [35] celebrated theorem on the matching polytope.

For a graph G, define the *matching polytope* $M(G) \subseteq \mathbb{R}^{E(G)}$ to be the convex hull of the characteristic vectors of matchings. Recall that a graph G is *factor-critical* if $G - v$ has a perfect matching for every vertex v.

Theorem 6.3.1 (Edmonds [35], Pulleyblank and Edmonds [77]). *Let G be a graph and $x \in \mathbb{R}^{E(G)}$. Then $x \in M(G)$ if and only if*

$$x \geq 0 \tag{6.3}$$

$$\sum_{e \in E(v)} x_e \leq 1 \qquad \text{for each } v \in V(G) \tag{6.4}$$

$$\sum_{e \in E(F)} x_e \leq \lfloor \frac{|V(F)|}{2} \rfloor \quad \text{for each 2-connected factor-critical } F \subseteq G. \tag{6.5}$$

We say that G has a *proper odd ear decomposition* if there is a sequence $G_0, G_1, \ldots, G_n$ so that G_0 is an odd cycle, $G_n = G$ and G_k is obtained from G_{k-1} for $k = 1, \ldots, n$ by adding an odd path between two (distinct) vertices of G_{k-1} whose interior vertices are disjoint from G_{k-1}.

Theorem 6.3.2 (Lovász [61]). *A graph is 2-connected and factor-critical if and only if it has a proper odd ear-decomposition.*

For the proof of the next two lemmas, we define C_5^+ to be the 5-cycle plus an added chord, and a *totally odd subdivision of C_5^+* to be a subdivision of C_5^+ in which every edge is replaced by a path of odd length.

Lemma 6.3.3. *Let H be a line graph (of a simple graph). Then H is t-perfect if and only if H does not contain K_4 as a t-minor.*

Proof. One direction is clear, so assume that H does not contain K_4 as a t-minor, and let G be such that $L(G) = H$. Since $M(G)$ =SSP(H), all we have to show is that TSTAB(H) is a subset of the polytope described by (6.3), (6.4), and (6.5) from Theorem 6.3.1. That is, we have to check that the inequalities from Theorem 6.3.1 are valid for TSTAB(H).

Condition (6.3) is clear, and for (6.4), pick a (non-isolated) vertex v of G. If v has degree 2 then (6.4) follows from an edge inequality in H, and if $d(v) = 3$ then (6.4) follows from an odd-cycle inequality for a triangle. This shows (6.4), since clearly, $\Delta(G) \leq 3$ as otherwise H contains K_4 as a subgraph.

For (6.5), suppose that G contains a 2-connected factor-critical subgraph F, which, by Theorem 6.3.2, has an odd ear-decomposition. So either F is an odd cycle, or F contains a totally odd subdivision X of C_5^+. But in the latter case, $L(X)$ is an induced subgraph of H, from which we obtain a K_4 as t-minor by performing t-contractions at vertices of degree 2, a contradiction.

Hence F is an odd cycle, and (6.5) follows from some odd-cycle inequality in H. Thus, we have shown that SSP(H) coincides with TSTAB(H), as desired. □

Let G be a claw-free graph with an edge-colouring, and let i, j be two colours. Denote the subgraph consisting of the edges coloured i or j together with their incident vertices by $G_{i,j}$. Note that the components of $G_{i,j}$ are paths or cycles.

Lemma 6.3.4. *Let H be a t-perfect line graph of a graph. Then $\chi(H) \leq 3$.*

Proof. Let G be a graph such that $H = L(G)$. We do induction on $|E(G)|$. Pick an edge $e = uv$. Then clearly, we may apply the induction hypothesis to the t-perfect line graph $L(G - e)$ to deduce that the edges of $G - e$ can be coloured with three colours.

So, let c be a colouring of the edges of $G-e$ with colours $\{1, 2, 3\}$. If there is a colour that is not used by the edges adjacent to e, then we can colour e with that colour and we are done. Thus, assume that all colours $\{1, 2, 3\}$ are used by edges adjacent to e. Since H does not contain K_4 as a subgraph we know that $\Delta(G) \leq 3$. We may therefore assume that u is incident with two edges f_1, f_2 with $c(f_i) = i$ and that v is incident either with one edge g_3, or with two edges g_1, g_3, so that $c(g_i) = i$. We suppose that $E(G)$ cannot be coloured with three colours, which will lead to a contradiction.

Let P' be the component of $G_{2,3}$ containing f_2. If g_3 does not lie in P', then we can swap colours along P', such that e is no longer incident with any edge coloured 2, a contradiction. Thus, $g_3 \in E(P')$, and the subpath $P := uP'v$ has even length. Hence $P + e$ is an odd cycle.

Next, let Q_1 be the component of $G_{1,3}$ containing f_1. In fact, Q_1 is a path. Suppose Q_1 meets P outside u, and let w be the first vertex after u in Q_1 that also lies in P. Then the last edge of Q_1w is coloured 1, and Q_1w therefore of odd length. We see that $(P + e) \cup Q_1w$ is a totally odd subdivision of C_5^+, which in H induces K_4 as a t-minor, which is impossible as H is t-perfect. Therefore, $V(Q_1 \cap P) = \{u\}$.

We swap colours along Q_1 and denote the resulting colouring by c'. Note that $c'(f_1) = 3$ and that P' is still coloured with $\{2, 3\}$. Now, if g_1 does not exist, then we can colour e with 1. On the other hand, if g_1 exists, then in the same way as before for Q_1, we deduce that the component Q_2 of $G_{1,2}$ (with respect to c') containing g_1 meets P only in v. In particular, by recolouring along Q_2 we obtain a colouring of $E(G-e)$ where no edge incident with e is coloured with 1, yielding a 3-edge-colouring of G. □

The proof of the lemma can easily be turned into an algorithm with running time $O(n^3)$, where n is the number of vertices. (We are cheating here a bit. The proof supposes that we know the graph G of which H is the line graph. However, with minor complications, the same induction can be performed directly in H.)

Corollary 6.3.5. *A t-perfect line graph can be coloured with three colours in polynomial time.*

6.4 Squares of cycles

As a preparation for our main lemma we show in this section that most squares of cycles are t-imperfect. In fact, the only t-perfect squares of cycles are C_3^2, which is a triangle, and C_6^2, the line graph of K_4.

Recall that C_n^2 denotes the square of a cycle of order n. We shall always assume that $V(C_n^2) = \{v_1, \ldots, v_n\}$ where the vertices are labelled in cyclic order.

Lemma 6.4.1. *Let $n \geq 4$, and let $n \notin \{6, 7, 10\}$. Then K_4 is a t-minor of C_n^2. Moreover, for $n \geq 8$ the K_4-t-minor is already contained in $C_n^2 - v_5$.*

Proof. Since $C_4^2 = K_4$ we only need to concern ourselves with C_n^2 for $n \geq 5$. Depending on n mod 4 we perform vertex-deletions and then t-contractions as indicated in Figure 6.2 until the only vertices left are $v_1, \ldots, v_4$. In particular, we delete the grey vertices in the initial segment (marked by a dashed box). Outside this segment we delete every other vertex until we reach the first vertex v_1 again. Finally, we contract the odd path between v_4 and v_1 to a single edge.

The length of the initial segment poses a constraint on the minimal size of the graph. For $n \equiv 0 \pmod 4$ the construction is possible for $n \geq 8$, for $n \equiv 1$ we need $n \geq 5$, for $n \equiv 2$ we need $n \geq 14$, and $n \geq 11$ is necessary for $n \equiv 3$. So the only cases we have not dealt with are $n = 6, 7, 10$, which are precisely the exceptions. The second part of the assertion follows directly from the construction of the subdivision of K_4. □

Lemma 6.4.2. *No square of a cycle of length at least 7 is t-perfect.*

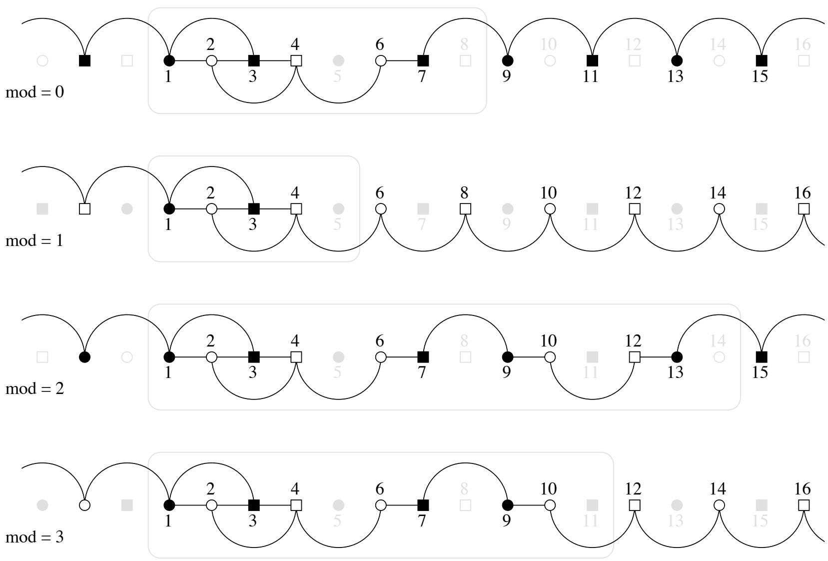


Figure 6.2: K_4-t-minors in C_n^2 depending on n mod 4.

Proof. By Lemma 6.4.1, we only need to check C_7^2 and C_{10}^2. However, neither C_7^2 nor C_{10}^2 is t-perfect. Indeed, the vector $x \in \mathbb{R}^{V(C_7^2)}$ defined by $x_v = 1/3$ for each $v \in V(C_7^2)$ clearly lies in TSTAB(C_7^2) but not in SSP(C_7^2) as $\mathbf{1}^T x = 7/3 > 2 = \alpha(C_7^2)$. We get a similar contradiction by assigning a value of 1/3 to every vertex in C_{10}^2. $\square$

6.5 The main lemma

Before we prove our main lemma, which will play an essential part in the proof of both Theorem 6.1.2 and Theorem 6.1.3, we quickly note two facts.

Lemma 6.5.1. *Let G be a claw-free graph. If $\Delta(G) \geq 5$ then G contains K_4 or W_5 as an induced subgraph.*

Proof. Consider a vertex v of G. If v has at least six neighbours, then, by Ramsey theory, $N(v)$ contains a triangle or three independent vertices. The former leads to a K_4, and the latter to a claw, which is impossible. If $|N(v)| = 5$ then $G[v \cup N(v)]$ is a 5-wheel, or contains a K_4, since the 5-cycle is the only triangle-free graph on five vertices with $\alpha \leq 2$. $\square$

We call a triangle T *odd* if there is a vertex v outside T that is adjacent to an odd number of the vertices in T. We need the following theorem.

Theorem 6.5.2 (Harary [53, Theorem 8.4])**.** *Let G be a claw-free graph. Then G is a line graph if and only if every pair of odd triangles that shares exactly one edge induces a K_4.*

Let us now state the main lemma. It shows that the structure of a claw-free t-perfect graph is rather restricted, provided the graph is 3-connected.

Lemma 6.5.3 (Main lemma)**.** *Let G be a 3-connected claw-free graph with $\Delta(G) \leq 4$. If G does not contain K_4 as t-minor then one of the following statements holds true:*

(i) *G is a line graph;*

(ii) $G \in \{C_6^2 - v_1v_6, C_7^2 - v_7, C_{10}^2 - v_{10}, C_7^2, C_{10}^2\}$.

Proof. We shall repeatedly make use of the following argument. Assume that in the neighbourhood of a vertex u we find a path xyz, and assume that u has a fourth neighbour $v \notin \{x, y, z\}$. As K_4 is not a subgraph of G we know that $xz \notin E(G)$. Then, because G is claw-free, v has to be adjacent to x or to z or to both.

First of all, we shall show that

$$P_6^2 \text{ is a subgraph of } G. \tag{6.6}$$

Recall that P_k denotes a path on k vertices.

Indeed, as we may assume that G is not a line graph, there exist by Theorem 6.5.2 two odd triangles that share exactly one edge, say $u_1u_2u_3$ and $u_2u_3u_4$. As G is 3-connected, $\{u_1, u_4\}$ does not separate G, and thus one of u_2 and u_3 has a neighbour $u_5 \notin \{u_1, u_2, u_3, u_4\}$. By symmetry, we may assume that $u_3u_5 \in E(G)$ and by the argument outlined at the beginning of this proof, we deduce from $u_1u_2u_4 \subseteq G[N(u_3)]$ that u_5 is adjacent to u_1 or to u_4 (or to both). Symmetry, again, allows us to assume that u_5 is adjacent to u_4.

As K_4 is not a subgraph of G, u_1 and u_5 each send exactly two edges to the triangle $u_2u_3u_4$. That triangle, however, is odd. Thus there exists a vertex $u_6 \notin \{u_1, \ldots, u_5\}$ that is adjacent to an odd number of vertices of the triangle. Since u_3 has four neighbours already among the u_i, it follows that u_6 is either adjacent to u_2 or to u_4. By forgetting that $u_1u_2u_3$ is odd, we obtain again a symmetric graph on $u_1, \ldots, u_5$, which means that we may, without loss of generality, assume that $u_6u_4 \in E(G)$, and that $u_6u_2 \notin E(G)$. The path $u_2u_3u_5$ that is contained in the neighbourhood of u_4 together with $u_6u_2 \notin E(G)$ ensures that u_6 is adjacent to u_5. This proves (6.6).

Next, we prove that

if $k \geq 6$ *so that* $P_k^2 \subseteq G$*, then either* $P_{k+1}^2 \subseteq G$ *as well, or* $V(G) = V(P_k)$*.* (6.7)

Assume that G has a vertex outside $P_k = v_1 \dots v_k$. Because G is 3-connected and $\Delta(G) \leq 4$, one of v_2 and v_{k-1}, let us say the latter, has a neighbour $v_{k+1} \notin V(P_k)$; if not then v_1 and v_k would separate $V(P_k)$ from the rest of the graph. From the fact that the path $v_{k-3}v_{k-2}v_k$ is contained in the neighbourhood of v_{k-1} we deduce that v_{k+1} is adjacent to v_{k-3} or to v_k. However, v_{k-3} is already adjacent to four vertices, namely to $v_{k-5}, v_{k-4}, v_{k-2}, v_{k-1}$ (recall that $k \geq 6$). Thus, $\Delta(G) \leq 4$ implies that v_{k+1} is in fact adjacent to v_k. Thus $P_{k+1}^2 \subseteq G$ and we have proved (6.7).

Now, by repeated application of (6.7) we arrive at a path $P_n = v_1 \dots v_n$, for some $n = |V(G)| \geq 6$, whose square is a subgraph of G. Observe that in the square of P_n every vertex has degree 4, except v_2 and v_{n-1}, which have degree 3, and except v_1 and v_n, which have degree 2. Since $\Delta(G) \leq 4$, the square of P_n and G may only differ in the presence or absence of the edges v_1v_{n-1}, v_1v_n, v_2v_{n-1} and v_2v_n in G. As G is 3-connected, each of v_1 and v_n is incident with at least one of these edges.

First, assume that $v_1v_n \notin E(G)$, which immediately entails that $v_1v_{n-1} \in E(G)$ and $v_2v_n \in E(G)$, and hence, as $\Delta(G) \leq 4$, that $v_2v_{n-1} \notin E(G)$. Since $v_1v_3v_4$ is a path in the neighbourhood of v_2, the fourth neighbour v_n of v_2 must be adjacent to v_4. This is only possible if $n = 6$, and we find that then $G = C_6^2 - v_1v_6$, which is as desired.

So, from now on, let us assume that

$$v_1v_n \in E(G). \tag{6.8}$$

Next, suppose that v_2v_{n-1} is an edge of G. Then $n > 6$ as otherwise $v_2, v_3, v_4, v_5 = v_{n-1}$ span a K_4. On the other hand, we find the path $v_{n-3}v_{n-2}v_n$ in the neighbourhood of v_{n-1}, which implies that v_2 is adjacent to v_{n-3} or to v_n. Since v_2 already has already four neighbours, namely v_1, v_3, v_4 and v_{n-1}, and since $n > 6$ it follows that $v_{n-3} = v_4$ and $n = 7$.

Consequently, G is isomorphic to $\tilde{C}_7^2$, which we define as the square of P_7 plus the edges v_1v_7 and v_2v_6. However, Figure 6.3 A shows that $\tilde{C}_7^2$ contains K_4 as a t-minor, a contradiction. (Alternatively, we might have argued that $\tilde{C}_7^2$ is the line graph of the graph obtained from K_4 by subdividing one edge.)

Thus,

$$v_2v_{n-1} \notin E(G). \tag{6.9}$$

So, by (6.8) and (6.9), G is isomorphic to one of the following graphs: $G = C_n^2$, $C_n^2 - v_1v_{n-1}$, and $C_n^2 - v_1v_{n-1} - v_2v_n$. Let us check these cases seperately.

First, assume $G = C_n^2$. Since $C_6^2 = L(K_4)$ and since by Lemma 6.4.1, for $n \geq 7$ every C_n^2 except C_7^2 and C_{10}^2 contains K_4 as a t-minor, we find that $G = C_7^2$ or $G = C_{10}^2$, which are two of the allowed outcomes of Lemma 6.5.3.

Next, assume that $G = C_n^2 - v_1v_{n-1}$. Observe that $(C_n^2 - v_1v_{n-1}) - v_1$ is isomorphic to $C_n^2 - v_5$. Hence, unless $n \in \{6, 7, 10\}$, Lemma 6.4.1 asserts that G contains K_4 as a t-minor. For $n = 7$ and $n = 10$, Figure 6.3 B and C indicate K_4-t-minors of G. So, $n = 6$, that is, $G = C_6^2 - v_1v_5$ which is isomorphic to $C_6^2 - v_1v_6$, and thus one of the allowed outcomes of the lemma.

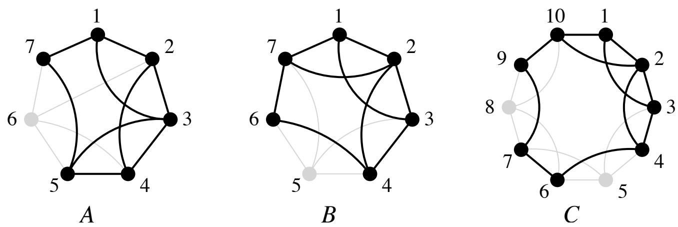


Figure 6.3: K_4 as a t-minor of $\tilde{C}_7^2$, $C_7^2 - v_1v_6$, $C_{10}^2 - v_1v_9$.

Finally, we treat the case when $G = C_n^2 - v_1v_{n-1} - v_2v_n$. Observe that then G is isomorphic to $C_{n+1}^2 - v_{n+1}$, and thus we may employ Lemma 6.4.1 again to deduce that $n+1 \in \{6, 7, 10\}$. Of these cases, $n+1 = 6$ is impossible as $n \geq 6$ by (6.6). Therefore, either $G = C_7^2 - v_7$ or $G = C_{10}^2 - v_{10}$, which is as desired. □

6.6 Colouring claw-free t-perfect graphs

We now prove the first of our two main results.

Proof of Theorem 6.1.2. Let G be claw-free and t-perfect. As every colouring of all the blocks yields a colouring of G, we may assume that G is 2-connected. We proceed by induction on $|V(G)|$.

Observe that we are able to 3-colour G if G is 3-connected. Indeed, by Lemma 6.5.1, we can then apply Lemma 6.5.3, which implies that either G is a line graph, or $G \in \{C_7^2, C_{10}^2\}$ or G is a subgraph of C_6^2 (recall that $C_7^2 - v_5$ is isomorphic to $C_6^2 - v_1v_5 - v_2v_6$), or $G = C_{10}^2 - v_{10}$. But $G \in \{C_7^2, C_{10}^2\}$ is impossible, by Lemma 6.4.2. If G is a line graph then it it follows from

Lemma 6.3.4 that G is 3-colourable. Finally, C_6^2 as well as $C_{10}^2 - v_{10}$ are easily seen to be 3-colourable.

If G has at most three vertices, we clearly have $\chi(G) \leq 3$, too. So, let us assume now that G is not 3-connected and has at least four vertices. Then there are distinct vertices u, v, and induced proper subgraphs L and R of G so that $V(L) \cap V(R) = \{u, v\}$ and $L \cup R = G$. As $|V(L)| < |V(G)|$ there is, by induction, a 3-colouring c_L of L. Permuting colours, if necessary, we may assume that $c_L(u) = 1$ and $c_L(v) \in \{1, 2\}$.

Define $\tilde{R}$ to be the graph we obtain from R by identifying u and v. Observe that at least one of $\tilde{R}$ and $R + uv$ is a proper t-minor of G, and thus has a 3-colouring by induction. This colouring can be extended to a 3-colouring c_R of R (with $c_R(u) = c_R(v)$, or with $c_R(u) \neq c_R(v)$, depending on the t-minor we found). Now, we can combine c_L and c_R, if necessary swapping colours on one side, to a 3-colouring of G. The only two situations where this is not possible is when $c_L(v) = 1$ and $\tilde{R}$ is not a t-minor of G, or when $c_L(v) = 2$ and $R + uv$ is not a t-minor of G.

In the former case, this means that there is no induced even u–v path in L. In particular, the Kempe-chain K at u in colours $1, 2$ does not contain v. We can thus recolour along K to obtain a colouring c'_L of L with $c'_L(u) \neq c'_L(v)$ that combines with c_R to a 3-colouring of G. In the latter case, i.e. when $c_L(v) = 2$ and $R + uv$ is not a t-minor of G, we proceed similarly, recolouring along a Kempe-chain at u to obtain a colouring c'_L with $c'_L(u) = c'_L(v)$ that combines with c_R to a 3-colouring of G □

Corollary 6.6.1. *Every claw-free t-perfect graph on n vertices can be coloured with three colours in polynomial time in n.*

Proof. Let us sketch how the proof of Theorem 6.1.2 can be turned into an algorithm. First observe that, by Lemma 6.5.1, Lemma 6.5.3 and Corollary 6.3.5 we can 3-colour any 3-connected claw-free t-perfect graph in polynomial time.

Now, starting with $L^0 := G$ we construct a sequence of graphs L^i and R^i. Indeed, if L^{i-1} fails to be 3-connected and has at least four vertices, we split L^{i-1} into two proper induced subgraphs L^i and R^i as in the proof above (with L^{i-1} in the role of G). Among all choices for L_i and R_i, we choose R^i so that $|V(R^i)|$ is minimal. (This can be accomplished by considering all of the $O(n^2)$ vertex sets of cardinality at most 2.)

We recursively compute a 3-colouring c_{L^i} of L^i and in order to check which of the graphs $R^i + uv$ and $\tilde{R}^i$ is a t-minor of G, we pick an induced u-v path P^i in L^i (for instance a shortest path). As we chose R_i minimal, the t-minor $R^i + uv$, respectively $\tilde{R}^i$, is 3-connected. Hence, we can compute its 3-colouring directly with the help of Lemma 6.5.3 and Corollary 6.3.5. If necessary, we then recolour L_i along a Kempe-chain at u. Since $L^{i-1} - L^i \neq \emptyset$, the procedure stops after at most n steps. □

Let us now turn for a moment to h-perfect graphs, to which our result on colourings easily carries over. Sbihi and Uhry [79] introduced h-perfect graphs as a common generalisation of perfect and t-perfect graphs. For the definition of h-perfect graphs we use the same inequalities as for t-perfect graphs, only that the edge inequalities are replaced with clique inequalities. So, a graph is called *h-perfect* if the SSP is determined by

$$\begin{aligned} & x \geq 0 \\ & x(K) \leq 1 \text{ for every clique } K \\ & x(C) \leq \lfloor |V(C)|/2 \rfloor \text{ for every induced odd cycle } C \end{aligned}$$

The proof of the following corollary is due to Sebő [72]. As it has not been published but contains a nice and useful technique we present it here.

Corollary 6.6.2. *Let G be a claw-free h-perfect graph. Then*

(i) $\chi(G) = \lceil \chi^*(G) \rceil$*; and*

(ii) $\chi(G) = \omega(G)$ *if* $\omega(G) \geq 3$.

Here, χ^* denotes the fractional chromatic number. More formally, if $\mathcal{S}$ denotes the set of all stable sets:

$$\begin{array}{rcl} \chi^*(G) & = & \min \mathbf{1}^T y, \; y \in \mathbb{R}^{\mathcal{S}} \\ \text{subject to} & & y \geq 0 \quad \text{and} \sum_{S \in \mathcal{S}, \, v \in S} y_S \geq 1 \text{ for all } v \in V \end{array} \tag{6.10}$$

Proof of Corollary 6.6.2. Define the polytope

$$P = \{x \in \mathbb{R}^V : x(S) \leq 1 \text{ for each stable set } S, \, x \geq 0\}.$$

Observe that $\max_{x \in P} \mathbf{1}^T x$ is the dual program of (6.10), so that we get $\chi^*(G) = \max_{x \in P} \mathbf{1}^T x$. Moreover, it is not hard to check that the anti-blocking polytope of P coincides with $\mathrm{SSP}(G)$. As G is h-perfect, Theorem 2.1 in Fulkerson [41] (see also [40]) yields therefore that every vertex $\neq \mathbf{0}$ of P is either the characteristic vector χ_K of a clique K of G or the vertex is of the form $\frac{2}{|C|-1} \chi_C$ for an odd cycle C.

First, assume that $\omega(G) \geq 3$. We show that

$$\text{\textit{there is a stable set S which intersects every clique of size $\omega(G)$.}} \tag{6.11}$$

Since $\omega(G) \geq 3 > \mathbf{1}^T(\frac{2}{|C|-1}\chi_C)$ for every odd cycle C of length ≥ 5, we see that $\max_{x \in P} \mathbf{1}^T x = \omega(G)$ is attained in every clique of size $\omega(G)$. Consider an optimal solution y of (6.10) and a clique K of size $\omega(G)$. Then

$$\omega(G) = \mathbf{1}^T \chi_K \leq \sum_S y_S \, \chi_S^T \chi_K = \sum_S y_S |S \cap K| \leq \sum_S y_S = \omega(G).$$

Thus, each stable set S with $y_S > 0$ must meet each such clique K, which proves (6.11).

Next, we find with (6.11) stable sets $S_1, \ldots, S_k$ where $k = \omega(G) - 3$ such that $G' := G - S_1 - \ldots - S_k$ has no clique of size 4. Thus, G' is t-perfect and therefore, by Theorem 6.1.2, colourable with three stable sets, $S_{k+1}, S_{k+2}, S_{k+3}$ say. Now, we can colour G with $S_1, \ldots, S_{\omega(G)}$. This proves assertion (ii), and (i), too, for $\omega(G) \geq 3$ as $\omega(G)$ is a lower bound for $\chi^*(G)$.

Finally, assume $\omega(G) < 3$. If G is not bipartite, in which case we are done, then $\chi^*(G) = \max_{x \in P} \mathbf{1}^T x$ is attained in $\frac{2}{|C|-1}\chi_C$ for some odd cycle C. Thus, $\chi^*(G) > 2$. On the other hand, G is t-perfect, and we can consequently, by Theorem 6.1.2, colour it with three colours. □

We remark that Sebő developed the arguments above to show that Conjecture 6.1.1 on the 4-colourability of t-perfect graphs is implied by the following claim.

Conjecture 6.6.3 (Sebő [72]). *Every triangle-free t-perfect graph is 3-colourable.*

6.7 Characterising claw-free t-perfect graphs

Lemma 6.5.3 together with Lemma 6.3.3 provides already a full characterisation of claw-free t-perfect graphs if, in addition, the graph is 3-connected. The task of the next few lemmas is to show that minimally t-imperfect claw-free graphs are, in fact, 3-connected.

The first of these lemmas is quite similar to Lemma 12 in Gerards and Shepherd [45]. As that lemma, however, is assembled from results of various authors, its proof is not easily verified. We therefore give a direct proof that draws on only two fairly simple facts.

Lemma 6.7.1. *Let G be a minimally t-imperfect graph, and assume $u, v \in V(G)$ to separate G. Then $G - \{u, v\}$ has exactly two components, one of which is a (possibly trivial) path. Moreover, $uv \notin E(G)$.*

Proof. Let $G = G_1 \cup G_2$ so that $\{u, v\}$ separates $G_1 - \{u, v\}$ from $G_2 - \{u, v\}$. Suppose that neither of G_1 and G_2 is a path. Let z be a non-integral vertex of TSTAB(G), denote by $\mathcal{I}$ the set of those non-negativity, edge and odd-cycle inequalities that are satisfied with equality by z. We define z^1 resp. z^2 to be the restriction of z to G_1 resp. G_2.

As in the proof of Theorem 1 in Gerards and Shepherd [45] we can deduce that

$$0 < z_w < 1 \textit{ for all } w \in V(G) \tag{6.12}$$

and

$$\textit{every odd cycle whose inequality is in } \mathcal{I} \textit{ fails to separate } G. \tag{6.13}$$

The last fact implies, in particular, that each odd cycle in $\mathcal{I}$ lies either completely in G_1 or in G_2 (recall that neither of G_1 and G_2 is a path). Thus, we can partition $\mathcal{I}$ in $(\mathcal{I}_1, \mathcal{I}_2)$ so that $\mathcal{I}_k$ pertains only to G_k. Now, if there is a $j \in \{1, 2\}$ so that $\dim \mathcal{I}_j = |V(G_j)|$ then z^j is a vertex of TSTAB(G_j) =SSP(G_j). Since z^j is non-integral we obtain a contradiction.

Therefore, we have $\dim \mathcal{I}_k = |V(G_k)| - 1$ for $k = 1, 2$, which means that $\mathcal{I}_k$ describes an edge of TSTAB(G_k). Denote the endvertices of this edge by s^k and t^k, i.e. $z^k = \lambda_k s^k + (1 - \lambda_k)t^k$ for some $0 \leq \lambda_k \leq 1$. As TSTAB(G_k) =SSP(G_k) by assumption, it follows that s^k is the characteristic vector of a stable set S_k of G_k; the same holds for t^k and a stable set T_k.

By (6.12), $z^1_u = z^2_u > 0$ and thus for each $k = 1, 2$ one of S_k and T_k needs to contain u. By renaming if necessary we may assume that $u \in S_1$ and $u \in S_2$. Then $u \notin T_k$ for $k = 1, 2$ as otherwise we obtain $z^k_u = \lambda_k + (1 - \lambda_k) = 1$ in contradiction to (6.12). This implies that

$$\lambda_1 = z^1_u = z_u = z^2_u = \lambda_2. \tag{6.14}$$

If $S_1 \cap \{v\} = S_2 \cap \{v\}$ then also $T_1 \cap \{v\} = T_2 \cap \{v\}$ as (6.12) implies as above that $v \in S_k$ if and only if $v \notin T_k$. In this case, $S := S_1 \cup S_2$ and $T := T_1 \cup T_2$ are stable sets of G and we obtain $z = \lambda_1 \chi_S + (1 - \lambda_1)\chi_T$, contradicting the choice of z as a non-integral vertex of TSTAB(G).

So, let us assume that S_1 and S_2 differ on $\{v\}$. Without loss of generality, let $v \in S_1$ but $v \notin S_2$. Then

$$\begin{aligned} S_1 \cap \{u, v\} &= \{u, v\}, & T_1 \cap \{u, v\} &= \emptyset, \\ S_2 \cap \{u, v\} &= \{u\} \quad \textit{and} & T_2 \cap \{u, v\} &= \{v\}. \end{aligned}$$

So, $\lambda_1 = z^1_v = z^2_v = 1-\lambda_2$, and hence, by (6.14), $\lambda_1 = \lambda_2 = 1/2$. In particular, it follows with (6.12) again that $z_w = 1/2$ for all $w \in V(G)$.

Now, since bipartite graphs are t-perfect by (6.2), G contains an odd cycle of length $2k+1$, say. However, adding up z along the cycle yields $k+1/2$, contradicting the odd-cycle inequalities. □

Next, let us prove that a minimally t-imperfect claw-free graph has minimum degree at least three. We start with a lemma that is a variant of Theorem 2.5 in Barahona and Mahjoub [2], and can be proved in a very similar way

Lemma 6.7.2 (Barahona and Mahjoub [2])**.** *Let G be a graph, and let uvw be a path in G so that $\deg(v) = 2$ and $uw \notin E(G)$. Furthermore, let $a^T x \le \alpha$ be a facet-defining inequality of SSP(G) so that $a_u = a_v = a_w$. Denote by G' the graph obtained from G by contracting uv and vw, and let $\tilde{v}$ be the resulting vertex, i.e. $V(G') \setminus V(G) = \{\tilde{v}\}$. If $a' \in \mathbb{R}^{V(G')}$ is defined by $a'_p = a_p$ for $p \in V(G' - \tilde{v})$ and $a_{\tilde{v}} = a_v$ then $a'^T x \le \alpha - a_v$ is a facet-defining inequality of SSP(G').*

The following lemma serves to guarantee that $a_u = a_v = a_w$ as in Lemma 6.7.2.

Lemma 6.7.3. *Let G be a graph and assume that for $a \in \mathbb{R}^{V(G)}$, $a > 0$ the inequality $a^T x \le \alpha$ is facet-defining in SSP(G), and that it is not a multiple of an edge inequality or of an odd-cycle inequality.*

(i) *If G contains a path uvw so that $\deg(v) = 2$ then $a_v \le a_w$.*

(ii) *If G contains a triangle wpq and a neighbour $v \notin \{p, q\}$ of w so that $\deg(w) = 3$ then $a_v \ge a_w$.*

Assertion (i) appears in Mahjoub [71].

Proof. For both cases, observe that as the SSP is full-dimensional there exists a set $\mathcal{S}$ of $|V(G)|$ affinely independent stable sets that satisfy $a^T x \le \alpha$ with equality. Since $a > 0$ it follows that $\alpha \ne 0$, which, in turn, implies that the characteristic vectors of the stable sets in $\mathcal{S}$ are even linearly independent. In particular, any inequality satisfied with equality by all $S \in \mathcal{S}$ is a multiple of $a^T x \le \alpha$.

(i) Since $a^T x \le \alpha$ is not a multiple of the edge inequality $x_u + x_v \le 1$ there must exist an $S_0 \in \mathcal{S}$ so that $u \notin S_0$ and $v \notin S_0$. As $a > 0$ this implies that $w \in S_0$. Clearly, $S'_0 := S_0 \setminus \{w\} \cup \{v\}$ is a stable set and thus $a^T \chi_{S'_0} \le \alpha = a^T \chi_{S_0}$. Hence $a_v \le a_w$.

(ii) Since $a^T x \leq \alpha$ is not a multiple of the triangle inequality $x_w + x_p + x_q \leq 1$ there must exist an $S_1 \in \mathcal{S}$ so that $\{w, p, q\} \cap S_1 = \emptyset$. Then, as $a > 0$ and $N(w) = \{v, p, q\}$, we have that $v \in S_1$ and that $S_1' := S_1 \setminus \{v\} \cup \{w\}$ is stable. Again, we obtain $a^T \chi_{S_1'} \leq \alpha = a^T \chi_{S_1}$ and therefore $a_w \leq a_v$. □

Lemma 6.7.4. *Let G be a minimally t-imperfect claw-free graph. Then G has minimum degree ≥ 3.*

Proof. It is easy to see that no vertex can have degree 1. Indeed, such a vertex would lead to a violation as in (6.12). So suppose there is a path $P = w_1 \ldots w_k$ with $k \geq 3$ so that all internal vertices have degree 2 in G but w_1 and w_k have degree > 2. Since G is claw-free and does not properly contain a K_4 we deduce that $\deg(w_1) = \deg(w_k) = 3$, and in fact there are neighbours p_1, q_1 of w_1 and p_k, q_k of w_k so that $w_1 p_1 q_1$ and $w_k p_k q_k$ are triangles in G.

As G is t-imperfect there exists a facet-defining inequality $a^T x \leq \alpha$ of $\mathrm{SSP}(G)$ with $a \geq 0$ that is not a multiple of a non-negativity, edge or odd-cycle inequality. Since G is minimally t-imperfect under vertex deletion it follows furthermore that $a > 0$.

Now, applying (i) of Lemma 6.7.3 we get that $a_{w_2} = \ldots = a_{w_{k-1}} \leq \min\{w_1, w_k\}$. Then, (ii) yields that $a_{w_1} = a_{w_2} = \ldots = a_{w_k}$.

Denote by G' the graph obtained from G by performing a t-contraction at w_2, and let $\tilde{w}$ be the resulting new vertex. Define $a'_u = a_u$ for $u \in V(G' - \tilde{w})$ and $a'_{\tilde{w}} = a_{w_2}$. Then, by Lemma 6.7.2, $a'^T x \leq \alpha - a_{w_2}$ is facet-defining for $\mathrm{SSP}(G')$. However, as $a' > 0$ and as G' is t-perfect it follows that G' consists of a single vertex, a single edge or of a single odd cycle. Then G is such a graph, too, and thus t-perfect, a contradiction. □

In Section 6.8 we will show in Lemma 6.8.1 that $C_7^2 - v_7$ as well as $C_{10}^2 - v_{10}$ are (strongly t-perfect and thus) t-perfect. Considering Figure 6.4 we see that $C_6^2 - v_1 v_6$ is a t-minor of $C_{10}^2 - v_{10}$. Hence, (assuming Lemma 6.8.1) the following lemma holds:

Lemma 6.7.5. *$C_7^2 - v_7$, $C_{10}^2 - v_{10}$ and $C_6^2 - v_1 v_6$ are t-perfect.*

We now prove our second main result.

Proof of Theorem 6.1.3. As neither of K_4, W_5, C_7^2 and C_{10}^2 is t-perfect (note Lemma 6.4.2), necessity is obvious. To prove sufficiency, consider a claw-free and minimally t-imperfect graph G.

Lemmas 6.7.1 and 6.7.4 ensure that G is 3-connected. Moreover, as we are done if G contains K_4 or W_5 as a t-minor, we obtain with Lemma 6.5.1

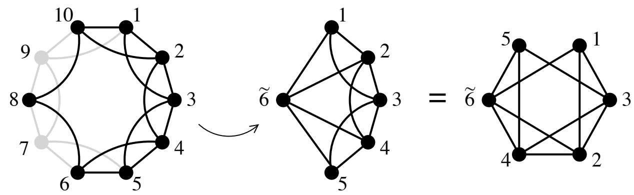


Figure 6.4: $C_6^2 - v_1v_6$ is a t-minor of $C_{10}^2 - v_{10}$.

that $\Delta(G) \leq 4$. Consequently, all preconditions of Lemma 6.5.3 are satisfied, and we may assume that G is of type (i) or (ii) as listed in Lemma 6.5.3.

Now, if G is a line graph then Lemma 6.3.3 forces G to contain K_4 as a t-minor, as desired. It remains to check the types listed in (ii). As by Lemma 6.7.5, the graphs $C_6^2 - v_1v_6$, $C_7^2 - v_7$ and $C_{10}^2 - v_{10}$ are t-perfect, it follows that $G \in \{C_7^2, C_{10}^2\}$, as desired. □

6.8 C_7^2 and C_{10}^2 are minimally t-imperfect

To conclude the proof of Theorem 6.1.3 it still remains to prove one last lemma (since we needed for the proof of Lemma 6.7.5 that $C_7^2 - v_7$ and $C_{10}^2 - v_{10}$ are t-perfect). We take this opportunity to show something slightly stronger, namely that C_7^2 and C_{10}^2 are minimally t-imperfect, and, moreover, minimally strongly t-imperfect.

In order to define strong t-perfection, consider a graph G and $w \in \mathbb{Z}^{V(G)}$, and denote by $\alpha_w(G)$ the maximum $w(S)$ over all stable sets S in G. Call a family $\mathcal{K}$ of edges and odd cycles a *w-cover of G* if every vertex v lies in at least $w(v)$ members of $\mathcal{K}$. If $\mathcal{K}$ consists of the subfamily of edges $\mathcal{E}$ and the subfamily of cycles $\mathcal{C}$ then it has *cost*

$$|\mathcal{E}| + \sum_{C \in \mathcal{C}} \frac{|C| - 1}{2}.$$

We say that G is *strongly t-perfect* if for every $w \in \mathbb{Z}^{V(G)}$ there is a w-cover of cost at most $\alpha_w(G)$. (Clearly, any w-cover has cost at least $\alpha_w(G)$.) This is equivalent to defining strongly t-perfect graphs as those for which (6.1) is totally dual integral.

Observe that it suffices to check the existence of the desired cover for all non-negative vectors w. Moreover, one can show that vertex deletion as well as t-contraction maintain strong t-perfection.

Strongly t-perfect graphs have been studied by Gerards [44] and Schrijver [80]; see also Schrijver [81, Chapter 68]. They show that bad resp. odd K_4-free graphs are strongly t-perfect. It is not known whether a t-perfect graph is necessarily strongly t-perfect, but the converse is true. So, the t-perfectness of $C_7^2 - v_7$ and $C_{10}^2 - v_{10}$ follows from the following lemma.

Lemma 6.8.1. *For $j \in \{7, 10\}$, the graph $C_j^2 - v_j$ is strongly t-perfect.*

The proof of this lemma is a bit involved and given below. Let us first get to the main result of this section:

Proposition 6.8.2. [13] *The graphs C_7^2 and C_{10}^2 are minimally t-imperfect as well as minimally strongly-t-imperfect.*

Proof. Lemma 6.4.2 yields that C_7^2 and C_{10}^2 are t-imperfect, and by Lemma 6.8.1, we know that the deletion of one vertex makes these graphs strongly t-perfect. Hence, as strong t-perfection implies t-perfection, our proposition follows. $\square$

Proof of Lemma 6.8.1. In both cases, $j = 7$ and $j = 10$, we proceed by induction on the total weight $w(V)$, where $V := V(C_j^2 - v_j)$ and w is the given non-negative vector in $\mathbb{Z}^V$ for which we have to find a w-cover. So, let $G \in \{C_7^2 - v_7, C_{10}^2 - v_{10}\}$. As the case when $w(V) = 0$ is trivial we will assume that w is given with $w(V) > 0$, and that the desired cover exists for all w' with $w'(V) < w(V)$.

Let $\{v_1, \ldots, v_{j-1}\}$ be the vertices of $V(G)$ in circular order, so that v_1, v_2, v_{j-2} and v_{j-1} have degree 3. Denote by $\mathcal{S}$ the set of all stable sets of weight $\alpha_w := \alpha_w(G)$, and write w_i for $w(v_i)$.

First of all, if there is triangle T so that every $S \in \mathcal{S}$ meets T, then we define $w'(v) := w(v) - 1$ for $v \in T^+ := V(T) \cap \bigcup_{S \in \mathcal{S}} S$ and $w'_i = w_i$ otherwise. As each $v \in T^+$ has positive weight $w(v)$—otherwise $S \setminus \{v\}$ would be in $\mathcal{S}$ and miss T—we conclude that w' is non-negative. Since $T^+ \neq \emptyset$ by assumption, the total weight $w'(V)$ is smaller than $w(V)$. Hence, by induction there is a w'-cover $\mathcal{K}'$ of cost $\alpha_{w'}$. Since $\alpha_{w'} = \alpha_w - 1$ the family $\mathcal{K}' \cup T$ is a w-cover of cost α_w, as desired.

We can argue similarly if every $S \in \mathcal{S}$ meets the edge $v_1 v_{j-1}$. So, let us assume from now on that for each triangle T in G there is a $S_T \in \mathcal{S}$ avoiding T, and that there exists a $S_{v_1 v_j}$ that is disjoint from $\{v_1, v_{j-1}\}$.

In the case when $G = C_7^2 - v_7$, the stable set $S_{v_4 v_5 v_6}$ of weight α_w needs to consist of a single vertex v_k with $k \in \{1, 2, 3\}$ as $v_1 v_2 v_3$ forms a triangle in G. Hence, $w_k = \alpha_w$. In the same way, we get that for some $l \in \{4, 5, 6\}$ the vertex v_l has weight α_w, too. Moreover, v_k and v_l have to be adjacent. If $(k, l) = (1, 6)$, then all other vertices have weight 0, and α_w times the edge

v_1v_6 is a w-cover of G. On the other hand, if $k \in \{2,3\}$ and $l \in \{4,5\}$, then $w_1 = w_6 = 0$. Furthermore, as $\{v_2\} = S_{v_3v_4v_5}$ and $\{v_5\} = S_{v_2v_3v_4}$ have weight α_w, the stable set $\{v_2, v_5\}$ has weight $2\alpha_w$, a contradiction.

Now, let us consider the case of $G = C_{10}^2 - v_{10}$. Let K be a triangle in G, or let K be the subgraph consisting of the edge v_1v_9. Suppose that $k \in V(K)$.

If $w(k) > 0$ and k has only one neighbour s outside K then, as $w(S_K) = \alpha_w$, S_K contains s, since otherwise we could increase the weight of S_K by including k. Since $S_K \setminus \{s\} \cup \{k\}$ is stable, it follows that $w(k) \leq w(s)$. Observe that this inequality trivially holds too, if $w(k) = 0$. We use this rule to obtain a number of inequalities that are listed in the table below.

K		
$v_1v_2v_3$	(a) $w_1 \leq w_9$	(b) $w_2 \leq w_4$
$v_7v_8v_9$	(c) $w_9 \leq w_1$	(d) $w_8 \leq w_6$
$v_2v_3v_4$	(e) $w_2 \leq w_1$	
$v_6v_7v_8$	(f) $w_8 \leq w_9$	

Now assume that the vertex $k \in V(K)$ has two adjacent neighbours s and t outside K (and then no other neighbours outside K). Because S_K can only contain one of s and t, we deduce as above that $w(k) \leq \max\{w(s), w(t)\}$. Using this argumentation, we obtain

K	
$v_3v_4v_5$	(g) $w_3 \leq \max\{w_1, w_2\}$
$v_5v_6v_7$	(h) $w_7 \leq \max\{w_8, w_9\}$
v_1v_9	(i) $w_1 \leq \max\{w_2, w_3\}$
v_1v_9	(j) $w_9 \leq \max\{w_7, w_8\}$

From (a) and (c), we get that $w_1 = w_9$, and (g) together with (e) yields $w_3 \leq w_1$. Symmetrically, we obtain $w_7 \leq w_9$, and with (e), (f), (i) and (j) this results in

$$\max\{w_2, w_3\} = w_1 = w_9 = \max\{w_7, w_8\}. \tag{6.15}$$

Now, take two stable sets $S, S' \in \mathcal{S}$ of cardinality 2 that avoid $v_4v_5v_6$ (such sets exist, as we may, for example, take $S_{v_4v_5v_6}$, after adding a vertex, if necessary). Observe that by (6.15), and since a stable set may meet each of the triangles $v_1v_2v_3$ and $v_7v_8v_9$ at most once, we may choose S and S' so that $S = \{v_1, s\}$ for some $s \in \{v_7, v_8\}$ and so that $S' = \{v_9, s'\}$ for some $s' \in \{v_2, v_3\}$.

Comparing the stable set $\{v_1, s, v_4\}$ to S we get $w_1+w(s)+w_4 \leq w(S) = w_1+w(s)$ and thus $w_4 = 0$. Hence, $w_2 = 0$ too, by (b), and $w_3 = w_1$, by (6.15). Symmetrically, comparing $\{v_9, s', v_6\}$ to S', we get that $w_6 = w_8 = 0$.

To sum up, we have discovered that $w_1 = w_3 = w_7 = w_9$ and that $w_2 = w_4 = w_6 = w_8 = 0$. Furthermore, $\alpha_w = w(S) = 2w_1$.

Finally, as $\{v_1, v_5\}$ is stable, it follows that $w_5 \leq w_1$. We conclude the proof by choosing a w-cover consisting of w_1 times the 5-cycle $v_1v_3v_5v_7v_9$ at a cost of $2w_1$. □

Chapter 7

Strongly t-perfect graphs

7.1 Strong t-perfection

In this chapter, which is based on [15], we shall show that for claw-free graphs t-perfection and strong t-perfection coincide. In the previous section we already discussed t-perfect graphs, that where defined as those graphs for which polytope TSTAB(G) described by the inequalities (6.1) is integral. Also, *strongly t-perfect* graphs have been mentioned: these were those for which (6.1) is totally dual integral (TDI). Clearly, strong t-perfection implies t-perfection but whether the converse is true as well is not clear at all.

Question 7.1.1. *Is every t-perfect graph also strongly t-perfect?*

The question is briefly discussed in Schrijver [81, Vol. B, Ch. 68], where also more details about strong and ordinary t-perfection can be found. Our main theorem in this chapter answers Question 7.1.1 affirmatively, provided the graphs are, in addition, claw-free.

Theorem 7.1.2.[15] *A claw-free graph is t-perfect if and only if it is strongly t-perfect.*

Strongly t-perfect graphs have been investigated by Gerards [44] and Schrijver [80]. The class of strongly t-perfect graphs encompasses bipartite, as well as almost bipartite graphs, that is, graphs whose odd cycles all share a common vertex. The most wide-reaching criterion that certifies strong t-perfection is due to Gerards. Call a subdivision of K_4 *odd* if every triangle of K_4 becomes an odd cycle in the subdivision.

Theorem 7.1.3 (Gerards [44])**.** *Every graph that does not contain an odd-K_4-subdivision as a subgraph is strongly t-perfect.*

The theorem has been strengthened by Schrijver in [80]. See also Gijswijt and Schrijver [46] for a more general result.

In our proof we shall use the more convenient alternative definition of strong t-perfection already used in the last section of Chapter 6. For technical reasons the definiton used in this chapter will be slightly different, however, it is easy to see that they are equivalent. Let G be a graph and let $\mathcal{K} = \mathcal{V} \cup \mathcal{E} \cup \mathcal{C}$ be a family of vertices, edges and odd cycles of G. We say $\mathcal{K}$ has *cost*

$$|\mathcal{V}| + |\mathcal{E}| + \sum_{C \in \mathcal{C}} \frac{|V(C)| - 1}{2}.$$

We say that $\mathcal{K}$ *covers a vertex* v k *times* if v lies in at least k members of $\mathcal{K}$. For a weight function $w \in \mathbb{Z}^{V(G)}$, we call $\mathcal{K}$ a *w-cover of* G if every vertex v is covered at least w_v times by $\mathcal{K}$.

We observe that every w-cover can be turned into an exact cover with the same or lower cost, i.e. into a cover that covers every vertex v exactly w_v times (provided $w \geq 0$). Indeed, this can easily be achieved by replacing odd cycles incident with an overly covered vertex v by a maximal matching of the cycle that misses v, by replacing incident edges by the other endvertex and/or by omitting v itself from the cover, if present.

For a subset S of $V(G)$, write $w(S) := \sum_{s \in S} w_s$, and define the *weighted stability number* of G

$$\alpha_w(G) := \max\{w(S) : S \subseteq V(G) \text{ is stable}\}.$$

By linear programming duality, G is strongly t-perfect if and only if there is a w-cover of cost $\alpha_w(G)$ for all $w \in \mathbb{Z}^{V(G)}$. Moreover, it is easy to see that we need only consider non-negative w.

7.2 Strong t-perfection and t-minors

We quickly repeat the definition of t-minors given in the previous chapter. Let v be a vertex of a graph G so that its neighbourhood $N(v)$ forms a stable set. Then $\tilde{G}$ is obtained from G by a *t-contraction at v* if $\tilde{G} = G/E(v)$, i.e. if $\tilde{G}$ is the result of contracting all the edges incident with v. We say that G' is a *t-minor of* G if G' can be obtained from G by a sequence of vertex deletions and t-contractions. It is not hard to check that t-perfection is stable under taking t-minors. The same holds for strong t-perfection:[1]

[1]While this result was known before [43], it does not appear to have been published anywhere.

Proposition 7.2.1.[15] *Every t-minor of a strongly t-perfect graph is strongly t-perfect.*

Proof. Let G be strongly t-perfect. It is straightforward to see that induced subgraphs of G are strongly t-perfect, too. It remains to show, therefore, that for every vertex v with stable neighbourhood $N(v)$ the graph $\tilde{G} := G/E(v)$ is strongly t-perfect as well.

Denote the new vertex of $\tilde{G}$ by $\tilde{v}$. Given a non-negative weight $\tilde{w} \in \mathbb{Z}^{V(\tilde{G})}$, we have to find a $\tilde{w}$-cover $\tilde{K}$ of $\tilde{G}$ that has cost $\alpha_{\tilde{w}}(G)$.

Set $\beta := \tilde{w}(V(\tilde{G})) + 1$, and define $w \in \mathbb{Z}^{V(G)}$ as $w_u := \tilde{w}_u$ for $u \in V(\tilde{G} - \tilde{v})$, $w_p := \beta$ for $p \in N(v)$ and $w_v := d(v) \cdot \beta - \tilde{w}_{\tilde{v}}$. Note that by the choice of β, every stable set of maximal weight with respect to w either contains v, or all of $N(v)$. In either case,

$$\alpha_w(G) \leq \alpha_{\tilde{w}}(\tilde{G}) + d(v) \cdot \beta - \tilde{w}_{\tilde{v}} = \alpha_{\tilde{w}}(\tilde{G}) + w_v. \tag{7.1}$$

As G is strongly t-perfect, there exists a w-cover $\mathcal{K}$ of cost $\alpha_w(G)$, which we may assume to cover v exactly w_v times. Moreover, we may require all the cycles in $\mathcal{K}$ to be induced.

Let $\mathcal{K}_v \subseteq \mathcal{K}$ consist of all $K \in \mathcal{K}$ that are incident with v. For each cycle $C \in \mathcal{K}_v$ contract the two edges incident with v. Note that this gives a cycle in $\tilde{G}$ as C was induced by assumption. Denote the family of the thus obtained cycles by $\tilde{\mathcal{K}}_v$. Since every cycle in $\tilde{\mathcal{K}}_v$ is two edges shorter than the corresponding cycle in $\mathcal{K}_v$, it follows that $\tilde{\mathcal{K}}_v$ costs w_v less than $\mathcal{K}_v$.

Next, we turn $\mathcal{K} \setminus \mathcal{K}_v$ into a family $\tilde{\mathcal{K}}'$ of vertices, edges and odd cycles in $\tilde{G}$. For this, interpret all the elements of $\mathcal{K} \setminus \mathcal{K}_v$ that do not meet $N(v)$ as a subgraph of $\tilde{G}$ and put them (with repetitions) in $\tilde{\mathcal{K}}'$. For every occurrence of a vertex in $N(v)$ add $\{\tilde{v}\}$ to $\tilde{\mathcal{K}}'$, and for every occurrence of an edge rs with $s \in N(v)$ add the edge $r\tilde{v}$ to $\tilde{\mathcal{K}}'$. For every cycle C in $\mathcal{K} \setminus \mathcal{K}_v$ that is incident with a vertex in $N(v)$, the edge set $E(C)$ can be partitioned in the edge sets of cycles in $\tilde{G}$. Add all the odd cycles to $\tilde{\mathcal{K}}'$ and every other edge from every even cycle. This yields a family $\tilde{\mathcal{K}}'$ of the same cost as $\mathcal{K} \setminus \mathcal{K}_v$ that covers every vertex in $V(\tilde{G} - \tilde{v})$ as often as $\mathcal{K} \setminus \mathcal{K}_v$, and which covers $\tilde{v}$ as often as $N(v)$ is covered in total by $\mathcal{K} \setminus \mathcal{K}_v$.

Thus the cost of $\tilde{\mathcal{K}} := \tilde{\mathcal{K}}_v \cup \tilde{\mathcal{K}}'$ is at most the cost of $\mathcal{K}$ minus w_v, that is, $\alpha_w(G) - w_v$. By (7.1), this is at most $\alpha_{\tilde{w}}(\tilde{G})$. Hence, it only remains to show that $\tilde{\mathcal{K}}$ is a $\tilde{w}$-cover of $\tilde{G}$. By construction, every vertex $u \neq \tilde{v}$ is covered adequately by $\tilde{\mathcal{K}}$, so we only have to check how often we covered $\tilde{v}$. Clearly $\tilde{v}$ is covered by $\tilde{\mathcal{K}}$ at least as often as $\mathcal{K}$ covered $N(v)$ minus $|\mathcal{K}_v|$, since all we lose are the edges in $\mathcal{K}_v$, and for each cycle $C \in \mathcal{K}_v$ we observe that while C

covered two vertices in $N(v)$ its counterpart in $\tilde{\mathcal{K}}$ still covers $\tilde{v}$ once. Hence, $\tilde{\mathcal{K}}$ covers $\tilde{v}$ at least $d(v) \cdot \beta - w_v = \tilde{w}_{\tilde{v}}$ times, as desired. as desired. □

A graph G is *minimally (strongly) t-imperfect* if it is (strongly) t-imperfect but every proper t-minor of G is (strongly) t-perfect. Examples of minimally strongly t-imperfect graphs are K_4, the 5-wheel, as well as the squares C_7^2 and C_{10}^2 of the 7-cycle and the 10-cycle, see [13] and the last section. These graphs are also minimally t-imperfect. Thus, if a graph contains, for instance, K_4 as a t-minor then it is strongly t-imperfect as well as t-imperfect. This observation enabled a succinct characterisation of t-perfection in claw-free graphs [13], and will be helpful below.

7.3 Strongly t-perfect and claw-free

In order to prove Theorem 7.1.2, we only have to show that every claw-free strongly t-imperfect graph G is t-imperfect. For this, we may clearly suppose G to be minimally strongly t-imperfect. Our first step is then to show that that G is 3-connected:

Lemma 7.3.1. *Let G be a minimally strongly t-imperfect graph. If G is claw-free then G is 3-connected.*

We postpone the lengthy proof of this lemma to the end of the section. Once equipped with Lemma 7.3.1 we may apply the following amalgamation of Lemmas 6.4.2, 6.5.1 and 6.5.3 from Chapter 6:

Lemma 7.3.2. *Let G be a 3-connected claw-free graph. If G is t-perfect then one of the following statements holds true:*

(i) *G is a line graph;*

(ii) $G \in \{C_6^2 - v_1v_6, C_7^2 - v_7, C_{10}^2 - v_{10}\}$.

Here and in the rest of this chapter, we denote by C_i the cycle of length i, and assume C_i to be defined on the vertex set $\{v_1, \dots, v_i\}$ so that they occur in this order in the cycle. With C_i^2 we denote the square of C_i, which is obtained from C_i by adding an edge between any two vertices of distance 2.

We need one further ingredient for the proof of Theorem 7.1.2. The following theorem describes a TDI system for the matching polytope of a graph G – this polytope is the convex hull in $\mathbb{R}^{E(G)}$ of matchings in G.

Theorem 7.3.3 (Cook [26]). *For every graph H the following system of inequalities is TDI:*

$$\begin{aligned} & y \in \mathbb{R}^{E(H)},\ y \geq 0 & & \\ & \sum_{e \in E(v)} y_e \leq 1 & & \text{for every } v \in V(H) \quad (7.2) \\ & \sum_{e \in E(F)} y_e \leq \lfloor \frac{|V(F)|}{2} \rfloor & & \text{for every 2-connected factor-critical } F \subseteq H. \end{aligned}$$

Proof of Theorem 7.1.2. We only need to show that a claw-free graph G that is minimally strongly t-imperfect is also t-imperfect. By Lemma 7.3.1, G is 3-connected. Thus, Lemma 7.3.2 is applicable and G therefore either t-imperfect (as desired), or a line graph, or one of the graphs in Lemma 7.3.2 (ii). Since C_7^2 and C_{10}^2 are minimally strongly t-imperfect [13], we only need to consider the cases when $G = C_6^2 - v_1v_6$ or when G is a line graph.

Suppose that $G = C_6^2 - v_1v_6$, and pick a weight $w \in \mathbb{Z}^{V(G)}$ so that G has no w-cover of cost $\alpha_w(G)$ of minimal total weight $w(V(G))$. Since G is supposed to be minimally strongly t-imperfect it follows that $w > 0$. Then for $w' := w - \mathbf{1}_{v_1v_2v_3}$ there exists a w'-cover $\mathcal{K}'$ of cost $\alpha_{w'}(G)$. However, every stable set of S of weight $w(S) = \alpha_w(G)$ meets the triangle $v_1v_2v_3$, which implies that $\mathcal{K}' \cup \{v_1v_2v_3\}$ is a w-cover of cost $\alpha_w(G)$, a contradiction.

So assume that G is a line graph, of a graph H say. First of all, note that H has maximal degree ≤ 3 since G, as a minimally strongly t-imperfect graph, cannot contain K_4 as a proper subgraph. Now, if the only 2-connected factor-critical subgraph F of H are odd cycles, then system (6.1) becomes (7.2) – which is TDI by Theorem 7.3.3, a contradiction to the strong t-perfection of G. On the other hand, if H has a 2-connected factor-critical subgraph F that is not an odd cycle, then H contains the subdivision S of a 5-cycle plus an added chord in which every edge may be replaced by an odd path. Viewed in G, such an S leads to K_4 as a t-minor; see [13, Lemma 7] for more details, if necessary. Again, G is t-imperfect. □

The only missing link in our proof of Theorem 7.1.2 is Lemma 7.3.1, i.e. the fact that every claw-free minimally strongly t-imperfect graph G is 3-connected. We will show this in two steps. First, we will see that G is 2-connected and that one side of every 2-separation is a path. In the second step, we will prove that the minimum degree of G is at least three.

For any graph G, we say that (G_1, G_2) is a *separation of order k of G*, or a *k-separation of G*, if G_1, G_2 are proper induced subgraphs of G with $G = G_1 \cup G_2$ and $|V(G_1 \cap G_2)| = k$.

For the first step we make use of a notion of Gerards [44]. Let (G_1, G_2) be a 2-separation of an arbitrary graph G, and denote by u and v the two vertices contained in both G_1 and G_2. Given $w \in \mathbb{Z}^{V(G)}$, define $s^i_w(X)$ to be the maximum $w(S)$ among all stable sets S in G_i with $S \cap \{u, v\} = X$. If no confusion is likely we omit the subscript w. Moreover, we denote by $G_i + P_2$ the graph G_i with an u–v path of length 2 added, and by $G_i + P_3$ the graph G_i plus an u–v path of length 3.

The next two lemmas exclude already a good number of types of 2-separations in a minimally strongly t-imperfect graph. We mention that the lemmas do not appear explicitly in [44] but may, without effort, be extracted from the proof of Theorem 1.8.

Lemma 7.3.4 (Gerards [44])**.** *Let G be a graph, and let (G_1, G_2) be a separation of order ≤ 2. If $G_1 \cap G_2$ forms a complete subgraph, and if G_1 and G_2 are strongly t-perfect, then G is strongly t-perfect.*

Lemma 7.3.5 (Gerards [44])**.** *Let G be a graph, and let (G_1, G_2) be a 2-separation so that $V(G_1) \cap V(G_2)$ consists of two non-adjacent vertices u and v. Then for every non-negative weight $w \in \mathbb{Z}^{V(G)}$ it holds thats:*

(i) *If $s^2(u, v) + s^2(\emptyset) \geq s^2(u) + s^2(v)$ and if $G_1 + P_2$ as well as $G_2 + P_3$ are strongly t-perfect then G has a w-cover of cost $\alpha_w(G)$.*

(ii) *If $s^2(u, v) + s^2(\emptyset) \leq s^2(u) + s^2(v)$ and if $G_1 + P_3$ as well as $G_2 + P_2$ are strongly t-perfect then G has a w-cover of cost $\alpha_w(G)$.*

Next, we relate the inequalities in (ii) and (iii) in the previous lemma with the existence of odd or even induced u–v paths.

Lemma 7.3.6. *Let (G_1, G_2) be a 2-separation of a graph G, and denote the two vertices common to both G_1 and G_2 by u, v. For every $w \in \mathbb{Z}^{V(G)}$ it holds that:*

(i) *If every induced u–v path in G_2 has even length then $s^2(u, v) + s^2(\emptyset) \geq s^2(u) + s^2(v)$.*

(ii) *If every induced u–v path in G_2 has odd length then $s^2(u, v) + s^2(\emptyset) \leq s^2(u) + s^2(v)$.*

Proof. (i) Pick a stable set S_u in G_2 with $u \in S_u$ but $v \notin S_u$ so that $w(S_u) = s^2(u)$, and choose a stable set S_v in G_2 with $v \in S_v$, $u \notin S_v$ and $w(S_v) = s^2(v)$. Denote by K the vertex set of the component of $G_2[S_u \cup S_v]$ containing u. Then, as every induced u–v path in G_2 has even length, it follows that $v \notin K$.

The symmetric difference $S_u \triangle K$ is a stable set, and hence misses $\{u, v\}$, while the stable set $S_v \triangle K$ contains $\{u, v\}$. Since no vertex from K lies in both of S_u and S_v, we get

$$s^2(u) + s^2(v) = w(S_u) + w(S_v) = w(S_u \triangle K) + w(S_v \triangle K) \leq s^2(\emptyset) + s^2(u, v).$$

(ii) Same as (i), only starting with stable sets $S_\emptyset$ and $S_{u,v}$ missing, respectively containing, $\{u, v\}$. □

For a 2-separation (G_1, G_2) of a graph G, there is one case that is not addressed by Lemma 7.3.5, namely the case when every induced u–v path in G_1 *and* in G_2 is even, or if every such path is odd.

Lemma 7.3.7. *Let (G_1, G_2) be a 2-separation of a graph G so that $V(G_1 \cap G_2) = \{u, v\}$, and let G_1 and G_2 be strongly t-perfect. If every induced u–v path in G is even, or if every such path is odd, then also G is strongly t-perfect.*

Proof. Given a non-negative weight function $w : V(G) \to \mathbb{Z}$ we shall show for $i = 1, 2$ that there are non-negative weights $w^i : V(G) \to \mathbb{Z}$ with $w^i | V(G_{3-i} - G_i) = 0$ so that

(i) $w^1 + w^2 = w$, and

(ii) $\alpha_{w^1}(G_1) + \alpha_{w^2}(G_2) \leq \alpha_w(G)$.

This then establishes the lemma, as we can combine the w^i-covers of G that are given by the strong t-perfection of the G^i to a w-cover of G of cost $\alpha_w(G)$.

In order to prove that such w^i exist, we proceed by induction on the sum $w_u + w_v$. Clearly, if $w_u + w_v = 0$, then the restrictions of w to G^i satisfy (i) and (ii). So assume w.l.o.g. that $w_u > 0$, and set $\tilde{w} := w - \mathbf{1}_u$. (Here, and below, $\mathbf{1}_Z$ denotes the characteristic vector of the set $Z \subseteq V(G)$, where we abbreviate $\mathbf{1}_{\{z\}}$ by $\mathbf{1}_z$.) By induction, we know that there exist $\tilde{w}^1$ and $\tilde{w}^2$ satisfying (i) and (ii).

In particular, there is a set $X \subseteq \{u, v\}$ such that $\alpha_{\tilde{w}_1}(G_1) = s^1_{\tilde{w}^1}(X)$ and $\alpha_{\tilde{w}_2}(G_2) = s^2_{\tilde{w}^2}(X)$. Now, if $\alpha_{\tilde{w}_1 + \mathbf{1}_u}(G_1) = s^1_{\tilde{w}^1 + \mathbf{1}_u}(X)$ then we may set $w_1 := \tilde{w}^1 + \mathbf{1}_u$ and $w^2 := \tilde{w}^2$ and are done. Hence we may assume that $\alpha_{\tilde{w}^1 + \mathbf{1}_u}(G_1) \neq s^1_{\tilde{w}^1 + \mathbf{1}_u}(X)$. This can only happen if $u \notin X$, and if, moreover, there is a set $Y_1 \subseteq \{u, v\}$ which contains u, such that $\alpha_{\tilde{w}^1}(G_1) = s^1_{\tilde{w}^1}(Y_1)$. (Then, we have that $\alpha_{\tilde{w}_1 + \mathbf{1}_u}(G_1) = s^1_{\tilde{w}_1 + \mathbf{1}_u}(Y_1)$.) Arguing in the same way for $\tilde{w}^2$, we find that there is a set $Y_2 \subseteq \{u, v\}$ which contains u, such that $\alpha_{\tilde{w}^2}(G_2) = s^2_{\tilde{w}^2}(Y_2)$. By symmetry of G_1 and G_2, we may suppose that $Y_1 = \{u\}$ and $Y_2 = \{u, v\}$, since we are done if $Y_1 = Y_2$.

So, depending on whether $X = \emptyset$ or $X = \{v\}$, we arrive at one of the following two cases:

(a) $\alpha_{\tilde{w}^1}(G_1) = s^1_{\tilde{w}^1}(\emptyset) = s^1_{\tilde{w}^1}(u)$ and $\alpha_{\tilde{w}^2}(G_2) = s^2_{\tilde{w}^2}(\emptyset) = s^2_{\tilde{w}^2}(u,v)$, or

(b) $\alpha_{\tilde{w}^1}(G_1) = s^1_{\tilde{w}^1}(v) = s^1_{\tilde{w}^1}(u)$ and $\alpha_{\tilde{w}^2}(G_2) = s^2_{\tilde{w}^2}(v) = s^2_{\tilde{w}^2}(u,v)$.

First, assume that case (a) holds. Now, if every induced u–v path in G is odd, then Lemma 7.3.6 (ii) implies that $\alpha_{\tilde{w}^2}(G_2) = s^2_{\tilde{w}^2}(u) = s^2_{\tilde{w}^2}(v) = s^2_{\tilde{w}^2}(\emptyset) = s^2_{\tilde{w}^2}(u,v)$. Thus, setting $w^1 := \tilde{w}_1 + \mathbf{1}_u$ and $w^2 := \tilde{w}^2$ will ensure (i) and (ii), as $s^1_{w^1}(u) = \alpha_{w^1}(G_1)$ and $s^2_{w^2}(u) = \alpha_{w^2}(G_2)$. So, in case (a), we may restrict our attention to the situation that every induced u–v path in G is even.

Then, by Lemma 7.3.6 (i), we have

$$s^1_{\tilde{w}^1}(v) \leq s^1_{\tilde{w}^1}(u,v). \tag{7.3}$$

Furthermore, as we may otherwise set $w^1 := \tilde{w}_1$ and $w^2 := \tilde{w}^2 + \mathbf{1}_u$, we see that

$$s^2_{\tilde{w}^2}(u) < \alpha_{\tilde{w}^2}(G_2). \tag{7.4}$$

Set

$$w^1 := \tilde{w}^1 + \mathbf{1}_v \quad \textit{and} \quad w^2 := \tilde{w}^2 + \mathbf{1}_u - \mathbf{1}_v.$$

Note that $\tilde{w}^2_v > 0$ since $s^2_{\tilde{w}^2}(u) < \alpha_{\tilde{w}^2}(G_2) = s^2_{\tilde{w}^2}(u,v)$. By (7.4), it is clear that $\alpha_{w^2}(G_2) = s^2_{w^2}(\emptyset) = s^2_{w^2}(u,v)$. On the other hand, (7.3) together with the fact that $s^1_{\tilde{w}^1}(u) = s^1_{\tilde{w}^1}(\emptyset)$ implies that $\alpha_{w^1}(G_1) \in \{s^1_{w^1}(\emptyset), s^1_{w^1}(u,v)\}$. Hence, our choice of w^1 and w^2 ensures (i) and (ii), as desired.

Now assume that case (b) above holds. If every induced u–v path in G is even, then Lemma 7.3.6 (i) implies that $\alpha_{\tilde{w}^1}(G_1) = s^1_{\tilde{w}^1}(\emptyset) = s^1_{\tilde{w}^1}(u,v)$. Thus, setting $w^1 := \tilde{w}_1$ and $w^2 := \tilde{w}^2 + \mathbf{1}_u$ will ensure (i) and (ii). So, we will suppose from now on that every induced u–v path in G is odd.

By Lemma 7.3.6 (ii), we have

$$s^2_{\tilde{w}^2}(\emptyset) \leq s^2_{\tilde{w}^2}(u), \tag{7.5}$$

and (as we may otherwise set $w^1 := \tilde{w}_1 + \mathbf{1}_u$ and $w^2 := \tilde{w}^2$) we see that

$$s^1_{\tilde{w}^1}(u,v) < \alpha_{\tilde{w}^1}(G_1) \textit{ and } s^2_{\tilde{w}^2}(u) < \alpha_{\tilde{w}^2}(G_2). \tag{7.6}$$

Observe that $\tilde{w}^2_v > 0$ by (7.6) and (b). Hence, setting

$$w^1 := \tilde{w}^1 + \mathbf{1}_u + \mathbf{1}_v \quad \textit{and} \quad w^2 := \tilde{w}^2 - \mathbf{1}_v.$$

resolves our problem, as (7.6) implies that $\alpha_{w^1}(G_1) = s^1_{w^1}(u) = s^1_{w^1}(v)$, and (7.5) implies that $\alpha_{w^2}(G_2) \in \{s^2_{w^2}(u), s^2_{w^2}(v)\}$. $\square$

Lemma 7.3.8. *Let G be a minimally strongly t-imperfect graph. Then G is 2-connected, and if (G_1, G_2) is a 2-separation of G then one of G_1 and G_2 is a path.*

Proof. That G is 2-connected is immediate from Lemma 7.3.4. Suppose that G has a 2-separation (H_1, H_2) with $V(H_1) \cap V(H_2) = \{u, v\}$. By Lemma 7.3.4, u and v are not adjacent.

If every induced u–v path in G is even or if every such path is odd then Lemma 7.3.7 implies that one of H_1 or H_2 is strongly t-imperfect, a contradiction, since G is minimally strongly t-imperfect.

So we may assume that one of the H_i, say H_1 contains an even induced u–v path, and the other, H_2 contains an odd induced u–v path. By minimality of G, this implies that $H_1 + P_3$ and $H_2 + P_2$ are strongly t-perfect. Now, pick a non-negative weight $w \in \mathbb{Z}^{V(G)}$ so that G has no w-cover of cost $\alpha_w(G)$.

Applied to $(G_1, G_2) := (H_1, H_2)$, Lemma 7.3.6 (ii) in combination with Lemma 7.3.5 (ii) imply that H_2 also contains an even induced u–v path. Moreover, Lemma 7.3.6 (i) and Lemma 7.3.5 (i) applied to $(G_1, G_2) := (H_2, H_1)$ yield that H_1 has an odd induced u–v path. Hence, for all $i = 1, 2$ and $j = 2, 3$ the graph $H_i + P_j$ is a t-minor of G. For contradiction, assume that the $H_i + P_j$ are proper t-minors of G, and thus strongly t-perfect. Now, whichever value $s^2(u, v) + s^2(\emptyset)$ takes, either Lemma 7.3.5 (i) or (ii) is applicable in order to obtain the final contradiction. □

We turn now to proving that claw-free minimally strongly t-imperfect graphs do not possess any vertices of degree less than three.

Lemma 7.3.9. *Let $G = (V, E)$ be a graph, let $w \in \mathbb{Z}^V$, and assume v to be a vertex with exactly two neighbours, p and q, so that $w_p = w_v = w_q$. Set $\tilde{G} = G/E(v)$, denote the new vertex by $\tilde{v}$ and define $\tilde{w} \in \mathbb{Z}^{V(\tilde{G})}$ by setting $\tilde{w}_u := w_u$ for $u \in V(\tilde{G} - \tilde{v})$ and $\tilde{w}_{\tilde{v}} := w_v$. If $\tilde{G}$ has a $\tilde{w}$-cover of cost $\alpha_{\tilde{w}}(\tilde{G})$ then G has a w-cover of cost $\alpha_w(G)$.*

Proof. Consider a stable set $\tilde{S}$ in $\tilde{G}$ with $\tilde{w}(\tilde{S}) = \alpha_{\tilde{w}}(\tilde{G})$. If $\tilde{v} \in \tilde{S}$ then $S := (\tilde{S} \setminus \{\tilde{v}\}) \cup \{p, q\}$ is a stable set in G with $w(S) = \alpha_{\tilde{w}}(\tilde{G}) + w_v$. If, on the other hand, $\tilde{v} \notin \tilde{S}$ then $S := \tilde{S} \cup \{v\}$ is stable in G, and $w(S) = \alpha_{\tilde{w}}(\tilde{G}) + w_v$. Thus, we get

$$\alpha_{\tilde{w}}(\tilde{G}) + w_v \leq \alpha_w(G). \tag{7.7}$$

By assumption, there is a $\tilde{w}$-cover $\tilde{\mathcal{K}}$ of $\tilde{G}$, which we may choose to cover $\tilde{v}$ exactly $\tilde{w}_{\tilde{v}} = w_v$ times. Observe that we may view $E(\tilde{G})$ as a subset of $E(G)$; for an edge $x\tilde{v}$ so that x is a neighbour of p as well as of q we arbitrarily pick one of xp and xq and identify it with $x\tilde{v}$. Thus, viewed in

G, the subfamily of $\tilde{\mathcal{K}}$ consisting of edges and odd cycles becomes a family of edges, odd cycles and odd p–q paths; denote the latter subfamily of $\tilde{\mathcal{K}}$ by $\tilde{\mathcal{P}}$. By completing every $P \in \tilde{\mathcal{P}}$ to an odd cycle through v, and by replacing every occurrence of $\{\tilde{v}\}$ in $\tilde{\mathcal{K}}$ by one of $\{p\}$ and $\{q\}$ we obtain from $\tilde{\mathcal{K}}$ a family $\mathcal{K}'$ of vertices, edges and odd cycles in G.

Set $\tilde{\gamma} := |\tilde{\mathcal{P}}|$ and observe that as $\tilde{\mathcal{K}}$ covers $\tilde{v}$ exactly w_v times, we get that $\tilde{\gamma} \leq w_v$. Moreover, it follows that each of p and q is covered by $\mathcal{K}'$ at most w_v times, while together they are covered $w_v + \tilde{\gamma}$ times since every $P \in \tilde{\mathcal{P}}$ leads to a cycle in $\mathcal{K}'$ that meets p as well as q. Since v is contained in these cycles as well, it is covered $\tilde{\gamma}$ times. Hence, by adding $w_v - \tilde{\gamma}$ edges, vp or vq, we can complete $\mathcal{K}'$ to a w-cover $\mathcal{K}$.

The cost of $\mathcal{K}$ is the cost of $\tilde{\mathcal{K}}$ plus the cost of extending the $P \in \tilde{\mathcal{P}}$ to cycles plus the cost of the additional edges incident with v. In other words, $\mathcal{K}$ costs

$$\alpha_{\tilde{w}}(\tilde{G}) + \tilde{\gamma} + w_v - \tilde{\gamma} = \alpha_{\tilde{w}}(\tilde{G}) + w_v \leq \alpha_w(G),$$

where the last inequality follows from (7.7). □

The following lemma uses an idea of Mahjoub [71].

Lemma 7.3.10. *Let G be a graph, and let $w \in \mathbb{Z}^{V(G)}$, $w > 0$, so that there is no w-cover of cost $\alpha_w(G)$ but for every $w' \leq w$ with one strictly smaller entry there is a w'-cover with cost $\alpha_{w'}(G)$.*

(i) *If G contains a path pvq so that $d(v) = 2$ then $w_v \leq w_q$.*

(ii) *If G contains a triangle prs and a neighbour $v \notin \{r, s\}$ of p so that $d(p) = 3$ then $w_p \leq w_v$.*

Proof. Suppose there is an edge or triangle X that is hit by every stable set S of weight $w(S) = \alpha_w(G)$. Set $w' := w - \mathbf{1}_X$, and observe that $\alpha_{w'}(G) = \alpha_w(G) - 1$. Hence, by assumption there is a w'-cover $\mathcal{K}'$ of cost $\alpha_w(G) - 1$, which together with X yields a w-cover of cost $\alpha_w(G)$, a contradiction. This proves that for every edge or triangle X there is a stable set S_X of weight $\alpha_w(G)$ that misses X.

(i) Consider the stable set S_{pv} of weight $\alpha_w(G)$ that misses the edge pv. Since $w_v > 0$, it follows that $q \in S_{pv}$. Then $S := S_{pv} \setminus \{q\} \cup \{v\}$ is a stable set with weight $w(S) = \alpha_w(G) - w_q + w_v \leq \alpha_w(G)$, which implies $w_v \leq w_q$, as desired.

(ii) Consider the stable set S_{prs} of maximal weight that misses prs, and note that $v \in S_{prs}$. Then the stable set $S_{prs} \setminus \{v\} \cup \{p\}$ has weight $\alpha_w(G) - w_v + w_p \leq \alpha_w(G)$, which implies $w_p \leq w_v$, as desired. □

We are finally prepared to prove Lemma 7.3.1.

Proof of Lemma 7.3.1. By Lemma 7.3.8, we only need to convince ourselves that $G = (V, E)$ does not contain any vertices of degree 2. So suppose otherwise, i.e. suppose there is a path $P = u \dots v$ in G with all interior vertices of degree 2 in G but with endvertices u, v of higher degree, and suppose that P does indeed contain an interior vertex. Note that by the minimality of G it cannot contain K_4 as a subgraph, as K_4 is strongly t-imperfect. Since G is claw-free it follows that both u and v have degree 3 and are incident with a triangle.

Among all non-negative $w \in \mathbb{Z}^V$ for which there is no w-cover of cost $\alpha_w(G)$ choose one, w say, so that $w(V)$ is minimal. Since G is strongly t-imperfect there is such a w and, moreover, it holds that $w > 0$ by the minimality of G. We may now apply Lemma 7.3.10 to the vertices in P plus the two triangles incident with u and v. This yields that w is constant on P. Let r be an interior vertex of P and set $\tilde{G} := G/E(r)$. Define $\tilde{w}$ as in Lemma 7.3.9 with r in the role of v. Then, $\tilde{G}$ is a proper t-minor of G and has thus a $\tilde{w}$-cover of cost $\alpha_{\tilde{w}}(\tilde{G})$. Now, however, Lemma 7.3.9 asserts that G has a w-cover of cost $\alpha_w(G)$, a contradiction to the choice of w. $\square$

7.4 Minimally strongly t-imperfect

Theorem 7.1.2 lends credibility to the conjecture that t-perfection is always strong. One way to prove the conjecture would consist in verifying whether the minimally t-imperfect graph coincide with the minimally strongly t-imperfect graphs. Unfortunately, a complete list of minimal elements is neither known for t-perfection nor for strong t-perfection.

So far, the only known minimally t-imperfect graphs are the odd wheels, the even Möbius ladders (see Schrijver [81]), and two additional graphs, the squares C_7^2 and C_{10}^2 of the 7-cycle and the 10-cycle, see Figure 7.1. All these are minimally strongly t-imperfect as well, and no others are known. In fact, that the odd wheels and the even Möbius ladders are minimally (strongly) t-imperfect can easily be deduced from the fact that *almost bipartite graphs* are strongly t-imperfect, which follows from Theorem 7.1.3. (A graph is almost bipartite if it can be made bipartite by deleting some vertex.) That C_7^2 and C_{10}^2 are minimally (strongly) t-imperfect is proved in [13]. We remark that the squares of other cycles do not have this property.

In this section, we will find seven more graphs that turn out to be minimal under both strong and ordinary t-perfection. Thus, those graphs can be seen as further evidence for the conjecture that t-perfection is always strong.

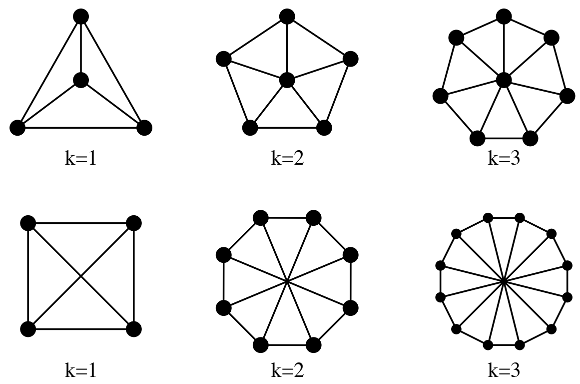


Figure 7.1: Minimally (strongly) t-imperfect graphs

Lemma 7.4.1. *Let G be a strongly t-perfect graph, and let $u, v \in V(G)$ be such that $N(u)$ and $N(v)$ partition $V(G) - \{u, v\}$. Then also $G + uv$ is strongly t-perfect.*

Proof. Suppose otherwise, and let $w \in \mathbb{Z}^{V(G)}$ be a witness of the fact that $G + uv$ is strongly t-imperfect with minimal total weight $w(V(G))$. Observe that this implies $\alpha_w(G + uv) < \alpha_w(G)$; otherwise a w-cover of G with cost $\alpha_w(G)$ is a w-cover of $G + uv$ with cost $\alpha_w(G + uv)$. This means that in G, every stable set of weight $\alpha_w(G)$ contains both u and v. Thus as $V(G) \setminus \{u, v\} = N(u) \cup N(v)$, the only stable set of weight $\alpha_w(G)$ is $\{u, v\}$. Hence, since the neighbourhoods of u and v are disjoint, we have that

$$\text{for every stable set } D \subseteq N(u) \text{ it holds that } w(D) < w_u. \tag{7.8}$$

Now, assume that there is a stable set S in $G + uv$ of weight $\alpha_w(G + uv)$ that avoids both u and v. Then, by (7.8), the stable set $S \cup \{u\} \setminus N(u)$ outweighs S, a contradiction. We have thus shown that every stable set of maximal weight in $G + uv$ meets either u or v.

Define $\bar{w}$ as $\bar{w}_x := w_x$ for all $x \notin \{u, v\}$, and $\bar{w}_x := w_x - 1$ for $x \in \{u, v\}$. (Observe that (7.8) ensures in particular that $w_u \neq 0$, and analogously we can show that $w_v \neq 0$.)

Since every stable set of maximal weight (with respect to w) in $G + uv$ meets u or v it follows that $\alpha_{\bar{w}}(G + uv) \leq \alpha_w(G + uv) - 1$. By the choice of w, $G + uv$ has a $\bar{w}$-cover of cost $\alpha_{\bar{w}}(G + uv)$. Add uv to obtain a w-cover of $G + uv$ of cost $\alpha_w(G + uv)$, a contradiction to the choice of w. □

Lemma 7.4.1 enables us to obtain new minimally strongly t-imperfect graphs from C_{10}^2 by adding any number of diagonals. See Figure 7.2.

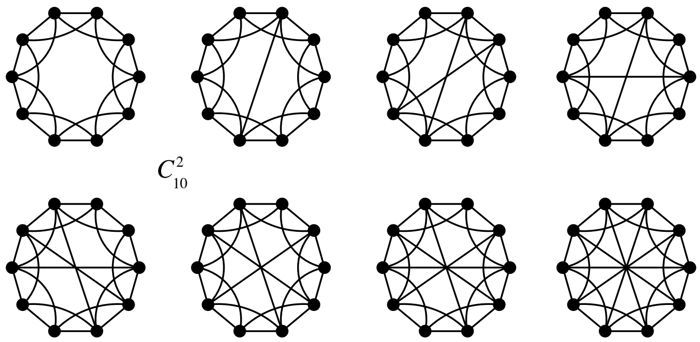


Figure 7.2: C^2_{10} plus diagonals

Proposition 7.4.2. [15] *The graphs C^2_{10}, $C^2_{10} + v_1v_6$, $C^2_{10} + v_1v_6 + v_2v_7$, $C^2_{10} + v_1v_6 + v_3v_8$, $\overline{C^2_{10}} - v_1v_6 - v_2v_7$, $\overline{C^2_{10}} - v_1v_6 - v_3v_8$, $\overline{C^2_{10}} - v_1v_6$ and $\overline{C^2_{10}}$ are minimally strongly t-imperfect as well as minimally t-imperfect.*

Proof. For $0 \leq i \leq 5$, denote by $\mathcal{C}^i$ the family of all graphs we obtain from C^2_{10} by adding exactly i diagonals, that is, edges of the form v_jv_{j+5}, and set $\mathcal{C} := \bigcup \mathcal{C}^i$. The graphs listed in the lemma are precisely the graphs in $\mathcal{C}$ (up to isomorphism). First note that, for every $G \in \mathcal{C}$, assigning a value of $\frac{1}{3}$ to each vertex yields a point z that lies in TSTAB(G). However, as $\alpha(G) = 3$ the point z lies outside the stable set polytope of G. Therefore, no graph in $\mathcal{C}$ is t-perfect, and thus every $G \in \mathcal{C}$ is strongly t-imperfect.

Next, we prove by induction on i that every proper t-minor of a $G \in \mathcal{C}^i$ is strongly t-perfect. Since C^2_{10} is minimally strongly t-imperfect [13], and since $\mathcal{C}^0 = \{C^2_{10}\}$, the induction start is assured. Now, assume the claim to be true for $i \leq 4$ and pick a graph $H \in \mathcal{C}^{i+1}$. Let j be such that v_jv_{j+5} is one of the diagonals of H, and set $G := H - v_jv_{j+5}$. As every proper t-minor of H is a (not necessarily proper) t-minor of $H - v_i$ for some i, it suffices to prove that $H - v_i$ is strongly t-perfect. If $i = j$ or $i = j + 5$ then $H - v_i = G - v_i$, and the claim is true as $G \in \mathcal{C}^i$. So, let $i \notin \{j, j + 5\}$, which implies $v_jv_{j+5} \in E(H - v_i)$. By induction, $G - v_i$ is strongly t-perfect, and using Lemma 7.4.1 it follows that $(G - v_i) + v_jv_{j+5} = H - v_i$ is strongly t-perfect, too. □

As a final observation, let us remark that we cannot obtain, in the same way, minimally strongly t-imperfect graphs by adding diagonals to C^2_7. Indeed, introducing any number of diagonals in C^2_7 leads to K_4 as a t-minor. The same happens if other edges than diagonals are added to C^2_{10}.

Chapter 8

Infinite extremal graph theory

8.1 An introduction to infinite extremal graph theory

From now on, we will shift our focus to infinite graphs. In this chapter, we will take a first approach to infinite extremal graph theory. Necessary concepts of infinite graph theory will be introduced along the way.

We will take our first steps in infinite extremal graph theory accompanied by a well-known result of Kostochka [31], which will serve as an example of the difficulties one encounters when trying to extend a 'finite' result to infinite graphs. The function $f(k)$ in the theorem is essentially the best possible bound [89].

Theorem 8.1.1.[Kostochka] *There is a constant c so that, for every $k \in \mathbb{N}$, if G is a finite graph of average degree at least $f(k) := ck\sqrt{\log k}$, then G has a complete minor of order k.*

How might this result extend to infinite graphs? First of all we have to note that it is not clear what the average degree of an infinite graph should be. We shall thus stick to the minimal degree as our 'density-indicating' parameter. A minor, on the other hand, is defined in same way as for finite graphs, only that the branch-sets may now be infinite.[1]

In rayless graphs we will then get a verbatim extension of Theorem 8.1.1 (see Chapter 10). In graphs with rays, however, large minimal degree at the vertices is too weak to force any interesting substructure. This is so because

[1]As long as our minors are locally finite, however (which will always be the case here), it does not make any difference whether we allow infinite branch-sets or not. It is easy to see that in this case any infinite branch-set may be restricted to a finite one.

infinite trees may have arbitrarily large degrees, but they do not even have any 2-connected subgraphs.

So at first sight, our goal seems unreachable. At second thought, however, the example of the infinite tree just shows that we did not translate the term 'large local densities' in the right way to infinite graphs. Only having every finite part of an infinite graph send out a large number of edges will not produce large overall density, if we do not require something to 'come back' from infinity.

The most natural way to do this is to impose a condition on the ends of the graph. Ends are defined as the equivalence classes of rays (one-way infinite paths), under the equivalence relation of not being separable by any finite set of vertices. Ends have a long history, see [31, 59].

In [17] and in [86], *end degrees* were introduced (see also [31, 84]). In fact, two notions have turned out useful (for different purposes): the vertex-degree and the edge-degree of an end ω. The *vertex-degree* of ω is defined as the maximum cardinality of a set of (vertex)-disjoint rays in ω, and the *edge-degree* is defined as the maximum cardinality of a set of edge-disjoint rays in ω. These maxima exist [49].

Do these notions help to force density in infinite graphs? To some extent they do: A large minimum degree at the vertices together with a large minimum vertex-/edge-degree at the ends implies a certain dense substructure, which takes the form of a highly connected or edge-connected subgraph.

More precisely, there is a function f_v such that every graph of minimum degree resp. vertex-degree $f_v(k)$ at the vertices and the ends has a k-connected subgraph, and there is also a function f_e such that every graph of minimum degree/edge-degree $f_e(k)$ at the vertices and the ends has a k-edge-connected subgraph. While f_e is linear, f_v is quadratic, and this is almost best possible. All these results are from [86] and will be presented in Chapter 9.

Some related results will already be discussed in Sections 8.3 and 8.4 of the present chapter. In Section 8.3 we shall see that independently of the degrees at the vertices, large vertex-degrees at the ends force an interesting planar substructure: An end of infinite vertex-degree produces the $\mathbb{N} \times \mathbb{N}$-grid as a minor [49], and an end of vertex-degree at least $\frac{3}{2}k - 1$ forces a $[k] \times \mathbb{N}$-grid-minor. We shall also show that this bound is best possible. In Section 8.4, we shall see that in locally finite vertex-transitive graphs, k-connectivity is implied by much weaker assumptions. In fact, the k-(edge)-connectivity of a locally finite vertex-transitive graph is equivalent to all its ends having vertex-(resp. edge)-degree k.

However, our notion of vertex-/edge-degrees is not strong enough to make an extension of Theorem 8.1.1 possible. This can be seen by taking the infinite r-regular tree and inserting the edge set of some spanning subgraph at each level (Example 10.3.1). With a little more effort we can transform our example into one with infinitely many ends of large but finite vertex-/edge-degree (Example 10.3.2).

To overcome this problem, we introduce in Section 10.4 a new end degree notion, the *relative degree*, that makes an extension of Theorem 8.1.1 to infinite locally finite graphs possible (Theorem 10.4.3). Moreover, every locally finite graph of minimum degree/relative degree k has a finite subgraph of average degree k (Theorem 10.4.2). An application of Theorem 10.4.3 is investigated in Section 10.5, where we ask whether as in finite graphs, large girth can be used to force large complete minors. A partial answer is given by Proposition 10.5.2.

8.2 Terminology for infinite graphs

All our notation is as in [31], but we take the oppotunity here to remind the reader of the few less standard concepts.

One of the main concepts in infinite graph theory is that of the *ends* of a graph G. An end of G is an equivalence class of rays (i.e. one-way infinite paths) of G, where we say that two rays are equivalent if no finite set of vertices separates them. We denote the set of ends of a graph G by $\Omega(G)$.

The *vertex-degree* and the *edge-degree* of an end $\omega \in \Omega(G)$ were introduced in [17] resp. in [85]. Sometimes, one refers to both at the same time speaking informally of the *end degree*. The vertex-degree $d_v(\omega)$ of ω is defined as the maximum cardinality of a set of (vertex)-disjoint rays in ω, and the edge-degree $d_e(\omega)$ of ω is defined as the maximum cardinality of a set of edge-disjoint rays in ω. These maxima exist [49], see also [31]. Cleary, the vertex-degree of an end is at most its edge-degree. We shall encounter a third end degree notion in Section 10.4.

For a subgraph H of a graph G, we write $\partial_v H := N(G - H)$ for its *vertex-boundary*. Similarly, $\partial_e H := E(H, G - H)$ is the *edge-boundary* of H.

An induced connected subgraph H of an infinite graph that has a finite vertex-boundary is called a *region*. If H contains rays of an end ω, we will say that H is a *region of* ω.

For $k \in \mathbb{N}$, a separator of a graph of size k will often be called a *k-separator*, and *k-cuts* are defined analogously. We say that a separator (or cut) S of a graph G separates some set $A \subseteq V(G)$ *from* an end $\omega \in \Omega(G)$, if the component of $G - S$ that contains rays of ω does not meet A.

8.3 Grid minors

From now on, we will deal with graphs that may have rays. We have already seen in the introduction to this chapter that then large degrees at the vertices are not enough to forcesomething as simple as cycles. We shall thus use additionally the end degrees in order to force interesting substructures in infinite graphs. In this section, we start modestly by asking for minors that are planar.

Particularly interesting planar graphs are the grids. The *infinite grid* $\mathbb{Z} \times \mathbb{Z}$ is the graph on $\mathbb{Z}^2$ having all edges of the form $(m, n)(m + 1, n)$ and of the form $(m, n)(m, n + 1)$, for $m, n \in \mathbb{Z}$. The *half-grid* $\mathbb{N} \times \mathbb{Z}$, the *quarter-grid* $\mathbb{N} \times \mathbb{N}$, and the $[k] \times \mathbb{N}$-grid are the induced subgraphs of $\mathbb{Z} \times \mathbb{Z}$ on the respective sets.

A well-known result in infinite graph theory concerns the quarter-grid[2], which is a minor of every graph that has an end of infinite vertex-degree (this is a classical result of Halin [49] who called such ends *thick ends*).

Theorem 8.3.1 (Halin [49])**.** *Let G be graph which has an end ω of infinite vertex-degree. Then the $\mathbb{N} \times \mathbb{N}$-grid is a minor of G.*

From Halin's proof it follows that the rays of the subgraph of G that can be contracted to $\mathbb{N} \times \mathbb{N}$ belong to ω (see also the proof in Diestel's book [31]). On the other hand, it is clear that if a subdivision of the quarter-grid appears as a subgraph of some graph G, the its rays belong to an end of infinite vertex-degree in G.

Thus, it is not surprising that assuming large (but not infinite) degrees and vertex-degrees we cannot force a quarter-grid minor. One example is the graph G'_k from Example 10.3.2, whcih has the additional quality of being planar.

However, both graphs contain something quite similar to a quarter-grid: a $[k] \times \mathbb{N}$ grid, where k depends on the minimum vertex-degree we required at the ends. In fact, such a grid always appears in a graph with an end ω of large enough vertex-degree. It will follow from the proof that the rays corresponding to the rays of the minor, in G belong to ω.

Theorem 8.3.2. **[83]** *Let $k \in \mathbb{N}$ and let G be graph which has an end ω of vertex-degree at least $\frac{3}{2}k - 1$. Then the $[k] \times \mathbb{N}$-grid is a minor of G.*

---

[2] Observe that when considering minors, it makes no difference whether we work with the half-grid or the quarter-grid, since, as one easily checks, each of the two is a minor of the other.

The bound on the vertex-degree is sharp. This is illustrated by Example 8.3.3, after the proof of Theorem 8.3.2.

Proof of Theorem 8.3.2. We shall proceed by induction on k. For $k = 1$ and $k = 2$, the assertion clearly holds, so assume that $k \geq 3$ and that ω is an end of a graph G with $d_v(\omega) \geq \frac{3}{2}k - 1$.

Choose a set $\mathcal{R}$ of $d_v(\omega)$ disjoint rays from ω. Consider the auxiliary graph H with $V(H) := \mathcal{R}$ where two vertices R and R' are adjacent if there exists an infinite set of disjoint $V(R)$–$V(R')$ paths in G which avoid all $R'' \in \mathcal{R}$ with $R'' \neq R, R'$. Let T be a spanning tree of H. Clearly, if T happens to be a path, it is easy to construct the desired minor.

So suppose otherwise. Then T has (at least) three leaves R_1, R_2, R_3. Observe that the graph $G' := G - V(\bigcup_{j=1,2,3} R_j)$ has an end ω' of degree

$$d_v(\omega') = d_v(\omega) - 3 \geq \frac{3}{2}k - 4 = \frac{3}{2}(k-2) - 1$$

whose rays, when viewed in G, belong to ω. Hence, by induction, the $[k - 2] \times \mathbb{N}$-grid is a minor of G'. In other words, G' contains a set of rays $Q_1, Q_2, \ldots Q_{k-2} \in \omega'$, and furthermore, each Q_i is linked to Q_{i+1} by infinitely many disjoint paths, which do not meet any other Q_j.

In G, the Q_i belong to ω. Thus, since $|\mathcal{R}| = d_v(\omega)$, each Q_i meets $\bigcup \mathcal{R}$ infinitely often. Hence each Q_i meets (at least) one of the rays in $\mathcal{R}$, which we shall denote by $R(Q_i)$, infinitely often.

The tree T from above contains three paths P_i, $i = 1, 2, 3$, so that P_i starts in $V(R_i)$ and ends in $\bigcup_{i=1}^{k-2} V(R(Q_i))$. Since R_1, R_2 and R_3 are leaves of T, the P_i can be chosen so that they are disjoint except possibly in their endvertices. Using the path systems in G represented by the P_i, it is now easy to see that for each R_j, $j = 1, 2, 3$, there is a Q_{i_j} among the Q_i such that there exist an infinite family of disjoint $V(R_j)$–$V(Q_{i_j})$ paths which avoid all other $Q_{i'}$ and $R_{j'}$. Say $i_1 \leq i_2 \leq i_3$.

In order to see that the $[k] \times \mathbb{N}$-grid is a minor of G, we shall now define a family of rays $\tilde{Q}_1, \tilde{Q}_2, \ldots \tilde{Q}_k \in \omega$ so that $\tilde{Q}_i$ and $\tilde{Q}_{i+1}$ are connected by infinitely many disjoint paths which do not meet any other $\tilde{Q}_i$. For $i < i_1$ set $\tilde{Q}_i := Q_i$, and for $i > i_3 + 2$ set $\tilde{Q}_i := Q_{i-2}$. Set $\tilde{Q}_{i_2+1} := R_2$. For $i \neq i_2 + 1$ with $i_1 < i < i_3 + 2$, we choose $\tilde{Q}_i$ as a suitable ray which alternatively visits Q_{i-1} and Q_i, if $i \leq i_2$, or Q_{i-2} and Q_{i-1}, if $i > i_2 + 1$. Finally, $\tilde{Q}_{i_1}$ and $\tilde{Q}_{i_3+2}$ are chosen so that they alternate between R_1 and Q_{i_1}, respectively between Q_{i_3} and R_3. Clearly this choice of the rays $\tilde{Q}_i$ ensures that, together with suitable connecting paths, the $\tilde{Q}_i$ may be contracted to a $[k] \times \mathbb{N}$-grid. □

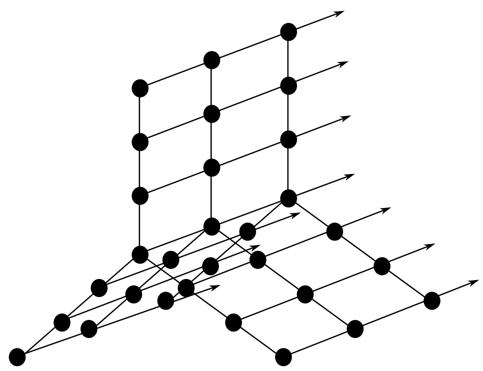


Figure 8.1: The graph $Y(3)$ from Example 8.3.3.

Example 8.3.3. *Denote by $K_{1,3}(\ell)$ the graph that is obtained by replacing each edge of $K_{1,3}$ with a path of length ℓ. Define $Y(\ell) := K_{1,3}(\ell) \times \mathbb{N}$. (That is, for each $i \in \mathbb{N}$, we take a copy of $K_{1,3}(\ell)$ and add an edge between every ith and $(i+1)$th copy of each vertex in $K_{1,3}(\ell)$.)*
Clearly, the vertex-degree of the unique end of Y is $3\ell + 1$. We shall show in Lemma 8.3.4 that the $[k] \times \mathbb{N}$-grid is not a minor of $Y(\ell)$, for $k = 2\ell + 2$.

Lemma 8.3.4. *Let $\ell \in \mathbb{N}$ and let $k = 2\ell + 2$. Then the graph $Y(\ell)$ from Example 8.3.3 has an end of vertex-degree $\frac{3}{2}k - 2$, but the $[k] \times \mathbb{N}$-grid is not a minor of $Y(\ell)$.*

Proof. Suppose otherwise. Then the graph $Y(\ell)$ contains a family of rays $\mathcal{R} := \{R_1, R_2, \dots R_k\}$ such that for $i = 1, 2, \dots, k-1$, there are infinitely many finite paths connecting R_i with R_{i+1}, such that all these paths are all disjoint, except possibly in their endvertices, and such that they avoid all $R_{i'}$ with $i' \neq i, i+1$.
Let $n \in \mathbb{N}$ be such that all R_i meet $Y_n := K_{1,3}(\ell) \times \{n\}$, the nth copy of $K_{1,3}(\ell)$ in $Y(\ell)$. Write $V(Y_n)$ as $\{v_0, v_1^1, v_2^1, \dots, v_\ell^1, v_1^2, v_2^2, \dots, v_\ell^2, v_1^3, v_2^3, \dots, v_\ell^3\}$ where each $v_0 v_1^j v_2^j \dots v_\ell^j$ induces a path in Y_n.
For each $j = 1, 2, 3$ consider that ray $R(j) \in \mathcal{R}$ that meets a v_m^j with largest index m. Observe that (at least) one of these three rays, say $R(1)$ is neither equal to R_1 nor to R_k. Let $R'(1)$ be the ray in $\mathcal{R}$ that meets v_m^1 with the second largest index m, or, if there is no such, let $R'(1)$ be the ray that meets v_0 (which then exists, since $|\mathcal{R}| = k > 2\ell + 1$ and since each ray of $\mathcal{R}$ meets Y_n).
We claim that $S := V(R'(1)) \cup V(\bigcup_{h \leq n} Y_h)$ separates $R(1)$ from the rest of the R_i, which clearly leads to the desired contradiction, since $R(1) \neq R_1, R_k$,

and thus has to be connected to two of the R_i by infinitely many disjoint finite paths that avoid all other R_i. So suppose otherwise, and let P be a path that connects $R(1)$ in $Y(\ell) - S$ with some $R_{i^*} \in \mathcal{R}$.
By construction of $Y(\ell)$, this is only possible if $R'(1)$ uses vertices of the type v_m^2 or v_m^3. Let $\tilde{n}$ be the smallest index $\geq n$ such that this occurs, say the $\tilde{n}$th copy of v_1^2 lies on $R'(1)$. Then also the $\tilde{n}$th copy of v_0 lies on $R'(1)$, and furthermore, all other R_i (with the exception of $R(1)$) have to pass through the $\tilde{n}$th copies of the vertices $v_2^2, v_3^2, \ldots, v_\ell^2, v_1^3, v_2^3, \ldots, v_\ell^3$. Hence the total number of rays in $\mathcal{R}$ cannot exceed $2\ell + 1$, a contradiction, as $k = 2\ell + 2$. □

8.4 Connectivity of vertex-transitive graphs

In the next chapter, we shall se that large degrees at the vertices together with large vertex-/edge-degree at the ends ensure the existence of highly vertex-/edge-connected subgraphs. In this section, we shall already investigate the same problem for vertex-transitive graphs. As vertex-transitive graphs are regular, we need no longer use the term 'mimimum degree'. Thus our question reduces to the following in vertex-transitive graphs: Which degree at each vertex do we need in order to ensure that our graph has a k-(edge)-connected subgraph?

It is known that in finite graphs a degree of k is enough, and moreover the subgraph will be the graph itself. In fact, every finite vertex-transitive k-regular connected graph is k-edge-connected [64]. It is even k-connected, as long as it does not contain K^4 as a subgraph [62].

In infinite graphs, this is no longer true, if we only require degree k at the vertices, because of the trees. However, if we require a vertex-/edge-degree of at least k at the ends (which is conversely implied by the k-(edge)-connectivity, see below), we can obtain analogous results for infinite locally finite graphs. We may even drop the condition on the degrees of the vertices.

Proposition 8.4.1. **[83]** *Let G be an infinite locally finite graph, let $k \in \mathbb{N}$. Suppose that G is vertex-transitive and connected.*

(a) G is k-connected if and only if all ends of G have vertex-degree at least k.

(b) G is k-edge-connected if and only if all ends of G have edge-degree at least k.

In fact, the forward implications in Proposition 8.4.1 are easily implied by the following result, whose proof is not very difficult and can be found in [17] for the edge-case (the vertex-case is analogous).

Lemma 8.4.2. *Let $k \in \mathbb{N}$, let G be a locally finite graph, and let $\omega \in \Omega(G)$. Then*

(i) $d_v(\omega) = k$ if and only if k is the smallest integer such that every finite set $S \subseteq V(G)$ can be separated from ω with a k-separator, and

(ii) $d_e(\omega) = k$ if and only if k is the smallest integer such that every finite set $S \subseteq V(G)$ can be separated from ω with a k-cut.

Proof of Proposition 8.4.1. Because of Lemma 8.4.2 we only need to prove the backward implications. Let us only prove the implication for (a), for (b) this is analoguous.

Suppose the implication is not true, and let S be an ℓ-separator of G, for some $\ell < k$. Choose a vertex w at distance at least $\max\{dist(u, v) : u, v \in S\} + 1$ from all $v \in S$. (Observe that such a vertex w exists, since G is infinite, locally finite and connected.) Now, let ϕ be an automorphism of G that maps some vertex from S to w. Then $\phi(S)$ is contained in one component of $G - S$.

Next, choose an automorphism ϕ' that maps $\phi(w)$ 'far away' from $\phi(S)$ to a component of $G - \phi(S)$ that does not contain S. Continuing in this manner, we arrive at a sequence S, $\phi(S)$, $\phi'(\phi((S))$, $\phi''(\phi'(\phi(S)))$, ... of ℓ-separators of G. It is not difficult to construct a ray that meets each of these separators and hence defines an end of vertex-degree $\ell < k$. This contradicts our assumption that all ends have vertex-degree at least k. □

Chapter 9

Highly connected subgraphs of infinite graphs

9.1 The results

In this chapter, which is based on work from [86], we shall state and prove the results on highly connected subgraphs mentioned in the previous chapter. More precisely, we shall prove an infinite analogue of the following well-known theorem by Mader.

Theorem 9.1.1 (Mader [66]). *Any finite graph G of average degree at least $4k$ has a $(k+1)$-connected subgraph.*

As we already mentioned earlier, in infinite graphs it does not seem to be clear what an adequate concept of 'average degree' should be, and we will thus restrict ourselves to investigate the consequences of the (in finite graphs stronger) assumption of 'high minimum degree'.

But as we have seen, simply requiring high degree for the vertices is not enough, as the counterexample of the infinite r-regular tree T^r demonstrates. Now, since an infinite tree has rather 'thin' ends (which seem to play the role of the leaves of the infinite tree), this suggests, as conjectured by Diestel [30], that a minimum degree condition has to be imposed also on the ends of the graph.

In fact, if we require large vertex-degree at the ends, then T^r ceases to be a counterexample, as each of its ends has vertex-degree 1. And indeed, with this further condition on the vertex-degrees of the ends, highly connected subgraphs can be forced also in infinite graphs. As our main theorem in this chapter we prove the following infinite analogue:

Theorem 9.1.2. [86] *Let $k \in \mathbb{N}$ and let G be a graph such that each vertex has degree at least $2k(k+3)$, and each end has vertex-degree at least $2k(k+1)+1$. Then every infinite region of G has a $(k+1)$-connected region.*

Observe that while in Theorem 9.1.1, the bound on the degrees is linear in k, in Theorem 9.1.2 we require quadratic degree in k. This is in fact near to best possible:

Theorem 9.1.3. [86] *For each $k = 5\ell$, where $\ell \in \mathbb{N}$ is even, there exists a locally finite graph whose vertices have degree at least 2^ℓ and whose ends have vertex-degree at least $\ell \log \ell$, and which has no $(k+1)$-connected subgraph.*

In the previous chapter we also defined the *edge-degree*. It seems that the two concepts vertex-degree/edge-degree reflect for ends different aspects of the degree of a vertex. The vertex-degree of an end is the analogue of the size of the neighbourhood of a vertex, while the edge-degree corresponds to the number of incident edges.

This point of view suggests that for forcing highly (*vertex-*)connected subgraphs, high vertex-degree is a more natural requirement than high edge-degree. And in fact, it turns out that high edge-degrees at the ends and high degrees at the vertices together are not sufficient to force highly connected subgraphs, or even highly connected minors, in infinite graphs. In Section 9.4 we exhibit for all $r \in \mathbb{N}$ a locally finite graph of minimum degree and minimum edge-degree r that has no 4-connected subgraph and no 6-connected minor.

But, the assumption of high degree and high edge-degree does suffice to force highly edge-connected subgraphs in arbitrary graphs, with a lower bound on the (edge)-degrees that is only linear in k:

Theorem 9.1.4. [86] *Let $k \in \mathbb{N}$ and let G be a graph such that each vertex has degree at least $2k$ and each end has edge-degree at least $2k$. Then G has a $(k+1)$-edge-connected region.*

Moreover, highly edge-connected subgraphs can be found in every infinite region (Theorem 9.3.2).

In general, it is not possible to force *finite* highly vertex-/edge-connected subgraphs in infinite graphs by assuming high minimum degree and vertex- resp. edge-degree. Neither can one force *infinite* highly vertex- or edge-connected subgraphs. Counterexamples in this respect are provided in the discussion after Corollary 9.3.3, near the end of Section 9.3. However, any graph which obeys the (vertex-/edge)-degree bounds of Theorem 9.1.2 resp. Theorem 9.1.4, has either an infinite $(k+1)$-vertex-/edge-connected subgraph or infinitely many finite such (see Corollary 9.3.3/Corollary 9.5.2).

9.2 End degrees and more terminology

The terminology we use is standard, and can be found for example in [31]. A 1-way infinite path is called a *ray*, and the subrays of a ray are its *tails*. Two rays in a graph G are *equivalent* if no finite set of vertices separates them; the corresponding equivalence classes of rays are the *ends* of G. We denote the set of the ends of G by $\Omega(G)$.

Let H be a (possibly empty) subgraph of G, and write $H \subseteq G$. The *boundary* ∂H of H is the set $N(G - H)$ of all neighbours in H of vertices of $G - H$. We call H a *region* (of G) if H is a connected induced subgraph with finite boundary. Then $H' \subseteq H$ is a region of G if and only if it is a region of H.

Call a region H *profound*, if $V(H) \neq \partial H$. For example, all infinite regions are profound, and a profound region is not empty.

As in finite graphs, we call H *k-connected* for some $k \in \mathbb{N}$, if $|H| > k$ and no set of fewer than k vertices separates H. Similarly, H is *k-edge-connected* if $|H| > 1$ and no set of fewer than k edges separates H. Hence, if H is not k-edge-connected (and non-trivial), then it has a cut of cardinality less than k.

We shall consider two different extensions of the degree notion to ends. The *vertex-degree* (also known as the *multiplicity*, or *thickness*) of an end $\omega \in \Omega(G)$ is the maximum cardinality of a set of (vertex)-disjoint rays in ω. The *edge-degree* of ω (as suggested in [17]) is the maximum cardinality of a set of edge-disjoint rays in ω. It can be shown that these two degree concepts are well-defined, i.e. the considered maxima do indeed exist.[1]

9.3 Forcing highly edge-connected subgraphs

The main result of this section (Theorem 9.1.4) is that any graph G of large enough minimal degree and edge-degree contains a highly edge-connected subgraph, which in fact will be a region. We shall then see that such a region H can be found in any infinite region of G, and that there are either infinitely many such regions H, or one of infinite order.

Theorem 9.1.4 is best possible in the sense that high edge-degree is not

[1] Halin [49] proves the existence of an infinite set of disjoint rays if the number of disjoint rays in the considered graph is unbounded: with slight modifications, his proof yields the same result for rays of a fixed end. In [17], it is shown that the supremum of the cardinalities of sets of edge-disjoint rays in a given end is attained in locally finite graphs: this proof carries over similarly to arbitrary graphs.

sufficient to force highly connected subgraphs, as we shall see in the next section.

For the proof, we need the following lemma, which basically assures that if a graph contains some region with small cut to the outside world, then there is either a minimal such, or we have an infinite nested sequence of such regions so that their cuts are all disjoint.

Lemma 9.3.1. *Let $D \neq \emptyset$ be a region of a graph G so that $|E(D, G-D)| < m$ and so that $|E(D', G-D')| \geq m$ for every non-empty region $D' \subseteq D - \partial D$ of G. Then there is an inclusion-minimal region $H \subseteq D$ with $|E(H, G-H)| < m$ and $H \neq \emptyset$.*

Proof. If there is no such H, then we can construct an infinite sequence of distinct regions $D =: D_0 \supsetneq D_1 \supsetneq D_2 \supsetneq \ldots$ such that all cuts $F_i := E(D_i, G - D_i)$ have cardinality less than m. Note that any edge that lies in some F_i, but not in F_{i+1}, lies outside $E(D_{i+1}) \cup F_{i+1}$, and hence will not appear in any F_j with $j > i$.

By assumption, every region $D' \subseteq D$ which is not incident with any edge of F_0, sends at least m edges to the outside. Thus, there is an edge e in F_0 that appears in all F_i for $i \geq 0$. Let E be the set of all edges e_j for which there exists an index j such that $e_j \in F_i$ for all $i \geq j$. Clearly, $e \in E$, and $|E| < m$, where the latter follows from the boundedness of the cuts F_i.

Let n be so that $E \subseteq F_n$. Now, as $D_{n+1} \subsetneq D_n$, there is a vertex $x \in V(D_n - D_{n+1})$. Since D_n is connected, it contains a (finite) path P that connects x with y, the endvertex of e in D_n. All D_i with $i > n$ contain y, but not x, thus each F_i with $i > n$ must contain one of the edges on P. This implies that there is an edge e_j on P which for some $j > n$ lies in all F_i with $i \geq j$. Thus, $e_j \in E \subseteq F_n$, but $F_n \cap E(P) = \emptyset$, a contradiction. □

We now prove Theorem 9.1.4.

Proof of Theorem 9.1.4. First of all, we shall show that there exists a region $C \neq \emptyset$ such that

(a) $|E(C, G - C)| < 2k$, *and*

(b) *for every non-empty region $C' \subseteq C - \partial C$ we have that* $|E(C', G - C')| \geq 2k$.

Indeed, let us construct a sequence $C_0 \supseteq C_1 \supseteq C_2 \ldots$ of non-empty regions such that for $i \geq 0$ the following hold

(i) $|E(C_i, G - C_i)| < 2k$, and

(ii) $C_{i+1} \subseteq C_i - \partial C_i$.

Choose C_0 as any component of G. Now, if after finitely many, say j, steps of our construction we cannot go on, i.e. find a suitable C_{j+1}, it is because C_j has property (b). Property (a) is then ensured by (i).

So assume that we end up with an infinite sequence $C_0, C_1, C_2, \ldots$ of regions. Observe that, since C_0 is a region, $\partial C_i \neq \emptyset$ for each $i \geq 1$. As each of the C_i is connected, there is a sequence $(P_i)_{i\in\mathbb{N}}$ of ∂C_i–∂C_{i+1} paths such that for $i \geq 1$ the path P_{i+1} starts in the last vertex of P_i. By (ii), the paths P_i are non-trivial, and by construction, each P_i meets P_{i-1} and P_{i+1} only in its first respectively last vertex, and is disjoint from all the other P_j. Hence, their union $P := \bigcup_{i=1}^{\infty} P_i$ is a ray which has a tail in each of the C_i.

Let ω be the end of G that contains P. As, by assumption, ω has edge-degree at least $2k$, there is a family $\mathcal{R}$ of $2k$ edge-disjoint ω-rays in G. For each ray $R \in \mathcal{R}$ let n_R denote the distance its starting vertex has to ∂C_1. Set $n := \max\{n_R : R \in \mathcal{R}\} + 2$. Then by (ii), all of the $2k$ disjoint rays in $\mathcal{R}$ start outside C_n. But each ray in $\mathcal{R}$ is equivalent to P, and hence eventually enters C_n, contradicting the fact that $|E(C_n, G - C_n)| < 2k$ by (i). This proves the existence of a region $C \neq \emptyset$ with the properties (a) and (b).

Thus, Lemma 9.3.1 yields an inclusion-minimal non-empty region $H \subseteq C$ with $|E(H, G - H)| < 2k$. We claim that H is the desired $(k + 1)$-edge-connected region of G. In fact, otherwise the bound on the degrees of the vertices of G implies that $|H| \geq 2$, and so, H has a cut F with $|F| \leq k$. We may assume that F is a minimal cut, i.e. splits H into two (non-empty) regions H' and H''. For one of the two, say H', the cut $E(H', G - H')$ meets $E(H, G - H)$ in at most $\frac{|E(H,G-H)|}{2} < k$ edges. Hence

$$|E(H', G - H')| \leq |E(H', G - H') \cap E(H, G - H)| + |F| < 2k$$

and $H' \subsetneq H$, contradicting the minimality of H. □

Note that our proof yields a $(k+1)$-edge-connected region in every region C of G with $|E(C, G - C)| < 2k$ (simply start with $C_0 := C$ instead of taking any component of G). With slightly more effort (and slightly higher edge-degree), one can prove that every *infinite* region of G contains a $(k+1)$-edge-connected region:

Theorem 9.3.2. *Let $k \in \mathbb{N}$, and let G be a graph such that each vertex has degree at least $2k$, and each end has edge-degree at least $2k + 1$. Then every infinite region of G contains a $(k + 1)$-edge-connected region.*

Proof. Let D be an infinite region of G. If there is a region $D' \subseteq D$ with $|E(D', G - D')| \leq 2k$ and $D \neq \emptyset$, then we proceed as in the proof of Theorem 9.1.4 to find an inclusion-minimal non-empty region H with this property, which then turns out to be the desired $(k+1)$-edge-connected region.[2]

So, we can assume that D contains no non-empty region which sends less than $2k+1$ edges to the outside. Now, let $H \subseteq D$ be an infinite region with $|E(H, G-H)|$ minimal. If we can prove H to be $(k+1)$-edge-connected, we are done. But otherwise there is a cut F with $|F| \leq k$ that splits H into two regions H' and H''. At least one of these, say H', is infinite. By the choice of H, the number of edges H' sends to the rest of the graph is at least $|E(H, G - H)|$; hence, $E(H', G - H')$ contains all but at most $|F|$ edges of $E(H, G - H)$. Thus,

$$|E(H'', G - H'')| = |F \cup (E(H, G - H) - E(H', G - H'))| \leq 2k,$$

contradicting our assumption on D. □

Theorem 9.3.2 has two interesting corollaries.

Corollary 9.3.3. *Let $k \in \mathbb{N}$ and let G be a graph in which all vertices have degree at least $2k$, and all ends have edge-degree at least $2k+1$. Then any infinite region C of G has either infinitely many disjoint finite $(k+1)$-edge-connected regions, or an infinite $(k+1)$-edge-connected region.*

Proof. Take an inclusion-maximal set $\mathcal{D}$ of disjoint finite $(k+1)$-edge-connected regions of C (which exists by Zorn's Lemma), and assume that $|\mathcal{D}| < \infty$. Since any infinite component of $C' := C - \bigcup_{D \in \mathcal{D}} D \subseteq C$ is an infinite region of G, we may use Theorem 9.3.2 to obtain a $(k+1)$-edge-connected region H of C. Then H is infinite by the choice of $\mathcal{D}$. □

The two configurations of Corollary 9.3.3 of which one necessarily appears in any given graph of large enough minimal (edge)-degree, need not both exist, not even in locally finite graphs. Indeed, for given $r \in \mathbb{N}$, it is easy to construct an infinite locally finite graph G which has minimum degree and vertex- (and thus edge)- degree r but no infinite 3-edge-connected subgraph. We obtain G from the $r \times \mathbb{N}$ grid by joining each vertex to r disjoint copies of K^{r+1}. Any infinite subgraph of G which is at least 2-edge-connected is also a subgraph of the $r \times \mathbb{N}$ grid, and hence is at most 2-edge-connected.

[2] Observe that, starting with $|E(D', G-D')| \leq 2k$ instead of $< 2k$, we will have to adjust our inequalities, and the final contradiction is obtained by finding that $|E(H', G - H')| \leq 2k$.

On the other hand, there are also locally finite graphs of high minimum degree and vertex-degree that have no finite highly edge-connected subgraphs. To see this, we reuse an example from the introduction: for given $r \in \mathbb{N}$, add some edges to each level S_i of the r-regular tree T^r so that in the obtained graph $\tilde{T}^r$ each S_i induces a path. The only end of $\tilde{T}^r$ has infinite vertex- and edge-degree, and the vertices of $\tilde{T}^r$ have degree at least r. Now, for every finite subgraph H of $\tilde{T}^r$ there is a last level of $\tilde{T}^r$ that contains a vertex v of H. Then v has degree at most 3 in H, and hence, H is not 4-edge-connected.

Our second corollary of Theorem 9.1.4 describes how the graph G decomposes into subgraphs that either are highly edge-connected or are so that all their subgraphs send many edges to the outside. For this, we have to push the lower bound on the degree of the vertices a little:

Corollary 9.3.4. *Let $k \in \mathbb{N}$, and let G be a graph whose vertices have degree at least $4k+1$, and whose ends have edge-degree at least $2k+1$. Then there is a set $\mathcal{D}$ of disjoint $(k+1)$-edge-connected regions of G such that $|E(H, G - H)| \geq \max\{4k, |H|\}$ for each non-empty subgraph H of $G - \bigcup_{D \in \mathcal{D}} D$.*

For the proof, we need the following lemma:

Lemma 9.3.5. *Let $m \in \mathbb{N}$ and let G be a graph such that each of its vertices has degree at least m. Then every non-empty region H of G with $|E(H, G - H)| < m$ contains at least $m + 1$ vertices.*

Proof. We may assume that $m > 1$. Now, we can estimate the number of edges of H in two ways. On one hand,

$$\|H\| \geq \frac{m|H| - |E(H, G - H)|}{2} > \frac{m}{2}(|H| - 1),$$

as by assumption, each vertex of H has degree at least m in G. On the other hand, H cannot have more edges than the complete graph on $|H|$ vertices. This leaves us with the inequality $\frac{m}{2}(|H| - 1) < \frac{|H|(|H|-1)}{2}$, implying that $|H| > m$. $\square$

Proof of Corollary 9.3.4. Let $\mathcal{D}$ be an inclusion-maximal set $\mathcal{D}$ of disjoint $(k+1)$-edge-connected regions of G (which again exists by Zorn's Lemma).

Observe that it suffices to show $|E(H, G - H)| \geq \max\{4k, |H|\}$ for induced connected non-empty subgraphs H of $G - \bigcup_{D \in \mathcal{D}} D$, and consider such an H. If H is infinite, then Theorem 9.3.2 and the (maximal) choice of $\mathcal{D}$ imply that H is not a region of G, i.e. that $|E(H, G - H)|$ is infinite, as desired.

So assume that H is finite. In the case that $|H| < 4k$, Lemma 9.3.5 ensures that $|E(H, G-H)| \geq 4k$. In the case that $|H| \geq 4k$, suppose that $|E(H, G-H)| < |H|$. Then, H has average degree $d(H) \geq \delta_V - 1 \geq 4k$, and hence H has a k-edge-connected subgraph by Theorem 9.1.1, contradicting the choice of $\mathcal{D}$. □

9.4 High edge-degree but no highly connected subgraphs

In this section, we shall show that with high edge-degree in the ends and high degree in the vertices we cannot ensure the existence of highly connected subgraphs. Indeed, even highly connected minors need not be present.

More precisely, for given $r \in \mathbb{N}$ we will construct a locally finite graph G_r of minimum degree r at the vertices and minimum edge-degree at least r at the ends that has no 4-connected subgraph and no 6-connected minor. The idea is to 'thicken' the ends of the tree T_r, in the sense of augmenting their edge-degree, which we do by adding many edges but only a few vertices in order to keep the separators small.

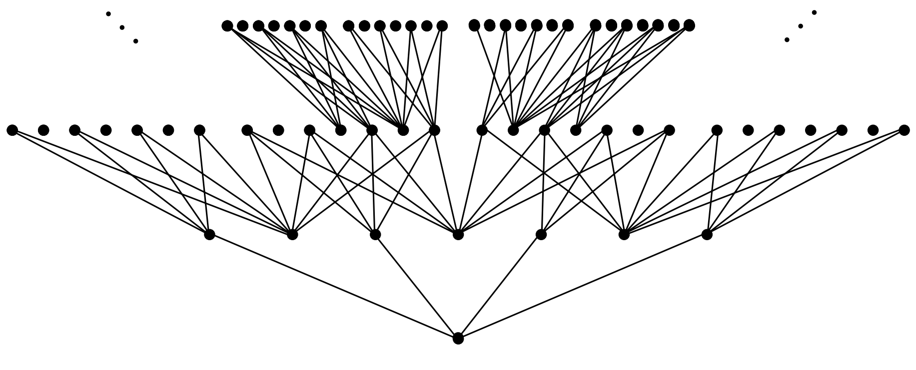


Figure 9.1: The graph G_4.

We start with an infinite rooted tree T_r in which each vertex sends r edges to the next level. The graph G_r will be obtained from T_r in the following manner. Let S_0 consist of the root r_0 of T_r and for $i \geq 1$ denote by S_i the i-th level of T_r. Now, successively for $i \geq 1$, we shall add some vertices to S_i, which results in an enlarged ith level S_i', and then add some edges between $S_i' - S_i$ and S_{i+1}.

For each vertex $x \in S_{i-1}$, add $r-1$ new vertices $v_1^x, v_2^x, \dots, v_{r-1}^x$ to its neighbourhood $S^x = \{s_1^x, s_2^x, \dots, s_r^x\}$ in S_i. Denote by S_i' the set thus obtained from S_i.

Then for each $j \leq r-1$ and each $x \in S_{i-1}$ add all edges between v_j^x and $N_{S_{i+1}}(\{s_j^x, s_{j+1}^x\})$. This yields a graph G_r on the disjoint union of sets $S_1', S_2', \dots$ as depicted in Figure 9.1 for $r = 4$.

The ends of G_r correspond to the ends of the underlying tree T_r, i.e. every two disjoint rays in T_r belong to different ends of G_r, and each end of G_r contains a ray from T_r. Indeed, two rays from T_r which in T_r are separated by $\bigcup_{i=0}^{j} S_i$ for some $j \in \mathbb{N}$, can be separated in G_r by the set $\bigcup_{i=0}^{j+1} S_i$. On the other hand, every ray $R \subseteq G_r$ has, for any fixed $j \in \mathbb{N}$, a tail in exactly one of the components of $G - \bigcup_{i=0}^{j} S_i$. This tail meets S^{x_j}, for some $x_j \in S_j$. Hence R is equivalent to the ray $x_0 x_1 x_2 \ldots \subseteq T_r$.

Lemma 9.4.1. *G_r has minimum degree r at the vertices and minimum edge-degree at least r at the ends.*

Proof. The definition of G clearly ensures the desired degree at the vertices. We show that the ends of G_r have edge-degree at least r by constructing a set of r edge-disjoint rays in each. Given an end ω of G_r, there is exactly one ray $R = r_0 r_1 r_2 \ldots \subseteq T_r$ in it (since the ends of G_r correspond to those of T_r, as remarked above).

Now, construct $r-1$ edge-disjoint ω-rays R_i, where $i = 1, \dots, r-1$; these will also be edge-disjoint from R. Each R_i starts in r_0, its second vertex is the ith neighbour of r_0 in S_1 which is unused by R. Next, it goes along some path that switches between S_2 and S_1' until it reaches r_2. Note that we can choose these paths edge-disjoint for different i, for example by letting R_i use only vertices $v \in S_2$ with $v = s_i^x$ for some $x \in S_1$ (or $v = r_2$). Similarly, we continue the R_i going from r_2 to the ith unused neighbour in S_3, and from there along edge-disjoint paths to r_4, and so on. Since the R_i agree on $r_0, r_2, r_4 \ldots$, they all belong to ω. □

Observe that every finite set A of vertices can be separated from any end ω by at most three vertices (namely by the neighbours of the unique component of $G_r - S_i'$ that contains a ray in ω, where j is large enough so that $A \subseteq \bigcup_{i=0}^{j} S_i'$). Hence, each end of G_r has vertex-degree at most 3.

In fact, Theorem 9.1.2 ensures that every graph of high minimum degree (at the vertices) has either an end of small vertex-degree or a highly connected subgraph. We shall see now that the latter is not the case for G_r.

Lemma 9.4.2. *G_r has no 4-connected subgraph.*

Proof. Suppose otherwise, and let H be a 4-connected subgraph of G. Let $i \in \mathbb{N}$ so that $V(H) \cap S'_i \neq \emptyset$. Now, if there is a vertex $v \in V(H) - \bigcup_{j=0}^{i+1} S'_i$, then it can be separated in G_r (and thus also in H) from $V(H) \cap S'_i$ by at most three vertices, namely by the neighbours of the component of $G_r - S'_{i+1}$ that contains v. So, as H is 4-connected, $V(H) - \bigcup_{j=0}^{i+1} S'_i$ must be empty. Then, there is a maximal $j \in \mathbb{N}$ such that $V(H) \cap S'_j \neq \emptyset$. But then by construction of G_r, any vertex in $V(H) \cap S'_j$ has degree at most three in H, contradicting the 4-connectedness of H. □

It is only slightly more difficult to prove that G_r has no highly connected minor:

Lemma 9.4.3. *G_r has no 6-connected minor.*

Proof. Suppose that G_r has a 6-connected minor M. Then there is an $n \in \mathbb{N}$ so that each branch-set of M has a vertex in $\bigcup_{i=0}^{n} S'_i$. Since M is 6-connected, each separator $T \subseteq \bigcup_{i=0}^{n} S'_i$ of G_r with $|T| \leq 5$ leaves a component C of $G_r - T$ such that $V(C) \cup T$ meets one and hence every branch-set of M. So as each S'_i can be separated in G_r from any component of $G - S'_i$ by at most three vertices, there is an $i < n$ such that each branch-set of M meets $S'_i \cup S'_{i+1}$. Moreover, there is a vertex $x \in S_i$ such that for $S := N_{S_{i+1}}(x)$ we have that each branch-set of M has a vertex in $S' := S \cup N_{S'_i}(S) \cup \{v_1^x, v_2^x, \ldots, v_{r-1}^x\}$. Then $|S' \cap S'_i| \leq 3$.

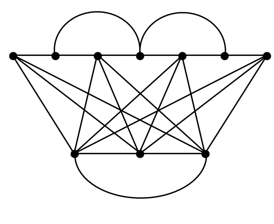


Figure 9.2: The graph G'_4 for $|S' \cap S'_i| = 3$.

We claim that M is also a minor of the finite graph G'_r (see Figure 9.2) which is obtained from $G_r[S']$ by adding an edge between every two vertices that are neighbours of the same component of $G_r - S'$. Indeed, each component C of $G_r - S'$ has at most three neighbours in S'. Hence, since M is 6-connected, C meets only (if at all) those branch-sets of M that also meet $N_{S'}(C)$. It is easy to see that M is still a minor of the graph we obtain from G_r by deleting C and adding all edges between vertices in $N_{S'}(C)$. Arguing analogously for the other components of $G_r - S'$, we see that M is also a minor of G'_r.

As $|S' \cap S'_i| \leq 3$, all but at most 3 branch-sets of M in G'_r have all their vertices in $|S' \cap S'_{i+1}|$. Then these give rise to a 3-connected minor of $G'_r - S'_i$. But each non-trivial block of $G'_r - S'_i$ is a triangle and hence has no 3-connected minor, yielding the desired contradiction. □

Note that the two latter results are best possible, since G_r has a 3-connected subgraph, the complete graph on 4 vertices, and a 5-connected minor, the complete graph on 6 vertices.

9.5 Forcing highly connected subgraphs

In this section, we shall prove our main result, Theorem 9.1.2. At first, we shall proceed similarly[3] as in the proof of Theorem 9.1.4 (resp. Theorem 9.3.2), until we arrive at an infinite region $C' \subseteq C$ with the property that $|\partial H| \geq \delta_\Omega$ holds for all regions $H \subsetneq C'$. This is achieved in Lemma 9.5.1 below.

But then, we see ourselves confronted with new difficulties. The region C' need not be highly connected, or even 2-connected; the reason is that we lost control on the degrees of the vertices in C'. (And the situation only changes for the worse if instead of C' we consider $C' - \partial C'$, which needs be neither connected, nor a region.)

Hence, we shall prefer a region $H \subseteq C'$ over C', if the vertices of ∂H have 'much' higher degree in H than those of $\partial C'$ have in C', even if ∂H has 'slightly' greater cardinality than $\partial C'$. This will be formalised below.

Our combination of measurements on the suitability of H, on one hand $|\partial H|$, and on the other, $d_H(\partial H)$, is responsible for the quadratic lower bounds on the degrees which this proof of Theorem 9.1.2 yields. We shall see in Section 9.6 that these bounds are indeed close to best possible.

Lemma 9.5.1. *Let G be a graph such that all its ends have vertex-degree at least $\delta_\Omega \in \mathbb{N}$. Let C be an infinite region of G. Then there exists a profound region $C' \subseteq C$ for which one of the following holds:*

[3]The situation here is a little more complicated, because a vertex-version of Lemma 9.3.1, replacing 'edges' with 'vertices' and 'cuts' with 'separators', fails, unless we make use of the high vertex-degree assumed in Theorem 9.1.2. To see this, take a ray $v_0v_1v_2\ldots$, to which we add all edges v_0v_i, and consider the region D which consists of all v_i but v_1.

(a) C' is finite and $|\partial C'| < \delta_\Omega$, or

(b) C' is infinite and $|\partial C''| \geq \delta_\Omega$ for every profound region $C'' \subsetneq C'$.

Proof. Suppose otherwise. Then in particular, (b) does not hold for C, i.e. there is a profound region $C_1 \subsetneq C$ with $|\partial C_1| < \delta_\Omega$. We shall construct a sequence $C_1 \supsetneq C_2 \supsetneq \ldots$ of profound regions of C so that $|\partial C_n| < \delta_\Omega$ for all $n \geq 1$.

So assume the region C_n with small boundary has been found. We may assume C_n to be such that $C_n - \partial C_n$ is connected and also, that $N(C_n - \partial C_n) = \partial C_n$. Indeed, if C_n is not so, we can take any component H of $C_n - \partial C_n$ (such a component exists, as C_n is profound) and add $N(H) \subseteq \partial C_n$. Then $H \subseteq C_n$ is a profound region with the desired small boundary which we may use instead of C_n.

As we suppose that (a) does not hold for C_n, we can assume that C_n is infinite. Since C_n is not as required in (b), there is a profound region $C_{n+1} \subsetneq C_n$ with $|\partial C_{n+1}| < \delta_\Omega$. In this manner, we obtain an infinite sequence $(C_n)_{n\in\mathbb{N}}$.

Denote by V the (possibly empty) set of those vertices v that from some $j \in \mathbb{N}$ on appear in all ∂C_n with $n \geq j$. Since all ∂C_n have size at most δ_Ω, the set V is finite. Furthermore, as $C_{n+1} \subsetneq C_n$ for every $n \geq 1$, and we chose the C_n so that $C_n - \partial C_n$ is connected, we have that $\partial C_n = V$ for at most one $n \in \mathbb{N}$. Let J be such that $V \subsetneq \partial C_n$ for all $n \geq J$.

Observe that for each $w \in \partial C_J - V$ there is an index j such that $w \notin \partial C_n$ for all $n \geq j$. Hence, there is an index J' so that $\partial C_J \cap \partial C_{J'} = V$. Set $C'_1 := C_J$, and set $C'_2 := C_{J'}$. Continuing in this manner, we arrive at an infinite subsequence $C'_1 \supsetneq C'_2 \supsetneq \ldots$ of profound regions of C, whose boundaries pairwisely meet only in V. Let us summarize the properties which the regions C'_n have, for each $n \in \mathbb{N}$:

(i) $|\partial C'_n| < \delta_\Omega$,

(ii) $C'_n - \partial C'_n$ is connected, and $N(C'_n - \partial C'_n) = \partial C'_n$,

(iii) $V \subsetneq \partial C'_n$, and

(iv) $\partial C'_{n+1} \subseteq (C'_n - \partial C'_n) \cup V$.

We claim that there is a ray R that has a tail in each C'_n. Indeed, by (ii) and (iii), there exists for each $n \in \mathbb{N}$ a $(\partial C'_n - V)$–$(\partial C'_{n+1} - V)$ path P_n such that each P_{n+1} starts in the last vertex of P_n. By (iv), the paths P_i are non-trivial, hence, their union is the desired ray R. Denote by ω the end of G that contains R.

As, by assumption, ω has vertex-degree at least δ_Ω, there is a set $\mathcal{R}$ of δ_Ω disjoint ω-rays in G. The starting vertices of the rays in $\mathcal{R}$ either lie outside $C_1' - \partial C_1'$, or have in $C_1' - V$ a finite distance to $\partial C_1' - V$. Hence, by (iv), there is an $N \in \mathbb{N}$ so that all rays of $\mathcal{R}$ start outside $C_N' - \partial C_N'$. But (being equivalent to R) each of these disjoint rays eventually enters $C_N' - \partial C_N'$, and thus meets $\partial C_N'$, a contradiction because by (i), $|\partial C_N'| < \delta_\Omega$. This completes the proof of Lemma 9.5.1. □

We are now ready to prove the main result of this chapter.

Proof of Theorem 9.1.2. Given an infinite region C of G, we shall find a $(k+1)$-connected region $H \subseteq C$. Theorem 9.1.2 obviously holds for $k = 1$, since the ends of a tree have vertex-degree $1 < \delta_\Omega$. We can thus assume that $k > 1$.

Suppose there exists a profound finite region $D \subseteq C$ with $|\partial D| < \delta_\Omega$. Then $D - \partial D$ has minimum degree at least $\delta(D - \partial D) \geq \delta_V - \delta_\Omega + 1 = 4k$. Hence Theorem 9.1.1 yields a finite $(k+1)$-connected subgraph of $D \subseteq C$, and we are done. Let us therefore assume that there is no such region D.

We may thus apply Lemma 9.5.1 to obtain an infinite region $C' \subseteq C$ with the property that

$$|\partial C''| \geq \delta_\Omega \textit{ for every profound region } C'' \subsetneq C'. \tag{9.1}$$

For a region $H \subseteq C'$ write

$$\Sigma_H := \sum_{v \in V(H)} \max\{0, \delta_V - d_H(v)\}.$$

Observe that this sum is finite, since all vertices of H but the finitely many in ∂H have degree at least δ_V in H. Now, choose an infinite region $H \subseteq C'$ such that $(k+3)|\partial H| + \Sigma_H$ is minimal.

Assume that there is a vertex $v \in V(H)$ that has degree at most $2k-3$ in H. Then clearly, $v \in \partial H$. Observe that $d_{H-v}(w) = d_H(w) - 1$ for each of the at most $2k-3$ neighbours w of v in H, and $d_{H-v}(w') = d_H(w')$ for all other vertices w' in H. Therefore,

$$\begin{aligned}(k+3)|\partial(H-v)| + \Sigma_{H-v} &\leq (k+3)|\partial H| + (k+3)(2k-4)\\ &\quad + \Sigma_H + (2k-3) - (\delta_V - d_H(v))\\ &< (k+3)|\partial H| + \Sigma_H + 2k(k+3) - \delta_V\\ &= (k+3)|\partial H| + \Sigma_H.\end{aligned}$$

So any infinite component of $H-v$ is a better choice than H, a contradiction. We thus have shown that

$$d_H(v) \geq 2(k-1) \textit{ for each vertex } v \in V(H). \tag{9.2}$$

Let us prove now that H is the desired $(k+1)$-connected region of C. Indeed, suppose otherwise. Then H has a separator T of cardinality at most k, which we may assume to be a minimal separator. Note that each such separator leaves a component D of $H-T$ such that $H' := H-D$ is an infinite region of C.

Suppose that $|V(D) \cap \partial H| \geq \delta_\Omega - |T|$. Then we obtain for the infinite region $H' \subseteq C'$ that

$$\begin{aligned} |\partial H'| &= |(\partial H - V(D)) \cup T| \\ &\leq |\partial H| - |V(D) \cap \partial H| + |T| \\ &\leq |\partial H| - \delta_\Omega + 2k. \end{aligned}$$

Furthermore,

$$\begin{aligned} \Sigma_{H'} &\leq \Sigma_H + \sum_{v \in T} \max\{0, \delta_V - d_{H'}(v)\} \\ &\leq \Sigma_H + k\delta_V, \end{aligned}$$

and so

$$\begin{aligned} (k+3)|\partial H'| + \Sigma_{H'} &\leq (k+3)|\partial H| - (k+3)(\delta_\Omega - 2k) + \Sigma_H + k\delta_V \\ &< (k+3)|\partial H| - 2k^2(k+3) + \Sigma_H + 2k^2(k+3) \\ &= (k+3)|\partial H| + \Sigma_H, \end{aligned}$$

contradicting the choice of H.

Hence,

$$|V(D) \cap \partial H| < \delta_\Omega - |T|.$$

Thus for the region $\tilde{D} := G[V(D) \cup T] \subseteq C'$, we have

$$|\partial \tilde{D}| = |(V(D) \cap \partial H) \cup T| \leq |V(D) \cap \partial H| + |T| < \delta_\Omega.$$

Observe that $\tilde{D} \neq H$. So, by (9.1), the region $\tilde{D}$ is not profound, i.e. $V(\tilde{D}) = \partial \tilde{D}$, implying that $V(D) \subseteq \partial H$. In particular, $|D| < \delta_\Omega - |T|$. Now, for any vertex $v \in V(D)$, we can estimate its degree in H as follows.

$$d_H(v) \leq |(D \cup T) - \{v\}| < \delta_\Omega - 1 = \delta_V - 4k.$$

Then $\delta_V - d_H(v) > 4k$, implying that

$$\begin{aligned}\Sigma_{H'} &\leq \Sigma_H - \sum_{v \in V(D)} \max\{0, \delta_V - d_H(v)\} + \sum_{v \in T} (d_H(v) - d_{H'}(v)) \\ &< \Sigma_H - 4k|D| + |T||D| \\ &\leq \Sigma_H - 3k.\end{aligned}$$

On the other hand, (9.2) ensures that $|D| \geq 1 + 2(k-1) - |T| \geq k - 1$. So

$$|\partial H'| \leq |\partial H| - |D| + |T| \leq |\partial H| + 1,$$

and thus (as $k > 1$ by assumption)

$$\begin{aligned}(k+3)|\partial H'| + \Sigma_{H'} &< (k+3)|\partial H| + (k+3) + \Sigma_H - 3k \\ &\leq (k+3)|\partial H| + \Sigma_H,\end{aligned}$$

again contradicting the choice of H. □

We finish this section with two corollaries of Theorem 9.1.2. The proof of the first is analogous to that of Corollary 9.3.3.

Corollary 9.5.2. *Let $k \in \mathbb{N}$ and let C be an infinite region of a graph G of minimum degree at least $2k(k+3)$ at the vertices and minimum vertex-degree at least $2k(k+1)+1$ at the ends. Then C has either infinitely many disjoint finite $(k+1)$-connected regions or an infinite $(k+1)$-connected region.*

Again, these two configurations need not both exist, as the examples following Corollary 9.3.3 illustrate. (Observe that if a graph has no k-edge-connected subgraph then it clearly has no k-connected subgraph.)

The second corollary of Theorem 9.1.2 is an analogue of Corollary 9.3.4.

Corollary 9.5.3. *Let $k \in \mathbb{N}$, and let G be a graph whose vertices have degree at least $\delta_V = 2k(k+3)$ and whose ends have vertex-degree at least $\delta_\Omega = 2k(k+1)+1$. Then there is a set $\mathcal{D}$ of disjoint $(k+1)$-connected regions of G such that $|\partial H| \geq \max\{\delta_\Omega, \frac{k-2}{k}|H|+1\}$ for each profound subgraph H of $G - \bigcup_{D \in \mathcal{D}} D$.*

Proof. Similarly as in the proof of Corollary 9.3.4, take an inclusion-maximal set $\mathcal{D}$ of disjoint $(k+1)$-connected regions of G, and observe that we only need to consider induced connected profound subgraphs H of $G - \bigcup_{D \in \mathcal{D}} D$. So let H be a such. If H is infinite, then Theorem 9.1.2 and the choice of $\mathcal{D}$ imply that H is not a region, i.e. that $|\partial H|$ is infinite, as desired.

So assume that H is finite. Then $|\partial H| \geq \delta_\Omega$, as otherwise $H - \partial H$ has minimum degree $\delta(H - \partial H) \geq \delta_V - \delta_\Omega + 1 \geq 4k$, and hence H has a $(k+1)$-connected subgraph by Theorem 9.1.1, contradicting the choice of $\mathcal{D}$.

Also, $|\partial H| > \frac{k-2}{k}|H|$. Indeed, suppose otherwise. Then H has average degree

$$\begin{aligned} d(H) &\geq \frac{\delta_V|H - \partial H| + |\partial H|}{|H|} \\ &\geq \delta_V - (\delta_V - 1)\frac{|\partial H|}{|H|} \\ &\geq \frac{2\delta_V + k - 2}{k} \\ &\geq 4k. \end{aligned}$$

Thus again, Theorem 9.1.1 yields a $(k+1)$-connected subgraph of H, a contradiction to the choice of $\mathcal{D}$. □

9.6 Linear degree bounds are not enough

Unlike in Mader's original theorem, and in Theorem 9.1.4, the bounds on the degrees and vertex-degrees we require in Theorem 9.1.2 are quadratic in k. It seems that our method of proof cannot yield better bounds, because the region H we find has to be best possible in two ways: small boundary on one hand, high in-degree of its vertices on the other. But the quadratic bounds we give are in fact not far from best possible: a minimum degree and vertex-degree only linear in k is insufficient to ensure $(k+1)$-connected subgraphs.

Proof of Theorem 9.1.3. Set $m := \lceil \log \ell \rceil$. The vertex set of our graph G will be that of a tree T, which is rooted in v_0. The root has 2ℓ neighbours in the first level S_1 of T, and for $i \geq 1$ each vertex in the ith level S_i sends two edges to the next level S_{i+1}. Set $S_0 := \{v_0\}$. Let $\leq$ be the order induced by the tree T on the vertex set $V(G) = V(T)$, that is, $x \leq y$ for $x, y \in V(G)$ if and only if x lies on the unique v_0–y path in T.

Now, for each $i \geq 0$ and each $x \in S_i$, add to T all edges xy, where $y \in S_{i+\ell}$ and $y \geq x$. Note that each $x \neq v_0$ has exactly 2^ℓ such 'new neighbours' y (while v_0 has at least that many). Hence, in the thus obtained graph G', each vertex v has degree $d_{G'}(v) \geq 2^\ell$.

In order to achieve a high vertex-degree in the ends of the graph, we shall add a few more edges to G'. For this, let us have a closer look at

T. For $j \in \mathbb{N}$, we inductively define sets $S^{(s)} \subseteq S_j$ for each 01-string s of length $j \geq 1$. Divide S_1 arbitrarily into two sets $S^{(0)}, S^{(1)}$ of equal size ($= \ell$). Then for each $j \geq 2$, and for each 01-string s of length $j-1$, partition the neighbourhood of $S^{(s)}$ in S_j into two sets $S^{(s0)}, S^{(s1)}$ of equal size ($= \ell$), in a way that the neighbourhoods of $S^{(s0)}$ and $S^{(s1)}$ in S_{j-1} are disjoint. Then S_j is the disjoint union of all $S^{(s)}$, where s varies over all 01-strings of length j. Now, for each 01-string s of any length, and for each 01-string t of length m match $S^{(s0)}$ with $S^{(s1t)}$, and match $S^{(s1)}$ with $S^{(s0t)}$.

This yields a graph G, which we claim to have the desired properties. Indeed, we have seen that already in G' the vertices have the required degree. Let us now investigate the end structure of G.

We claim that G does not have 'more' ends than T, i.e. every end of G contains a ray from T. Indeed, consider a ray R of G: we shall show that there is a ray in T which is equivalent to R. Let C_0 be the (unique) component of $G - \bigcup_{i=1}^{m+1} S_i$ that contains a tail of R. There is a path P_0 in T that connects this tail with the unique vertex $x_0 \in S_1$ for which $x_0 \leq c$ for all $c \in C_0$. Now, choose j large enough so that $V(P_0) \subseteq \bigcup_{i=0}^{j-1} S_i$, and let C_1 be the component of $G - \bigcup_{i=1}^{j+m} S_i$ that contains a tail of R. Again, there is a path P_1 in T that connects this tail with the unique vertex $x_1 \in S_j$ which satisfies $x_1 \leq c$ for all $c \in C_1$.

Continuing in this manner, we obtain an infinite set of disjoint paths P_i, where each P_i connects x_i with $V(R)$. Clearly, since $x_0 \leq x_1 \leq x_2 \leq \ldots$, there is a ray R' in T that contains all vertices x_i. The ray R' cannot be finitely separated from R, and thus is equivalent to R. Hence, every end of G contains a ray of T, as desired.[4]

Next, let us show that the ends of G have vertex-degree at least ℓm. Given an end $\omega \in \Omega(G)$, and a ray $R = v_0 v_1 v_2 \ldots \in \omega$, with $R \subseteq T$ and $v_i \in S_i$ for all i, we shall find a set of disjoint rays $R^i_j \in \omega$, where $i = 1, 2, \ldots, m$; and $j = 1, 2, \ldots, \ell$. These rays will exclusively use edges from $E(G) - E(G')$. Let $s = s_1 s_2 s_3 \ldots$ be a 01-string of infinite length so that $v_n \in S^{(s_1 s_2 \ldots s_n)}$ for each $n \geq 1$. Denote by $S(n)$ the set $S^{(s_1 s_2 \ldots s_{n-1}(1-s_j))}$. Now, for fixed $i \in \{1, \ldots, m\}$, the ℓ disjoint rays R^i_j will pass through all sets $S(n)$, where $n = i, i{+}m, i{+}2m, i{+}3m, \ldots$, using the $S(n)$–$S(n{+}m)$ edges of the matching from the definition of G. This is illustrated in Figure 9.3.

We thus obtain the desired rays R^i_j for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, \ell$. Observe that for each i, j, there are infinitely many disjoint $V(R)$–$V(R^i_j)$ paths in T, namely those that connect v_n with the vertex of R^i_j in $S(n+m)$.

[4]Moreover, as any two distinct rays of T that start in v_0 can be finitely separated in G, we have that the ends of G correspond to the ends of the tree T.

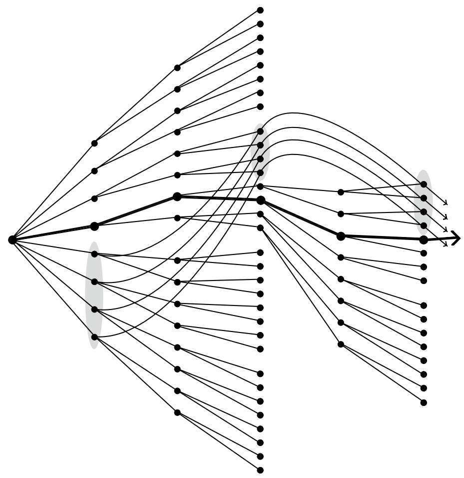


Figure 9.3: The equivalent rays $R_1^1, \ldots, R_4^1$ and R in the underlying tree T, for $\ell = 4$.

Hence the R_j^i are equivalent to R, and hence, ω has vertex-degree at least ℓm.

Let us finally prove that the graph G has no $(k+1)$-connected subgraph. Indeed, suppose otherwise, and let $H \subseteq G$ be $(k+1)$-connected. Now, for any given 01-string s, denote by $T^{(s)}$ the set of all vertices y that are comparable in T with one of the elements of $S^{(s)}$, and which, in the case that $y < x \in S^{(s)}$, in T have distance less than ℓ to x. Formally,

$$\begin{aligned} T^{(s)} :=& \{y \in V(G) : \textit{ there is an } x \in S^{(s)} \textit{ such that } y < x \textit{ and } d_T(x, y) < \ell\} \\ & \cup \{y \in V(G) : \textit{ there is an } x \in S^{(s)} \textit{ such that } y \geq x\}. \end{aligned}$$

We claim that for each $n \geq 1$

$$\textit{there is a } 01\textit{-string } s \textit{ of length } n \textit{ so that } V(H) \subseteq T^{(s)}. \tag{9.3}$$

Then, for every $i \in \mathbb{N}$ and for any vertex $v \in S_i$, we may apply (9.3) with $n = i + \ell$ to obtain that $v \notin V(H)$. Hence $H = \emptyset$, a contradiction.

It remains to show (9.3), which we do using induction on n. For $n = 1$, observe that $S_0 \cup S_1$ separates in G the sets $T^{(0)} - (S_0 \cup S_1)$ and $T^{(1)} - (S_0 \cup S_1)$. Hence, because $|S_0 \cup S_1| \leq k$, for either $s = 0$ or $s = 1$ we have that $V(H) \subseteq T^{(s)} \cup (S_0 \cup S_1) = T^{(s)} \cup S_1$. Furthermore, as by construction of G the vertices in $S^{(1-s)}$ each send only $2^m \leq 2\ell \leq k$ edges to $T^{(s)}$, it follows that $V(H) \subseteq T^{(s)}$, as desired.

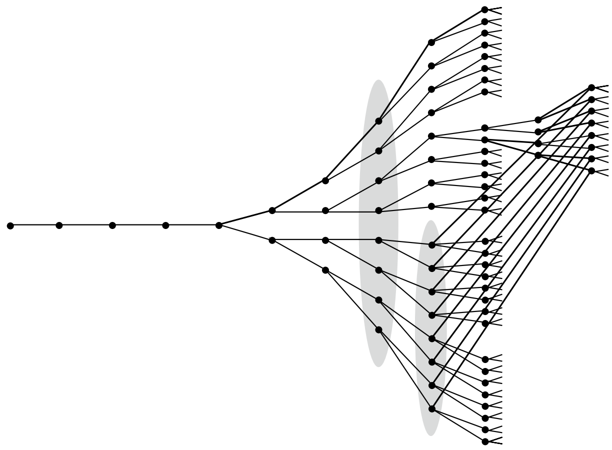


Figure 9.4: The subgraph of G induced by $T^{(s')}$, for $\ell = 8$. For simplicity, the edges of $G' - E(T)$ are not drawn.

For $n > 1$, we proceed similarly. The induction hypothesis provides us with a string s' of length $n-1$ such that $V(H) \subseteq T^{(s')}$. Now, the set

$$S := T^{(s')} \cap \bigcup_{i=n-\ell+1}^{n} S_i$$

has size at most

$$(\ell - m) + \sum_{i=0}^{m} 2^i + |T^{(s')} \cap S_n| \leq \ell + 2\ell + 2\ell = k.$$

Moreover, S separates $G[T^{(s')}]$ into the three sets $T^{(s')} \cap S_{n-\ell}$, $T^{(s'0)} - S$ and $T^{(s'1)} - S$, the first of which consists of one vertex t only. Thus, for either $s = s'0$ or $s = s'1$, say for $s = s'0$, we have that $V(H) \subseteq T^{(s)} \cup S$.

Observe that we can write

$$S - T^{(s)} = S^{(s'1)} \cup U$$

where

$$U = (T^{(s')} - T^{(s)}) \cap \bigcup_{i=n-m}^{n-1} S_i.$$

Now, by construction of G, each vertex of U has at most 3 neighbours in $T^{(s)} \cup S$. Therefore, $V(H) \cap U = \emptyset$. Moreover, as the vertices in $S^{(s'1)}$ each send only $2^m \leq k$ edges to $T^{(s)} \cup S^{(s'1)}$, it follows that $V(H) \cap S^{(s'1)} = \emptyset$, and hence, $V(H) \subseteq T^{(s)}$, as desired. This completes the proof of (9.3), and thus the proof of the theorem. $\square$

Chapter 10

Large complete minors in infinite graphs

10.1 An outline of this chapter

In this chapter, which is based on parts of [83], we shall give a satisfactory extension of Theorem 8.1.1 for two important classes of infinite graphs. These classes are the rayless and the locally finite graphs.

In infinite graphs, forcing large complete minors only with assumptions on the degrees of the vertices of the graph will not work, as has been explained in Chapter 8. If we additionally make assumptions on the vertex-/edge-degrees of the ends, then we can force highly (edge)-connected subgraphs, this has been discussed in Chapter 9, but not more: countexexamples shall be given in Section 10.3.

An exception are the rayless graphs. Proposition 10.2.1 is a verbatim extension of Theorem 8.1.1. This proposition will follow from a useful reduction theorem (Theorem 10.2.2), which states that every rayless graph of minimum degree k has a finite subgraph of minimum degree k. These results will be presented in Section 10.2.

For infinite locally finite graphs we solve our problem in a different way. As the usual vertex-/edge-degrees do not suffice, we suggest a new notion of an *end degree*, namely the *relative degree*, which will be introduced in Section 10.4. With large relative degree, we can force large complete minors in infinite locally finite graphs (Theorem 10.4.3). Moreover, every locally finite graph of minimum degree/relative degree k has a finite subgraph of average degree k (Theorem 10.4.2).

An application of Theorem 10.4.3 is investigated in Section 10.5, where we ask whether as in finite graphs, large girth can be used to force large complete minors. A partial answer is given by Proposition 10.5.2.

10.2 Large complete minors in rayless graphs

In this section on substructures we shall extend Theorem 8.1.1 to infinite rayless graphs:

Proposition 10.2.1. [28] *Let G be a rayless graph such that each vertex has degree at least $f(r)$. Then K^r is a minor of G.*

In fact, Proposition 10.2.1 follows at once from Theorem 8.1.1 together with the following reduction theorem:

Theorem 10.2.2. [28] *Let G be a rayless graph of minimum degree m. Then G has a finite subgraph of minimum degree m.*

In order to prove Theorem 10.2.2, we shall need Kőnig's infinity lemma:

Lemma 10.2.3. [31] *Let G be a graph on the union of disjoint finite non-empty sets S_i, $i \in \mathbb{N}$, so that each $v \in S_i$ has a neighbour in S_{i-1}. Then G has a ray.*

Proof of Theorem 10.2.2. We start with any finite set S_0. For $i \geq 1$ we shall choose for each vertex in S_{i-1} a set of $\max\{0, m\}$ neighbours in $V(G) \setminus \bigcup_{j<i} S_j$. Let S_i be the union of all these sets. Observe that if at some point we set $S_i = \emptyset$, then, as by assumption each vertex has degree at least m in G, we have found the desired subgraph of G. On the other hand, if we manage to define S_i for all $i \in \mathbb{N}$, we may apply Lemma 10.2.3 to find a ray, contradicting the assumption that G is rayless. □

10.3 Two counterexamples

This section is dedicated to two examples which show that large degree and large vertex-degree together are not strong enough assumptions to force large complete minors. The difference between the two examples is that the latter does not have ends of infinite vertex-degree.

Example 10.3.1. *For given k, we take the k-regular tree T_k with levels $L_0, L_1, L_2, \ldots$ and insert the edge set of a spanning cycle C_i at each level L_i of T_k. This can be done in a way so that the obtained graph G_k is still*

planar.
Clearly, G_k has one end of infinite vertex- and edge-degree, and furthermore, all vertices of G_k have degree at least k. It is easy to see that G_k is k-connected, but being planar, G_k has no complete minor of order greater than 4.

By deleting some (carefully chosen) edges from G_k, we obtain a planar graph of high minimal degree and vertex-degree whose (continuum many) ends all have finite vertex-degree:

Example 10.3.2. *Let $k \in \mathbb{N}$ be given, and consider the graph G_k from Example 10.3.1. Now, for each $i \in \mathbb{N}$, delete the edge $vw \in E(C_i)$ from $E(G_k)$, if v and w have no common ancestors in the levels $L_{i-k+2}, L_{i-k+3}, \ldots, L_{i-1}$. Denote the obtained graph by G'_k.*
As G_k is planar, also G'_k is. Clearly, $k \leq d(v) \leq k+2$ for each $v \in V(G)$. We show in Lemma 10.3.3 that the ends of G'_k have large, but finite vertex-degree.

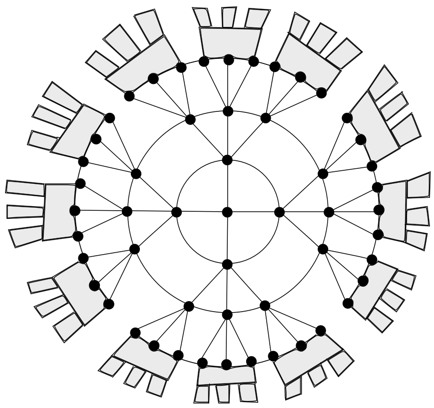


Figure 10.1: The graph G'_k from Example 10.3.2 for $k = 4$.

Lemma 10.3.3. *The ends of the graph G'_k from Example 10.3.2 all have vertex-degree between $k-2$ and $2k-3$.*

Proof. Consider, for each $x \in V(G'_k)$ the set

$$S_x := \{x\} \cup \bigcup_{i=1,\ldots,k-2} N^i(x),$$

where $N^i(x)$ here denotes the ith neighbourhood of x in level L_{m+i}, supposing that x lies in the mth level (of T_k).

Clearly for each $x \in V(G'_k)$, the set S_x separates G'_k. Hence, already $\partial_v S_x$, which has order between $k-1$ and $2k-3$, separates G'_k.
Let us use the sets S_x in order to show that the ends of G'_k correspond to the ends of T_k. In fact, all we have to show is that for each ray $R \in G'_k$ there is a ray R_T in T_k that is equivalent to R in G'_k. We can find such a ray R_T by considering for each i large enough the last vertex v_i of R in $V(L_i)$. Now, $v_i \in S_{w_i}$ for exactly one $w_i \in V(L_{i-k+2})$. By definition of the v_i, the w_i are adjacent to their successors $w_{i+1} \in V(L_{i-k+3})$. So, $R_T := w_k w_{k+1} w_{k+2} \dots$ is a ray in T_k as desired.

Thus G'_k has continuum many ends, all of which have vertex-degree at most $2k-3$, because of the separators $\partial_v S_x$. It remains to show that each end ω of G'_k has vertex-degree at least $k-2$.

For this, fix $\omega \in \Omega(G)$ and consider the union S_ω of the sets $\partial_v S_{w_i}$ for the ray $R = w_0 w_1 w_2 w_3 \dots$ of T_k that lies in ω, where we assume that R starts in $L_0 = \{w_0\}$. By Lemma 11.4.3, in order to see that ω has vertex-degree at least $k-2$ in $G[S]$ (and thus in G) we only have to show that no set of less than $k-2$ vertices separates L_0 from ω in $G[S]$.

So suppose otherwise, and let T be such a separator. Since every vertex of S has at least $k-1$ neighbours in the next level, we can reach the 2nd, 3rd, $\dots k-2$th level from w_0 in $G[S]-T$. By definition of G'_k, these levels contain spanning cycles, and thus, as $|T| > k-2$, there is a w_i with $i \in \{1, 2, \dots, k-2\}$ which can be reached from w_0 in $G[S]-T$. We repeat the argument with w_i in the role of w_0, observing that in $G[S] \cap (L_{i+1} \cup L_{i+2} \cup \dots \cup L_{i+k-2})$, each level contains spanning paths, by construction of G'_k. □

10.4 Large relative degree forces large complete minors

In the previous sections we explored which substructures may or may not be forced in an infinite graph if we assume large (vertex-)degree at both vertices and ends. In particular we saw that Theorem 8.1.1 (with the average degree replaced by the minimum degree) does not extend to infinite graphs that have rays.

In the present section we shall overcome this problem. We will see that with a different, more appropriate notion of the end degree a satisfactory extension of Theorem 8.1.1 is possible.

For this let us first take a closer look at the graph G'_k from Example 10.3.2. Why do the large (vertex-)degrees not interfere with the planarity? Observe that, for each finite set $S \subseteq V(G)$, the edge-boundary of the subgraph $G'_k - S$ has about the same size as its vertex-boundary. So locally the density is never large enough to force non-planarity. Similar as in the tree T_k, the density that the high degrees should generate gets lost towards infinity.

In order to avoid this behaviour, we have to prohibit regions R of an end ω which have the property that $|\partial_e R|/|\partial_v R|$ is small, or at least we should prohibit sequences of such regions converging to ω. This is not unnatural: applied to vertices this gives the usual degree, as each vertex v is contained in a smallest region, namely $R = \{v\}$, for which $|\partial_e R|/|\partial_v R| = d(v)$.

Let us make our idea more precise. Suppose that G is a locally finite graph. Write $(H_i)_{i\in\mathbb{N}} \to \omega$ if $(H_i)_{i\in\mathbb{N}}$ is an infinite sequence of regions of G such that $H_{i+1} \subseteq H_i - \partial H_i$ and $\omega \in \overline{H_i}$ for each $i \in \mathbb{N}$. Observe that such sequences always exist (as G is locally finite). Define the relative degree of an end as

$$d_{e/v}(\omega) := \inf_{(H_i)_{i\in\mathbb{N}}\to\omega} \liminf \frac{|\partial_e H_i|}{|\partial_v H_i|}.$$

Note that by Lemma 11.4.3, in locally finite graphs, we can express our earlier notions, the vertex- and the edge-degree also using converging sequences of regions:

$$d_v(\omega) = \inf_{(H_i)_{i\in\mathbb{N}}\to\omega} \liminf |\partial_v H_i|, \text{ and} \tag{10.1}$$

$$d_e(\omega) = \inf_{(H_i)_{i\in\mathbb{N}}\to\omega} \liminf |\partial_e H_i|. \tag{10.2}$$

Our three end degree concepts relate as follows:

Lemma 10.4.1. *Let G be a locally finite graph, let $\omega \in \Omega(G)$. Then*

$$d_{e/v}(\omega) \le \frac{d_e(\omega)}{d_v(\omega)} \le d_e(\omega).$$

Proof. The second inequality in trivial. For the first one, let S be the set of starting vertices of a set of disjoint rays of ω that has cardinality $d_v(\omega)$. Then for each region $H \subseteq G - S$ of ω we have that $|\partial_v H| \ge d_v(\omega)$.

Observe that for the determination of $d_e(\omega)$ using (10.2) it suffices to consider only sequences of regions H_i that lie in $G-S$. As each such sequence $(H_i)_{i\in\mathbb{N}}$ may be used also to determine $d_{e/v}(\omega)$, and since for each H_i from such a sequence it holds that

$$\frac{|\partial_e H_i|}{|\partial_v H_i|} \le \frac{|\partial_e H_i|}{d_v(\omega)},$$

the desired inequality follows. □

With the notion of the relative degree, we can prove a very useful reduction theorem:

Theorem 10.4.2. [28] *Let G be a locally finite graph such that each vertex has degree at least k, and for each end ω we have $d_{e/v}(\omega) \geq k$. Then G has a finite subgraph H of average degree at least k.*

Proof. Choose a vertex $v \in V(G)$ and set $S_0 := \{v\}$. Inductively we shall construct a sequence $(S_i)_{i\in\mathbb{N}}$ of finite vertex sets. In each step $i \geq 0$ we consider the set $\mathcal{C}^i$ of all components C of $G - S_i$ with

$$\frac{|\partial_v C|}{|\partial_e C|} < f(r),$$

and set

$$S_{i+1} := S_i \cup \bigcup_{C\in\mathcal{C}^i} \partial_v C.$$

Now, there are two possibilities: either from some i_0 on all S_i are the same, or they all differ. In the first case observe that $G[S_{i_0} \cup N(S_{i_0})]$ is the desired subgraph H. In the second case we may apply Kőnig's infinity lemma to find a sequence $(C_i)_{i\in\mathbb{N}}$ with $C_i \in \mathcal{C}^i$ and $C_i \subseteq C_{i-1} - \partial_v C_{i-1}$ for $i \geq 1$.

It is easy to construct a ray R that passes exactly once through each $\partial_v C_i$, and hence there is an end $\omega \in \bigcap_{i\in\mathbb{N}} \overline{C}_i$. By construction of the C_i, it follows that $d_{e/v}(\omega) < f(r)$, a contradiction. □

We may now use Theorem 10.4.2 as a black box in order to translate to infinite locally finite graphs any kind of results from finite graph theory that make assumptions only on the average or minimum degree. For example, Theorem 10.4.2 together with Theorem 8.1.1 yields at once the desired extension of Theorem 8.1.1 to locally finite graphs.

Theorem 10.4.3. [28] *Let G be a locally finite graph such that each vertex has degree at least $f(r)$, and for each end ω we have $d_{e/v}(\omega) \geq f(r)$. Then K^r is a minor of G.*

Let us remark that we may not weaken the assumption of Theorem 10.4.3 in the following sense. Denote by $d'_{e/v}$ the the fraction of the edge- and the vertex-degree, that is, we set $d'_{e/v}(\omega) := d_e(\omega)/d_v(\omega)$. By Lemma 10.4.1, we have $d'_{e/v} \geq d_{e/v}$.

Now, there is no function f' such that all graphs with $d'_{e/v}(\omega), d(v) > f'(k)$ for all ends ω and vertices v contain a complete minor of order k. This can be seen by considering the example from Section 9.4 in the previous chapter.

10.5 Using large girth

In finite graphs, we can force large complete minors by assuming large girth, and a minimal degree of 3. More precisely, every finite graph of minimal degree at least 3 and girth at least $g(k) := 8k + 3$ has a complete minor of order k (see [31]).

If we do not take the ends into account, then it is easy to see that this fact does not extend to infinite graphs. Clearly, the 3-regular infinite tree T_3 has infinite girth and no large complete minors, and even if finite girth was required, we might simply add an edge to T_3, and still have a counterexample.

But, the ends of our example have end degree 1 in each of our three end degree notions. Now, we shall see that requiring large minimum vertex-degree at the ends, together with large girth, and minimum degree at least 3 at the vertices, will still not suffice to force large complete minors.

Example 10.5.1. *For all $g \in \mathbb{N}$, we construct a planar graph H_g with finite girth g, minimal degree 3 at the vertices and a unique end, which has infinite vertex-degree.*
Take the union of the cycles of length g^n, over all $n \in \mathbb{N}$. We shall add edges between each C_{g^n} and $C_{g^{n+1}}$, one for each vertex in $V(C_{g^n})$, in a way that their new neighbours lie at distance g on $C_{g^{n+1}}$. Clearly, this can be done in a way so that we obtain a planar graph H_g (cf. Figure 10.5). Being planar, H_g has no complete minor of order greater than 4.

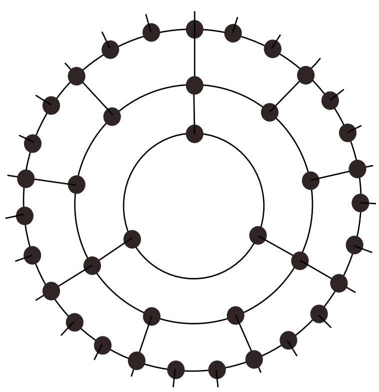


Figure 10.2: The graph H_g from Example 10.5.1 for $g = 3$.

However, the relative degree of the end of H is relatively small (in fact, it is 1). Is this a necessary feature of any counterexample? That is, does every graph of minimum degree 3 and large girth and without large complete minors have to have an end of small relative degree? At least the relative degrees cannot be too large:

Proposition 10.5.2. [**28**] *Every locally finite graph G of minimal degree at least 3 at the vertices, minimal relative degree at least $r(k) = ck\sqrt{\log k}$ at the ends and girth at least $g(k) = 8k + 3$ has a complete minor of order k.*

Proof. One may employ the same proof as for finite graphs, as given e.g. in [31]. The strategy there is to construct first a minor M of G that has large minimal degree, and then apply Theorem 8.1.1 to M. In an infinite graph, we can construct the minor M in exactly the same way, and it is not overly difficult to see that M does not only have large degree at the vertices, but also has at least the same relative degree at the ends as G. It suffices to apply Theorem 10.4.3 to obtain the desired minor. □

How much can this bound be lowered? May we take $r(k)$ to be constant, even $r(k) = 3$? It would be interesting to find the smallest number $r(k)$ so that every locally finite graph with $d(v) \geq 3$ and $d_{e/v}(\omega) \geq r(k)$ for all vertices v and ends ω, and of girth at least $g(k)$ has a complete minor of order k.

Chapter 11

Minimal k-(edge)-connectivity in infinite graphs

11.1 Four notions of minimality

The minimal degree and bounds on the number of vertices that attain the minimal degree have been much studied for finite graphs that are in certain ways minimally k-connected/k-edge-connected. This minimality will have two meanings here: minimality upon vertex deletion and upon edge deletion (see below for the precise definition).

In this chapter, which is based on [82], we shall give an overview of the results known, and show that most carry over to infinite graphs. This is made possible by considering the (vertex/edge)-degree of the ends as well as the degree of the vertices.

Four notions will be of interest. For $k \in \mathbb{N}$, we shall call a graph G *edge-minimally k-connected*, resp. *edge-minimally k-edge-connected* if G is k-connected resp. k-edge-connected, but $G - e$ is not, for every edge $e \in E(G)$. Analogously, call G *vertex-minimally k-connected*, resp. *vertex-minimally k-edge-connected* if G is k-connected resp. k-edge-connected, but $G - v$ is not, for every vertex $v \in V(G)$.

These four classes of graphs often appear in the literature under the names of k-minimal/k-edge-minimal/k-critical/k-edge-critical graphs. Overviews of results known for finite graphs and also for digraphs can be found in [5, 39].

11.2 The situation in finite graphs

It is known that finite graphs which belong to one of the classes defined above have vertices of small degree. In fact, three of the four cases the trivial lower bound of k on the minimum degree is attained. We summarise the known results in the following theorem:

Theorem 11.2.1. *Let G be a finite graph, let $k \in \mathbb{N}$. Then*

(a) **(Halin [50])** *If G is edge-minimally k-connected, then G has a vertex of degree k,*

(b) **(Lick et al [21], Mader [63])** *If G is vertex-minimally k-connected, then G has a vertex of degree $\frac{3}{2}k - 1$,*

(c) **(Lick [60])** *If G is edge-minimally k-edge-connected, then G has a vertex of degree k,*

(d) **(Mader [69])** *If G is vertex-minimally k-edge-connected, then G has a vertex of degree k.*

Note that in Theorem 11.2.1 (b), the bound of $3k/2 - 1$ on the degree is best possible. For even k, this can be seen by replacing each vertex of C_ℓ, a circle of some length $\ell \geq 4$, with a copy of $K^{k/2}$, the complete graph on $k/2$ vertices, and adding all edges between two copies of $K^{k/2}$ when the corresponding vertices of C_ℓ are adjacent. This procedure is sometimes called the strong product[1] of C_ℓ and $K_{k/2}$. For odd values of k similar examples can be constructed, using $K^{(k+1)/2}$'s instead of $K^{k/2}$'s, and in the end deleting two vertices which belong to two adjacent copies of $K^{(k+1)/2}$.

In all four cases of Theorem 1, the minimal degree is attained by more than one vertex:

Theorem 11.2.2. *Let G be a finite graph, let $k \in \mathbb{N}$. Then*

(a) **(Mader [65])** *In case (a) of Theorem 11.2.1, G has at least $c_k|G|$ vertices of degree k, where c_k is a constant depending only on k, unless $k = 1$, in which case G has at least two vertices of degree k,*

(b) **(Hamidoune [52])** *In case (b) of Theorem 11.2.1, if $k > 1$ then G has at least two vertices of degree $\frac{3}{2}k - 1$,*

[1]The *strong product* of two graphs H_1 and H_2 is defined in [55] as the graph on $V(H_1) \times V(H_2)$ which has an edge $(u_1, u_2)(v_1, v_2)$ whenever $u_i v_i \in E(H_i)$ for $i = 1$ or $i = 2$, and at the same time either $u_{3-i} = v_{3-i}$ or $u_{3-i}v_{3-i} \in E(H_{3-i})$.

(c) **(Mader [68])** *In case (c) of Theorem 11.2.1, G has at least $c'_k|G|$ vertices of degree k, where c'_k is a constant depending only on k, unless $k = 1$ or $k = 3$, in which case G has at least two resp. four vertices of degree k,*

(d) **(Mader [69])** *In case (d) of Theorem 11.2.1, if $k > 1$ then G has at least two vertices of degree k.*

In case (a), actually more than the number of vertices of small degree is known: If we delete all the vertices of small degree, we are left with a forest (this was shown in [65], see also [5]).

The constant c_k from (a) can be chosen as $c_k = \frac{k-1}{2k-1}$, and this is best possible [65]. Actually one can ensure that G has at least $\max\{c_k|G|, k+1, \Delta(G)\}$ vertices of degree k [65], where $\Delta(G)$ denotes the maximum degree of G. In (c), the constant c'_k may be chosen as about $1/2$ as well (for estimates, see [6, 18, 19, 70]).

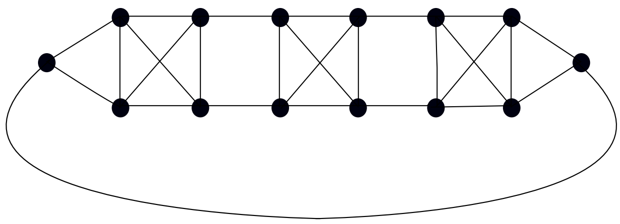


Figure 11.1: A finite vertex-minimally k-connected graph with only two vertices of degree $< 2(k-1)$, for $k = 3$.

The bounds on the number of vertices of small degree are best possible in (b) and (d), for $k \neq 2$[2]. Indeed, for $k \geq 3$ consider the following example which we borrow from [69]. Take any finite number $\ell \geq 2$ of copies H_i of the complete graph $K^{2(k-1)}$, and join every two consecutive H_i with a matching of size $k-1$, in a way that all these matchings are disjoint. Join a new vertex a to all vertices of H_1 that still have degree $2(k-1)-1$, and analogously join a new vertex b to half of the vertices of H_ℓ. Finally join a and b with an edge. See Figure 11.1.

The obtained graph is vertex-minimally k-connected as well as vertex-minimally k-edge-connected. However, all vertices but a and b have degree $2(k-1)$, which, as $k \geq 3$, is greater than $\max\{k, \frac{3}{2}k - 1\}$.

[2]And for $k = 2$ one might not find more than 4 vertices of small degree, as the so-called ladder graphs show. As for $k = 1$, the only vertex-minimally 1-(edge)-connected graphs is the trivial graph K_1, since every other connected graph has non-separating vertices.

11.3 The situation in infinite graphs

For infinite graphs, a positive result for case (a) of Theorem 11.2.1 has been obtained by Halin [51] who showed that every infinite locally finite edge-minimally k-connected graph has infinitely many vertices of degree k, provided that $k \geq 2$. Mader [67] extended the result showing that for $k \geq 2$, every infinite edge-minimally k-connected graph G has in fact $|G|$ vertices of degree k (see Theorem 11.3.1 (a) below). It is clear that for $k = 1$, we are dealing with trees, which, if infinite, need not have vertices of degree 1.

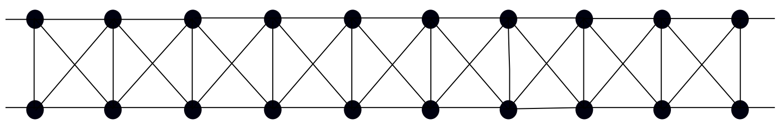


Figure 11.2: An infinite vertex-minimally k-connected graph without vertices of degree $3k/2 - 1$, for $k = 2$.

For the other three cases of Theorem 11.2.1, the infinite version fails. In fact, for case (b) this can be seen by considering the strong product of the double-ray (i.e. the two-way infinite path) with the complete graph K^k (cf. Figure 11.2). The obtained graph is $(3k - 1)$-regular, and vertex-minimally k-connected. If instead of the double-ray we take the r-regular infinite tree T_r, for any $r \in \mathbb{N}$, the degrees of the vertices become unbounded in k (see Figure 11.3).

The example from Figure 11.3 also works for case (d) of Theorem 11.2.1 (then we get a vertex-minimally k^2-edge-connected $((r + 1)k - 1)$-regular graph). Counterexamples for an infinite version of (c) will be given now.

For the values 1 and 3 this is particularly easy, as for $k = 1$ we may consider the double ray D, and for $k = 3$ its square D^2. All the vertices of these graphs have degree 2 resp. 4, but D and D^2 are edge-minimally 1- resp. 3-edge-connected.

For arbitrary values $k \in \mathbb{N}$, we construct a counterexample as follows. Choose $r \in \mathbb{N}$ and take the rk-regular tree T_{rk}. For each vertex v in T_{rk}, insert edges between the neigbourhood N_v of v in the next level so that N_v spans r disjoint copies of K^k (cf. Figure 11.5). This procedure gives an edge-minimally k-edge-connected graph, as one easily verifies. However, the vertices of this graph all have degree at least rk.

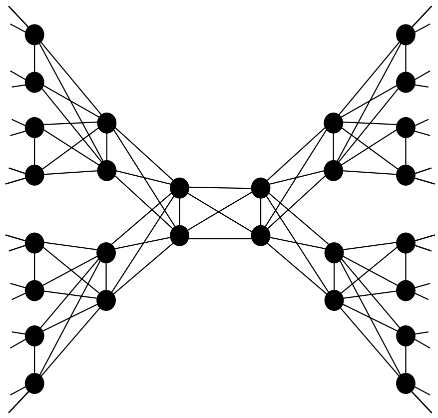


Figure 11.3: The strong product of T_3 with K^2.

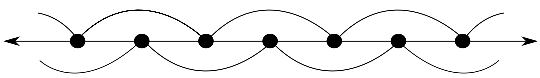


Figure 11.4: The square of the double-ray.

Hence a literal extension of Theorems 11.2.1 and 11.2.2 to infinite graphs is not true, except for part (a). The reason can be seen most clearly comparing Figures 11.1 and 11.2: Where in a finite graph we may force vertices of small degree just because the graph has to end somewhere, in an infinite graph we can just 'escape to infinity'. So an adequate extension of Theorem 1 should also measure the something like 'the degree at infinity'.

Of course, with these 'points at infinity' we mean nothing else but the ends of graphs, as defined in Chapter 8. Recall that the set of all ends of a graph G is denoted by $\Omega(G)$. Also recall that we defined the vertex-degree $d_v(\omega)$ of an end ω as the supremum of the cardinalities of the set of (vertex)-disjoint rays in ω, and the edge-degree $d_e(\omega)$ of an end ω is defined as the supremum of the cardinalities of the set of edge-disjoint rays in ω, and that this definition implies at once that $d_e(\omega) \geq d_v(\omega)$.

In light of this, we observe at once what happens in the case $k = 1$ of the infinite version of Theorem 1 (a) above. Infinite trees aka edge-minimally 1-connected graphs need not have vertices that are leaves, but if not, then they must have 'leaf-like' ends, that is, ends of vertex-degree 1. In fact, it is easy to see that in a tree T, with root r, say, every ray starting at r corresponds to an end of T, and that all ends of T have vertex- and edge-degree 1.

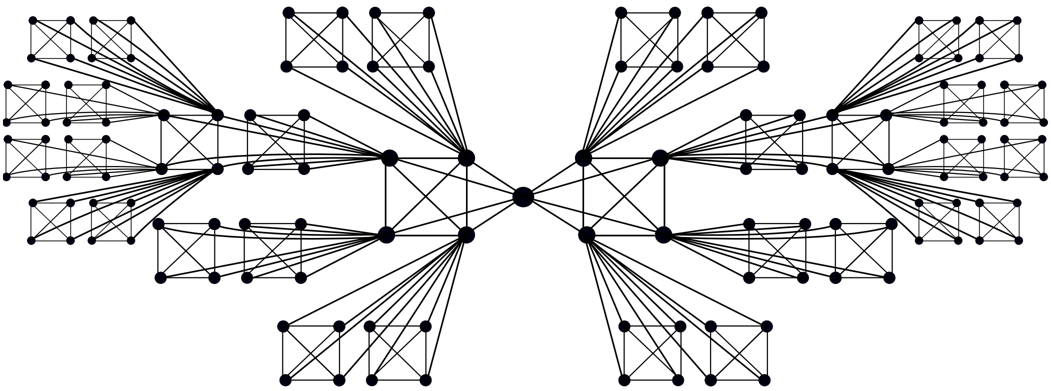


Figure 11.5: An edge-minimally 4-edge-connected graph without vertices of degree 4.

This observation gives case (a') in the following generalisation of Theorem 11.2.1 to infinite graphs. Cases (b)–(d), respectively their quantative versions in Theorem 4, are the topic of this chapter. These results appear in [82].

Theorem 11.3.1. *Let G be a graph, let $k \in \mathbb{N}$. Then*

(a) **(Mader [67])** *If G is edge-minimally k-connected and $k \geq 2$, then G has a vertex of degree k,*

(a') **[82]** *If G is edge-minimally 1-connected, then G has a vertex of degree 1 or an end of edge-degree 1,*

(b) **[82]** *If G is vertex-minimally k-connected, then G has a vertex of degree $\leq \frac{3}{2}k - 1$ or an end of vertex-degree $\leq k$,*

(c) **[82]** *If G is edge-minimally k-edge-connected, then G has a vertex of degree k or an end of edge-degree $\leq k$,*

(d) **[82]** *If G is vertex-minimally k-edge-connected, then G has a vertex of degree k or an end of vertex-degree $\leq k$.*

We can even give bounds on the number of vertices/ends of small degree:

Theorem 11.3.2. *Let G be a graph, let $k \in \mathbb{N}$. Then*

(a) **(Mader [67])** *In case (a) of Theorem 11.2.1, the cardinality of the set of vertices of degree k is $|G|$,*

(a') [**82**] *In case (a') of Theorem 11.2.1, the cardinality of the set of vertices and ends of (edge)-degree* 1 *is* $|G|$ *unless* $|G| \leq \aleph_0$*, in which case there are at least* 2 *vertices/ends of (edge)-degree* 1*,*

(b) [**82**] *In case (b) of Theorem 11.2.1,*
$|\{\omega \in \Omega(G) : d_v(\omega) \leq k\} \cup \{v \in V(G) : d(v) \leq \frac{3}{2}k - 1\}| \geq 2,$

(c) [**82**] *In case (c) of Theorem 11.2.1,*
$|\{\omega \in \Omega(G) : d_e(\omega) \leq k\} \cup \{v \in V(G) : d(v) = k\}| \geq 2,$

(d) [**82**] *In case (d) of Theorem 11.2.1,*
$|\{\omega \in \Omega(G) : d_v(\omega) \leq k\} \cup \{v \in V(G) : d(v) = k\}| \geq 2.$

Concerning part (c) we remark that one may replace graphs with multigraphs (see Corollary 11.5.2). Also, in (a') and (c), one may replace the edge-degree with the vertex-degree (as this yields a weaker statement).

We shall prove Theorem 11.3.2 (b)–(d) in Sections 11.4, 11.5 and 11.6 respectively. Statement (a') is fairly simple, in fact, it follows from what we remarked above that every tree has at least two leafs/ends of vertex-degree 1. In general, this is already the best bound (because of the finite paths and the double ray). For trees T of uncountable order we get more, as these have to contain a star with degree $|G|$ leaves, which we can extend to a union of $|G|$ almost disjoint paths and rays, thus finding $|G|$ vertices/ends of (edge)-degree 1.

In analogy to the finite case, the bounds on the degrees of the vertices in (b) cannot be lowered, even if we allow the ends to have slightly larger vertex-degree. An example for this is given at the end of Section 11.4.

Also, the bound on the number of vertices/ends of small degree in Theorem 11.3.2 (b) and (d) is best possible: for (b), this can be seen by considering again the strong product of the double ray with the complete graph K^k (see Figure 11.2 for $k = 2$). For (d), we may consider the Cartesian product[3] of the double ray with the complete graph K^k (for $k = 2$ that is the double-ladder).

As for Theorem 11.3.2 (c), it might be possible that the bound of Theorem 11.2.2 (c) extends. For infinite graphs G, the positive proportion of the vertices there should translate to an infinite set S of vertices and ends of small degree/edge-degree. More precisely, one would wish for a set S of cardinality $|V(G)|$, or even stronger, $|S| = |V(G) \cup \Omega(G)|$.

[3]The Cartesian product of two graphs H_1 and H_2 is defined as the graph on $V(H_1) \times V(H_2)$ which has an edge $(u_1, u_2)(v_1, v_2)$ if for for $i = 1$ or $i = 2$ we have that $u_i v_i \in E(H_i)$ and $u_{3-i} = v_{3-i}$ [31, 55].

Observe that it is necessary to exclude also in the infinite case the two exceptional values $k = 1$ and $k = 3$, as there are graphs (namely D and D^2, see above) with only two vertices/ends of (edge)-degree 1 resp. 3. For other values of k, an extension might hold:

Question 11.3.3. *Are there $k \in \mathbb{N}$ for which every infinite edge-minimally k-edge-connected graph G contains infinitely many vertices or ends of (edge)-degree k? Does G have $|V(G)|$ (or even $|V(G) \cup \Omega(G)|$) such vertices or ends?*

Another interesting question is which k-(edge)-connected graphs have vertex- or edge-minimally k-(edge)-connected subgraphs. Finite graphs trivially do, but for infinite graphs this is not always true. This observation leads to the study of vertex-/edge-minimally k-(edge)-connected *(standard) subspaces* rather than graphs. For more on this, see [29, 83], the latter of which contains a version of Theorem 3 (a) for standard subspaces.

11.4 Vertex-minimally k-connected graphs

In this section we shall show part (b) of Theorem 11.3.2. For the proof, we need two lemmas. The first of these lemmas may be extracted from [21] or from [67], and at once implies Theorem 11.2.1 (b). For completeness, we shall give a proof.

Lemma 11.4.1. *Let $k \in \mathbb{N}$, $k > 0$, let G be a vertex-minimally k-connected graph, and let H be a profound finite k-region of G. Then G has a vertex v of degree at most $\frac{3}{2}k - 1$.*
Moreover, if $|G - H| > |H - \partial_v H|$, then $v \in V(H)$.

Proof. Assume that H was chosen inclusion-minimal among all profound finite k-regions of G. Set $T := \partial_v H$, set $C_1 := H - T$, and set $C_2 := G - H$. Let $x \in V(C_1)$, and observe that since G is vertex-minimally k-connected, there is a k-separator T' of G with $x \in T'$. Let D_1 be a component of $G - T'$, set $D_2 := G - T' - D_1$, and set $T^* := T \cap T'$. Furthermore, for $i, j = 1, 2$ set $A^i_j := C_i \cap D_j$ and set $T^i_j := (T' \cap C_i) \cup (T \cap D_j) \cup T^*$. Observe that $N(A^i_j) \subseteq T^i_j$.

We claim that there are i_1, i_2, j_1, j_2 with either $(i_1, j_1) = (i_2, 3 - j_2)$ or $i_1, j_1) = (3 - i_2, j_2)$ such that for $(i, j) \in \{(i_1, j_1), (i_2, j_2)\}$:

$$|T^i_j| \leq k \text{ and } A^i_j = \emptyset. \qquad (11.1)$$

In fact, observe that for $j = 1, 2$ we have that $|T^1_j| + |T^2_j| \leq |T| + |T'| = 2k$. Thus either $|T^1_j| \leq k$, which by the minimality of H implies that A^1_j is empty, or $|T_j^2| < k$, which by the k-connectivity of G implies that A_j^2 is empty. This proves (11.1).

We hence know that there is an $X \in \{C_1, C_2, D_1, D_2\}$ such that $V(X) \subseteq T \cup T'$. Moreover, if $X \in \{C_1, C_2\}$ then $|X| \cup |T \cap D_1| \leq k$ and $|X| \cup |T \cap D_2| \leq k$. Since $|T| = k$, this implies that $2|X| \leq 2k - |T| = k$. In the same way we get that if $X \in \{D_1, D_2\}$ then $2|X| \leq 2k - |T'| = k$. Hence, in any case there is a vertex $v \in X$ of degree at most

$$\max\{|T| + |X| - 1, |T'| + |X| - 1\} \leq k + k/2 - 1.$$

It is easy to see that we may choose $v \in V(H)$ unless both $|T_1^1|$ and $|T_2^1|$ are greater than k. But then by (11.1) $V(C_2) \subseteq T'$, and thus $|C_2| \leq k/2 \leq |T' \cap C_1| \leq |C_1|$, as desired. □

We also need Lemma 9.5.1 from Chapter 9. Observe that the outcome of Lemma 9.5.1 is invariant under modifications of the structure of $G - C$. Hence we may always assume that $d_v(\omega) \geq m$ only for ends ω of G that have rays in C. Let us restate the thus modified lemma:

Lemma 11.4.2. *Let G be a graph such that all its ends have vertex-degree at least $m \in \mathbb{N}$. Let C be an infinite region of G. Then there exists a profound region $C' \subseteq C$ for which one of the following holds:*

(a) C' is finite and $|\partial_v C'| < m$, or

(b) C' is infinite and $|\partial_v C''| \geq m$ for every profound region $C'' \subsetneq C'$.

We are now ready to prove Theorem 11.3.2 (b).

Proof of Theorem 11.3.2 (b). First of all, we claim that for every infinite region H of G it holds that

There is a vertex $v \in V(H)$ of degree $\leq \frac{3}{2}k - 1$ or an end of vertex-degree $\leq k$ with rays in H. (11.2)

In order to see (11.2), we assume that there is no end as desired and apply Lemma 11.4.2 to H with $m := k+1$. This yields a profound region $H' \subseteq H$. We claim that (a) of Lemma 11.4.2 holds; then we may use Lemma 11.4.1 to find a vertex $w \in V(H')$ with $d(w) \leq 3k/2 - 1$.

So, assume for contradiction that (b) of Lemma 11.4.2 holds. Since G is k-connected there exists a finite family $\mathcal{P}$ of finite paths in G such that each pair of vertices from $\partial_v H'$ is connected by k otherwise disjoint paths from $\mathcal{P}$. Set

$$S := \partial_v H' \cup V(\bigcup \mathcal{P}),$$

and observe that $H' - S$ is still infinite. In particular, $H' - S$ contains a vertex v. Since G is vertex-minimally k-connected, v lies in a k-separator T' of G. By the choice of $v \notin S$, all of $\partial_v H'$ is contained in one component of $G - T'$. Let C'' be a component of $G - T'$ that does not contain $\partial_v H'$. Then $H'' := G[C'' \cup T']$ is a profound region with $H'' \subsetneq H'$. Thus, because of (b), $k + 1 \leq |T'| = k$, a contradiction as desired. This proves (11.2).

Now, let $T \subseteq V(G)$ be any separator of G of size k (which exist by the vertex-minimality of G). First suppose that $G - T$ has at least one infinite component C. Then we apply Lemma 11.4.1 or (11.2) to any component of $G - C$ and find an end of vertex-degree k with no rays in C, or a vertex $v \in V(G - C)$ of degree at most $3k/2 - 1$. Apply (11.2) to C respectively to $C - v$ to find the second end/vertex of small (vertex)-degree.

It remains to treat the case when all components of $G - T$ are finite. As we otherwise apply Theorem 2 (b), we may assume that $G - T$ has infinitely many components. Hence, as G has no $(k-1)$-separators, each $x \in T$ has infinite degree. This means that we may apply Lemma 11.4.1 to any two components of $G - T$ in order to find two vertices of degree $\leq 3k/2 - 1$. $\square$

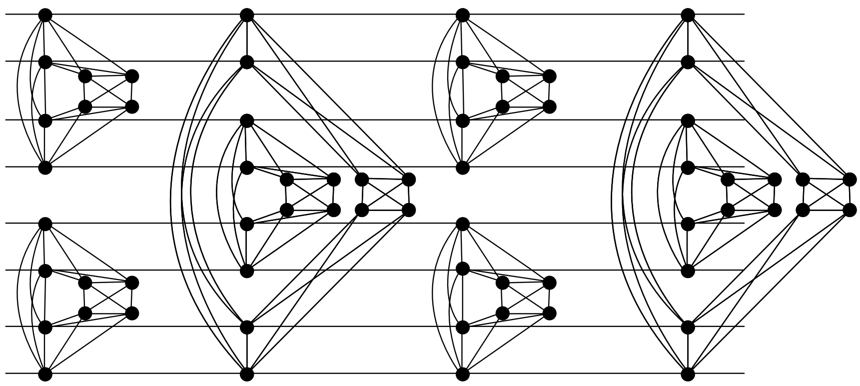


Figure 11.6: A vertex-minimally k-connected graph with $d(v) \geq \frac{3}{2}k - 1$ and $d_v(\omega) \gg k$ for all $v \in V(G)$ and $\omega \in \Omega(G)$.

Observe that the bound on the degree given by Theorem 11.3.1 (b) is best possible. Indeed, by the following lemma from [17][4], the vertex-degree of the ends of a k-connected locally finite graph has to be at least k.

[4]Actually, [17] only contains the edge-version of Lemma 11.4.3, however, the proof is analoguous.

Lemma 11.4.3. *Let $k \in \mathbb{N}$, let G be a locally finite graph, and let $\omega \in \Omega(G)$. Then $d_v(\omega) = k$ if and only if k is the smallest integer such that every finite set $S \subseteq V(G)$ can be separated[5] from ω with a k-separator.*

Moreover, even if we allow a larger vertex-degree of the ends, we cannot expect a lower bound on the degrees of the vertices. This is illustrated by the following example for even k (and for odd k there are similar examples).

Let $\ell \in \mathbb{N} \cup \{\aleph_0\}$, and take the disjoint union of ℓ double-rays $R_1, \ldots, R_\ell$. For simplicity, assume that k divides ℓ. For each $i \in \mathbb{Z}$, take ℓ/k copies of the strong product of C_4 with $K^{k/2}$, and identify the vertices that belong to the first or the last copy of $K^{k/2}$ with the ith vertices the R_j. This can be done in a way that the obtained graph, which is easily seen to be vertex-minimally k-connected, has two ends of vertex-degree ℓ, while the vertices have degree either $3k/2 - 1$ or $3k/2 + 1$.

11.5 Edge-minimally k-edge-connected graphs

We now prove part (c) of Theorem 11.3.2. For this, we shall need Lemma 9.3.1 from 9. which will yield a lemma similar to Lemma 11.4.2 from the previous section:

Lemma 11.5.1. *Let $D \neq \emptyset$ be a region of a graph G so that $|\partial_e D| < m$ and so that $d_e(\omega) \geq m$ for every end $\omega \in \Omega(G)$ with rays in D. Then there is an inclusion-minimal non-empty region $H \subseteq D$ with $|\partial_e H| < m$.*

Proof. Set $D_0 := D$ and inductively for $i \geq 1$, choose a non-empty region $D_i \subseteq D_{i-1} - \partial_v D_{i-1}$ such that $|\partial_e D_i| \leq k$ (if such a region D_i exists). If at some step i we are unable to find a region D_i as above, then we apply Lemma 9.3.1 to D_{i-1} to find the desired region H. On the other hand, if we end up defining an infinite sequence of regions, then these regions define an end of G that has edge-degree at most k. In fact, the connectivity of the D_i guarantees that there is and end with rays in each D_i. If ω had more than k edge-disjoint rays, starting in some (finite) set S, say, then for some $i \in \mathbb{N}$ we have that $S \cap V(D_i) = \emptyset$ which leads to the desired contradiction. □

We can now prove part (c) of our main theorem:

Proof of Theorem 11.3.2. Since G is edge-minimally k-edge-connected, G has a non-empty region D such that $|\partial_e D| = k$, and such that $G - D \neq \emptyset$. We

[5] We say a set $T \subseteq V(G)$ separates a set $S \subseteq V(G)$ from an end $\omega \in \Omega(G)$ if the unique component of $G - T$ that contains rays of ω does not contain vertices from S.

shall find a vertex or end of small (edge)-degree in D and then repeat the procedure for $G - D$ in order to find the second point.

First, we apply Lemma 11.5.1 with $m := k + 1$ to obtain an inclusion-minimal non-empty region $H \subseteq D$ with $|\partial_e H| \leq k$. If $V(H)$ should consist of only one vertex, then this vertex has degree k, as desired. So suppose that $V(H)$ has more than one vertex, that is, $E(H)$ is not empty.

Let $e \in E(H)$. By the edge-minimal k-edge-connectivity of G we know that e belongs to some minimal cut F of G. Say $F = E(A, B)$ where $A, B \neq \emptyset$ partition $V(G)$. Since $e \in F$, neither $A_H := A \cap V(H)$ nor $B_H := B \cap V(H)$ is empty.

So, $|\partial_e A_H| > k$ and $|\partial_e B_H| > k$, by the minimality of H. But then, since $|\partial_e H| \leq k$ and $|F| \leq k$, we obtain that

$$\begin{aligned} |\partial_e(A \setminus A_H)| + |\partial_e(B \setminus B_H)| &\leq 2|\partial_e H| + 2|F| - |\partial_e A_H| - |\partial_e B_H| \\ &< 4k - 2k \\ &= 2k. \end{aligned}$$

Hence, either $|\partial_e(A \setminus A_H)|$ or $|\partial_e(B \setminus B_H)|$, say the former, is strictly smaller than k. Since G is k-edge-connected, this implies that $A \setminus A_H$ is empty. But then $A \subsetneq V(H)$, a contradiction to the minimality of H. □

Let us now turn to multigraphs, that is, graph with parallel edges, which sometimes appear to be the more appropriate objects when studying edge-connectivity. Note that we may apply the proof of Theorem 11.3.2 (c) with only small modifications[6] to multigraphs. Defining the edge-degree of an end ω of a multigraph in the usual way as the supremum of the cardinalities of the sets of edge-disjoint rays from ω, we thus get:

Corollary 11.5.2. *Let G be an edge-minimally k-edge-connected multigraph. Then $|\{v \in V(G) : d(v) = k\} \cup \{\omega \in \Omega(G) : d_e(\omega) \leq k\}| \geq 2$.*

In particular, this yields that every finite edge-minimally k-edge-connected multigraph has at least two vertices of degree k.

However, a statement in the spirit of Theorem 11.2.2 (c) does not hold for multigraphs, no matter whether they are finite or not. For this, it suffices to consider the graph we obtain by multiplying the edges of a finite or infinite path by k. This operation results in a multigraph which has no more than the two vertices/ends of (edge)-degree k which were promised by Corollary 11.5.2.

[6] We will then have to use a version of Lemma 9.3.1 for multigraphs. Observe that such a version holds, as we may apply Lemma 9.3.1 to the simple graph obtained by subdividing all edges of the multigraph.

11.6 Vertex-minimally k-edge-connected graphs

In this section we shall prove Theorem 11.3.2 (d). The proof is based on Lemma 11.6.2, which at once yields Theorem 11.2.2 (d), the finite version of Theorem 11.3.2 (d). The proof of this lemma is very much inspired by Mader's original proof of Theorem 11.2.2 (d) in [69].

We need two auxiliary lemmas before we get to Lemma 11.6.2. For a set $X \subseteq V(G) \cup E(G)$ in a graph G write $X_V := X \cap V(G)$ and $X_E := X \cap E(G)$.

Lemma 11.6.1. *Let $k \in \mathbb{N}$. Let G be a graph, let $S \subseteq V(G) \cup E(G)$ with $|S| \leq k$, and let C be a component of $G - S$ so that in G, every vertex of C has a neighbour in $G - S - C$. Then C contains a vertex of degree at most k.*

Proof. Suppose that the vertices of C all have degree at least $k+1$. Then each sends at least $k + 1 - |S_V| - (|C| - 1)$ edges to $G - S - C$. This means that

$$|C|(k + 1 - |S_V| - (|C| - 1)) \leq |S_E| = k - |S_V|.$$

So $|C|(k - |S_V| - |C| + 1) \leq k - |S_V| - |C|$, which, as $|C| \geq 1$, is only possible if $|C| > k - |S_V|$. But this is impossible, because each vertex of C is incident with an edge in S_E, and hence $|C| \leq |S_E| = k - |S_V|$. □

As usal, the edge-connectivity of a graph G is denoted by $\lambda(G)$. Also, in order to make clear which underlying graph we are referring to, it will be useful to write $\partial_e^G H = \partial_e H$ where a H is a region of a graph G.

Lemma 11.6.2. *Let $k \in \mathbb{N}$, let G be a k-edge-connected graph, let $x \in V(G)$ and let C be an inclusion-minial region of G with the property that C has a vertex x so that $|\partial_e^{G-x} C| = \lambda(G - x) < k$. Suppose for each $y \in V(C)$, the graph $G - y$ has a cut of size $< k$. Then $C - x$ contains a vertex of degree k (in G).*

Proof. If every vertex of C has a neighbour in $D := G - S - C$ then we may apply Lemma 11.6.1 and are done. So let us assume that there is a vertex y all of whose neighbours lie in $C \cup x$. By assumption, $G - y$ has a cut F of size $\lambda(G - y) < k$, which splits $G - y$ into A and B, with $x \in V(A)$, say.

Since G is k-edge-connected, F is not a cut of G. Hence y has neighbours in both A and B. Thus, as $N(y) \subseteq V(C) \cup x$ has no neighbours in D, and $x \in V(A)$, it follows that $B \cap C \neq \emptyset$. By the choice of C and x we may thus assume that $|\partial_e^{G-x}(B \cap C)| > \lambda(G - x) = |F|$. So,

$$|\partial_e^{G-x}(A \cap D)| \leq |\partial_e^{G-x} C| + |F| - |\partial_e(B \cap C)| < |\partial_e^{G-x} C| = \lambda(G - x),$$

implying that $A \cap D = \emptyset$. That is, $A \cup y \subsetneq C \cup x$, a contradiction to the choice of C. □

Finite graphs clearly do contain inclusion-minimal regions C as in Lemma 11.6.2, which hence implies Theorem 1 (d). We then apply the lemma to any inclusion-minimal region with the desired properties that is contained in $G - (C - x)$ in order to find a second vertex of small degree. We thus get Theorem 2 (d):

Corollary 11.6.3 (Theorem 2 (d))**.** *Let G be a finite vertex-minimally k-edge-connected graph. Then G has at least two vertices of degree k.*

This means that for a proof of Theorem 11.3.2 (d) we only need to worry about the infinite regions, which is accomplished in the next lemma.

Lemma 11.6.4. *Let $k \in \mathbb{N}$, let G be a vertex-minimally k-edge-connected graph and let D be a region of G. Let $x \in V(D)$ such that $|\partial_e^{G-x} D| = \lambda(G-x) < k$. Suppose G has no inclusion-minimal region C with the property that C contains a vertex y so that $|\partial_e^{G-y} C| = \lambda(G-y) < k$ and $C \subseteq D$. Then G has an end of vertex-degree $\leq k$ with rays in D.*

Proof. We construct a sequence of infinite regions D_i, starting with $D_0 := D$ which clearly is infinite. Our regions will have the property that $D_i \subseteq D_{i-1} - \partial D_{i-1}$.

In step $i \geq 1$, for each pair of vertices in $\partial_v^G D_{i-1}$, take a set of k edge-disjoint paths joining them: the union of all these paths gives a finite subgraph H of G. Since D_{i-1} was infinite, $D_{i-1} - H$ still is, and thus contains a vertex y. Since G is vertex-minimally k-edge-connected, $G - y$ has a cut of size less than k, which splits $G - y$ into A and B, say, which we may assume to be connected. Say A contains a vertex of $\partial_v^G D_{i-1}$. Then $\partial_v^G D_{i-1} \subseteq V(A)$, and thus $B \subseteq D_{i-1}$. Observe that $D_i := B \cup y$ is infinite, as otherwise it would contain an inclusion-minimal region C as in the statement of the lemma.

As all the D_i are connected, it is easy to construct a ray R which has a subray in each of the D_i. Say R belongs to the end $\omega \in \Omega(G)$. We will show that $d_v(\Omega) \leq k$, which in turn proves the lemma.

Suppose otherwise. Then ω contains a set of $k+1$ disjoint rays. Let S be the set of starting vertices of these rays. Since $D_i \subseteq D_{i-1} - \partial D_{i-1}$ for all i, there is an $n \in \mathbb{N}$ such that $S \cap V(D_n) = \emptyset$. (To be precise, one may take $n := \max_{s \in S, v \in \partial_v^G D_0} dist(s, v) + 1$.) But then, it is impossible that all rays of $\mathcal{R}$ have subrays in D_n. □

We finally prove Theorem 11.3.2 (d).

Proof of Theorem 11.3.2 (d). Let $x \in V(G)$, and let F be a cut of $G - x$ with cardinality $|F| = \lambda(G - x)$. Say F splits $G - x$ into A and B. First suppose that one of A and B, say A, contains an inclusion-minimal region C such that C has a vertex y with the property that $|\partial_e^{G-y} C| = \lambda(G - y) < k$. Then Lemma 11.6.2 finds a vertex of degree at most k in $C - y$.

Now, if also $D := G - (C - y)$ contains an inclusion-minimal region C' such that C' has a vertex y' with the property that $|\partial_e^{G-y'} C'| = \lambda(G-y') < k$, then we may apply Lemma 11.6.2 again to find a second vertex of degree at most k in G. On the other hand,if D does not contains such a region, we use Lemma 11.6.4 to find an end of the desired degree. Finally, if both A and B do not have an inclusion-minimal region as above, we apply Lemma 11.6.4 to both to find the desired end of small degree. □

Chapter 12

Duality of ends

12.1 Duality of graphs

In 1932 Whitney [94] introduced the concept of dual graphs: a graph[1] G^* is a dual of a finite graph G if there exists a bijection $* : E(G) \to E(G^*)$ so that a set $F \subseteq E(G)$ is a circuit of G precisely when F^* is a bond in G^*.

Nowadays graph duality is a standard subject, treated in any textbook (see e.g. [31]). The main properties of the dual is its symmetry (i.e. that G is a dual of G^*), and its uniqueness for 3-connected planar graphs. Another important well known feature is the fact that a finite graph is planar if and only if its has a dual. This is a theorem of Whitney [94].

Building on work by Thomassen [90, 91], Bruhn and Diestel [11] extended duality to (a superclass of) locally finite graphs. They showed that with their notion, which we shall also use in this chapter, many properties of dual graphs are retained in infinite graphs. These include the three aspects mentioned in the previous paragraph.

This chapter, which is based on [14], contains a study of the relation between the end space of a graph and the end space of its dual. The first result we present states that there exists a homeomorphism between these two spaces that arises in a natural way from the bijection $*$ on the edges.

More precisely, we will demonstrate that, given a pair G, G^* of (infinite) duals, the endvertices of a set $F \subseteq E(G)$ converge towards an end ω of G if and only if the endvertices of F^* converge towards the dual end ω^*. For this, we shall define a topological space $|G|$ on the point set of G together with

[1]Throughout this chapter, let us allow all our graphs are allowed to have loops and parallel edges, with the exception of 2-connected graphs, which we require to be loopless, and of 3-connected graphs, which, in addition, cannot have parallel egdes.

its set of ends $\Omega(G)$ (and analogously for G^*). This space will be defined in Section 12.2, see also Chapter 8 of [31]. Further discussion of Theorem 12.1.1 can be found in Section 12.4.

Theorem 12.1.1. [14] *Let G and G^* be 2-connected dual graphs. Then there is a homeomorphism $* : \Omega(G) \to \Omega(G^*)$, where the two spaces are endowed with the subspace topology of $|G|$ resp. $|G^*|$, so that for all $F \subseteq E(G)$ and ends ω it holds that*

$$\omega \in \overline{F} \text{ if and only if } \omega^* \in \overline{F^*}. \tag{12.1}$$

Thick ends are defined as those ends that contain an infinite set of disjoint rays. They play an important role in the study of the automorphism group of a graph, see for instance Halin [49]. We will prove that thickness is preserved in the dual end:

Theorem 12.1.2. [14] *Let G, G^* be a pair of dual graphs, and let ω be an end of G. Then ω is thick if and only if ω^* is thick.*

In fact, we shall prove something stronger. Exclusively in this chapter we shall use the notion of the *degree* $d_G(\omega)$ of an end ω, which will be a slightly modified version of the vertex-degree used earlier (for a locally finite graph they will coincide). For the precise definition see Section 12.7.

Theorem 12.1.2 will be a consequence of the following result:

Theorem 12.1.3. [14] *Let G and G^* be a pair of 2-connected dual graphs, and let ω be an end of G. Then $d_G(\omega) = d_{G^*}(\omega^*)$.*

In order to prove Theorem 12.1.3, we make use of a notion of connectivity, introduced by Tutte [92], that coincides with the connectivity of the cycle-matroid of the graph. As a by-product we obtain the following result, which for finite graphs is a theorem of Tutte [92].

Theorem 12.1.4. [14] *Let G and G^* be a pair of dual graphs, and let $k \geq 2$. Then G is k-Tutte-connected if and only if G^* is k-Tutte-connected.*

We will define Tutte-connectivity in Section 12.6 (all other definitions can be found in the next section), but let us remark here that a graph is 3-Tutte-connected if and only if it is 3-connected. Therefore, Theorem 12.1.4 has the following consequence:

Corollary 12.1.5 (Thomassen [91])**.** *Let G and G^* be a pair of dual graphs. Then G is 3-connected if and only if G^* is 3-connected.*

Duality for infinite graphs was first explored by Thomassen. Faced with the incongruity that an infinite graph may have infinite cuts as well as finite ones but (in the traditional definition) only finite circuits he chose to ignore infinite cuts. Consequently, G^* is a dual of G, in the sense of Thomassen, if for all *finite* sets $F \subseteq E(G)$, F is a circuit precisely when F^* is a bond. This concept allowed him to prove an infinite version of Whitney's planarity criterion: a 2-connected graph G has a (Thomassen-)dual if and only if it is planar and satisfies

every two vertices of G can be separated by finitely many edges. (†)

However, Thomassen's definition is not completely satisfactory, as the symmetry in taking duals is lost, as well as the uniqueness of the duals of 3-connected graphs. These deficits are ultimately due to the disregard of infinite cuts.

Infinite circuits, which have been proposed by Diestel and Kühn [32, 33, 34], promise a way out of this dilemma. Taking infinite circuits into account led to the more restrictive definition of duals in [11]: there, a set $F \subseteq E(G)$, finite or infinite, is a circuit if and only if F^* is a bond. These duals overcome the drawbacks of Thomassen's definition, i.e. they retain the basic properties of finite duals. We will define and very briefly discuss infinite circuits in the next section.

12.2 The cycle space of an infinite graph

In this chapter, we shall introduce the cycle space of an infinite graph. For this we have to define a topological space on the point set of the graph plus its ends. These notions have been first suggested by Diestel and Kühn [32, 33, 34]. A circuit of some fixed graph G will be defined as the edge set of a homeomorphic image of the unit circle in a certain topological space based on G.

We shall introduce the topology in two steps. First we define a topological space $|G|$, whose points are the vertices and ends of G, as well as the interior points of edges of G. In the second step we shall identify some of the points of $|G|$.

So, in order to define $|G|$, see G as endowed with the topology of a 1-complex, so every edge is homeomorphic to the unit interval and a basic open neighbourhood of a vertex consists of the union of half-open edges, one for each incident edge. In order to describe the neighbourhoods of an end ω, pick a finite vertex set S, and denote the component of $G - S$ that contains a ray of ω (and thus a subray for every ray in ω) by $C(S,\omega)$. We say that ω *belongs to* $C(S,\omega)$. A basic open neighbourhood of ω now consists of $C(S,\omega)$, all ends that have a ray in $C(S,\omega)$ and the union of all interior points of edges between S and $C(S,\omega)$. In the case of a locally finite graph the resulting space $|G|$ is called the *Freudenthal compactification* of G.

In non-locally finite graphs, we say that a vertex v *dominates* an end ω, if there are infinitely many paths between v and a ray in ω that pairwise only meet in v. We define an equivalence relation $\sim$ on $|G|$ as follows. For two ends ω and ω', let $\omega \sim \omega'$ if both ω and ω' are dominated by the same vertex. For a vertex v and an end ω, let $v \sim \omega$ if v dominates ω. We denote by $\tilde{G}$ the quotient space of $|G|$ under the equivalence relation $\sim$. In particular, if G is locally finite, then $\tilde{G} = |G|$. Observe furthermore that, if G satisfies (†), then no two vertices of G are identified in $\tilde{G}$.

We shall need to work within both spaces $|G|$ and $\tilde{G}$. In order to distinguish between closures of sets $X \subseteq V(G) \cup E(G)$ in the two spaces, we write $\overline{X}$ for the closure of X in $|G|$, and $\widetilde{X}$ for the closure of X in $\tilde{G}$.

Next, we define *circles* in $\tilde{G}$ as the homeomorphic images of the unit circle. If a circle contains an interior point of an edge then it contains the whole edge. Thus it makes sense to speak of the edge set of a circle, which is called a *circuit*. The homeomorphic image of the unit interval $[0,1]$ in $\tilde{G}$ is an *arc*. Observe that circuits as well as arcs must contain edges.

For the merits of infinite circuits and the topological cycle space, which is based on this definition, see the overview article by Diestel [30]. Let us just mention here that $\mathcal{C}(G)$ retains all the basic properties of the cycle space of a finite graph [10, 32, 33, 34] (which the space obtained considering only finite cycles does not), and, using infinite cycles, many well-known theorems for finite graphs have verbatim extensions to locally finite graphs[2]. These extensions include MacLane's theorem, Tutte's/Kelmans' planarity criterion, Nash-Williams' Arboricity theorem, Gallai's theorem about cycle-cocycle partitions, Tutte's generating theorem, Tutte's/Nash-Williams' tree-packing theorem, and the already mentioned Whitney's planarity criterion [9, 11, 12, 16, 85]. Let us also remark that a more general approach to cycle spaces has been pursued by Richter and Vella [93], who define (infinite) circuits for a wider range of topological spaces.

[2]For most, the restriction to locally finite graphs can be lowered by only demanding that no two vertices are connected by infinitely many paths. But then, the cycle space notion (or rather the underlying topology on the graph) has to be modified.

12.3 Duality for infinite graphs

In order to define duals for infinite graphs, let us first remark that we have to restrict ourselves to graphs G that satisfy (†). In fact, as Thomassen [91] observed, this is a necessary condition for a graph to have a dual (in Thomassen's and thus in our sense as well).

Now, we call a graph G^* a *dual* of G if there is a bijection $* : E(G) \to E(G^*)$ so that a (finite or infinite) set $F \subseteq E(G)$ is a circuit of G precisely when F^* is a bond in G^*. (A *bond* is a minimal non-empty cut.)

The dual G^* then can be seen to satisfy (†) as well. So, the class of graphs with (†) is closed under taking duals, unlike the class of locally finite graphs. Whenever we speak of duals we will therefore tacitly assume that the original graph (and then automatically the dual too) satisfies (†). We refer to [11] for more details.

We list two properties of duals, that will be needed throughout the paper.

Lemma 12.3.1. *Let G and G^* be a pair of dual graphs. Then G is 2-connected if and only if G^* is 2-connected.*

The lemma follows easily from the fact that a every two edges lie in a common circuit if and only if the graph is 2-connected, which is the case precisely when every two edges lie in a common bond. Variants of this lemma can be found in Thomassen [90] as well as in [11].

Theorem 12.3.2. **[11]** *Let G^* be a dual graph of a graph G. Then G is also a dual of G^*.*

As a convenience we will, for a set F of edges, write $V[F]$ to denote the set of endvertices of the edges in F.

12.4 Discussion of our results

Before turning to the proofs of our main results, we shall discuss why their statements have the precise forms they do. Let us start with the bijection we wish to define between the end spaces of two dual graphs G and G^*. Our mapping will be an extension of the bijection $* : E(G) \to E(G^*)$ on the edges (and we will therefore, slightly abusing notation, denote it with $*$ as well). More precisely, we aim at a bijection $*$ between $\Omega(G)$ and $\Omega(G^*)$, so that for all $F \subseteq E(G)$, the endvertices of F converge against an end ω of G if and only if the endvertices of F^* converge against ω^*.

In the space $\tilde{G}$, which is instrumental in the definition of duality, the accumulation points of vertex sets are the identification classes of ends. Recall that any two ends that cannot be separated by finitely many edges are identified, giving rise to larger equivalence classes of rays called *edge-ends* by some authors (e.g. Hahn, Laviolette and Širáň [48]). So, should we not search for a bijection of the edge-ends rather than of the ends?

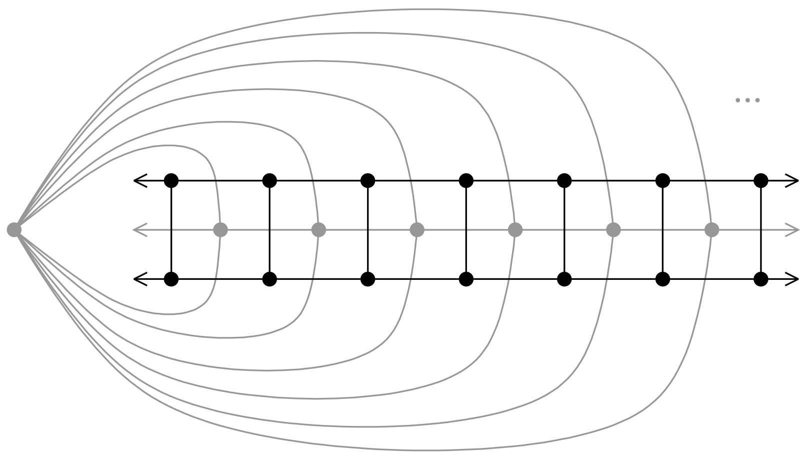


Figure 12.1: No correspondence between edge-ends of duals

Figure 12.1 demonstrates that there is no hope for a bijection between edge-ends (even without any structural requirements). The double ladder has two edge-ends, while its dual graph has only one edge-end.

The reason that this attempt fails lies in the nature of duals. The existence of finite edge-cuts between (edge-)ends will not be preserved in the dual. In fact, such a (minimal) cut corresponds to a circuit in the dual, which need not separate anything. By contrast, a vertex-separation whose deletion results in two sufficiently large sides does, in some sense, carry over to the dual graph; this is the essence of Theorem 12.1.4 and will be further explored in Section 12.6.

Our bijection will thus be between the ends of G and G^*. This means that we will work in $|G|$, since any two identified ends cannot be distinguished topologically in $\tilde{G}$. Endowing $\Omega(G)$ resp. $\Omega(G^*)$ with the subspace topology of $|G|$ resp. $|G^*|$, we will show the existence of a bijection $\Omega(G) \to \Omega(G^*)$ that is structure-preserving in the sense above. Moreover, Theorem 12.1.1 ensures that $*$ is a homoeomorphism:

Theorem 12.1.1. [14] *Let G and G^* be 2-connected dual graphs. Then there* *is a homeomorphism* $*: \Omega(G) \to \Omega(G^*)$*, where the two spaces are endowed with the subspace topology of* $|G|$ *resp.* $|G^*|$*, so that*

$$\text{for all } F \subseteq E(G) \text{ and ends } \omega \text{ it holds that } \omega \in \overline{F} \text{ if and only if } \omega^* \in \overline{F^*}. \tag{12.1}$$

We remark that the requirement that G and G^* are 2-connected cannot be dropped. This is illustrated by the example of the double ray. Every dual of the double ray is a graph whose edge set is the union of countably many loops, and thus contains no end at all.

We shall prove Theorem 12.1.1 in the next section.

Let us now turn to our second objective: showing that our bijection $*$ preserves thickness. This will be achieved in Theorem 12.1.2. Again, we are confronted with the question why focus on preserving (vertex-)thickness instead of "edge-thickness", i.e. the existence of infinitely many edge-disjoint rays in an end.

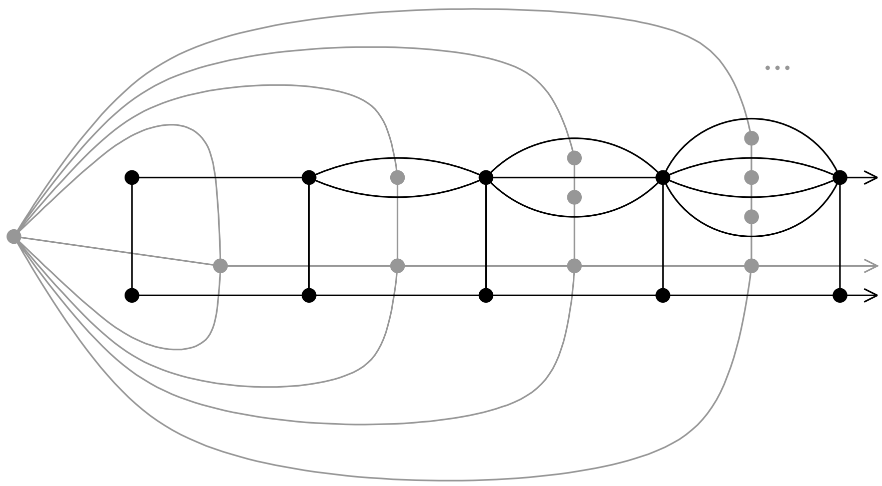


Figure 12.2: Edge-thick end with edge-thin dual end

This is answered by Figure 12.2, which shows a graph that has a single edge-thick end while the unique end of its dual graph does not even possess two edge-disjoint rays. The reason is the same as above: although (or because) the notion of duals is based on edges and operations with edges, the existence of (small) edge-separators is not preserved in the dual.

Since not all vertex-separators are preserved in the dual, connectivity is not an invariant of (finite or infinite) duals, as we have already remarked in the introduction. But, the related notion of *Tutte-connectivity* is. We defer to Section 12.6 for the definition; suffice it to say here that there are two reasons why a graph may have low Tutte-connectivity: Either it has a small vertex-separator or it contains a small circuit. In Section 12.6, we prove that Tutte-connectivity is an invariant of infinite duals, too (Theorem 12.1.4).

Theorem 12.1.4 is an important milestone on our way to proving Theorem 12.1.3 and thus Theorem 12.1.2. Our proof of Theorem 12.1.4 differs from the usual proof of Tutte's finite version, which is done in two steps. First, one shows that Tutte-connectivity coincides with the connectivity of the cycle-matroid of the graph. Then one observes that matroid connectivity is invariant under duality.

If we want to use this approach for Theorem 12.1.4 as well, we first have to answer two questions. Which notion of infinite matroids should we use? And how do we define higher connectivity in a matroid?

The first question is easy to answer. Although it is sometimes claimed that there is no proper concept of an infinite matroid that provides duality and the existence of bases at the same time, B-matroids, as defined by Higgs [56], accomplish that (see also Oxley [73]). Moreover, one can prove that duality in B-matroids is compatible with taking dual graphs. While the second problem, the definition of higher connectivity, can also be overcome in a satisfactory way, its solution together with the introduction of B-matroids would take quite a bit of time and effort. Therefore, we will, in Section 12.6, present a matroid-free proof of Theorem 12.1.4.

12.5 $*$ induces a homeomorphism on the ends

Before we are able to prove Theorem 12.1.1, we need three lemmas.

The proof of the first of these lemmas is not hard, and also can be found in [31, Lemma 8.2.2]. We only need the lemma for the proof of Lemma 12.5.2 below.

Lemma 12.5.1. *Let G be a connected graph, and let U be an infinite subset of $V(G)$. Then G contains a ray R with infinitely many disjoint R–U paths or a subdivided star with infinitely many leaves in U.*

Lemma 12.5.2. *Let G be a 2-connected graph satisfying (†). If U is an infinite set of vertices then $\overline{U}$ contains an end of G.*

Proof. Suppose otherwise. Then there is no ray R in G with infinitely many disjoint R–U paths. So, an application of Lemma 12.5.1 yields a subdivided star S that contains an infinite subset U' of U. We delete the centre of S and apply Lemma 12.5.1 again, this time to U', which yields another subdivided star S' with infinitely many leaves in U'. But then, the centre of S and the centre of S' are infinitely connected, contradicting (†). □

Lemma 12.5.3. *Let G be a 2-connected graph, and let X and Y be two sets of edges such that $\overline{X} \cap \overline{Y} \cap \Omega(G) \neq \emptyset$. Then there are infinitely many (edge-)disjoint finite circuits each of which meets both X and Y.*

Proof. Let $\mathcal{Z}$ be an $\subseteq$-maximal set of finite disjoint circuits so that each $C \in \mathcal{Z}$ meets both X and Y, and suppose that $|\mathcal{Z}|$ is finite. Putting $Z := \bigcup \mathcal{Z}$, we pick for every two $x, y \in V[Z]$ for which it is possible an x–y path $P_{x,y}$ that is edge-disjoint from Z. Denote by Z' the union of Z with the edge sets of all these paths, and observe that still $|Z'| < \infty$.

We claim that for every component K of $G - V[Z']$ it holds that

$$\text{for every } v, w \in N(K) \text{ there is a } v\text{–}w \text{ path in } (V[Z'], Z' \setminus Z). \tag{12.2}$$

Indeed, by construction, there are $x, y \in V[Z]$ and (possibly trivial) v–x resp. w–y paths Q_v resp. Q_w with edges in $Z' \setminus Z$. Then x and y are connected through $K \cup Q_v \cup Q_w \subseteq G - Z$. Hence in $P_{x,y} \cup Q_v \cup Q_w \subseteq (V[Z'], Z' \setminus Z)$ we find a v–w path. This proves (12.2).

Now, because $\overline{X} \cap \overline{Y}$ contains an end, there exists a component K of $G - V[Z']$ which contains infinitely many vertices of both $V[X]$ and $V[Y]$. Choose edges $e_X, e_Y \in E(K) \cup E(K, G - K)$ so that $e_X \in X$, and $e_Y \in Y$. Since G is 2-connected, there is a finite circuit C which contains both e_X and e_Y. The maximality of $\mathcal{Z}$ implies that C meets Z in at least one edge. In particular, C contains the edge sets of (possibly identical) $N(K)$-paths P_X and P_Y so that $e_X \in E(P_X)$, and $e_Y \in E(P_Y)$.

Being connected, K contains a $V(P_X)$–$V(P_Y)$ path P. Thus, we find in $P \cup P_X \cup P_Y$ an $N(K)$-path P' with $e_X, e_Y \in E(P')$. By (12.2), there exists a path R in $(V[Z'], Z' \setminus Z)$ between the endvertices of P'. Now, $E(P') \cup E(R)$ is a circuit that meets both X and Y but is edge-disjoint from Z, a contradiction to the maximality of $\mathcal{Z}$. □

We can now prove the first main result of this chapter.

Proof of Theorem 12.1.1. We start by claiming that for each $F \subseteq E(G)$ and each end ω of G the following is true:

$$\text{if } \overline{F} \cap \Omega(G) = \{\omega\} \text{ then } \overline{F^*} \text{ contains exactly one end.} \tag{12.3}$$

Suppose the claim is not true. By Lemma 12.5.2, this cannot be because $\overline{F^*}$ fails to contain an end; rather there must be (at least) two ends, α_1 and α_2, in $\overline{F^*}$. Take a finite connected subgraph T of G^* so that $V(T)$ separates α_1 and α_2 in G^*. For $i = 1, 2$, denote by K_i the component of $G^* - T$ to which α_i belongs, and set $X_i^* := (E(K_i) \cup E(K_i, T)) \cap F^*$. Since each of the X_i^* is infinite, it follows from Lemma 12.5.2 that $\overline{X_i}$ contains an end. As $\overline{X_i} \subseteq \overline{F}$, this end must be ω. Hence, Lemma 12.5.3 yields disjoint finite circuits $C_1, C_2, \dots$ in G each of which meets X_1 as well as X_2.

We claim that each of the bonds C_i^* contains an edge of T. Indeed, let M_1 and M_2 be the two components of $G^* - C_i^*$. Since C_i^* meets both X_1^* and X_2^*, each M_j contains a vertex in $K_1 \cup T$ and a vertex in $K_2 \cup T$. As, for $j = 1, 2$, M_j is connected it follows that $V(M_j) \cap V(T) \neq \emptyset$. So, since T is connected, there is an M_1–M_2 edge in $E(T)$, i.e. $C_i^* \cap E(T) \neq \emptyset$, for each $i \in \mathbb{N}$. This yields a contradiction since the C_i^* are disjoint but T is finite. Therefore, Claim (12.3) is established.

Now, we define $* : \Omega(G) \to \Omega(G^*)$. Given an end $\omega \in \Omega(G)$, pick any set $F \subseteq E(G)$ with $\overline{F} \cap \Omega(G) = \{\omega\}$ (choose, for instance, the edge set of a ray in ω). Define $\omega^* = \omega^*(F)$ to be the, by (12.3), unique end in $\overline{F^*}$. To see that this mapping is well-defined, i.e. that it does not depend on the choice of F, consider a second set $D \subseteq E(G)$ as above, and observe that $\omega^*(D) = \omega^*(D \cup F) = \omega^*(F)$. Since G is a dual of G^* (Theorem 12.3.2), we may apply (12.3) to G^* and see that $*$ is a bijection and satisfies (12.1).

Next, we prove that $* : \Omega(G) \to \Omega(G^*)$ is continuous. For this, let an end $\omega^* \in \Omega(G^*)$ and an open neighbourhood $U^* \subseteq \Omega(G^*)$ of ω^* be given. Then there exists a finite vertex set $S \subseteq V(G^*)$, and a component K of $G^* - S$ so that $W^* := K \cap \Omega(G^*) \subseteq U^*$.

Setting $F^* := E(G^*) \setminus (E(K) \cup E(S, K))$, we observe that $W^* = \Omega(G^*) \setminus \overline{F^*}$. Hence, by (12.1), $W = \Omega(G) \setminus \overline{F}$. So, W is an open neighbourhood of ω whose image is contained in U^*. Finally, by interchanging the roles of G and G^* we see that the inverse of $*$ is continuous as well. □

12.6 Tutte-connectivity

In this and in the next section, we are concerned with how (Tutte-)connectivity is preserved in the dual. The main idea underlying our proofs is the duality of spanning trees: given a pair of finite connected dual graphs G and G^*, a set D is the edge set of a spanning tree of G, if and only if $E(G^*) \setminus D^*$ is the edge set of a spanning tree in G^*.

For a pair of infinite graphs, the situation is slightly more complicated.

In fact, if $E(G^*)\setminus D^*$ is the edge set of a spanning tree, then $(V(G), D)$ might very well be disconnected—topologically, however, $\widetilde{D}$ (the closure of D in $\tilde{G}$) is always connected.

Moreover, $\widetilde{D}$ forms a *topological spanning tree* (TST for short) of $\tilde{G}$: a path-connected circuit-free subspace of $\tilde{G}$ that contains all vertices of G, and every edge of which it contains an interior point. For more on the relation between spanning trees in G and G^* see [11]. TSTs were first introduced by Diestel and Kühn in [34], where it is proved that $\tilde{G}$ always has a TST provided G is connected.

We will use the tree duality implicitly in the key lemma, Lemma 12.6.4, below. The next two lemmas help to relate the tree duality to vertex separations.

Lemma 12.6.1. *Let G be a graph satisfying (†), let T be a subgraph that does not contain any circuits, and let $U \subseteq V(T)$ such that $0 < |U| < \infty$. Then there exists a set $F \subseteq E(T)$ of size at most $|U| - 1$ so that every arc in $\widetilde{T}$ between two vertices in U meets F.*

Proof. We use induction on $|U|$. The assertion is trivial for $|U| = 1$, so for the induction step asume that $|U| > 1$. Choose $v \in U$, then by the induction assumption there is a set $D \subseteq E(T)$ such that each vertex w of $U \setminus \{v\}$ lies in a different path-component K_w of $\widetilde{T} - D$. If there is no vertex $w \in U \setminus \{v\}$ such that $v \in K_w$, we are done, so assume there is such a w.

Observe that there exists exactly one v–w arc A in $\widetilde{T} - D$. Indeed, if there were two, then it is easy to see that the edge set of their union would contain a circuit. Now, choose any edge e on A, and set $F := D \cup \{e\}$. Clearly, F is as desired, which completes the proof. □

Lemma 12.6.2. *Let H be a connected graph, let $F \subseteq E(H)$, and let $W \subseteq V(H)$. If every W-path in H meets F then $|F| \geq |W| - 1$.*

Proof. Since no two vertices of W can lie in the same component of $H - F$, we deduce that $H - F$ has at least $|W|$ components. As each deletion of a single edge increases the number of components by at most one, $H - F$ can have at most $|F| + 1$ components. □

Let us now introduce the notion of Tutte-connectivity, see Tutte [92]. For finite graphs, the Tutte-connectivity coincides with the connectivity of the cycle-matroid of the graph. We remark that for $k \in \{2, 3\}$, a graph is k-Tutte-connected if and only if it is k-connected. For greater k the two notions of connectivity are not equivalent.

Definition 12.6.3. *A k-*Tutte-separation *of a graph G is a partition (X, Y) of $E(G)$ so that $|X|, |Y| \geq k$ and so that at most k vertices of G are incident with edges in both of X and Y.*
*We say that a graph G is k-*Tutte-connected *if G has no ℓ-Tutte-separation for any $\ell < k$.*

Consider a k-Tutte-separation (X, Y) in a (2-connected) graph G with a dual G^*. To prove that Tutte-connectivity is invariant under taking duals, we would ideally like to see that (X^*, Y^*) is a k-Tutte-separation in G^*. This, however, is not always true—if the two sides of the separation do not induce connected subgraphs of G^*, then the number of vertices in $V[X^*] \cap V[Y^*]$ can be much higher than k. Thus we will strengthen the requirements and lessen our expectations. By demanding the subgraph $(V[Y], Y) - V[X]$ to be connected, we shall be able to guarantee that at least $(V[Y^*], Y^*)$ is connected. Moreover, we will be content with finding an ℓ-Tutte-separation of G^* for some $\ell \leq k$ that is derived from (X^*, Y^*).

The statement of the next lemma, which accomplishes just that, is a bit more general than we need for Theorem 12.1.4, as we shall reuse it for Theorem 12.1.3.

Lemma 12.6.4. *Let G and G^* be a pair of 2-connected dual graphs, and let (X, Y) be a k-Tutte-separation such that $C_Y := (V[Y], Y) - V[X]$ is non-empty and connected, and such that $Y = E(C_Y) \cup E(C_Y, V[X])$. Then*

(i) *there exists a component L of $(V[X^*], X^*)$ so that $(E(L), E(G^*) \setminus E(L))$ is an ℓ-Tutte-separation for some $\ell \leq k$; and*

(ii) *for each component K of $(V[X^*], X^*)$ with $|E(K)| \geq k$ it holds that $(E(K), E(G^*) \setminus E(K))$ is a k-Tutte-separation.*

Proof. First, we prove that

$$\widetilde{Y^*} \text{ is path-connected in } \tilde{G}^*. \tag{12.4}$$

Suppose that this is not the case. Then we can write Y as the disjoint union of two sets Y_1 and Y_2 so that there is no Y_1^*–Y_2^* arc in $\tilde{G}^*$ that only uses edges from Y^*.

In particular, there is no circle in $\tilde{G}^*$ that only uses edges from Y^* and meets both Y_1^* and Y_2^*. Equivalently, there is no bond in G that only uses edges from Y, and meets both Y_1 and Y_2.

However, since C_Y is connected and since every edge in Y is incident with a vertex in C_Y, there is a vertex $x \in V(C_Y)$ which is incident with both

Y_1 and Y_2. Observe that the cut B_x of G, which consists of all edges incident with x, is a subset of Y. As G is 2-connected, B_x is a bond, which yields the desired contradiction and thus proves (12.4).

Now, set $U := V[X] \cap V[Y]$ and $W := V[X^*] \cap V[Y^*]$. Observe that each vertex in W is incident with both X^* and Y^*. So, if $|W|$ is infinite, then Lemma 12.5.2 implies that $\overline{X^*} \cap \overline{Y^*}$ contains an end, while $\overline{X} \cap \overline{Y}$ does not (as X and Y are finitely separated by U). This contradicts Theorem 12.1.1. We have thus shown that

$$|W| \text{ is finite.} \tag{12.5}$$

Let T_X be the edge set of a maximal topological spanning forest of $\widetilde{X}$, i.e. the union of TSTs of the spaces $\tilde{C}$ corresponding to the components C of $(V[X], X)$. We point out that every circuit of G that lies entirely in X is a circuit of $(V[X], X)$. It follows that T_X does not contain any circuits of G.

Next, we prove that

$$\textit{every } W\textit{-path in } (V[X^*], X^*) \textit{ meets } T_X^*. \tag{12.6}$$

Suppose there is a W-path whose edge set D^* lies in $X^* \setminus T_X^*$. By (12.4), there is a circuit C^* of G^* with $C^* \cap X^* = D^*$. Thus, C is a bond in G, and hence D is a finite cut of the subgraph $(V[X], X)$ of G. Consequently, D contains a bond B of $(V[X], X)$, which then is completely contained in a component K_B of $(V[X], X)$. As $B \subseteq D \subseteq X \setminus T_X$, the intersection of B with T_X is empty. Thus, B is a finite cut of K_B that is disjoint from T_X but that separates two vertices incident with T_X. Since, on the other hand, $\widetilde{T_X}$ restricted to $\widetilde{K_B}$ is path-connected, we obtain a contradiction. This proves (12.6).

Next, Lemma 12.6.1 yields a set $F \subseteq T_X$ of at most $|U| - 1$ edges so that every U-arc in $\widetilde{T_X} \subseteq \widetilde{X}$ meets F. This means that every circuit C of G with $C \cap X \subseteq T_X$ meets F. Thus, every bond B^* of G^* with $B^* \cap X^* \subseteq T_X^*$ meets F^*. Hence, denoting by $\mathcal{K}$ the set of components of $(V[X^*], X^*)$, we obtain that

$$\textit{for every } K \in \mathcal{K}\textit{, the graph } H_K := K - (T_X^* \setminus F^*) \textit{ is connected.} \tag{12.7}$$

Now, for every $K \in \mathcal{K}$, observe that by (12.6), every W-path in H_K meets F^*. So, by (12.7), we may apply Lemma 12.6.2 to H_K. Doing so for each $K \in \mathcal{K}$, we obtain that $|F^*| \geq |W| - |\mathcal{K}|$. On the other hand, $|F^*| = |F| \leq |U| - 1$ by the choice of F, implying that

$$|W| \leq |U| + |\mathcal{K}| - 1. \tag{12.8}$$

Suppose that for every $K \in \mathcal{K}$, it holds that $|V(K) \cap W| > |E(K)|$. Then

$$|W| = \sum_{K \in \mathcal{K}} |V(K) \cap W| \geq \sum_{K \in \mathcal{K}} (|E(K)| + 1) = |X^*| + |\mathcal{K}|.$$

As $|X^*| = |X| \geq |U|$, we obtain that $|W| \geq |U| + |\mathcal{K}|$. This yields a contradiction to (12.8), since by (12.5), $|W|$ is finite. Therefore, there exists an $L \in \mathcal{K}$ with

$$\ell := |V(L) \cap W| \leq |E(L)|.$$

Observe that if we can show now that $\ell \leq k$, then it follows that $(E(L), E(G^*) \setminus E(L))$ is an ℓ-Tutte-separation of G^*, as desired for (i). So, in order to prove (i), and (ii), it suffices to prove that for each $K \in \mathcal{K}$ it holds that

$$|V(K) \cap W| \leq |U|.$$

Suppose otherwise. Then there exists an $M \in \mathcal{K}$ such that

$$|W| = \sum_{K \in \mathcal{K}} |V(K) \cap W| \geq (|U| + 1) + \sum_{K \in \mathcal{K},\, K \neq M} |V(K) \cap W|.$$

Because G is 2-connected, so is G^* (Lemma 12.3.1). Thus $|V(K) \cap W| \geq 1$ for every $K \in \mathcal{K}$, resulting again in $|W| \geq |U| + |\mathcal{K}|$, a contradiction, as desired. □

We now prove that Tutte-connectivity is invariant under taking duals.

Proof of Theorem 12.1.4. We show that if G has a k-Tutte-separation (X, Y), then G^* has an ℓ-Tutte-separation for some $\ell \leq k$. By Theorem 12.3.2, this is enough to prove the theorem.

First, assume that $V[Y] \setminus V[X] \neq \emptyset$. Let K be a component of $(V[Y], Y) - V[X]$, and set $Z := E(K) \cup E(K, G - K)$. As $E(K, G - K)$ contains at least one edge for each vertex in $N(K)$, it follows that $|Z| \geq |N(K)|$. Thus, $(Z, E(G) \setminus Z)$ is a k'-Tutte-separation of G for $k' := |N(K)| \leq k$. We can now apply Lemma 12.6.4 (i) to obtain the desired ℓ-Tutte-separation of G^*.

So, we may assume that $V[Y] \setminus V[X] = \emptyset$. Then, since $|Y| \geq k$, there is a circuit C in Y, say of length $\ell \leq k$. Hence, C^* is a bond of size ℓ in G^*; let K_1 and K_2 be the components of $G^* - C^*$. Now,

$$|E(K_1 \cup K_2)| = |X^*| + |Y^*| - |C^*| \geq 2k - \ell.$$

Thus, we can partition C^* into C_1^* and C_2^* so that each $Z_i^* := E(K_i) \cup C_i^*$ has cardinality at least ℓ.

In order to show that (Z_1^*, Z_2^*) is an ℓ-Tutte-separation of G^* it remains to check that $U := V[Z_1^*] \cap V[Z_2^*]$ has cardinality at most ℓ. To this end, consider a vertex $v \in U$, and let j be such that $v \in V(K_j)$. Then v is incident with an edge $e_v^* \in C_{3-j}^*$, whose other endvertex lies in K_{3-j}, because C^* is a cut. This defines an injection from $U \to C^*$, which implies $|U| \leq |C^*| \leq \ell$, as desired. □

12.7 The dual preserves the end degrees

In this section we will use Lemma 12.6.4 in order to prove Theorem 12.1.3 that relates the 'degree' of an end ω to the degree of its dual end ω^*. Let us first discuss the degree notion.

For an end ω, define $m(\omega)$ to be the supremum of the cardinalities of sets of disjoint rays in ω; Halin [49] showed that this supremum is indeed attained. In [17] and in [86] the number of vertex- (or edge-)disjoint rays in an end has been successfully used to serve as the degree of an end in a locally finite graph (whether vertex- or edge-disjoint rays should be considered depends on the application). This motivates the definition of the degree $d(\omega) := m(\omega)$ of an end ω of a locally finite graph.

Now, if G and G^* are a dual pair of 2-connected locally finite graphs, then it will turn out that $m(\omega) = m(\omega^*)$ for every end ω of G. In non-locally finite graphs we need to be a bit more careful: Figure 12.2 indicates that dominating vertices should be taken into account.

For an end $\omega \in \Omega(G)$ and a finite vertex set S, we say that $U \subseteq V(G)$ *separates S from ω* if U meets every ray in ω that starts in S. We define here the *degree* $d(\omega)$ of an end $\omega \in \Omega(G)$ to be the minimal number k such that for each finite set $S \subseteq V(G)$, we can separate S from ω in G by deleting at most k vertices from G. If there is no such k, we set $d(\omega) := \infty$. Lemma 12.7.1 will show that this definition is consistent with the one given above for locally finite graphs.

So, denote by $\mathrm{dom}(\omega)$ the number of vertices that dominate an end ω (possibly infinite). Note that the graphs we are interested in, namely those that satisfy (†), are such that $\mathrm{dom}(\omega) \in \{0, 1\}$ for every end ω.

Lemma 12.7.1. *Let G be a graph and let $\omega \in \Omega(G)$. Then $d(\omega) = m(\omega) + \mathrm{dom}(\omega)$.*

Proof. It is easy to see that $d(\omega)$ is at least $m(\omega) + \mathrm{dom}(\omega)$. For the other direction, we may assume that $\mathrm{dom}(\omega) < \infty$. Denote by D the set of vertices that dominate ω. As D is a finite set, there is an obvious bijection between the ends of $G - D$ and G, which we will tacitly use.

We observe first that for any finite vertex set T, there exists a finite T–ω separator T' in $G - D$ that is contained in $C_{G-D}(T, \omega)$. Indeed, otherwise, by Menger's theorem[3], $G[T \cup C(T, \omega)] - D$ contains infinitely many paths between T and some ray in ω that are pairwise disjoint except possibly in T. As T is finite, this implies that $T \setminus D$ contains a vertex which dominates ω, contradicting our choice of D.

Now, choose any finite $S \subseteq V(G)$. Starting with $S_0 := S \setminus D$ we can choose inductively finite vertex sets S_i so that $S_i \subseteq V(C_{G-D}(S_{i-1}, \omega))$ is an S_{i-1}–ω separator in $G - D$, and has minimal cardinality with that property. Since $S_i \subseteq V(C_{G-D}(S_{i-1}, \omega))$, all the S_i are pairwise disjoint.

Applying Menger's theorem repeatedly between S_{i-1} and S_i we obtain a set $\mathcal{R}$ of disjoint rays in ω of cardinality at least $|S_1|$. As $S_1 \cup D$ separates S from ω in G, we have shown that S can be separated from ω by at most $|S_1| + |D| \leq m(\omega) + \mathrm{dom}(\omega)$ vertices, thus proving the lemma. □

We remark that Lemma 12.7.1 can be obtained easily from results of Polat [76]; we chose to provide the proof nevertheless since the statement of Polat's results together with the necessary adaptions would have taken about as much time and space.

We finally turn to the proof of Theorem 12.1.3, which asserts that the degree of an end is preserved by taking duals. In conjunction with Lemma 12.7.1 the theorem immediately yields Theorem 12.1.2.

Proof of Theorem 12.1.3. First assume that $d(\omega) \leq k$, where $k \in \mathbb{N}$ is a finite number. We wish to show that ω^* has vertex-degree $\leq k$, too.

So, let a finite vertex set $T \subseteq V(G^*)$ be given. Pick a finite edge set F^* of cardinality at least k so that $T \subseteq V[F^*]$ and so that F^* induces a connected graph. Now, since $d(\omega) \leq k$ there is a set $U \subseteq V(G)$ of cardinality at most k that separates (the finite set) $V[F]$ from ω. If C is the component of $G - U$ to which ω belongs then set $Y := E(C) \cup E(C, U)$ and $X := E(G) \setminus Y$. Because $k \geq |U| = |V[X] \cap V[Y]|$, and because $|Y| = \infty$ and $|X| \geq |F| \geq k$, it follows that (X, Y) is a k-Tutte-separation.

Since $F^* \subseteq X^*$ induces a connected subgraph, there is a component K of $(V[X^*], X^*)$ that contains all of F^*. As $|F^*| \geq k$, Lemma 12.6.4 (ii) implies that $(E(K), E(G^*) \setminus E(K))$ is a k-Tutte-separation. Moreover, as $\omega \notin \overline{X}$, it

[3]We use here, and below, that the cardinality version of Menger's theorem holds in infinite graphs. This can easily be deduced from Menger's theorem for finite graphs, see for instance [31, Section 8.4]

follows that $\omega^* \notin \overline{K}$. Thus, $N_{G^*}(G^* - K)$ is a vertex set of cardinality $\leq k$ that separates $T \subseteq V[F^*]$ from ω^*, as desired.

In conclusion, since G is also a dual of G^* (Theorem 12.3.2), it follows that $d(\omega) = d(\omega^*)$ if either of ω and ω^* has finite degree. In the remaining case, we trivially have $d(\omega) = \infty = d(\omega^*)$. □

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