Exact and Parameterized Algorithms for Max Internal Spanning Tree⋆

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Abstract. We consider the $\mathcal{NP}$-hard problem of finding a spanning tree with a maximum number of internal vertices. This problem is a generalization of the famous Hamiltonian Path problem. Our dynamic-programming algorithms for general and degree-bounded graphs have running times of the form $\mathcal{O}^*(c^n)$ ($c \leq 3$). The main result, however, is a branching algorithm for graphs with maximum degree three. It only needs polynomial space and has a running time of $\mathcal{O}(1.8669^n)$ when analyzed with respect to the number of vertices. We also show that its running time is $2.1364^k n^{\mathcal{O}(1)}$ when the goal is to find a spanning tree with at least k internal vertices. Both running time bounds are obtained via a Measure & Conquer analysis, the latter one being a novel use of this kind of analysis for parameterized algorithms.

1 Introduction

Motivation. In this paper we investigate the following problem:

> Max Internal Spanning Tree (MIST)
> **Given:** A graph $G = (V, E)$ with n vertices and m edges.
> **Task:** Find a spanning tree of G with a maximum number of internal vertices.

MIST is a generalization of the well-studied Hamiltonian Path problem: find a path in a graph such that every vertex is visited exactly once. Clearly, such a path, if it exists, is also a spanning tree, namely one with a maximum number of internal vertices. Whereas the running time barrier of 2^n has not been broken for general graphs, there are faster algorithms for cubic graphs (using only polynomial space). It is natural to ask if for the generalization, MIST, this can also be obtained.

A second issue is to find an algorithm for MIST with a running time of the form $\mathcal{O}^*(c^n)$. [1] The naïve approach gives only an upper bound of $\mathcal{O}^*(2^m)$. A possible application could be the following scenario. Consider cities which should

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[1] $f(n) = \mathcal{O}^*(g(n))$ if $f(n) \leq p(n) \cdot g(n)$ for some polynomial $p(n)$.

be connected with water pipes. The possible connections between them can be represented by a graph G. It suffices to compute a spanning tree T for G. In T we may have high degree vertices that have to be implemented by branching pipes which cause turbulences and therefore pressure may drop. To minimize the number of branching pipes one can equivalently compute a spanning tree with the smallest number of leaves, leading to MIST. Vertices representing branching pipes should not be of arbitrarily high degree, motivating us to investigate MIST on degree-restricted graphs.

Previous Work. It is well-known that the more restricted problem, Hamiltonian Path, can be solved within $\mathcal{O}(n^2 2^n)$ steps and exponential space. This result has been independently obtained by Bellman [1], and Held and Karp [6]. The Traveling Salesman problem (TSP) is very closely related to Hamiltonian Path. Basically, the same algorithm solves this problem, but there has not been any improvement on the running time since 1962. The space requirements have, however, been improved and now there are $\mathcal{O}^*(2^n)$ algorithms needing only polynomial space. In 1977, Kohn *et al.* [9] gave an algorithm based on generating functions with a running time of $\mathcal{O}(2^n n^3)$ and space requirements of $\mathcal{O}(n^2)$ and in 1982 Karp [8] came up with an algorithm which improved storage requirements to $\mathcal{O}(n)$ and preserved this run time by an inclusion-exclusion approach.

Eppstein [4] studied TSP on cubic graphs. He could achieve a running time of $\mathcal{O}(1.260^n)$ using polynomial space. Iwama and Nakashima [7] could improve this to $\mathcal{O}(1.251^n)$. Björklund *et al.* [2] considered TSP with respect to degree-bounded graphs. Their algorithm is a variant of the classical 2^n-algorithm and the space requirements are therefore exponential. Nevertheless, they showed that for a graph with maximum degree d there is a $\mathcal{O}^*((2-\epsilon_d)^n)$-algorithm. In particular for $d = 4$ there is a $\mathcal{O}(1.8557^n)$- and for $d = 5$ a $\mathcal{O}(1.9320^n)$-algorithm.

MIST was also studied with respect to parameterized complexity. The (standard) parameterized version of the problem is parameterized by k, and asks whether G has a spanning tree with at least k internal vertices. Prieto and Sloper [11] proved a $\mathcal{O}(k^3)$-vertex kernel for the problem showing $\mathcal{FPT}$-membership. In [12] the kernel size has been improved to $\mathcal{O}(k^2)$ and in [5] to $3k$. Parameterized algorithms for MIST have been studied in [3, 5, 12]. Prieto and Sloper [12] gave the first FPT algorithm, with running time $2^{4k \log k} \cdot n^{\mathcal{O}(1)}$. This result was improved by Cohen *et al.* [3] who solve a more general directed version of the problem in time $49.4^k \cdot n^{\mathcal{O}(1)}$. The current fastest algorithm has running time $8^k \cdot n^{\mathcal{O}(1)}$ [5].

Salamon [14] studied the problem considering approximation. He could achieve a $\frac{7}{4}$-approximation. A $2(\Delta - 2)$-approximation for the node-weighted version is a by-product. Cubic and claw-free graphs were considered by Salamon and Wiener [13] introducing algorithms with approximation ratios $\frac{6}{5}$ and $\frac{3}{2}$, respectively.

Our Results. This paper gives two algorithms:

(*a*) A dynamic-programming algorithm solving MIST in time $\mathcal{O}^*(3^n)$. We extend this algorithm and show that for any degree-bounded graph a running time of $\mathcal{O}^*((3-\epsilon)^n)$ with $\epsilon > 0$ can be achieved. To our knowledge this is the first algorithm for MIST with a running time bound of the form $\mathcal{O}^*(c^n)$.[2]

(*b*) A polynomial-space branching algorithm solving the maximum degree 3 case in time $\mathcal{O}(1.8669^n)$. We also analyze the same algorithm from a parameterized point of view, achieving a running time of $2.1364^k n^{\mathcal{O}(1)}$ to find a spanning tree with at least k internal vertices (if possible). The latter analysis is novel in a sense that we use a potential function analysis—Measure & Conquer—in a way that, to our knowledge, is much less restrictive than any previous analysis for parameterized algorithms that were based on the potential function method.

Notions and Definitions. We consider only simple undirected graphs $G = (V, E)$. The *neighborhood* of a vertex $v \in V$ in G is $N_G(v) := \{u \mid \{u, v\} \in E\}$ and its *degree* is $d_G(v) := |N_G(v)|$. The *closed neighborhood* of v is $N_G[v] := N_G(v) \cup \{v\}$ and for a set $V' \subseteq V$ we let $N_G(V') := \left(\bigcup_{u \in V'} N_G(u)\right) \setminus V'$. We omit the subscripts of $N_G(\cdot)$, $d_G(\cdot)$, and $N_G[\cdot]$ when the graph is clear from the context. A *subcubic graph* has maximum degree at most three. For a (partial) spanning tree $T \subseteq E$ let $I(T)$ be the set of its internal (non-leaf) vertices and $L(T)$ the set of its leaves. An *i-vertex* u is a vertex with $d_T(u) = i$ with respect to some spanning tree T, where $d_H(u) := \{\{u, v\} \mid \{u, v\} \in H\}$ for any $H \subseteq E$. The *tree-degree* of some $u \in V(T)$ is $d_T(u)$. We also speak of the *T-degree* $d_T(v)$ when we refer to a specific spanning tree. A *Hamiltonian path* is a sequence of pairwise distinct vertices $v_1, \dots, v_n$ from V such that $\{v_i, v_{i+1}\} \in E$ for $1 \le i \le n-1$.

The Problem on General Graphs. By means of Dynamic Programming and the help of an upper bound on the number of connected vertex-subsets of degree bounded graph (shown by [2]) we show the next statement.

Lemma 1. MIST *can be solved in time $\mathcal{O}^*(3^n)$ and for graphs with maximum degree Δ,* MIST *can be solved in time $\mathcal{O}^*(3^{(1-\epsilon_\Delta)n})$ with $\epsilon_\Delta > 0$.*

2 Subcubic Maximum Internal Spanning Tree

2.1 Observations

Let t_i^T denote the number of vertices u such that $d_T(u) = i$ for a spanning tree T. Then the following proposition can be proved by induction on $n_T := |V(T)|$.

Proposition 1. *In any spanning tree T, $2 + \sum_{i \ge 3} (i-2) \cdot t_i^T = t_1^T$.*

[2] Before the camera-ready version of this paper was prepared, Nederlof [10] came up with a polynomial-space $\mathcal{O}^*(2^n)$ algorithm for MIST on general graphs, answering a question in a preliminary version of this paper.

Due to Proposition 1, MIST on subcubic graphs boils down to finding a spanning tree T such that t_2^T is maximum. Every internal vertex of higher degree would also introduce additional leaves.

Lemma 2. *[11] An optimal solution T_o to* Max Internal Spanning Tree *is a Hamiltonian path or the leaves of T_o are independent.*

The proof of Lemma 2 shows that if T_o is not a Hamiltonian path and there are two adjacent leaves, then the number of internal vertices can be increased. In the rest of the paper we assume that T_o is not a Hamiltonian path due to the next lemma which uses the $\mathcal{O}(1.251^n)$ algorithm for Hamiltonian Cycle on subcubic graphs [7] as a subroutine.

Lemma 3. Hamiltonian Path *can be solved in time $\mathcal{O}(1.251^n)$ on subcubic graphs.*

Lemma 4. *Let T be a spanning tree and $u, v \in V(T)$ two adjacent vertices with $d_T(u) = d_T(v) = 3$ such that $\{u, v\}$ is not a bridge. Then there is a spanning tree $T' \supset (T \setminus \{\{u, v\}\})$ with $|I(T')| \geq |I(T)|$ and $d_{T'}(u) = d_{T'}(v) = 2$.*

Proof. By removing $\{u, v\}$, T is separated into two parts T_1 and T_2. The vertices u and v become 2-vertices. As $\{u, v\}$ is not a bridge, there is another edge $e \in E \setminus E(T)$ connecting T_1 and T_2. By adding e we lose at most two 2-vertices. Then let $T' := (T \setminus \{\{u, v\}\}) \cup \{e\}$ and it follows that $|I(T')| \geq |I(T)|$. □

2.2 Reduction Rules

Let $E' \subseteq E$. Then, $\partial E' := \{\{u, v\} \in E \setminus E' \mid u \in V(E')\}$ are the edges outside E' that have a common end point with an edge in E' and $\partial_V E' := V(\partial E') \cap V(E')$ are the vertices that have at least one incident edge in E' and another incident edge not in E'. During the algorithm we will maintain an acyclic subset of edges F which will be part of the final solution. The following invariant will always be true: $G[F]$ consists of a tree T and a set P of *pending tree edges (pt-edges)*. Here a pt-edge $\{u, v\} \in F$ is an edge with one end point u of degree 1 and the other end point $v \notin V(T)$. $G[T \cup P]$ will always consist of $1 + |P|$ components. Next we present several reduction rules. The order in which they are applied is crucial: Before a rule is applied the preceding ones were carried out exhaustively.

Cycle: Delete any edge $e \in E$ such that $E(T) \cup \{e\}$ has a cycle.
Bridge: If there is a bridge $e \in \partial E(T)$, then add e to F.
Deg1: Let $u \in V \setminus V(F)$ with $d(u) = 1$. Then add its incident edge to F.
Pending: If a a vertex v is incident to $d_G(v) - 1$ pt-edges, then remove them.
ConsDeg2: If there are edges $\{v, w\}, \{w, z\} \in E \setminus E(T)$ such that $d_G(w) = d_G(z) = 2$, then delete $\{v, w\}, \{w, z\}$ from G and add the edge $\{v, z\}$ to G.
Deg2: If there is an edge $\{u, v\} \in \partial E(T)$ such that $u \in V(T)$ and $d_G(u) = 2$, then add $\{u, v\}$ to F.
Attach: If there are edges $\{u, v\}, \{v, z\} \in \partial E(T)$ such that $u, z \in V(T)$, $d_T(u) = 2$, $1 \leq d_T(z) \leq 2$, then delete $\{u, v\}$. See Fig. 1(a)

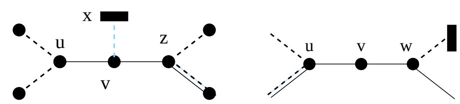


Fig. 1. Light edges may be not present. Double edges (dotted or solid, resp.) refer to edges which are either T-edges or not, resp. Edges attached to oblongs are pt-edges.

Attach2: If there is a vertex $u \in \partial_V E(T)$ with $d_T(u) = 2$ and $\{u, v\} \in E \setminus E(T)$ such that v is incident to a pt-edge, then delete $\{u, v\}$.

Special: If there are two edges $\{u, v\}, \{v, w\} \in E \setminus F$ with $d_T(u) \geq 1$, $d_G(v) = 2$, and w is incident to a pt-edge, then add $\{u, v\}$ to F. See Fig. 1(b).

We mention that **ConsDeg2** can create double edges. In this case simply delete one of them which is not in F (at most one can be part of F).

Lemma 5. *The reduction rules stated above are sound.*

Proof. Let $T_o \supset F$ be a spanning tree of G with a maximum number of internal vertices. The correctness of the first five reduction rules is easily verified.

Deg2 Since the preceding reduction rules do not apply, we have $d_G(v) = 3$ and there is one incident edge, say $\{v, z\}$, $z \neq u$, that is not pending. Assume u is a leaf in T_o. Define another spanning tree $T'_o \supset F$ by setting $T'_o = (T_o \cup \{\{u, v\}\}) \setminus \{v, z\}$. Since $|I(T_o)| \leq |I(T'_o)|$, T'_o is also optimal.

Attach If $\{u, v\} \in E(T_o)$ then $\{v, z\} \notin E(T_o)$ due to the acyclicity of T_o and as T is connected. Then by exchanging $\{u, v\}$ and $\{v, z\}$ we obtain a solution T'_o with at least as many 2-vertices.

Attach2 Suppose $\{u, v\} \in E(T_o)$. Let $\{v, p\}$ be the pt-edge and $\{v, z\}$ the third edge incident to v (that must exist and is not pending, since **Pending** did not apply). Since **Bridge** did not apply, $\{u, v\}$ is not a bridge. Firstly, suppose $\{v, z\} \in E(T_o)$. Due to Lemma 4, there is also an optimal solution $T'_o \supset F$ with $\{u, v\} \notin E(T'_o)$. Secondly, assume $\{v, z\} \notin E(T_o)$. Then $T' = (T_o \setminus \{\{u, v\}\}) \cup \{\{v, z\}\}$ is also optimal as u has become a 2-vertex.

Special Suppose $\{u, v\} \notin E(T_o)$. Then $\{v, w\}, \{w, z\} \in E(T_o)$ where $\{w, z\}$ is the third edge incident to w. Let $T'_o := (T_o \setminus \{\{v, w\}\}) \cup \{\{u, v\}\}$. In T'_o, w is a 2-vertex and hence T' is also optimal. □

2.3 The Algorithm

The algorithm we describe here is recursive. It constructs a set F of edges which are selected to be in every spanning tree considered in the current recursive step. The algorithm chooses edges and considers all relevant choices for adding them to F or removing them from G. It selects these edges based on priorities chosen to optimize the running time analysis. Moreover, the set F of edges will always be the union of a tree T and a set of edges P that are not incident to the tree and have one end point of degree 1 in G (pt-edges). We do not explicitly write in the algorithm that edges move from P to T whenever an edge is added to F that is incident to both an edge of T and an edge of P. To maintain the connectivity of T, the algorithm explores edges in the set $\partial E(T)$ to grow T.

If $|V| > 2$ every spanning tree T must have a vertex v with $d_T(v) \geq 2$. Thus initially the algorithm creates an instance for every vertex v and every possibility that $d_T(v) \geq 2$. Due to the degree constraint there are no more than $4n$ instances. After this initial phase, the algorithm proceeds as follows.

1. Carry out each reduction rule exhaustively in the given order.
2. If $\partial E(T) = \emptyset$ and $V \neq V(T)$, then G is not connected and does not admit a spanning tree. Ignore this branch.
3. If $\partial E(T) = \emptyset$ and $V = V(T)$, then return T.
4. Select $\{a, b\} \in \partial E(T)$ with $a \in V(T)$ according to the following priorities (if such an edge exists):
   a) there is an edge $\{b, c\} \in \partial E(T)$,
   b) $d_G(b) = 2$,
   c) b is incident to a pt-edge, or
   d) $d_T(a) = 1$.

   Recursively solve the two instances where $\{a, b\}$ is added to F or removed from G respectively, and return the spanning tree with most internal vertices.
5. Otherwise, select $\{a, b\} \in \partial E(T)$ with $a \in V(T)$. Let c, x be the other two neighbors of b. Recursively solve three instances where
   (i) $\{a, b\}$ is removed from G,
   (ii) $\{a, b\}$ and $\{b, c\}$ are added to F and $\{b, x\}$ is removed from G, and
   (iii) $\{a, b\}$ and $\{b, x\}$ are added to F and $\{b, c\}$ is removed from G.

   Return the spanning tree with most internal vertices.

By a Measure & Conquer analysis taking into account the degrees of the vertices, their number of incident edges that are in F, and to some extent the degrees of their neighbors, we obtain the following result.

Theorem 1. MIST *can be solved in time* $\mathcal{O}(1.8669^n)$ *on subcubic graphs.*

Let us provide the measure we use in the analysis. Let $D_2 := \{v \in V \mid d_G(v) = 2, d_F(v) = 0\}$, $D_3^\ell := \{v \in V \mid d_G(v) = 3, d_F(v) = \ell\}$ and $D_3^{2*} := \{v \in D_3^2 \mid N_G(v) \setminus N_F(v) = \{u\}$ and $d_G(u) = 2\}$. Then the measure we use is

$$\mu(G) = \omega_2 \cdot |D_2| + \omega_3^1 \cdot |D_3^1| + \omega_3^2 \cdot |D_3^2 \setminus D_3^{2*}| + |D_3^0| + \omega_3^{2*} \cdot |D_3^{2*}|$$

with the weights $\omega_2 = 0.3193$, $\omega_3^1 = 0.6234$, $\omega_3^2 = 0.3094$ and $\omega_3^{2*} = 0.4144$.The proof of the theorem uses the following result.

Lemma 6. *None of the reduction rules increase* μ *for the given weights.*

Let $\Delta_3^0 := \Delta_3^{0*} := 1 - \omega_3^1$, $\Delta_3^1 := \omega_3^1 - \omega_3^2$, $\Delta_3^{1*} := \omega_3^1 - \omega_3^{2*}$, $\Delta_3^2 := \omega_3^2$, $\Delta_3^{2*} := \omega_3^{2*}$ and $\Delta_2 = 1 - \omega_2$. We define $\tilde{\Delta}_3^i := \min\{\Delta_3^i, \Delta_3^{i*}\}$ for $1 \leq i \leq 2$, $\Delta_m^\ell = \min_{0 \leq j \leq \ell}\{\Delta_3^j\}$, and $\tilde{\Delta}_m^\ell = \min_{0 \leq j \leq \ell}\{\tilde{\Delta}_3^j\}$.

Proof. (of Theorem 1) As the algorithm deletes edges or moves edges from $E \setminus F$ to F, cases 1–3 do not contribute to the exponential function in the running time of the algorithm. It remains to analyze cases 4 and 5, which we do now. Note that after applying the reduction rules exhaustively, we have that for all $v \in \partial_V E(T)$, $d_G(v) = 3$ (**Deg2**) and for all $u \in V$, $d_P(u) \leq 1$ (**Pending**).

4.(a) Obviously, $\{a, b\}, \{b, c\} \in E \setminus E(T)$, and there is a vertex d such that $\{c, d\} \in E(T)$; see Figure 2(a). We have $d_T(a) = d_T(c) = 1$ due to the reduction rule **Attach**. We consider three cases.

$d_G(b) = 2$. When $\{a, b\}$ is added to F, **Cycle** deletes $\{b, c\}$. We get an amount of ω_2 and ω_3^1 as b drops out of D_2 and c out of D_3^1 (**Deg2**). Also a will be removed from D_3^1 and added to D_3^2 which amounts to a reduction of at least $\tilde{\Delta}_3^1$. When $\{a, b\}$ is deleted, $\{b, c\}$ is added to $E(T)$ (**Bridge**). By a symmetric argument we get a reduction of $\omega_2 + \omega_3^1 + \tilde{\Delta}_3^1$ as well. In total this yields a $(\omega_2 + \omega_3^1 + \tilde{\Delta}_3^1, \omega_2 + \omega_3^1 + \tilde{\Delta}_3^1)$-branch.

$d_G(b) = 3$ and there is one pt-edge incident to b. Adding $\{a, b\}$ to F decreases the measure by $\tilde{\Delta}_3^1$ (from a) and $2\omega_3^1$ (deleting $\{b, c\}$, then **Deg2** on c). By Deleting $\{a, b\}$ we decrease μ by $2\omega_3^1$ and by $\tilde{\Delta}_3^1$ (from c). This amounts to a $(2\omega_3^1 + \tilde{\Delta}_3^1, 2\omega_3^1 + \tilde{\Delta}_3^1)$-branch.

$d_G(b) = 3$ and no pt-edge is incident to b. Let $\{b, z\}$ be the third edge incident to b. In the first branch the measure drops by at least $\omega_3^1 + \tilde{\Delta}_3^1$ from c and a (**Deg2**), 1 from b (**Deg2**). In the second branch we get $\omega_3^1 + \Delta_2$. Observe that we also get an amount of at least $\tilde{\Delta}_m^1$ from $q \in N_T(a) \setminus \{b\}$ if $d_G(q) = 3$. If $d_G(q) = 2$ we get ω_2. It results a $(\omega_3^1 + \tilde{\Delta}_3^1 + 1, \omega_3^1 + \Delta_2 + min\{\omega_2, \tilde{\Delta}_m^1\})$-branch.

Note that from this point on, for all $u, v \in V(T)$ there is no $z \in V \setminus V(T)$ with $\{u, z\}, \{z, v\} \in E$.

4.(b) As the previous case does not apply, the other neighbor c of b has $d_T(c) = 0$, and $d_G(c) \geq 2$ (**Pending**), see Figure 2(b). Additionally, observe that $d_G(c) = 3$ due to **ConsDeg2** and that $d_P(c) = 0$ due to **Special**. We consider two subcases.

$d_T(a) = 1$. When we add $\{a, b\}$ to F, then $\{b, c\}$ is also added due to **Deg2**. The reduction is at least $\tilde{\Delta}_3^1$ from a, ω_2 from b and Δ_3^0 from c. When $\{a, b\}$ is deleted, $\{b, c\}$ becomes a pt-edge. There is $\{a, z\} \in E \setminus E(T)$ with $z \neq b$, which is subject to a **Deg2** reduction rule. We get at least ω_3^1 from a, ω_2 from b, Δ_3^0 from c and $\min\{\omega_2, \tilde{\Delta}_m^1\}$ from z. This is a $(\tilde{\Delta}_3^1 + \Delta_3^0 + \omega_2, \omega_3^1 + \Delta_3^0 + \omega_2 + \min\{\omega_2, \tilde{\Delta}_m^1\})$-branch.

$d_T(a) = 2$. Similarly, we obtain a $(\Delta_3^{2*} + \omega_2 + \Delta_3^0, \Delta_3^{2*} + \omega_2 + \Delta_3^0)$-branch.

4.(c) In this case, $d_G(b) = 3$ and there is one pt-edge attached to b, see Figure 2(c). Note that $d_T(a) = 2$ can be ruled out due to **Attach2**. Thus, $d_T(a) = 1$. Let $z \neq b$ be such that $\{a, z\} \in E \setminus E(T)$. Due to the priorities, $d_G(z) = 3$. We distinguish between the cases where c, the other neighbor of b, is incident to a pt-edge or not.

$d_P(c) = 0$. First suppose $d_G(c) = 3$. Adding $\{a, b\}$ to F allows a reduction of $2\Delta_3^1$ (due to case 4.(b) we can exclude Δ_3^{1*}). Deleting $\{a, b\}$ implies that we get a reduction from a and b of $2\omega_3^1$ (**Deg2** and **Pending**). As $\{a,z\}$ is added to F we reduce $\mu(G)$ by at least $\tilde{\Delta}_3^1$ as the state of z changes. Now due to **Pending** and **Deg1** we include $\{b,c\}$ and get Δ_3^0 from c. We have at least a $(2\Delta_3^1, 2\omega_3^1 + \tilde{\Delta}_3^1 + \Delta_3^0)$-branch.

If $d_G(c) = 2$ we consider the two cases for z also. These are $d_P(z) = 1$ and $d_P(z) = 0$. The first entails $(\omega_3^1 + \Delta_3^{1*}, 2\omega_3^1 + \tilde{\Delta}_3^1 + \omega_2 + \tilde{\Delta}_m^2)$. Note that when we add $\{a,b\}$ we trigger **Attach2**. The second is a $(\Delta_3^1 + \Delta_3^{1*}, 2\omega_3^1 + \Delta_3^0 + \omega_2 + \tilde{\Delta}_m^2)$-branch.

$d_P(c) = 1$. Let $d \neq b$ be the other neighbor of c that does not have degree 1. When $\{a,b\}$ is added to F, $\{b,c\}$ is deleted by **Attach2** and $\{c,d\}$ becomes a pt-edge (**Pending** and **Deg1**). The changes on a incur a measure decrease of Δ_3^{1*} and those on b,c a measure decrease of $2\omega_3^1$. When $\{a,b\}$ is deleted, $\{a,z\}$ is added to F (**Deg2**) and $\{c,d\}$ becomes a pt-edge by two applications of the **Pending** and **Deg1** rules. Thus, the decrease of the measure is at least $3\omega_3^1$ in this branch. In total, we have a $(\Delta_3^{1*} + 2\omega_3^1, 3\omega_3^1)$-branch here.

4.(d) Now, $d_G(b) = 3$, b is not incident to a pt-edge, and $d_T(a) = 1$. See Figure 2(c). There is also some $\{a,z\} \in E \setminus E(T)$ such that $z \neq b$. Note that $d_T(z) = 0$, $d_G(z) = 3$ and $d_P(z) = 0$. Otherwise either **Cycle** or cases 4.(b) or 4.(c) would have been triggered. From the addition of $\{a,b\}$ to F we get $\Delta_3^1 + \Delta_3^0$ and from its deletion ω_3^1 (from a via **Deg2**), Δ_2 (from b) and at least Δ_3^0 from z and thus, a $(\Delta_3^1 + \Delta_3^0, \omega_3^1 + \Delta_2 + \Delta_3^0)$-branch.

5. See Figure 2(d). The algorithm branches in the following way: 1) Delete $\{a,b\}$, 2) add $\{a,b\},\{b,c\}$, and delete $\{b,x\}$, 3) add $\{a,b\},\{b,x\}$ and delete $\{b,c\}$. Due to **Deg2**, we can disregard the case when b is a leaf. Due to Lemma 4 we also disregard the case when b is a 3-vertex. Thus by branching in this manner we find at least one optimal solution.

The reduction in the first branch is at least $\omega_3^2 + \Delta_2$. We get an additional amount of ω_2 if $d(x) = 2$ or $d(c) = 2$ from **ConsDeg2**. In the second we have to consider also the vertices c and x. There are exactly three situations for $h \in \{c,x\}$ α) $d_G(h) = 2$, β) $d_G(h) = 3$, $d_P(h) = 0$ and γ) $d_G(h) = 3$, $d_P(h) = 1$. We will only analyze branch 2) as 3) is symmetric. We first get a reduction of $\omega_3^2 + 1$ from a and b. We reduce μ due to deleting $\{b,x\}$ by: α) $\omega_2 + \tilde{\Delta}_m^2$, β) Δ_2, γ) $\omega_3^1 + \tilde{\Delta}_m^2$. Next we examine the amount by which μ will be decreased by adding $\{b,c\}$ to F. We distinguish between the cases α,β and γ: α) $\omega_2 + \tilde{\Delta}_m^2$, β) Δ_3^0, γ) $\tilde{\Delta}_3^1$.

For $h \in \{c,x\}$ and $W \in \{\alpha,\beta,\gamma\}$ let 1_W^h be the indicator function which is set to one if we have situation W at vertex h. Otherwise it is zero. Now the branching tuple can be stated the following way :

$(\omega_3^2 + \Delta_2 + (1_\alpha^x + 1_\alpha^c)\cdot\omega_2,$
$\omega_3^2+1+1_\alpha^x\cdot(\omega_2+\tilde{\Delta}_m^2)+1_\beta^x\cdot\Delta_2+1_\gamma^x\cdot(\omega_3^1+\tilde{\Delta}_m^2)+1_\alpha^c\cdot(\omega_2+\tilde{\Delta}_m^2)+1_\beta^c\cdot\Delta_3^0+1_\gamma^c\cdot\tilde{\Delta}_3^1),$
$\omega_3^2+1+1_\alpha^c\cdot(\omega_2+\tilde{\Delta}_m^2)+1_\beta^c\cdot\Delta_2+1_\gamma^c\cdot(\omega_3^1+\tilde{\Delta}_m^2)+1_\alpha^x\cdot(\omega_2+\tilde{\Delta}_m^2)+1_\beta^x\cdot\Delta_3^0+1_\gamma^x\cdot\tilde{\Delta}_3^1)$

The amount of $(1_\alpha^x + 1_\alpha^c)\cdot\omega_2$ comes from possible applications of **ConsDeg2**.

Observe that every instance created by branching is smaller than the original instance in terms of μ. Together with Lemma 6 we see that every step of the

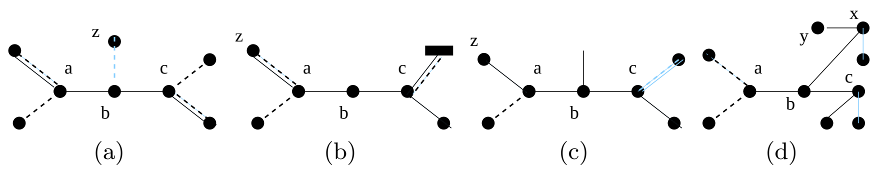


Fig. 2. Light edges may be not present. Double edges (dotted or solid, resp.) refer to edges which are either T-edges or not, resp. Edges attached to oblongs are pt-edges.

algorithm only decreases μ. Now if we evaluate the upper bound for every given branching tuple for the given weights we can conclude that MIST can be solved in time $\mathcal{O}^*(1.8669^n)$ on subcubic graphs. □

Let us now turn to a parameterized analysis of the algorithm. For general graphs, the smallest known kernel has size $3k$. This can be easily improved to $2k$ for subcubic graphs.

Lemma 7. *MIST on subcubic graphs has a* $2k$*-kernel.*

Applying the algorithm of Theorem 1 on this kernel for subcubic graphs would lead to a running time of $3.4854^k n^{\mathcal{O}(1)}$. However, we can achieve a faster parameterized running time by applying a Measure & Conquer analysis which is customized to the parameter k. We would like to put forward that our use of the technique of Measure & Conquer for a parameterized algorithm analysis goes beyond previous work as our measure is not restricted to differ from the parameter k by just a constant. We first demonstrate our idea with a simple analysis.

Theorem 2. *Deciding whether a subcubic graph has a spanning tree with at least* k *internal vertices can be done in time* $2.7321^k n^{\mathcal{O}(1)}$.

Proof. Consider the algorithm described earlier, with the only modification that the parameter k is adjusted whenever necessary (for example, when two pt-edges incident to the same vertex are removed), and that the algorithm stops and answers Yes whenever T has at least k internal vertices. Note that the assumption that G has no Hamiltonian path can still be made due to the $2k$-kernel of Lemma 7: the running time of the Hamiltonian path algorithm is $1.251^{2k} n^{\mathcal{O}(1)} = 1.5651^k n^{\mathcal{O}(1)}$. The running time analysis of our algorithm relies on the following measure: $\kappa := \kappa(G, F, k) := k - \omega \cdot |X| - |Y|$ where $X := \{v \in V \mid d_G(v) = 3, d_T(v) = 2\}$, $Y := \{v \in V \mid d_G(v) = d_T(v) \geq 2\}$ and $\omega = 0.45346$. Let $U := V \setminus (X \cup Y)$. Note that a vertex which has already been decided to be internal, but that still has an incident edge in $E \setminus T$, contributes a weight of $1 - \omega$ to the measure. Or equivalently, such a vertex has been only counted by ω. None of the reduction and branching rules increases κ and we have that $0 \leq \kappa \leq k$ at any time of the execution of the algorithm. By a simple case analysis and evaluating the branching factors, the proof follows. □

This analysis can be improved by also measuring vertices of degree 2 and vertices incident to pt-edges differently.

Theorem 3. *Deciding whether a subcubic graph has a spanning tree with at least k internal vertices can be done in time $2.1364^k n^{\mathcal{O}(1)}$.*

We consider a more detailed measure:
$\kappa := \kappa(G, F, k) := k - \omega_1 \cdot |X| - |Y| - \omega_2|Z| - \omega_3|W|$, where

- $X := \{v \in V \mid d_G(v) = 3, d_T(v) = 2\}$ is the set of vertices of degree 3 that are incident to exactly 2 edges of T,
- $Y := \{v \in V \mid d_G(v) = d_T(v) \geq 2\}$ is the set of vertices of degree at least 2 that are incident to only edges of T,
- $W := \{v \in V \setminus (X \cup Y) \mid d_G(v) \geq 2, \exists u \in N(v) \text{ st. } d_G(u) = d_F(u) = 1\}$ is the set of vertices of degree at least 2 that have an incident pt-edge, and
- $Z := \{v \in V \setminus W \mid d_G(v) = 2, N[v] \cap (X \cup Y) = \emptyset\}$ is the set of degree 2 vertices that do not have a vertex of $X \cup Y$ in their closed neighborhood, and are not incident to a pt-edge.

We immediately set $\omega_1 := 0.5485, \omega_2 := 0.4189$ and $\omega_3 := 0.7712$. Let $U := V \setminus (X \cup Y \cup Z \cup W)$. We first have to show that the algorithm can be stopped whenever the measure drops to 0 or less.

Lemma 8. *Let $G = (V, E)$ be a connected graph, k be an integer and $F \subseteq E$ be a set of edges that can be partitioned into a tree T and a set of pending edges P. If none of the reduction rules applies to this instance and $\kappa(G, F, k) \leq 0$, then G has a spanning tree $T^* \supseteq F$ with at least k internal nodes.*

We also show that reducing an instance does not increase its measure.

Lemma 9. *Let (G', F', k') be an instance resulting from the application of a reduction rule to an instance (G, F, k). Then, $\kappa(G', F', k') \leq \kappa(G, F, k)$.*

Proof. (of Theorem 3) Table 1 outlines how vertices a, b, and their neighbors move between U, X, Y, Z, and W in the branches where an edge is added to F or deleted from G in the different cases of the algorithm. For each case, the worst branching tuple is given.

The tight branching numbers are found for cases 4.(b) with $d_T(a) = 2$, 4.(c), 4.(d), and 5. with all of b's neighbors having degree 3. The respective branching numbers are $(2 - \omega_1 - \omega_2, 1 - \omega_1 - \omega_2 + \omega_3)$, $(2\omega_1 - \omega_3, 2)$, $(\omega_1, 1 + \omega_2)$, and $(1 - \omega_1 + \omega_2, 2 - \omega_1 + \omega_2, 2 - \omega_1 + \omega_2)$. They all equal 2.1364. □

3 Conclusion & Future Research

We have shown that Max Internal Spanning Tree can be solved in time $\mathcal{O}^*(3^n)$. In a preliminary version of this paper we asked if MIST can be solved in time $\mathcal{O}^*(2^n)$ and also expressed our interest in polynomial space algorithms for MIST. These questions have been settled very recently by Nederlof [10] by providing a $\mathcal{O}^*(2^n)$ polynomial-space algorithm for MIST which is based on the principle of Inclusion-Exclusion and on a new concept called "branching walks".

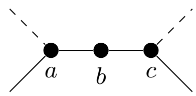
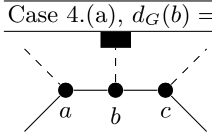
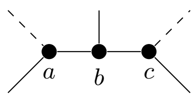
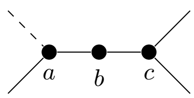
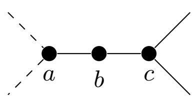
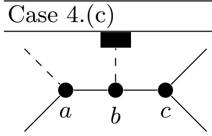
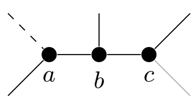
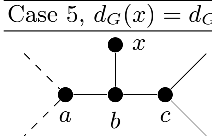
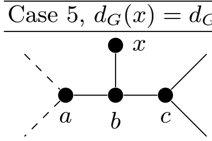
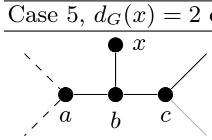
	add	delete	branching tuple
Case 4.(a), $d_G(b) = 2$			
	$a : U \to X$ $b : Z \to U$ $c : U \to Y$	symmetric	$(1 + \omega_1 - \omega_2, 1 + \omega_1 - \omega_2)$
Case 4.(a), $d_G(b) = 3$, b is incident to a pt-edge			
	$a : U \to X$ $b : W \to Y$ $c : U \to Y$	symmetric	$(2 + \omega_1 - \omega_3, 2 + \omega_1 - \omega_3)$
Case 4.(a), $d_G(b) = 3$, b is not incident to a pt-edge			
	$a : U \to X$ $b : U \to Y$ $c : U \to Y$	$a : U \to Y$ $b : U \to Z$	$(2 + \omega_1, 1 + \omega_2)$
Case 4.(b), $d_T(a) = 1$			
	$a : U \to X$ $b : Z \to Y$	$a : U \to Y$ $b : Z \to U$ $c : U \to W$	$(1 + \omega_1 - \omega_2, 1 + \omega_3 - \omega_2)$
Case 4.(b), $d_T(a) = 2$			
	$a : X \to Y$ $b : Z \to Y$	$a : X \to Y$ $b : Z \to U$ $c : U \to W$	$(2 - \omega_1 - \omega_2, 1 - \omega_1 - \omega_2 + \omega_3)$
Case 4.(c)			
	$a : U \to X$ $b : W \to X$	$a : U \to Y$ $b : W \to Y$ $c : U \to W$	$(2\omega_1 - \omega_3, 2)$
Case 4.(d)			
	$a : U \to X$	$a : U \to Y$ $b : U \to Z$	$(\omega_1, 1 + \omega_2)$
Case 5, $d_G(x) = d_G(c) = 3$ and there is $q \in (X \cap (N(x) \cup N(c))$, w.l.o.g. $q \in N(c)$			
	$a : X \to Y$ $b : U \to Y$ (2nd branch) $q : X \to Y$	$a : X \to Y$ $b : U \to Z$	$(2 - \omega_1, 3 - 2\omega_1, 1 - \omega_1 + \omega_2)$
Case 5, $d_G(x) = d_G(c) = 3$			
	$a : X \to Y$ $b : U \to Y$ $c/x : U \to Z$ There are 3 branches; 2 of them (add) are symmetric.	$a : X \to Y$ $b : U \to Z$	$(1 - \omega_1 + \omega_2, 2 - \omega_1 + \omega_2, 2 - \omega_1 + \omega_2)$
Case 5, $d_G(x) = 2$ or $d_G(c) = 2$			
	$a : X \to Y$ $b : U \to Y$ When $\{a, b\}$ is deleted, **ConsDeg2** additionally decreases k by 1 and removes a vertex of Z.	$a : X \to Y$ $b : U \to Z$	$(2 - \omega_1, 2 - \omega_1, 2 - \omega_1)$

Table 1. Analysis of the branching for the running time of Theorem 3

This paper focuses on algorithms for MIST that work for the degree-bounded case, in particular, for subcubic graphs. The main novelty is a Measure & Conquer approach to analyze our algorithm from a parameterized perspective (parameterizing by the solution size). We are not aware of many examples where this was successfully done without cashing the obtained gain at an early stage, see [15]. More examples in this direction would be interesting to see.

A related problem is the generalization to directed graphs: Find a directed tree, which consist of directed paths form the root to the leaves with as few leaves as possible. Which results can be carried over to the directed case?

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