A NEW MATHEMATICAL MODEL FOR SUPERCOOLING

by

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Abstract. In this article we study supercooling from a macroscopic point of view, by modeling the evolution of a supercooled body from its liquid state to its solid state. A ørst model, which would be expected to have discontinuous solutions, is regularized by introducing an intrinsic viscous dissipation. By applying the classical method of Faedo-Galerkin, this regularized model is shown to have a global smooth solution, which describes the state transition of the supercooled body approximately.

1. Introduction

Supercooling is the phenomenon that takes place when a substance, as water, remains in its liquid state after its temperature drops below its melting point. Experience shows this phenomenon -when it occurs- to be very unstable, and that the appearance of a small crystal suŒces to shift the system to the solid state. In spite of the very violent nature of this transition as observed with the bare eye, a slow motion camera reveals a surface of discontinuity that separates the liquid zone from the solid zone. This surface advances, with a speed that depend on the temperature of the local medium, until the whole system is solidiøed.

This subject has been widely studied by physicists from a microscopic point of view, focusing on the internal structure of the crystals that form as the freezing front advances. Kivelson [11] proposes a model, called Theory of Frustration-Limited Domains, which describes the formation of dendritic crystalline structures. In another context, Wang [16] investigates the eœect of carbon on a supercooled alloy. In that work, Wang shows that the presence of carbon increases both the range for which supercooling exists and the mechanical resistance of the alloy. Other interesting works on supercooling from a microscopic point of view were done by Wahnstrom [15] and by Sciortino [13].

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A macroscopic approach to supercooling was initiated by Frmond [8]. In his work, Frmond shows how classical thermodynamics of continuous media can be adapted to the study of supercooling. The laws of its behavior and the equations of evolution obtained by Frmond describe supercooling at an engineer's scale. In this article, following the line of reasoning introduced by Frmond, a new macroscopic model for supercooling is presented and mathematically analyzed.

The starting point is the Energy Equation for a material medium that evolves from a supercooled state to a solid state, under the hypothesis that the whole medium is at rest. In order to distinguish the solid zone from the supercooled zone, a state quantity β is introduced, representing the volumetric fraction of liquid material. This state quantity is constant on both sides of the solidifying front, and undergoes a jump discontinuity on the front itself. The original boundary free model in [8] for the evolution of the freezing front is replaced by a partial diœerential equation in which the moving front in not explicitly mentioned. In order to obtain this equation, we ørst assume that the evolution of the system is a reversible process. Thermodynamic arguments show that the non reversibility of the process can be modelled by the introduction of a viscosity term in β. The model is completed by an energy balance equation, which can be now written, with the help of the β quantity, in the whole domain. This new model allows a detailed mathematical analysis of the existence and regularity of its solutions. Moreover, it seems suitable for numerical computation since it can be stated as a nonlinear variational problem.

The main mathematical result of this article is Theorem (3.1), which guarantees the existence of at least one solution of the new supercooling model. The proof of the result is based on the known technique of Galerkin approximation, using a particular basis. This method has been used, for example, in the papers by Lions [12], Temam [14] and in Conca, Murat & Pironneau [2]. Let us recall that the method consists in discretizing the model, ønding adequate a-priori estimates for the approximated solutions, and passing to the limit with the help of some appropriate compacity result. In our application of Galerkin's method, besides of the need for a particular basis, the treatment of the non-linearities that occur in the model is of special interest.

A topic of interest for future research consists of studying the behavior of the solutions as the viscosity term tends to zero.

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2. Formulation of a mathematical model for supercooling

In this section, a mathematical model for supercooling is presented. This model aims at describing of the evolution of the freezing front in a material that initially is in a supercooled state with a small solidiøed portion (produced, say, by the spontaneous formation of crystals or by an external agent). In order to write the equations that describe this evolution, we assume that at each time t, the domain $\Omega \subseteq \mathbb{R}^3$ that is occupied by the whole body is divided into two zones: the liquid zone $\Omega_\ell(t)$, corresponding to the material that is still in the supercooled state, and the solid zone $\Omega_s(t)$, corresponding to the material that is already frozen. These two zones are separated by a surface of discontinuity. The situation is shown in Figure (2.1).

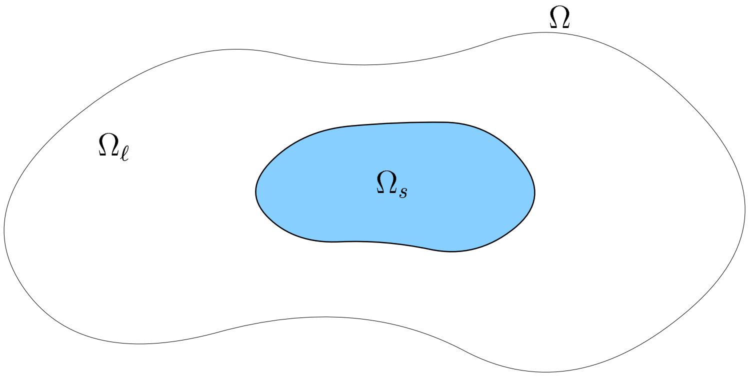

Figure 2.1. Progress of the solidification front in a supercooled medium

2.1. The reversible model. The equation that locally describes the evolution of the system from the supercooled state toward the frozen state is obtained from the conservation of energy in the system. In the absence of motion or external heat sources, the corresponding equation is as follows:

$$\frac{\partial e}{\partial t} + \operatorname{div} \vec{q} = 0 \ \text{ in } \Omega, \tag{2.1}$$

where $e(\vec{x}, t)$ corresponds to the density of internal energy of the medium at the point $\vec{x} \in \Omega$ and at time t, while $\vec{q}(\vec{x}, t)$ corresponds to the heat flow of the medium.

In one of its simplest forms, the medium's internal energy in each zone, Ω_ℓ and Ω_s, may be assumed to be an affine function of the local temperature u. The slope of these functions correspond to the specific heat of the medium, which in order to simplify the equations shall be assumed equal to 1 for both zones. The additive constants of the affine functions must be such that there is a jump shift between the supercooled zone and the solid zone equal to the phase change latent heat of the material. Here again, for the sake of simplification, we shall assume that this latent heat equals 1. Thus, the function $e(\vec{x}, t)$ can be written as:

$$e(\vec{x}, t) = \begin{cases} u(\vec{x}, t) & \text{if } \ \vec{x} \in \Omega_s \\ u(\vec{x}, t) + 1 & \text{if } \ \vec{x} \in \Omega_\ell \end{cases} .$$

To mathematically distinguish both zones $\Omega_\ell(t)$ and $\Omega_s(t)$ we introduce a state quantity $\beta(\vec{x}, t)$, representing the volume fraction of the supercooled material at each point. This state quantity β will thus take the value 1 in $\Omega_\ell(t)$ and 0 in $\Omega_s(t)$. With its help, the internal energy $e(\vec{x}, t)$ can be written in terms of the temperature as:

$$e = u + \beta.$$

To model the heat Æow $\vec{q}(\vec{x}, t)$, we shall adopt the Fourier law with unit conductivity, that is

$$\vec{q} = -\nabla u.$$

Therefore, the diœerential equation corresponding to the energy balance of the system during the evolution from the supercooled state to the solid state is:

$$\frac{\partial u}{\partial t} + \frac{\partial \beta}{\partial t} - \Delta u = 0 \quad \text{in } \Omega_\ell \cup \Omega_s. \tag{2.2}$$

In order to complete the diœerential equation model, we must add the initial conditions and the boundary conditions at the domain's external border and at the interphase between $\Omega_\ell(t)$ and $\Omega_s(t)$. The boundary conditions at the external border will be classical conditions of Neumann or Dirichlet type. At the interphase, the boundary conditions consist of the continuity of the temperature and the discontinuity of the state quantity β (which steps from 0 to 1). Additionally, since the freezing surface is moving, some law for its evolution must be provided. In the present case we adopt a law proposed by Frmond [8], which is in agreement with experiments and states that the normal surface velocity of the freezing front is a known function of the temperature. Notice that this latter assumption leads naturally to a free boundary problem. Now we shall see how to obtain from this assumption, using classical distribution theory, an equation in which the surface of discontinuity is not explicitly mentioned.

Before proceeding any further, let us introduce some additional notation.

- The evolution from the supercooled state to the solid state proceeds over a time interval $[0, T]$, where $T > 0$.
- At every time $t \in [0, T]$, the open domain $\Omega \subseteq \mathbb{R}^3$ divides into two open parts $\Omega_\ell(t)$ and $\Omega_s(t)$ separated by the moving discontinuity $\Gamma(t) = \partial\Omega_s(t) \cap \partial\Omega_\ell(t)$. This discontinuity advances at a normal speed $\vec{W}$, which is a function of the temperature, $\vec{W} = W(u)\vec{n}$, where $\vec{n}$ is the normal vector to $\Gamma(t)$ pointing to the liquid zone.
- We shall denote the space-time domain we work in by $Q = \Omega \times]0, T[\subseteq \mathbb{R}^4$. This domain, thus, divides into its supercooled part

  $$Q_\ell = \{(\vec{x}, t) \in Q / \vec{x} \in \Omega_\ell(t)\},$$

  and its solidiøed part

  $$Q_s = \{(\vec{x}, t) \in Q / \vec{x} \in \Omega_s(t)\}$$

  (see Figure 2.2). These two zones are separated, in $\mathbb{R}^4$, by the frontier

  $$G = \{(\vec{x}, t) \in Q / \vec{x} \in \Gamma(t)\}.$$

- Finally, we let $\mathcal{D}(Q)$ be the space of inønitely diœerentiable functions $\varphi : Q \to \mathbb{R}$ with compact support contained in Q, and we let $\mathcal{D}'(Q)$ be the space of distributions in Q, that is, the space of continuous linear functionals from $\mathcal{D}(Q)$ into $\mathbb{R}$.

With the help of this notation we can state the following property for the function β:

Proposition 2.1. If $\vec{W}(\vec{x}, t)$ is a C^∞ function over Q, such that $\vec{W}(\vec{x}, t) = W\vec{n}$ in G, then

$$\frac{\partial \beta}{\partial t} + \vec{W} \cdot \nabla \beta = 0 \quad \text{in } \mathcal{D}'(Q).$$

Remark 2.2. In this proposition, the function $\vec{W}(\vec{x}, t)$ is a regular extension of the freezing front velocity function (which has physical meaning only in G) to the whole domain Q.

Proof. In order to prove this property it suŒces to rewrite the quantities $\frac{\partial \beta}{\partial t}$ and $\nabla \beta$ in the distribution sense. To do so, let us consider an arbitrary test function $\varphi \in \mathcal{D}(Q)$. We may observe that:

$$\left\langle \frac{\partial \beta}{\partial t} + \vec{W} \cdot \nabla \beta, \varphi \right\rangle = -\left\langle \beta, \frac{\partial \varphi}{\partial t} + \operatorname{div}(\varphi \vec{W}) \right\rangle \tag{2.3}$$

$$= -\int_{Q_\ell} \frac{\partial \varphi}{\partial t} + \operatorname{div}(\varphi \vec{W}) dV \tag{2.4}$$

$$= -\int_{\partial Q_\ell} \left[\varphi \vec{W}, \varphi\right] \cdot \vec{N}_\ell dS \tag{2.5}$$

$$= -\int_{G} \left[\varphi \vec{W}, \varphi\right] \cdot \vec{N}_\ell dS. \tag{2.6}$$

In the above expressions, dV and dS are, respectively, a volume element and a surface element of $\mathbb{R}^4$, and the vector $\vec{N}_\ell$ is exterior and normal to the domain Q_ℓ. In (2.6), the integral extends over G only, since φ has compact support in Q.

The value of the normal vector $\vec{N}_\ell dS$ can be written as:

$$\vec{N}_\ell dS = \left[-\frac{\vec{W}}{W}, W \right] d\Gamma(t) dt.$$

By substituting this expression into (2.6) the proof is completed. $\square$

In order to apply the previous proposition, it is necessary to have an extension $\vec{W}(\vec{x}, t)$ of the velocity of the freezing front to all of Q. Since information is available only for the speed of this front, it is convenient to modify the property in such a way that only the norm of the vector $\vec{W}(\vec{x}, t)$ is involved. To achieve this, it suŒces to remark that

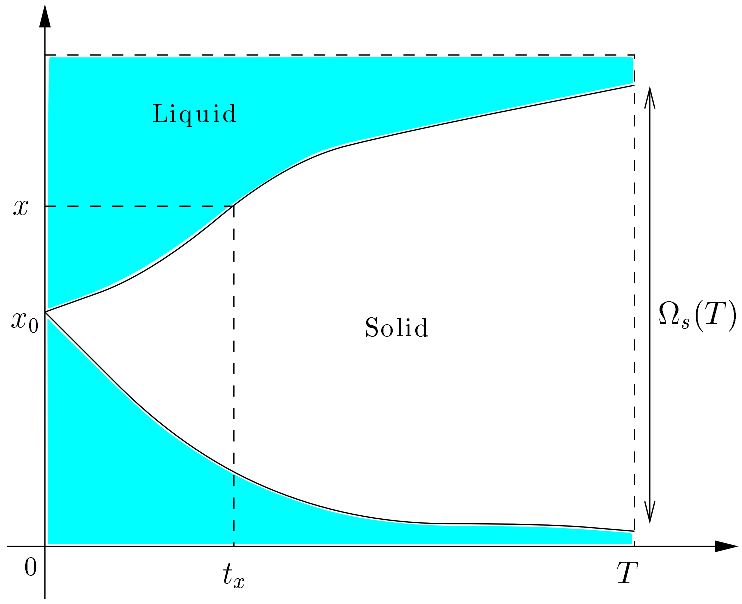

Figure 2.2. The domains Q_ℓ and Q_s

the vectors $\vec{W}(\vec{x}, t)$ and $\nabla\beta$ are parallel. With this remark, the previous proposition becomes:

$$\frac{\partial\beta}{\partial t} + W|\nabla\beta| \;=\; 0. \tag{2.7}$$

Remark 2.3. The parallelism between $\vec{W}(\vec{x}, t)$ and $\nabla\beta$ is clear since $\nabla\beta$ is a distribution concentrated at the front G, at which its norm is a Dirac delta and its direction is normal to the front pointing toward the liquid zone (just as $\vec{W}(\vec{x}, t)$).

This equation together with equation (2.2) provide a ørst model for the study of the freezing transition of a substance in supercooled state. The system of diœerential equations is:

$$\frac{\partial\beta}{\partial t} + W(u)|\nabla\beta| \;=\; 0 \qquad \text{in } \mathcal{D}'(Q) \tag{2.8}$$

$$\frac{\partial u}{\partial t} + \frac{\partial\beta}{\partial t} - \Delta u \;=\; 0 \qquad \text{in } \mathcal{D}'(Q). \tag{2.9}$$

In these equations the free boundary does not appear explicitly. Once the system is solved, the liquid and the solid zone can be totally recovered from the values taken by the state quantity β.

2.2. The irreversible model. The end of the supercooling of liquid water involves microscopic movements which have macroscopic eœects. The density and the volume change slightly. In some case the rapid transformation of water into ice is noisy thus involving movements. We decide to take the power of these microscopic movements into account but without introducing new state quantities. Thus the only quantity which is related to the microscopic velocities is $\frac{\partial \beta}{\partial t}$. The power of the interior forces we choose involves besides the usual macroscopic power $-\sigma : D(\vec{U})$, the new microscopic power $-B\frac{d\beta}{dt} - \vec{H}.\nabla\beta$. The principle of virtual power gives the classical equation of motion and a new one,

$$\begin{aligned} -B + \operatorname{div}\vec{H} = A \qquad & \text{in } \Omega, \\ \vec{H}.\vec{n} = a \qquad & \text{on } \partial\Omega, \end{aligned}$$

where A and a are the exterior energy applied to the domain Ω. The constitutive laws are given by;

$$\begin{aligned} B &= \frac{\partial \Psi}{\partial \beta} + \frac{\partial \Phi}{\partial(d\beta/dt)}, \\ H &= \frac{\partial \Psi}{\partial \nabla\beta}, \end{aligned}$$

where Ψ is the free energy depending on the state quantities β and $\nabla\beta$ and where Φ is the pseudo-potential of dissipation depending on the velocities of the state quantities.

A simple choice for these functions is,

$$\begin{aligned} \Psi(\beta, \nabla\beta) &= \frac{\varepsilon}{2}(\nabla\beta)^2 + I(\beta) \text{ and}, \\ \Phi(\frac{d\beta}{dt}, \nabla\frac{d\beta}{dt}) &= \frac{1}{2}\left(\frac{d\beta}{dt}\right)^2, \end{aligned}$$

where ε is a physical parameter quantifying the interactions of the water particles during the evolution of the supercooling. The function I is the indicator function of the segment $[0,1]$ making compulsory for β to be a volume fraction with value between 0 and 1. If it is assumed exterior actions $A = W(u)|\nabla\beta|$ and $a = 0$, the equation of motion and the constitutive laws give,

$$\begin{aligned} \frac{d\beta}{dt} - \varepsilon\Delta\beta + W(u)|\nabla\beta| + \partial I(\beta) \ni 0 \quad & \text{in } \Omega, \\ \varepsilon\frac{\partial \beta}{\partial n} = 0 \quad & \text{on } \partial\Omega. \end{aligned}$$

Adding to this equation, the energy balance equations (2.2) we obtain a second model for the supercooling evolution. This model take into account microscopic movements, and therefore leads to an irreversible process, trough the appearance of the ε coeŒcient.

In the case that $\varepsilon = 0$, this model is reduced to equations (2.8)ˇ(2.9), since in this case the indicator function I is no more necessary, provided that $\beta \in [0,1]$ at $t = 0$.

After adding boundary conditions and initial conditions, our problem is: Find functions $\beta, u \in \mathcal{D}'(Q)$, suŒciently smooth, such that:

$$\frac{\partial \beta}{\partial t} - \varepsilon \Delta \beta + W(u)|\nabla \beta| + \partial I(\beta) \ni 0 \qquad \text{in } Q \tag{2.10}$$

$$\frac{\partial u}{\partial t} + \frac{\partial \beta}{\partial t} - \Delta u = 0 \qquad \text{in } Q \tag{2.11}$$

$$\beta(x,0) = \beta_0(x) \qquad \text{in } \Omega \tag{2.12}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega \tag{2.13}$$

$$\frac{\partial \beta}{\partial n} = 0 \qquad \text{on } \partial\Omega \times]0,T[\tag{2.14}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega \times]0,T[\tag{2.15}$$

where the initial conditions β_0 and u_0 are known functions on Ω. Furthermore, $W(\cdot)$ is a continuous and compact support function on $\mathbb{R}$.

The variational formulation of the system of equations (2.10)˘(2.15) is the following problem which we will call problem (P) from now on.

For $\beta_0, u_0 \in H^1(\Omega)$,

$$(P) \left\| \begin{array}{rcl} \multicolumn{3}{c}{\text{Find } u,\ \beta \in L^\infty(0,T;H^1(\Omega)) \cap C^0(0,T;L^2(\Omega)) \ :} \\ \displaystyle\int_\Omega \frac{\partial \beta}{\partial t}\varphi + \varepsilon \int_\Omega \nabla\beta \cdot \nabla\varphi + \int_\Omega W(u)|\nabla\beta|\varphi + \int_\Omega \partial I(\beta)\varphi & \ni & 0 \quad \forall \varphi \in H^1(\Omega) \\ \displaystyle\int_\Omega \frac{\partial u}{\partial t}\varphi + \int_\Omega \frac{\partial \beta}{\partial t}\varphi + \int_\Omega \nabla u \cdot \nabla\varphi & = & 0 \quad \forall \varphi \in H^1(\Omega) \\ \beta(0) & = & \beta_0 \\ u(0) & = & u_0 \end{array} \right.$$

3. Existence of a global solution for the regularized model

In this section we will show that the regularized supercooling model admits at least one solution. More precisely, we shall prove the following theorem:

Theorem 3.1. Given β_0, $u_0 \in H^1(\Omega)$, there exists functions u, $\beta \in L^\infty(0,T;H^1(\Omega)) \cap C^0(0,T;L^2(\Omega))$ solution of the problem (P).

In order to prove this theorem we begin by replacing the indicator function I by a $C^2(\mathbb{R})$ function. For this, we deøne a ørst approximation I_1 by the C^2 function such that $I_1(0) = I_1'(0) = 0$ and:

$$I_1''(x) = \begin{cases} 0 & \text{if} \quad x \le -2 \\ x+2 & \text{if} \quad -2 \le x \le -1 \\ -x & \text{if} \quad -1 \le x \le 0 \\ 0 & \text{if} \quad 0 \le x \le 1 \\ (x-1) & \text{if} \quad 1 \le x \le 2 \\ (3-x) & \text{if} \quad 2 \le x \le 3 \\ 0 & \text{if} \quad 3 \le x \end{cases}$$

For each positive integer number p we deøne the approximation I_p by $I_p(x) = pI_1(x)$.

Some usefull properties of the function I_p are the following:

Lemma 3.2. I_p is a real convex function function in $C^2(\mathbb{R})$ such that, $\forall x \in \mathbb{R}$:

$$\begin{array}{rcl} I_p(x) & \ge & 0 \\ I_p(x) = 0 & \Leftrightarrow & x \in [0,1] \\ xI_p'(x) & \ge & 0 \\ |I_p'(x)| & \le & p \\ |I_p(x)| & \le & p|x| \end{array}$$

With this function I_p we deøne the following approximated problem (P_p)

$$(P_p) \left\| \begin{array}{rcl} \multicolumn{3}{l}{\text{Find } u,\ \beta \in L^\infty(0,T;H^1(\Omega)) \cap C^0(0,T;L^2(\Omega)) \ :} \\ \displaystyle\int_\Omega \frac{\partial \beta}{\partial t}\varphi + \varepsilon \int_\Omega \nabla\beta \cdot \nabla\varphi + \int_\Omega W(u)|\nabla\beta|\varphi + \int_\Omega I_p'(\beta)\varphi & = & 0\,\forall\varphi \in H^1(\Omega) \\ \displaystyle\int_\Omega \frac{\partial u}{\partial t}\varphi + \int_\Omega \frac{\partial \beta}{\partial t}\varphi + \int_\Omega \nabla u \cdot \nabla\varphi & = & 0\,\forall\varphi \in H^1(\Omega) \\ \beta(0) & = & \beta_0 \\ u(0) & = & u_0 \end{array} \right.$$

where β_0 and u_0 are the initial conditions.

The theorem (3.1) will be a consequence of the following result.

Theorem 3.3. Given β_0 and u_0 in $H^1(\Omega)$. For each positive integer number p, the problem (P_p) has a solution (β^p, u^p). Moreover, the sequence $\{(\beta^p, u^p)\}$ has a subsequence that converge to a solution of the problem (P).

To prove this latter theorem we use the method of Faedo-Galerkin. In order to apply the method, we begin by discretizing the model considering a basis for $H^1(\Omega)$. Next, we ønd a-priori estimates for the approximated solutions, and ønally, we pass to the limit in order to obtain functions $u,\ \beta \in L^\infty(0,T;H^1(\Omega)) \cap C^0(0,T;L^2(\Omega))$ with the desired properties. In what follows, each one of these steps is presented in detail.

3.1. Step 1. Discretization of the problem (P_p). Since $H^1(\Omega)$ is a separable space, there exists a sequence of linearly independent elements $\{v_i\}_{i\in\mathbb{N}}$ which is total in $H^1(\Omega)$. The v_i's can be chosen as in the subsection (3.6). Then this basis is orthogonal for the inner product on $H^1(\Omega)$ and orthonormal with respect to the inner product on $L^2(\Omega)$.

For each $m \in \mathbb{N}$ we denote by $V_m = [v_1, \dots, v_m]$ the space spawned by $v_1, \dots, v_m$ and by β_0^m and u_0^m the $H^1(\Omega)$ projection of β_0 and u_0, respectively, on the space V_m.

For each $m \in \mathbb{N}$ we deøne the approximate solution β^m and u^m of the problem (P_p) as the solution of the following diœerential equations system (P_m):

$$
(P_m)\left\|\begin{array}{lrcl}
 & \multicolumn{3}{c}{\text{Find } u(t),\ \beta(t) \in V_m \ :} \\
(a) & \displaystyle\int_\Omega \frac{\partial\beta}{\partial t}\varphi + \varepsilon\int_\Omega \nabla\beta\cdot\nabla\varphi + \int_\Omega W(u)|\nabla\beta|\varphi + \int_\Omega I_p'(\beta)\varphi & = & 0\,\forall\varphi\in V_m, \\
(b) & \displaystyle\int_\Omega \frac{\partial u}{\partial t}\varphi + \int_\Omega \frac{\partial\beta}{\partial t}\varphi + \int_\Omega \nabla u\cdot\nabla\varphi & = & 0\,\forall\varphi\in V_m \\
(c) & \beta(0) & = & \beta_0^m \\
(d) & u(0) & = & u_0^m.
\end{array}\right.
$$

The functions β^m and u^m can be written on the form:

$$
\begin{aligned}
\beta^m &= \sum_{j=1}^m \beta_j^m(t) v_j \\
u^m &= \sum_{j=1}^m u_j^m(t) v_j
\end{aligned}
$$

The equations (P_m) form a nonlinear diœerential system for the real functions $\beta_1^m(t), \dots, \beta_m^m(t)$ and $u_1^m(t), \dots, u_m^m(t)$:

$$
\begin{aligned}
\frac{d\beta_i^m}{dt} + \varepsilon\left|\nabla v_i\right|_{L^2}^2 \beta_i^m(t) + \int_\Omega v_i\left(W(u^m)|\nabla\beta^m| + I_p'(\beta^m)\right) &= 0 \\
\frac{du_i^m}{dt} + \frac{d\beta_i^m}{dt} + \left|\nabla v_i\right|_{L^2}^2 u_i^m(t) &= 0 \\
\beta_i^m(0) &= \int_\Omega \beta_0^m v_i \\
u_i^m(0) &= \int_\Omega u_0^m v_i
\end{aligned}
$$

where the orthonormality of the vectors $\{v_1, \dots, v_m\}$ in $L^2(\Omega)$ has been taken into account.

This system can be reduced to the compact form:

$$\frac{d\beta_i^m}{dt} = \varepsilon \mathrm{L}_i \beta_i^m + \mathrm{F}_i\left(\beta_1^m, \ldots, \beta_m^m, u_1^m, \ldots, u^m\right) \tag{3.1}$$

$$\frac{du_i^m}{dt} = \mathrm{L}_i u_i^m - \varepsilon \mathrm{L}_i \beta_i^m - \mathrm{F}_i\left(\beta_1^m, \ldots, \beta_m^m, u_1^m, \ldots, u_m^m\right) \tag{3.2}$$

$$\beta_i^m(0) = \int_\Omega \beta_0^m v_i \tag{3.3}$$

$$u_i^m(0) = \int_\Omega u_0^m v_i \tag{3.4}$$

where th L_i's are real constant and F_i's are non-linear functions deøned by:

$$\begin{aligned} \mathrm{F}_i(x_1, \ldots, x_m, y_1, \ldots, y_m) &= \int_\Omega v_i |x_1 \nabla v_1 + \cdots + x_m \nabla v_m| W(y_1 v_1 + \cdots + y_m v_m) \\ &+ \int_\Omega v_i I_p'(x_1 v_1 + \cdots + x_m v_m), \end{aligned}$$

which are continuous due to the continuity of W and I_p'.

Thus, the nonlinear system (3.1)˘(3.2) with initial conditions (3.3)-(3.4) admits a maximal solution deøned on some interval $[0, t^m[$ with $0 < t^m \leq \infty$. If $t^m < \infty$, then $\left|\beta^m\right|_{L^2} + \left|u^m\right|_{L^2} \to \infty$ as $t \to t^m$. The a-priori estimates that will be established in the next step show that this does not happen, whence $t^m = \infty$ and the system admits a maximal solution deøned on all of $[0, T]$.

3.2. Step 2. A-priori estimates. In this step we shall ønd a-priori estimates for β^m and u^m in order to ønd inequalities that provide information on whether these sequences are bounded. We use the classical technique of considering β_i^m, u_i^m and their time derivatives as the test function in the equations (a) and (b) of problem (P_m).

Lemma 3.4. (ørst a-priori estimate) The sequence $\{\beta^m\}$ is bounded in $L^2\left(0, T; H^1(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Omega)\right)$ and the sequence $\left\{I_p'(\beta^m)\beta^m\right\}$ is bounded in $L^1(0, T; L^1(\Omega))$.

Proof. Let us consider $\varphi = \beta^m$ in equation (a) of problem (P_m). We thereby obtain:

$$\int_\Omega \frac{\partial \beta^m}{\partial t}\beta^m + \varepsilon \int_\Omega \nabla\beta^m \cdot \nabla\beta^m + \int_\Omega W(u^m)|\nabla\beta^m|\beta^m + \int_\Omega I_p'(\beta^m)\beta^m = 0,$$

that is,

$$\frac{1}{2}\frac{d}{dt}\left|\beta^m\right|_{L^2}^2 + \varepsilon\left|\nabla\beta^m\right|_{L^2}^2 + \int_\Omega I_p'(\beta^m)\beta^m \leq -\int_\Omega W(u^m)|\nabla\beta^m|\beta^m. \tag{3.5}$$

Since the function $W(\cdot)$ is bounded in $L^\infty(\mathbb{R})$, if we denote $C_W = \left|W\right|_{L^\infty}$, we have that:

$$\begin{aligned} -\int_\Omega W(u^m)|\nabla\beta^m|\beta^m &\leq C_W \int_\Omega |\nabla\beta^m||\beta^m| \\ &\leq C_W \left|\nabla\beta^m\right|_{L^2} \left|\beta^m\right|_{L^2}. \end{aligned}$$

Considering these inequalities, (3.5) leads to:

$$\frac{1}{2}\frac{d}{dt}\left|\beta^m\right|_{L^2}^2 + \varepsilon\left|\nabla\beta^m\right|_{L^2}^2 + \int_\Omega I_p'(\beta^m)\beta^m \leq C_W \left|\nabla\beta^m\right|_{L^2}\left|\beta^m\right|_{L^2}, \tag{3.6}$$

But it is well known that:

$$\forall\alpha, x, y > 0, \;\; xy \leq \alpha x^2 + \frac{1}{4\alpha}y^2, \tag{3.7}$$

then for each $b > 0$ we can write (3.6) as:

$$\frac{1}{2}\frac{d}{dt}\left|\beta^m\right|_{L^2}^2 + \varepsilon\left|\nabla\beta^m\right|_{L^2}^2 + \int_\Omega I_p'(\beta^m)\beta^m \leq C_W\left(b\left|\nabla\beta^m\right|_{L^2}^2 + \frac{1}{4b}\left|\beta^m\right|_{L^2}^2\right),$$

Rearranging the previous expression, we get:

$$\frac{1}{2}\frac{d}{dt}\left|\beta^m\right|_{L^2}^2 - \frac{C_W}{4b}\left|\beta^m\right|_{L^2}^2 + (\varepsilon - bC_W)\left|\nabla\beta^m\right|_{L^2}^2 + \int_\Omega I_p'(\beta^m)\beta^m \leq 0. \tag{3.8}$$

Now if we choose the constant b small enough so that $c = (\varepsilon - bC_W)$ becomes positive, then the following Gronwall-type inequality for $\left|\beta^m\right|_{L^2}^2$ is obtained:

$$\frac{d}{dt}\left|\beta^m\right|_{L^2}^2 - \frac{C_W}{2b}\left|\beta^m\right|_{L^2}^2 \leq 0,$$

from which we infer that for all t in $[0, T]$ we have

$$\left|\beta^m\right|_{L^2}^2(t) \leq \left|\beta_0\right|_{L^2}^2 \exp\left(\frac{C_W}{2b}T\right).$$

Thus, we have shown that the sequence $\{\beta^m\}$ is bounded in $L^\infty(0, T; L^2(\Omega))$.

Taking again inequality (3.8) and integrating it over $[0, T]$ we obtain:

$$\frac{1}{2}\left|\beta^m(T)\right|_{L^2}^2 + c\int_0^T \left|\nabla\beta^m\right|_{L^2}^2 + \int_0^T\int_\Omega I_p'(\beta^m)\beta^m \leq \frac{1}{2}\left|\beta_0^m\right|_{L^2}^2 + \frac{C_W}{4b}\int_0^T \left|\beta^m\right|_{L^2}^2,$$

from which, by considering that the sequences $\{\beta_0^m\}$ and $\{\beta^m\}$ are bounded in $L^2(\Omega)$ and $L^2(0, T; L^2(\Omega))$, respectively, we conclude the assertion of lemma. $\square$

Lemma 3.5. (Second a-priori estimate) The sequence $\{\frac{\partial\beta^m}{\partial t}\}$ is bounded in $L^2(0, T; L^2(\Omega))$, the sequence $\{\nabla\beta^m\}$ is bounded in $L^\infty(0, T; L^2(\Omega))$ and the sequence $\{I_p(\beta^m)\}$ is bounded in $L^\infty(0, T; L^1(\Omega))$.

Proof. Let us now consider $\varphi = \dfrac{\partial \beta^m}{\partial t}$ in the equation (a) of problem (P_m). We then obtain:

$$\int_\Omega \frac{\partial \beta^m}{\partial t}\frac{\partial \beta^m}{\partial t} + \varepsilon \int_\Omega \nabla \beta^m \cdot \nabla \frac{\partial \beta^m}{\partial t} + \int_\Omega W(u^m)|\nabla \beta^m| \frac{\partial \beta^m}{\partial t} + \int_\Omega I_p'(\beta^m)\frac{\partial \beta^m}{\partial t} \;=\; 0,$$

that is,

$$\left|\frac{\partial \beta^m}{\partial t}\right|_{L^2}^2 + \frac{\varepsilon}{2}\frac{d}{dt}\left|\nabla \beta^m\right|_{L^2}^2 + \frac{d}{dt}\int_\Omega I_p(\beta^m) \;=\; -\int_\Omega W(u^m)|\nabla \beta^m|\frac{\partial \beta^m}{\partial t}.$$

But as we have:

$$\begin{aligned} -\int_\Omega W(u^m)|\nabla \beta^m|\frac{\partial \beta^m}{\partial t} &\leq C_W\left|\nabla \beta^m\right|_{L^2}\left|\frac{\partial \beta^m}{\partial t}\right|_{L^2} \\ &\leq C_W\left(b\left|\nabla \beta^m\right|_{L^2}^2 + \frac{1}{4b}\left|\frac{\partial \beta^m}{\partial t}\right|_{L^2}^2\right) \\ &= bC_W\left|\nabla \beta^m\right|_{L^2}^2 + \frac{C_W}{4b}\left|\frac{\partial \beta^m}{\partial t}\right|_{L^2}^2, \end{aligned}$$

then we get:

$$(1-\frac{C_W}{4b})\left|\frac{\partial \beta^m}{\partial t}\right|_{L^2}^2 + \frac{\varepsilon}{2}\frac{d}{dt}\left|\nabla \beta^m\right|_{L^2}^2 - bC_W\left|\nabla \beta^m\right|_{L^2}^2 + \frac{d}{dt}\int_\Omega I_p(\beta^m) \;\leq\; 0.$$

then, if we integrates over the interval $[0,t]$, where $t \in [0,T]$, we obtain that

$$\begin{aligned} &(1-\frac{C_W}{4b})\int_0^t \left|\frac{\partial \beta^m}{\partial t}\right|_{L^2}^2 + \frac{\varepsilon}{2}\left|\nabla \beta^m(t)\right|_{L^2}^2 + \int_\Omega I_p(\beta^m(t)) \\ &\qquad\qquad \leq \frac{\varepsilon}{2}\left|\nabla \beta_0^m\right|_{L^2}^2 + \int_\Omega I_p(\beta_0^m) + bC_W\int_0^t \left|\nabla \beta^m\right|_{L^2}^2 \end{aligned}$$

If in this inequality we choose the constant b large enough, the factor $1-\frac{C_W}{4b}$ becomes positive. Considering that the sequence $\{\nabla \beta_0^m\}$ is bounded in $L^2(\Omega)$, that the integral $\displaystyle\int_\Omega I_p(\beta_0^m)$ converges to $\displaystyle\int_\Omega I_p(\beta_0) = 0$ then is bounds, and that the sequence $\{\nabla \beta^m\}$ is bounded in $L^2(0,T;L^2(\Omega))$, it follows that the sequence $\left\{\dfrac{\partial \beta^m}{\partial t}\right\}$ is bounded in $L^2(0,T;L^2(\Omega))$, the sequence $\{\nabla \beta^m\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and the sequence $\{I_p(\beta^m)\}$ is bounded in $L^\infty(0,T;L^1(\Omega))$. $\square$

Remark 3.6. The bound of integral $\displaystyle\int_\Omega I_p(\beta_0^m)$ depends on p then also the bound found in this a-priori estimate depend on p. It is only for this reason that the bounds found in all the following a-priori estimates depend on p.

Lemma 3.7. (third a-priori estimate) The sequence $\{u^m\}$ is bounded in $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$.

Proof. Let us now consider $\varphi = u^m$ in the equation (b) of problem (P_m). We then obtain:

$$\int_\Omega \frac{\partial u^m}{\partial t}u^m + \int_\Omega \frac{\partial \beta^m}{\partial t}u^m + \int_\Omega \nabla u^m \cdot \nabla u^m = 0$$

that is

$$\frac{1}{2}\frac{d}{dt}\left|u^m\right|^2_{L^2} + \left|\nabla u^m\right|^2_{L^2} = -\int_\Omega \frac{\partial \beta^m}{\partial t}u^m.$$

If we integrate over the $[0,t]$ interval, we obtain:

$$\begin{aligned}\frac{1}{2}\left|u^m(t)\right|^2_{L^2} + \int_0^t \left|\nabla u^m\right|^2_{L^2} &= \frac{1}{2}\left|u_0^m\right|^2_{L^2} - \int_0^t\!\!\int_\Omega \frac{\partial \beta^m}{\partial t}u^m \\ &\leq \frac{1}{2}\left|u_0^m\right|^2_{L^2} + \frac{1}{2}\int_0^t \left|\frac{\partial \beta^m}{\partial t}\right|^2_{L^2} + \frac{1}{2}\int_0^t \left|u^m\right|^2_{L^2}\end{aligned}$$

that is, rearranging terms, for all $t \in [0,T]$ we have:

$$\frac{1}{2}\left|u^m(t)\right|^2_{L^2} - \frac{1}{2}\int_0^t \left|u^m\right|^2_{L^2} + \int_0^t \left|\nabla u^m\right|^2_{L^2} \leq \frac{1}{2}\left|u_0^m\right|^2_{L^2} + \frac{1}{2}\int_0^t \left|\frac{\partial \beta^m}{\partial t}\right|_{L^2}.$$

From this expression we deduce a Gronwall-type inequality for the function $F(t) = \frac{1}{2}\int_0^t \left|u^m\right|^2_{L^2}$. From this, we deduce that $F(t)$ is bounded.

If we look at the previous inequality, using the last result we can conclude that the sequence u^m is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. □

Lemma 3.8. (Fourth a-priori estimate) The sequence $\left\{\frac{\partial u^m}{\partial t}\right\}$ is bounded in $L^2(0,T;L^2(\Omega))$ and the sequence $\{\nabla u^m\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$.

Proof. Let us now consider $\varphi = \frac{\partial u^m}{\partial t}$ in the equation (b) of problem (P_m). We then obtain:

$$\int_\Omega \frac{\partial u^m}{\partial t}\frac{\partial u^m}{\partial t} + \int_\Omega \frac{\partial \beta^m}{\partial t}\frac{\partial u^m}{\partial t} + \int_\Omega \nabla u^m \cdot \nabla \frac{\partial u^m}{\partial t} = 0$$

that is:

$$\left|\frac{\partial u^m}{\partial t}\right|^2_{L^2} + \frac{1}{2}\frac{d}{dt}\left|\nabla u^m\right|^2_{L^2} = -\int_\Omega \frac{\partial \beta^m}{\partial t}\frac{\partial u^m}{\partial t}$$

Integrating this equation over $[0,t]$, where $t \in [0,T]$, we obtain:

$$\int_0^t \left|\frac{\partial u^m}{\partial t}\right|_{L^2}^2 + \frac{1}{2}\left|\nabla u^m(t)\right|_{L^2}^2 = \frac{1}{2}\left|\nabla u_0^m\right|_{L^2}^2 - \int_0^t\!\!\int_\Omega \frac{\partial \beta^m}{\partial t}\frac{\partial u^m}{\partial t} \tag{3.9}$$

$$\leq \frac{1}{2}\left|\nabla u_0^m\right|_{L^2}^2 + \frac{1}{2}\int_0^t \left|\frac{\partial \beta^m}{\partial t}\right|_{L^2}^2 + \frac{1}{2}\int_0^t \left|\frac{\partial u^m}{\partial t}\right|_{L^2}^2 \tag{3.10}$$

Hence, $\forall t \in [0,T]$ it follows that:

$$\frac{1}{2}\int_0^t \left|\frac{\partial u^m}{\partial t}\right|_{L^2}^2 + \frac{1}{2}\left|\nabla u^m(t)\right|_{L^2}^2 \leq \frac{1}{2}\left|\nabla u_0^m\right|_{L^2}^2 + \frac{1}{2}\int_0^t \left|\frac{\partial \beta^m}{\partial t}\right|_{L^2}^2$$

From this we deduce that the sequence $\left\{\dfrac{\partial u^m}{\partial t}\right\}$ is bounded in $L^2(0,T;L^2(\Omega))$ and that the sequence $\{\nabla u^m\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$. $\square$

Lemma 3.9. (øfth a-priori estimate) The sequence $\{\beta^m\}$ is bounded in $L^2\left(0,T;H^2(\Omega)\right)$.

Proof. Let us consider $\varphi = \Delta\beta^m$ in equation (a) of problem (P_m). We thereby obtain:

$$\int_\Omega \frac{\partial \beta^m}{\partial t}\Delta\beta^m + \varepsilon\int_\Omega \nabla\beta^m\cdot\nabla\Delta\beta^m + \int_\Omega W(u^m)|\nabla\beta^m|\Delta\beta^m + \int_\Omega I_p'(\beta^m)\Delta\beta^m = 0,$$

that is,

$$\int_\Omega \frac{\partial \beta^m}{\partial t}\Delta\beta^m - \varepsilon\left|\Delta\beta^m\right|_{L^2}^2 + \int_\Omega W(u^m)|\nabla\beta^m|\Delta\beta^m - \int_\Omega I_p''(\beta^m)\left|\nabla\beta^m\right|^2 = 0,$$

Integrating this equation over $[0,T]$, we obtain:

$$\begin{aligned}
&\varepsilon\left\|\Delta\beta^m\right\|_{L^2(0,T;L^2(\Omega))}^2 + \int_0^T\int_\Omega I_p''(\beta^m)\left|\nabla\beta^m\right|^2 \\
&\qquad = \int_0^T\!\!\int_\Omega \frac{\partial \beta^m}{\partial t}\Delta\beta^m + \int_0^T\!\!\int_\Omega W(u^m)|\nabla\beta^m|\Delta\beta^m \\
&\qquad \leq \left[\left\|\frac{\partial \beta^m}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + C_W\left\|\nabla\beta^m\right\|_{L^2(0,T;L^2(\Omega))}\right]\left\|\Delta\beta^m\right\|_{L^2(0,T;L^2(\Omega))}
\end{aligned}$$

Since all sequences in the right hand side are bounded and $I_p''(x) \geq 0$, we deduce that the sequence $\{\Delta\beta^m\}$ is also bounded in $L^2(0,T;L^2(\Omega))$ and then we conclude the assertion of the lemma. $\square$

3.3. Step 3. Limiting procedure. The a-priori estimates found in Step 2 show that:

1. $\|\beta^m\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla\beta^m\|_{L^2(0,T;L^2(\Omega))} + \left\|I_p'(\beta^m)\beta^m\right\|_{L^1(0,T;L^1(\Omega))} \le C$
2. $\left\|\dfrac{\partial\beta^m}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\beta^m\|_{L^\infty(0,T;L^2(\Omega))} + \|I_p(\beta^m)\|_{L^\infty(0,T;L^1(\Omega))} \le C(p)$
3. $\|u^m\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u^m\|_{L^2(0,T;L^2(\Omega))} \le C(p)$
4. $\left\|\dfrac{\partial u^m}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla u^m\|_{L^\infty(0,T;L^2(\Omega))} \le C(p)$
5. $\|\Delta\beta^m\|_{L^2(0,T;L^2(\Omega))} \le C(p)$

In order to pass to the limit we will use the Aubin's lemma, which is stated next:

Lemma. (Aubin's lemma.) Let X, H and Y be three Banach spaces such that the injective mappings $X \hookrightarrow H \hookrightarrow Y$ are continuous and the ørst one is compact. Let $\{s_n\}$ be a sequence bounded in $L^p(0,T;X)$ such that the sequence of its time derivatives $\{\dfrac{\partial s_n}{\partial t}\}$ is bounded in $L^q(0,T;Y)$, where $p, q \in [1, +\infty]$. Then:

If $p < \infty$, the sequence $\{s_n\}$ is compact in $L^p(0,T;H)$

If $p = \infty$, the sequence $\{s_n\}$ is compact in $C^0(0,T;H)$

In our case, the sequences $\{\beta^m\}$ and $\{u^m\}$ satisfy the hypotheses of Aubin's lemma for $X = H^1(\Omega)$, $H = Y = L^2(\Omega)$, $p = \infty$ and $q = 2$. Thus, these sequences are compacts in $C^0\left(0,T;L^2(\Omega)\right)$. Additionally the sequence $\{\beta^m\}$ satisfy the hypotheses of Aubin's lemma for $X = H^2(\Omega)$, $H = H^1(\Omega)$, $Y = L^2(\Omega)$ and $p = q = 2$. Thus, this sequence is also compact in $L^2\left(0,T;H^1(\Omega)\right)$.

Hence, there exist subsequences $\{\beta^{m'}\}$ and $\{u^{m'}\}$, and functions $\beta \in L^\infty\left(0,T;H^1(\Omega)\right) \cap C^0\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^2(\Omega)\right)$ and $u \in L^\infty\left(0,T;H^1(\Omega)\right) \cap C^0\left(0,T;L^2(\Omega)\right)$ such that:

$$
\begin{aligned}
\beta^{m'} &\rightharpoonup \beta \text{ weakly* in } L^\infty\left(0,T;H^1(\Omega)\right)\\
\beta^{m'} &\rightharpoonup \beta \text{ weakly in } L^2\left(0,T;H^2(\Omega)\right)\\
\beta^{m'} &\to \beta \text{ strongly in } C^0\left(0,T;L^2(\Omega)\right)\\
\beta^{m'} &\to \beta \text{ strongly in } L^2\left(0,T;H^1(\Omega)\right)\\
u^{m'} &\rightharpoonup u \text{ weakly* in } L^\infty\left(0,T;H^1(\Omega)\right)\\
u^{m'} &\to u \text{ strongly in } C^0\left(0,T;L^2(\Omega)\right)
\end{aligned}
$$

For the sake of convenience, the above subsequences shall be indexed by m instead of m'.

In order to analyze the limiting passage we must not forget that the subsequences $\{\beta^m\}$ and $\{u^m\}$ satisfy the equations:

$$\int_\Omega \frac{\partial \beta^m}{\partial t} v_i + \varepsilon \int_\Omega \nabla\beta^m \cdot \nabla v_i + \int_\Omega W(u^m)|\nabla\beta^m| v_i + \int_\Omega I_p'(\beta^m) v_i = 0 \tag{3.11}$$

$$\int_\Omega \frac{\partial u^m}{\partial t} v_i + \int_\Omega \frac{\partial \beta^m}{\partial t} v_i + \int_\Omega \nabla u^m \cdot \nabla v_i = 0 \tag{3.12}$$

$$\beta^m(0) = \beta_0^m \tag{3.13}$$

$$u^m(0) = u_0^m \tag{3.14}$$

for every v_i, $i = 1, \dots, m$.

Let us consider a real function $\phi(t)$ continuously diœerentiable in $[0, T]$ and such that $\phi(T) = 0$. By multiplying equations (3.11) and (3.12) by $\phi(t)$ and then integrating by parts, the following equations are obtained:

$$\begin{aligned} \int_\Omega \beta_0^m v_i \phi(0) - \int_0^T\!\!\int_\Omega \beta^m \phi' v_i + \varepsilon \int_0^T\!\!\int_\Omega \nabla\beta^m \cdot \nabla\phi v_i + \\ + \int_0^T\!\!\int_\Omega W(u^m)|\nabla\beta^m|\phi v_i + \int_0^T\!\!\int_\Omega I_p'(\beta^m)\phi v_i &= 0 \end{aligned} \tag{3.15}$$

$$\int_\Omega u_0^m v_i \phi(0) - \int_\Omega \beta_0^m v_i \phi(0) - \int_0^T\!\!\int_\Omega u^m \phi' v_i - \int_0^T\!\!\int_\Omega \beta^m \phi' v_i + \int_0^T\!\!\int_\Omega \nabla u^m \cdot \nabla\phi v_i = 0 \tag{3.16}$$

In accordance with the convergence properties of the sequences $\{\beta^m\}$ and $\{u^m\}$ we infer directly, for the linear terms, that:

$$\begin{aligned} -\int_\Omega \beta_0^m v_i \phi(0) &\to -\int_\Omega \beta_0 v_i \phi(0) \\ -\int_0^T\!\!\int_\Omega \beta^m \phi' v_i &\to -\int_0^T\!\!\int_\Omega \beta \phi' v_i \\ \int_0^T\!\!\int_\Omega \nabla\beta^m \cdot \nabla\phi v_i &\to \int_0^T\!\!\int_\Omega \nabla\beta \cdot \nabla\phi v_i \\ -\int_\Omega u_0^m v_i \phi(0) &\to -\int_\Omega u_0 v_i \phi(0) \\ -\int_0^T\!\!\int_\Omega u^m \phi' v_i &\to -\int_0^T\!\!\int_\Omega u \phi' v_i \\ \int_0^T\!\!\int_\Omega \nabla u^m \cdot \nabla\phi v_i &\to \int_0^T\!\!\int_\Omega \nabla u \cdot \nabla\phi v_i. \end{aligned}$$

To analyze the corresponding limiting procedure for the non linear terms we will use the following lemma.

Lemma 3.10. The sequences $\{W(u^m)\}$, $\{|\nabla\beta^m|\}$ and $\{I_p'(\beta^m)\}$ have strong converging subsequences in $L^2(0,T;L^2(\Omega))$ to $W(u)$, $|\nabla\beta|$ and $I_p'(\beta)$ respectively.

The proof of this lemma will be postponed until the end of the section.

From this lemma, we obtain:

$$\begin{aligned}\int_0^T\!\!\int_\Omega W(u^m)|\nabla\beta^m|\phi v_i &\to \int_0^T\!\!\int_\Omega W(u)|\nabla\beta|\phi v_i\\ \int_0^T\!\!\int_\Omega I_p'(\beta^m)\phi v_i &\to \int_0^T\!\!\int_\Omega I_p'(\beta)\phi v_i.\end{aligned}$$

Hence, taking limits in equations (3.15)-(3.16), we obtain that for each $i\in\mathbb{N}$:

$$\begin{aligned}-\int_\Omega\beta_0 v_i\phi(0)-\int_0^T\!\!\int_\Omega\beta\phi' v_i+\varepsilon\int_0^T\!\!\int_\Omega\nabla\beta\cdot\nabla\phi v_i+\int_0^T\!\!\int_\Omega W(u)|\nabla\beta|\phi v_i+\int_0^T\!\!\int_\Omega I_p'(\beta)\phi v_i &= 0\\ -\int_\Omega u_0 v_i\phi(0)-\int_\Omega\beta_0 v_i\phi(0)-\int_0^T\!\!\int_\Omega u\phi' v_i-\int_0^T\!\!\int_\Omega\beta\phi' v_i+\int_0^T\!\!\int_\Omega\nabla u\cdot\nabla\phi v_i &= 0\end{aligned}$$

Since these equalities are valid for each $i\in\mathbb{N}$, and the expressions are all linear and continuous with respect to v_i, it follows that they are valid for any function in $H^1(\Omega)$. More precisely, $\forall\varphi\in H^1(\Omega)$ the following holds:

$$\begin{aligned}-\int_\Omega\beta_0\varphi\phi(0)-\int_0^T\!\!\int_\Omega\beta\phi'\varphi+\varepsilon\int_0^T\!\!\int_\Omega\nabla\beta\cdot\nabla\phi\varphi+\int_0^T\!\!\int_\Omega W(u)|\nabla\beta|\phi\varphi+\int_0^T\!\!\int_\Omega I_p'(\beta)\phi\varphi &= 0\\ -\int_\Omega u_0\varphi\phi(0)-\int_\Omega\beta_0\varphi\phi(0)-\int_0^T\!\!\int_\Omega u\phi'\varphi-\int_0^T\!\!\int_\Omega\beta\phi'\varphi+\int_0^T\!\!\int_\Omega\nabla u\cdot\nabla\phi\varphi &= 0\end{aligned}$$

If we consider for these equations the special case $\phi(0)=0$, we obtain that $\forall\varphi\in H^1(\Omega)$ we have:

$$\frac{d}{dt}\int_\Omega\beta\varphi+\varepsilon\int_\Omega\nabla\beta.\nabla\varphi+\int_\Omega W(u)|\nabla\beta|\varphi+\int_\Omega I_p'(\beta)\varphi = 0 \tag{3.17}$$

$$\frac{d}{dt}\int_\Omega u\varphi+\frac{d}{dt}\int_\Omega\beta\varphi+\int_\Omega\nabla u.\nabla\varphi = 0. \tag{3.18}$$

This proves that the diœerential equations in problem (P_p) do have a solution. To prove that (β,u) is a solution of the problem (P_p), we still need to investigate whether β and u satisfy the initial conditions. To do so, let us once more multiply these latter equations by a real-valued function $\phi(t)$ that is continuously diœerentiable on $[0,T]$ and such that $\phi(T)=0$, and let us integrate by parts:

$$\begin{aligned}-\int_\Omega\beta(0)\varphi\phi(0)-\int_0^T\!\!\int_\Omega\beta\phi'\varphi+\varepsilon\int_0^T\!\!\int_\Omega\nabla\beta\cdot\nabla\phi\varphi+\int_0^T\!\!\int_\Omega W(u)|\nabla\beta|\phi\varphi+\int_0^T\!\!\int_\Omega I_p'(\beta)\phi\varphi &= 0\\ -\int_\Omega u(0)\varphi\phi(0)-\int_\Omega\beta(0)\varphi\phi(0)-\int_0^T\!\!\int_\Omega u\phi'\varphi-\int_0^T\!\!\int_\Omega\beta\phi'\varphi+\int_0^T\!\!\int_\Omega\nabla u\cdot\nabla\phi\varphi &= 0\end{aligned}$$

Comparing these equations with the previous ones we deduce that for all $\varphi \in H^1(\Omega)$:

$$\begin{aligned} \int_\Omega (\beta(0) - \beta_0)\,\varphi &= 0 \\ \int_\Omega (u(0) - u_0)\,\varphi + \int_\Omega (\beta(0) - \beta_0)\,\varphi &= 0 \end{aligned}$$

These equalities show that $\beta(0) = \beta_0$ and that $u(0) = u_0$. Thus, it is proven that there exists some solution to the (P_p) problem.

3.4. Limiting procedure on p. In the previous subsection we have proved that $\forall p > 0$ there is a solution of the problem (P_p). Let us denote by (β^p, u^p) that solution. In this subsection we will prove that if $p \to \infty$ there exist a subsequence that we will still denote (β^p, u^p) that converge to a solution of problem (P).

The a-priori estimates found in the subsection (3.2) for the sequences $\{\beta^m\}$ and $\{u^m\}$ pass to the limits and we have that:

1. $\|\beta^p\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla\beta^p\|_{L^2(0,T;L^2(\Omega))} + \left\|I'_p(\beta^p)\beta^p\right\|_{L^1(0,T;L^1(\Omega))} \leq C$
2. $\left\|\dfrac{\partial\beta^p}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\beta^p\|_{L^\infty(0,T;L^2(\Omega))} + \|I_p(\beta^p)\|_{L^\infty(0,T;L^1(\Omega))} \leq C(p)$
3. $\|u^p\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u^p\|_{L^2(0,T;L^2(\Omega))} \leq C(p)$
4. $\left\|\dfrac{\partial u^p}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla u^p\|_{L^\infty(0,T;L^2(\Omega))} \leq C(p)$
5. $\|\Delta\beta^p\|_{L^2(0,T;L^2(\Omega))} \leq C(p)$

Unfortunately, all these estimates, except the first one, depends on p. This makes them useless for the limit procedure on p.

Nevertheless, (β^p, u^p) is a solution of the problem (P_p), and it is possible to verify that a.e. $t \in [0,T]$:

$$\begin{aligned} \Delta\beta^p &\in L^2(\Omega) \\ \Delta u^p &\in L^2(\Omega) \\ \frac{\partial\beta^p}{\partial n} = 0 &\in \partial\Omega \\ \frac{\partial u^p}{\partial n} = 0 &\in \partial\Omega \end{aligned}$$

and consequently, for each $\varphi \in L^2(\Omega)$ we obtain that:

$$\begin{aligned} \int_\Omega \frac{\partial\beta^p}{\partial t}\varphi - \varepsilon\int_\Omega \Delta\beta^p.\varphi + \int_\Omega W(u^p)|\nabla\beta^p|\varphi + \int_\Omega I'_p(\beta^p)\varphi &= 0 \\ \int_\Omega \frac{\partial u^p}{\partial t}\varphi + \int_\Omega \frac{\partial\beta^p}{\partial t}\varphi - \int_\Omega \Delta u^p.\varphi &= 0. \end{aligned}$$

If for these equations we choose as test function $\dfrac{\partial \beta^p}{\partial t}$, u^p, $\dfrac{\partial u^p}{\partial t}$ and $\Delta\beta^p$ it is possible to obtain a-priori estimates for β^p and u^p, which are similar to those obtained for β^m and u^m, but this time the bounds do not depend on p, that is:

1. $\|\beta^p\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla\beta^p\|_{L^2(0,T;L^2(\Omega))} + \left\|I_p'(\beta^p)\beta^p\right\|_{L^1(0,T;L^1(\Omega))} \leq C$
2. $\left\|\dfrac{\partial \beta^p}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\beta^p\|_{L^\infty(0,T;L^2(\Omega))} + \|I_p(\beta^p)\|_{L^\infty(0,T;L^1(\Omega))} \leq C$
3. $\|u^p\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u^p\|_{L^2(0,T;L^2(\Omega))} \leq C$
4. $\left\|\dfrac{\partial u^p}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla u^p\|_{L^\infty(0,T;L^2(\Omega))} \leq C$
5. $\|\Delta\beta^p\|_{L^2(0,T;L^2(\Omega))} \leq C$

Hence, there exist subsequences $\left\{\beta^{p'}\right\}$ and $\left\{u^{p'}\right\}$, and functions $\beta \in L^\infty\left(0,T;H^1(\Omega)\right) \cap C^0\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^2(\Omega)\right)$ and $u \in L^\infty\left(0,T;H^1(\Omega)\right) \cap C^0\left(0,T;L^2(\Omega)\right)$ such that:

$$
\begin{aligned}
\beta^{p'} &\rightharpoonup \beta \text{ weakly* } \text{ in } L^\infty\left(0,T;H^1(\Omega)\right)\\
\beta^{p'} &\rightharpoonup \beta \text{ weakly } \text{ in } L^2\left(0,T;H^2(\Omega)\right)\\
\beta^{p'} &\rightarrow \beta \text{ strongly } \text{ in } C^0\left(0,T;L^2(\Omega)\right)\\
\beta^{p'} &\rightarrow \beta \text{ strongly } \text{ in } L^2\left(0,T;H^1(\Omega)\right)\\
u^{p'} &\rightharpoonup u \text{ weakly* } \text{ in } L^\infty\left(0,T;H^1(\Omega)\right)\\
u^{p'} &\rightarrow u \text{ strongly } \text{ in } C^0\left(0,T;L^2(\Omega)\right)
\end{aligned}
$$

For the sake of convenience, the above subsequences shall be indexed by p instead of p'.

In order to analyze the limiting passage we must not forget that the subsequences $\{\beta^p\}$ and $\{u^p\}$ satisfy the equations:

$$
\begin{aligned}
\int_\Omega \frac{\partial \beta^p}{\partial t}\varphi + \varepsilon\int_\Omega \nabla\beta^p.\nabla\varphi + \int_\Omega W(u^p)|\nabla\beta^p|\varphi + \int_\Omega I_p'(\beta^p)\varphi &= 0\\
\int_\Omega \frac{\partial u^p}{\partial t}\varphi + \int_\Omega \frac{\partial \beta^p}{\partial t}\varphi + \int_\Omega \nabla u^p.\nabla\varphi &= 0.\\
\beta^p(0) &= \beta_0\\
u^p(0) &= u_0
\end{aligned}
$$

The first equation can be restated as:

$$
\begin{aligned}
\int_\Omega \frac{\partial \beta^p}{\partial t}(\varphi-\beta^p) + \varepsilon\int_\Omega \nabla\beta^p.(\nabla\varphi-\nabla\beta^p) + \int_\Omega W(u^p)|\nabla\beta^p|(\varphi-\beta^p)&\\
+\int_\Omega I_p(\varphi) - \int_\Omega I_p(\beta^p) &\geq 0
\end{aligned}
$$

then, for each $\varphi \in K = \{v \in H^1(\Omega) \, : \, 0 \leq v(x) \leq 1, \text{ a.e. in}\, \Omega\}$,

$$\int_\Omega \frac{\partial \beta^p}{\partial t}(\varphi - \beta^p) + \varepsilon \int_\Omega \nabla \beta^p . (\nabla \varphi - \nabla \beta^p) + \int_\Omega W(u^p)|\nabla \beta^p|(\varphi - \beta^p) \geq \int_\Omega I_p(\beta^p) \geq 0.$$

Passing to the limit, as in the previous section, it is easily veriøed that the functions β and u satisfy the equations

$$\begin{aligned}
(3.19) \int_\Omega \frac{\partial \beta}{\partial t}(\varphi - \beta) + \varepsilon \int_\Omega \nabla \beta . (\nabla \varphi - \nabla \beta) + \int_\Omega W(u)|\nabla \beta|(\varphi - \beta) &\geq 0 \, \forall \varphi \in K \\
\int_\Omega \frac{\partial u}{\partial t}\varphi + \int_\Omega \frac{\partial \beta}{\partial t}\varphi + \int_\Omega \nabla u . \nabla \varphi &= 0 \, \forall \varphi \in H^1(\Omega) \\
\beta(0) &= \beta_0 \\
u(0) &= u_0
\end{aligned}$$

To complete the proof, it is necessary to prove that $\beta \in K$, that is, it is necessary to prove that $\beta(x,t) \in [0,1]$ a.e. in $\Omega \times [0,T]$. Indeed, as

$$\int_0^T \int_\Omega I_p(\beta^p) \leq C$$

it follows that

$$0 \leq \int_0^T \int_\Omega I_1(\beta^p) \leq \frac{C}{p}$$

and from where it is obtained, for $p \to \infty$, that

$$\int_0^T \int_\Omega I_1(\beta^p) \to 0.$$

But it is also true that

$$\int_0^T \int_\Omega I_1(\beta^p) \to \int_0^T \int_\Omega I_1(\beta)$$

and we can conclude that

$$\int_0^T \int_\Omega I_1(\beta) = 0.$$

Since $I_1(x) \geq 0$, we conclude that $I_1(\beta(x,t)) = 0$ a.e. in $\Omega \times [0,T]$. In other words, $\beta(x,t) \in [0,1]$ a.e. in $\Omega \times [0,T]$.

The equation (3.19) can be written as

$$\begin{aligned}
&\int_\Omega \frac{\partial \beta}{\partial t}(\varphi - \beta) + \varepsilon \int_\Omega \nabla \beta . (\nabla \varphi - \nabla \beta) + \int_\Omega W(u)|\nabla \beta|(\varphi - \beta) \\
&\qquad + \int_\Omega I(\varphi) - \int_\Omega I(\beta) \geq 0 \quad \forall \varphi \in H^1(\Omega)
\end{aligned}$$

that is,

$$\int_\Omega \frac{\partial \beta}{\partial t}\varphi + \varepsilon \int_\Omega \nabla\beta.\nabla\varphi + \int_\Omega W(u)|\nabla\beta|\varphi + \int_\Omega \partial I(\beta)\varphi \ni 0 \qquad \forall\varphi\in H^1(\Omega)$$

This concludes the proof of the existence of a solution for the problem (P) with the following regularity

$$\beta, u \in L^\infty\left(0,T;H^1(\Omega)\right)\cap L^2\left(0,T;H^2(\Omega)\right)\cap C^0\left(0,T;L^2(\Omega)\right).$$

3.5. Convergence of the non-linear term. In this subsection we will prove the lemma (3.10), that is:

Lemma. The sequences $\{W(u^m)\}$, $\{|\nabla\beta^m|\}$ and $\left\{I_p'(\beta^m)\right\}$ have subsequences that converge strongly in $L^2\left(0,T;L^2(\Omega)\right)$ to $W(u)$, $|\nabla\beta|$ and $I_p'(\beta)$ respectively.

Proof. Let us begin looking at the term $W(u^m)$. Since $u^m\to u$ strongly in $L^2(0,T;L^2(\Omega))$, there is a subsequence $\left\{u^{m'}\right\}$ that converges to u almost everywhere in $\Omega\times]0,T[$. Hence, due to the continuity of W it follows that

$$\left|W(u^{m'})-W(u)\right|^2\to 0$$

almost everywhere in $\Omega\times]0,T[$. On the other hand, since the function W and the domain Ω are bounded, we have that

$$\left|W(u^{m'})-W(u)\right|^2 \;\leq\; g$$

almost everywhere in $\Omega\times]0,T[$, where g is a constant and thus integrable in $\Omega\times]0,T[$.

In view of the Lebesgue Dominated Convergence Theorem it follows that

$$\int_{\Omega\times]0,T[}\left|W(u^{m'})-W(u)\right|^2\to\int_{\Omega\times]0,T[}0 \;=\; 0,$$

which completes the proof that $W(u^{m'})\to W(u)$ strongly in $L^2(0,T;L^2(\Omega))$.

For the term $|\nabla\beta^m|$ it is enough to remark that $\nabla\beta^m\to\nabla\beta$ strongly in $\left[L^2(0,T;L^2(\Omega))\right]^N$ and that

$$\big\||\nabla\beta^m|-|\nabla\beta|\big\|_{L^2(0,T;L^2(\Omega))}\leq\left\|\nabla\beta^m-\nabla\beta\right\|_{[L^2(0,T;L^2(\Omega))]^N}.$$

Finally, for the term $I_p'(\beta^m)$, as $\left|I_p'(x)\right|\leq p$, the same argument used for the sequence $W(u^m)$ can be applied. $\square$

3.6. Choosing an appropriate basis $\{v_i\}_{i\in\mathbb{N}}$. An appropriate basis $\{v_i\}_{i\in\mathbb{N}}$, together with the corresponding orthogonality conditions, can be found as the solutions of the spectral problem:

$$\begin{aligned} u-\Delta u &= \lambda u \qquad \text{on } \Omega \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Indeed, the solutions of this spectral problem form a basis for both $L^2(\Omega)$ and $H^1(\Omega)$ such that:

$$\big\langle v_i, v_j\big\rangle + \big\langle \nabla v_i, \nabla v_j\big\rangle = \lambda_i \big\langle v_i, v_j\big\rangle.$$

Since this equality must still hold if i and j are interchanged, it follows that:

$$\begin{aligned} \big\langle \nabla v_i, \nabla v_j\big\rangle &= 0 \\ \big\langle v_i, v_j\big\rangle &= 0, \end{aligned}$$

which are the properties we required for the basis.

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