

Some Results on Blossoming and Multivariate B-Splines

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Abstract

From the study of some smooth piecewise polynomial functions defined over particular partitions of the space, we obtain a characterization of B-spline functions. The key tools for this approach are blossoms of polynomial functions, and H.P. Seidel generalization of the De Boor - De Casteljau algorithm. Simplex splines appear as the natural framework for multivariate B-splines, and B-patches provide the right normalization. A new proof of the piecewise polynomial reproduction formula is given.

Key Words : Multivariate B-splines, polar forms, blossoms, piecewise polynomial functions, B-patches.

§1. Introduction

Since the work of Curry and Schoenberg [2], it is known that every C^{k-1} piecewise polynomial function S of degree k on a partition of $\mathbb{R}$ defined by distinct points t_i can be represented as a linear combination of the B -splines $B_i(u)$ based on $k + 2$ consecutive points $t_i, t_{i+1}, \dots, t_{i+k+1}$.

More recently, using the notion of *blossom* or *polar form*, L. Ramshaw ([8], [9]) gave an explicit expression for the coefficients :

$$S(u) = \sum_{i \in \mathbb{Z}} f_i(t_{i+1}, \dots, t_{i+k}) B_i(u) \quad (1)$$

where f_i is the blossom of F_i , the restriction of S to the interval $[t_i, t_{i+1}]$. Observe that :

$$f_i(t_{i+1}, \dots, t_{i+k}) = f_{i+1}(t_{i+1}, \dots, t_{i+k}) = \dots = f_{i+k}(t_{i+1}, \dots, t_{i+k})$$

To our knowledge, the preceding formula (1) cannot be directly extended to the multivariate setting.

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Following the work of Curry and Schoenberg, C. De Boor proposed in [1] a definition of multivariate B -splines : the simplex spline. Meanwhile, many authors have tried to produce useful linear combinations of simplex splines sharing some of the properties of the univariate B -splines, in particular, the polynomial or piecewise polynomial reproduction property. Dahmen-Micchelli and Höllig ([3], [7]), using combinatorial arguments, proposed convenient basis of simplex splines that reproduce polynomials of degree k . But the reproduction of C^{k-1} piecewise polynomial functions on a given triangulation could not be settled.

More recently, using the formalism of polar forms, Dahmen, Micchelli and Seidel [5], and Seidel [12] obtained a general result similar to (1), but in the s -dimensional space. Suppose that $\mathcal{T} = \{\Delta_I\}_{I \in \mathcal{I}}$ is a triangulation of $\mathbb{R}^2$ with $\Delta_I = \{t_0^I, t_1^I, t_2^I\}$ and that a set of $k+1$ points $t_{j,0}^I = t_j^I, t_{j,1}^I, \dots, t_{j,k}^I$ is associated with each vertex t_j^I of the triangulation. If $\beta = (\beta_0, \beta_1, \beta_2) \in (\mathbb{Z}_+)^3$, with $|\beta| = \beta_0 + \beta_1 + \beta_2 = k$, let us denote by t_β^I the following k -uplet :

$$t_\beta^I = t_{0,0}^I, \dots, t_{0,\beta_0-1}^I, t_{1,0}^I, \dots, t_{1,\beta_1-1}^I, t_{2,0}^I, \dots, t_{2,\beta_2-1}^I$$

and by V_β^I the following set of $k+3$ points :

$$V_\beta^I = \{t_{0,0}^I, \dots, t_{0,\beta_0}^I, t_{1,0}^I, \dots, t_{1,\beta_1}^I, t_{2,0}^I, \dots, t_{2,\beta_2}^I\}.$$

Then, if S is any C^{k-1} piecewise polynomial function of degree k on $\mathcal{T}$, it can be represented as the following double sum :

$$S(u) = \sum_{I \in \mathcal{I}} \sum_{|\beta|=k} f^I(t_\beta^I) \mathcal{N}_\beta^I(u) \quad (2)$$

where f^I is the blossom of the restriction F^I of S to Δ_I . And $\mathcal{N}_\beta^I$ is the simplex spline based on the set V_β^I for a convenient normalization

It can be observed that the formula (2) when particularized to the univariate case does not lead to the formula (1).

The proof of (2) given by Dahmen, Micchelli and Seidel ([5], [12]) is recurrent with respect to the degree and the multiplicity. The purpose of the present paper is to give a direct proof (cf Th. 13) which allows a different intuitive approach of this *phenomenon*. It also gives a result concerning the C^{k-1} joining of 2 polynomials through a particular net of lines (cf Lemma 7).

Note that we are considering more restrictive assumptions than [12]. All the knot-lines of the simplex splines $\mathcal{N}_\beta^I$ are distinct, and in particular, multiplicity is 1.

The paper is organized in two parts : the univariate case (§2) gives a simple presentation of the results and allows to introduce the notations in an easy way. Then, the bivariate case (§3) can be expressed in a very similar form.

§2. The Univariate Case

Let $[t_0, t_1]$ be an interval of $\mathbb{R}$ ($t_0 < t_1$). Around t_0 and t_1 we consider a “cloud” of k additional points $t_{i,1}, \dots, t_{i,k}$, $i = 0$ or 1 . Set $t_{i,0} = t_i$, $i = 0, 1$, for convenience. For each $\beta = (\beta_0, \beta_1) \in (\mathbb{Z}_+)^2$, such that $|\beta| = \beta_0 + \beta_1 \leq k$, we define :

$$\Delta_\beta := [t_{0,\beta_0}, t_{1,\beta_1}] \quad (3)$$

and suppose that

$$\Omega := \bigcap_{|\beta| \leq k} \text{int } \Delta_\beta \neq \emptyset \quad (4)$$

This is equivalent to

$$\max_{j=0,\dots,k} t_{0,j} < \min_{j=0,\dots,k} t_{1,j}$$

Observe that this assumption is satisfied if the additional points $t_{i,j}$ are chosen sufficiently close to the initial points t_i , $i = 0, 1$.

For a multi-integer $\beta \in \mathbb{Z}_+^2$, we shall use the notation :

$$t_\beta := (t_{0,0}, t_{0,1}, \dots, t_{0,\beta_0-1}, t_{1,0}, t_{1,1}, \dots, t_{1,\beta_1-1}) \in \mathbb{R}^{|\beta|}$$

and for each of the $k+1$ multi-integers β such that $|\beta| = k$, we consider the following linear functional defined on $\mathcal{P}_k$ (the space of polynomials of degree less or equal k) :

$$\mathcal{L}_\beta : F \in \mathcal{P}_k \longrightarrow f(t_\beta) \quad (5)$$

where f is the blossom of F .

Theorem 1. *The functionals $\mathcal{L}_\beta$ are linearly independent.*

Proof : This is a consequence of the extension of the De Boor - De Casteljau algorithm due to H.P. Seidel [10]. Let us define the following linear mapping :

$$L : F \in \mathcal{P}_k \longrightarrow \{\mathcal{L}_\beta(F), |\beta| = k\} \in \mathbb{R}^{k+1}$$

Obviously, the linear functionals $\mathcal{L}_\beta$ are linearly independent iff the linear mapping L is injective. This follows from the fact that, from the values

$$\mathcal{L}_\beta(F) = c_\beta, \quad |\beta| = k \quad (6)$$

it is possible to construct $F(u)$ at an arbitrary point u . Setting :

$$c_{\beta,\nu}(u) = f(t_\beta u^\nu) \quad (7)$$

for $\nu = 0, \dots, k$ and $|\beta| = k - \nu$, where u^ν means “ u repeated ν times”, we observe that :

$$c_{\beta,0}(u) = c_{\beta} \quad (8)$$

and

$$c_{0,k}(u) = F(u) \quad (9)$$

The condition (4) allows us to define $\lambda_{\beta,0}(u)$ and $\lambda_{\beta,1}(u)$ as the barycentric coordinates of u with respect to Δ_{β} , that is :

$$u = \lambda_{\beta,0}(u)t_{0,\beta_0} + \lambda_{\beta,1}(u)t_{1,\beta_1}, \text{ with } \lambda_{\beta,0}(u) + \lambda_{\beta,1}(u) = 1.$$

Using the fact that the blossom is affine with respect to each variable separately, $c_{\beta,\nu} = f(t_{\beta}u^{\nu-1}\mathbf{u})$ can be computed in terms of

$$c_{\beta+e_0,\nu-1}(u) = f(t_{\beta}u^{\nu-1}\mathbf{t}_{0,\beta_0}) \text{ and } c_{\beta+e_1,\nu-1}(u) = f(t_{\beta}u^{\nu-1}\mathbf{t}_{1,\beta_1})$$

(where $e_0 = (1, 0)$ and $e_1 = (0, 1)$) by :

$$c_{\beta,\nu}(u) = \lambda_{\beta,0}(u)c_{\beta+e_0,\nu-1}(u) + \lambda_{\beta,1}(u)c_{\beta+e_1,\nu-1}(u). \quad \blacksquare \quad (10)$$

Definition : Let us denote by B_{β} , $|\beta| = k$ the dual basis in $\mathcal{P}_k$ of the functionals $\mathcal{L}_{\beta}$, i.e. the $k+1$ polynomials of degree k , such that

$$\mathcal{L}_{\gamma}(B_{\beta}) = \begin{cases} 1 & \text{if } \gamma = \beta \\ 0 & \text{if } \gamma \neq \beta \end{cases} \quad (11)$$

for all β ($|\beta| = k$) and all γ ($|\gamma| = k$).

The $k+1$ polynomials B_{β} are called the B -weights associated with the $t_{0,j}$ and $t_{1,j}$, $j = 0, \dots, k-1$ (observe that they do not depend on $t_{0,k}$ and $t_{1,k}$). If the $t_{0,j}$ are all equal to $t_0 = 0$ and the $t_{1,j}$ are all equal to $t_1 = 1$, then the B_{β} are just the classical Bernstein polynomials.

By definition, for any polynomial $F \in \mathcal{P}_k$ we have :

$$F(u) = \sum_{|\beta|=k} \mathcal{L}_{\beta}(F) B_{\beta}(u)$$

i.e.

$$F(u) = \sum_{|\beta|=k} f(t_{\beta}) B_{\beta}(u) \quad (12)$$

where f is the blossom of F . The representation of a polynomial in this basis is called the B -patch representation. Observe the similarity of this formula and the formulas (1) and (2).

For simplicity, we now suppose that the $t_{0,j}$, $j = 1, \dots, k$, are all distinct. Let us denote by $\mathcal{S}$ the set of piecewise polynomial functions of degree k on the partition of $\mathbb{R}$ defined by the $k+1$ points $t_{0,j}$, $j = 0, \dots, k$, which belong to $C^{k-1}(\mathbb{R})$.

We shall denote by a_β the affine function defined on $\mathbb{R}$ such that

$$a_\beta(t_{0,\beta_0}) = 0 \quad \text{and} \quad a_\beta(t_{1,\beta_1}) = 1 \quad (13)$$

and by Φ_β the polynomial of degree k defined by :

$$\Phi_\beta(u) = a_\beta(u)^k$$

We observe that the blossom of Φ_β is just :

$$\phi_\beta(u_1, \dots, u_k) = \prod_{j=1}^k a_\beta(u_j)$$

Theorem 2. *For arbitrary given polynomials F^- and F^+ of degree k , there exists a unique function $S \in \mathcal{S}$, such that :*

- for $u \leq t_{0,j}$, $j = 0, \dots, k$: $S(u) \equiv F^-(u)$,
- for $u \geq t_{0,j}$, $j = 0, \dots, k$: $S(u) \equiv F^+(u)$.

Moreover, the function S can be written in the following form :

$$S(u) = F^-(u) + \sum_{|\beta|=k} \mu_\beta (a_\beta(u))_+^k \quad (14)$$

where the reals μ_β satisfy the following relations :

$$\sum_{\substack{|\beta|=k \\ \beta_0 \geq \gamma_0}} \mu_\beta \phi_\beta(t_\gamma) = f(t_\gamma) \quad (15)$$

for all γ , $|\gamma| = k$ (where f is the blossom of $F = F^+ - F^-$), which determine them uniquely.

Proof :

As the function $[a_\beta(u)]_+^k$ is proportional to the truncated power function $[u - t_{0,\beta_0}]_+^k$, any function belonging to $\mathcal{S}$ can be written in the form (14). Now, for $u \geq t_{0,j}$, $j = 0, \dots, k$, we have $[a_\beta(u)]_+^k = \Phi_\beta(u)$ and thus, the μ_β must satisfy :

$$\sum_{|\beta|=k} \mu_\beta \Phi_\beta(u) \equiv F(u),$$

or equivalently, using Th. 1 :

$$\mathcal{L}_\gamma \left(\sum_{|\beta|=k} \mu_\beta \Phi_\beta \right) = \mathcal{L}_\gamma(F), \text{ for all } \gamma, |\gamma| = k,$$

i.e., explicitly :

$$\sum_{|\beta|=k} \mu_\beta \phi_\beta(t_\gamma) = f(t_\gamma), \text{ for all } \gamma, |\gamma| = k.$$

Now, as the $t_{0,j}$ are distinct, we observe that (see (13)) :

$$\phi_\beta(t_\gamma) = 0 \text{ iff } \beta_0 < \gamma_0$$

and we obtain the relations (15).

These relations allow to compute the μ_β recursively. For $\gamma = (k, 0)$, we have explicitly :

$$\mu_\gamma = \frac{f(t_\gamma)}{\phi_\gamma(t_\gamma)} \quad (16)$$

Suppose that $\gamma = (\gamma_0, \gamma_1)$ satisfies $\gamma_1 > 0$ (i.e. $\gamma_0 < k$) and that all μ_β such that $\beta_0 > \gamma_0$ have been computed. Then μ_γ can be obtained by :

$$\mu_\gamma = \frac{f(t_\gamma) - \sum_{\substack{|\beta|=k \\ \beta_0 > \gamma_0}} \mu_\beta \phi_\beta(t_\gamma)}{\phi_\gamma(t_\gamma)} \quad \blacksquare \quad (17)$$

Corollary 3. *If F^- and F^+ are two polynomials of degree k which have a contact of order $k - 1$ at t_0 , then the function $S \in \mathcal{S}$ defined in Th. 2 is just given by :*

$$S(u) = \begin{cases} F^-(u) & \text{if } u \leq t_0 \\ F^+(u) & \text{if } u \geq t_0 \end{cases} \quad (18)$$

(i.e. S has no discontinuity of the k^{th} derivative at the knots $t_{0,j}$, $j = 1, \dots, k$).

Proof : If the polynomials F^- and F^+ have a contact of order $k - 1$ at t_0 , the function S defined by (18) obviously belongs to $\mathcal{S}$ and necessarily coincides with the unique function defined in Th. 2. $\blacksquare$

Remark : It can be observed that if F^- and F^+ have a contact of order $k - 1$ at t_0 , the blossom f of $F \equiv F^+ - F^-$ takes the value 0 for all k -uplets containing at least one time t_0 (see [8]), that means :

$$f(t_\beta) = 0, \text{ for all } \beta, |\beta| = k \text{ such that } \beta_0 > 0.$$

This implies, using (17), that :

$$\mu_\gamma = 0, \text{ for all } \gamma, |\gamma| = k \text{ such that } \gamma_0 > 0$$

and for $\gamma = (0, k)$:

$$\mu_\gamma = \frac{f(t_\gamma)}{\phi_\gamma(t_\gamma)}$$

Let us suppose now that the $2k + 2$ points $t_{i,j}$, $j = 0, \dots, k$, $i = 0, 1$, are distinct. For $\beta = (\beta_0, \beta_1)$ such that $|\beta| = k$, let us denote by V_β the following set of $k + 2$ points :

$$V_\beta = \{t_{0,0}, t_{0,1}, \dots, t_{0,\beta_0}, t_{1,0}, t_{1,1}, \dots, t_{1,\beta_1}\}$$

It is well known that there exists a piecewise polynomial function of degree k on the partition of $\mathbb{R}$ defined by V_β which belongs to $C^{k-1}(\mathbb{R})$ and whose support is $\text{co } V_\beta$. This function is unique up to a multiplicative factor. The following theorem shows that this factor can be chosen in such a way that this function coincides with B_β on the interval Ω (see (4)) between the two “clouds” of points $t_{0,j}$, $j = 0, \dots, k$ and $t_{1,j}$, $j = 0, \dots, k$.

Theorem 4. *For all β , $|\beta| = k$, there exists a unique function $\mathcal{N}_\beta$ piecewise polynomial of degree k on the partition of $\mathbb{R}$ defined by V_β , belonging to $C^{k-1}(\mathbb{R})$, with support $\text{co } V_\beta$ and whose restriction to Ω coincides with B_β .*

Proof : Consider a fixed α , $|\alpha| = k$, and use Th. 2 with $F^+ = B_\alpha$ and $F^- = 0$. If f denotes the blossom of $F = F^+ - F^- = B_\alpha$, then, by definition, the values of $f(t_\gamma)$ are all zero, except for $\gamma = \alpha$. Using the expression (17) of the μ_γ , we observe that it only uses the values of $f(t_\beta)$ for $\beta_0 \geq \gamma_0$. Hence μ_γ will be non zero only if $\alpha_0 \geq \gamma_0$, i.e. the expression (14) uses only the knots $t_{0,0}, \dots, t_{0,\alpha_0}$.

A similar argument around the point t_1 shows that F has discontinuities only around the points $t_{1,0}, \dots, t_{1,\alpha_1}$. ■

Remark : The set V_β contains $k + 2$ distinct points. Suppose that $V_\beta = \{\xi_0, \xi_1, \dots, \xi_{k+1}\}$ where $\xi_i < \xi_{i+1}$. The classical B -spline on the ξ_i is defined by :

$$(\xi_{k+1} - \xi_0)[\xi_0 \dots \xi_{k+1}](\cdot - \xi)_+^k$$

where $[\xi_0 \dots \xi_{k+1}]$ denotes the divided differences operator of order $k + 1$ based on the $k + 2$ points ξ_i . The function $\mathcal{N}_\beta$ is proportional to this function but not equal (except in the case where the $t_{0,j}$, $j = 0, \dots, \beta_0$ are decreasing and the $t_{1,j}$, $j = 0, \dots, \beta_1$ are increasing). In the same way, if M_β denotes the B -spline based on the points of V_β whose integral is equal to 1, then $\mathcal{N}_\beta$ is also proportional to M_β :

$$\mathcal{N}_\beta(u) = \frac{(t_{1,\beta_1} - t_{0,\beta_0})}{(k + 1)} M_\beta(u)$$

Theorem 5. *Let F be a polynomial of degree k and f its blossom. The function :*

$$\sum_{|\beta|=k} f(t_\beta) \mathcal{N}_\beta(u)$$

is piecewise polynomial of degree k on the partition of $\mathbb{R}$ defined by the $2k + 2$ points $t_{0,j}$ and $t_{1,j}$, $j = 0, \dots, k$. Its support is contained in $\text{co } \{t_{0,j}, j = 0, \dots, k; t_{1,j}, j = 1, \dots, k\}$ and its restriction to Ω is equal to F .

Proof : For each β , $\mathcal{N}_\beta$ is piecewise polynomial and

$$\text{Support } \mathcal{N}_\beta \subseteq \text{co} \{t_{0,j}, j = 0, \dots, k; t_{1,j}, j = 1, \dots, k\}$$

As for $u \in \Omega$, we have $\mathcal{N}_\beta(u) \equiv B_\beta(u)$, the result follows from (12). ■

Now, consider a bi-infinite sequence t_i , $i \in \mathbb{Z}$ such that $t_i < t_{i+1}$ and $\mathbb{R} = \cup_{i \in \mathbb{Z}} [t_i, t_{i+1}[$. Suppose that, around each t_i , we have a “cloud” of $k + 1$ distinct points $t_{i,0} = t_i, t_{i,1}, \dots, t_{i,k}$. We define :

$$\Delta_\beta^i := [t_{i,\beta_0}, t_{i+1,\beta_1}] \quad (19)$$

and suppose that for all $i \in \mathbb{Z}$:

$$\Omega_i := \bigcap_{|\beta| \leq k} \text{int } \Delta_\beta^i =] \max_{j=0,\dots,k} t_{i,j}, \min_{j=0,\dots,k} t_{i+1,j}[\neq \emptyset \quad (20)$$

For each interval $[t_i, t_{i+1}]$ we build (as above for the interval $[t_0, t_1]$) the B -weights $B_\beta^i(u)$, $|\beta| = k$ and the $k + 1$ normalized B -splines $\mathcal{N}_\beta^i(u)$, $|\beta| = k$. Then we have the following result (Seidel [12]) :

Theorem 6. *Let S be an arbitrary piecewise polynomial function of degree k defined on the partition of $\mathbb{R}$ defined by the t_i , $i \in \mathbb{Z}$, belonging to $C^{k-1}(\mathbb{R})$, $k \geq 1$. Let us denote by F^i the restriction of S to $[t_i, t_{i+1}]$ and f^i its blossom. Then :*

$$S(u) = \sum_{i \in \mathbb{Z}} \sum_{|\beta|=k} f^i(t_\beta^i) \mathcal{N}_\beta^i(u) \quad (21)$$

where $t_\beta^i = t_{i,0}, t_{i,1}, \dots, t_{i,\beta_0-1}, t_{i+1,0}, t_{i+1,1}, \dots, t_{i+1,\beta_1-1}$

Proof : By Th. 5 we have for $u \in \Omega_i$:

$$F^i(u) = \sum_{|\beta|=k} f^i(t_\beta^i) \mathcal{N}_\beta^i(u)$$

Thus, as the support of $\mathcal{N}_\beta^{i'}$ has an empty intersection with Ω_i for $i' \neq i$, the formula (21) is already satisfied for any $u \in \Omega_i$, $i \in \mathbb{Z}$.

The function $\tilde{S}$ defined by the double sum on the right side of (21) coincides with F^i on Ω_i and with F^{i-1} on Ω_{i-1} . As F^i and F^{i-1} have a contact of order $k - 1$ at t_i , applying Cor. 3, the function $\tilde{S}$ has no discontinuity of the k^{th} derivative at the knots $t_{i,j}$, $j = 1, \dots, k$, hence $\tilde{S} \equiv S$. ■

§3. The Bivariate Case

Let $\{t_0, t_1, t_2\}$ be a set of 3 affinely independent points of $\mathbb{R}^2$ and Δ_0 the corresponding triangle. Around each point t_i we consider a “cloud” of additional points $t_{i,j}$, $j = 1, \dots, k$, ($t_{i,0} = t_i$). For each $\beta = (\beta_0, \beta_1, \beta_2) \in (\mathbb{Z}_+)^3$, such that $|\beta| = \beta_0 + \beta_1 + \beta_2 \leq k$ we define :

$$\Delta_\beta = \text{co} \{t_{0,\beta_0}, t_{1,\beta_1}, t_{2,\beta_2}\} \quad (22)$$

and we suppose that

$$\Omega = \bigcap_{|\beta| \leq k} \text{int } \Delta_\beta \neq \emptyset \quad (23)$$

This assumption implies that all triangles with vertices $t_{0,\beta_0}, t_{1,\beta_1}, t_{2,\beta_2}$, ($|\beta| = k$) are non degenerate and have the same orientation (see [5]). Intuitively the assumption (23) is satisfied if the additional points $t_{i,j}$ are chosen sufficiently close to the t_i .

We shall use the notation :

$$t_\beta := (t_{0,0}, \dots, t_{0,\beta_0-1}, t_{1,0}, \dots, t_{1,\beta_1-1}, t_{2,0}, \dots, t_{2,\beta_2-1}) \in \mathbb{R}^{|\beta|} \quad (24)$$

On the space $\mathcal{P}_k(\mathbb{R}^2)$ of polynomials of degree less or equal k on $\mathbb{R}^2$, the dimension of which will be denoted by $d_k = (k+1)(k+2)/2$ we consider the d_k linear functionals $\mathcal{L}_\beta$ defined for each β ($|\beta| = k$) by :

$$\mathcal{L}_\beta : F \in \mathcal{P}_k(\mathbb{R}^2) \longrightarrow f(t_\beta) \quad (25)$$

where f is the blossom of F .

The functionals $\mathcal{L}_\beta$ ($|\beta| = k$) are linearly independent. Like for the one-dimensional case (Th. 1) this is a consequence of the Seidel-De Boor-De Casteljau algorithm : The value $F(u)$ of the polynomial $F \in \mathcal{P}_k(\mathbb{R}^2)$ satisfying $\mathcal{L}_\beta(F) = c_\beta$, $|\beta| = k$, can be obtained by the following recurrence relation :

$$\begin{aligned} c_{\beta,0}(u) &= c_\beta \\ c_{\beta,\nu}(u) &= \sum_{j=0}^2 \lambda_{\beta,j}(u) c_{\beta+e_j,\nu-1}(u) \\ c_{0,k}(u) &= F(u) \end{aligned} \quad (26)$$

where the $c_{\beta,\nu}(u)$ are defined as in (7), e_j denotes the canonical basis of $\mathbb{R}^3$ and the $\lambda_{\beta,j}(u)$ are the barycentric coordinates of u with respect to the triangle Δ_β .

Let us denote by B_β (B -weights), $|\beta| = k$ the dual basis in $\mathcal{P}_k(\mathbb{R}^2)$ of the d_k linear functionals $\mathcal{L}_\beta$, i.e., the d_k polynomials of degree k such that

$$\mathcal{L}_\gamma(B_\beta) = \begin{cases} 1 & \text{if } \gamma = \beta \\ 0 & \text{if } \gamma \neq \beta \end{cases} \quad (27)$$

for all β ($|\beta| = k$) and all γ ($|\gamma| = k$). If the $t_{j,l}$ are all equal to t_j , for $l = 1, \dots, k$, $j = 0, 1, 2$, then the B_β are just the classical Bernstein polynomials in two variables.

For any polynomial $F \in \mathcal{P}_k(\mathbb{R}^2)$ we have

$$F(u) \equiv \sum_{|\beta|=k} \mathcal{L}_\beta(F) B_\beta(u)$$

i.e.

$$F(u) \equiv \sum_{|\beta|=k} f(t_\beta) B_\beta(u) \quad (28)$$

where f is the blossom of F .

For $|\beta| = k$, let us denote by a_β the affine function on $\mathbb{R}^2$ defined by :

$$a_\beta(t_{0,\beta_0}) = 0 \quad a_\beta(t_{1,\beta_1}) = 0 \quad a_\beta(t_{2,\beta_2}) = 1$$

We define :

$$\begin{aligned} L_\beta &= \{u \mid a_\beta(u) = 0\} \\ L_\beta^+ &= \{u \mid a_\beta(u) > 0\} \\ L_\beta^- &= \{u \mid a_\beta(u) \leq 0\} \end{aligned}$$

The set L_β is the straight line spanned by t_{0,β_0} and t_{1,β_1} . It corresponds to the edge of Δ_β opposite to the vertex t_{2,β_2} . As $|\beta| = k$ implies that $\beta_0 + \beta_1 \leq k$, the number of lines L_β is equal to the dimension d_k of $\mathcal{P}_k(\mathbb{R}^2)$.

The assumption (23) implies that the set

$$\mathcal{D}^+ := \bigcap_{|\beta|=k} L_\beta^+ \quad (29)$$

is non empty.

If $\omega \in \Omega$, the straight line L spanned by ω and t_2 necessarily intersects each line L_β ($|\beta| = k$). Thus, the set

$$\mathcal{D}^- := \bigcap_{|\beta|=k} L_\beta^- \quad (30)$$

is also non empty.

Observe that every line parallel to L also intersects each line L_β .

From now, we shall assume that

$$\text{all the lines } L_\beta \text{ are distinct} \quad (31)$$

Then, it is clearly possible to choose a line parallel to L which intersects the lines L_β in d_k distinct points. More generally, one can find 2 lines L' and L'' , distinct and parallel to L , such that the open region $]L', L''[$ between them intersects the lines L_β in d_k pairwise disjoint segments I_β (cf Fig. 1).

Figure 1. Example in the quadratic case

The lines L_β ($|\beta| = k$) define a partition $\mathcal{P}$ of $\mathbb{R}^2$. Let us denote by $\mathcal{S}_{\mathcal{P}}$ the space of piecewise polynomial functions of degree k on $\mathcal{P}$ which belong to $C^{k-1}(\mathbb{R}^2)$.

We have the following result :

Lemma 7. *Given two polynomials of degree k , F^- and F^+ , there exists a unique function $S \in \mathcal{S}_{\mathcal{P}}$ such that :*

- for $u \in \mathcal{D}^-$: $S(u) \equiv F^-(u)$,
- for $u \in \mathcal{D}^+$: $S(u) \equiv F^+(u)$.

Moreover, the function S has the following form :

$$S(u) = F^-(u) + \sum_{|\beta|=k} \mu_\beta (a_\beta(u))_+^k \quad (32)$$

where the reals μ_β satisfy the following relations :

$$\sum_{\substack{|\beta|=k \\ \beta_0 \geq \gamma_0, \beta_1 \geq \gamma_1}} \mu_\beta \varphi_\beta(t_\gamma) = f(t_\gamma) \quad (33)$$

for all γ , $|\gamma| = k$, where f is the blossom of $F = F^+ - F^-$ and

$$\varphi_\beta(u_1, \dots, u_k) := \prod_{j=1}^k a_\beta(u_j)$$

is the blossom of

$$\Phi_\beta(u) = a_\beta(u)^k$$

Proof : Choose an open region $]L', L''[$ as above which intersects the d_k lines L_β in d_k disjoint intervals. On this domain the function S has the following form :

$$S(u) = F^-(u) + \sum_{|\beta|=k} \mu_\beta (a_\beta(u))_+^k$$

For $u \in \mathcal{D}^-$, we have $S(u) = F^-(u)$. Now, as for $u \in \mathcal{D}^+$, $a_\beta(u)_+^k = \Phi_\beta(u)$, the μ_β must satisfy :

$$\sum_{|\beta|=k} \mu_\beta \Phi_\beta(u) \equiv F(u)$$

or equivalently, using the fact that the d_k linear functionals $\mathcal{L}_\gamma$ are independent :

$$\mathcal{L}_\gamma \left(\sum_{|\beta|=k} \mu_\beta \Phi_\beta \right) \equiv \mathcal{L}_\gamma(F) , \text{ for all } \gamma, |\gamma| = k,$$

i.e. explicitly :

$$\sum_{|\beta|=k} \mu_\beta \varphi_\beta(t_\gamma) = f(t_\gamma), \text{ for all } \gamma, |\gamma| = k.$$

Now, observing that :

$$\varphi_\beta(t_\gamma) = 0 \text{ iff } \beta_0 < \gamma_0 \text{ or } \beta_1 < \gamma_1,$$

we obtain the relation (33).

The relation (33) allows to compute recursively the μ_β : for $\gamma = (\gamma_0, \gamma_1, \gamma_2)$ such that $\gamma_2 = 0$ (i.e., $\gamma_0 + \gamma_1 = k$), we have explicitly :

$$\mu_\gamma = \frac{f(t_\gamma)}{\varphi_\gamma(t_\gamma)} \quad (34)$$

Suppose now that $\gamma = (\gamma_0, \gamma_1, \gamma_2)$ satisfies $\gamma_2 > 0$ (i.e., $\gamma_0 + \gamma_1 < k$), and that all μ_β such that $\beta_0 + \beta_1 > \gamma_0 + \gamma_1$ have been computed. Then μ_γ can be obtained by :

$$\mu_\gamma = \frac{f(t_\gamma) - \sum_{\substack{|\beta|=k \\ \beta_0 > \gamma_0, \beta_1 > \gamma_1 \\ (\beta_0, \beta_1) \neq (\gamma_0, \gamma_1)}} \mu_\beta \varphi_\beta(t_\gamma)}{\varphi_\gamma(t_\gamma)} \quad (35)$$

Observe now that the expression of the μ_β does not depend on the chosen region $]L', L''[$. As the number of intersections between the lines L_β is finite, it is possible to build a finite number of such regions, the union of which is dense on $\mathbb{R}^2$. Hence, using a continuity argument, the expression (32) is true on all $\mathbb{R}^2$. ■

Corollary 8. *If F^- and F^+ are two polynomials of degree k which have a contact of order $k - 1$ along the line L_γ with $\gamma = (0, 0, k)$, then the function $S \in \mathcal{S}_\mathcal{P}$ defined by Lemma 7 is just given by :*

$$S(u) = \begin{cases} F^-(u) & \text{if } u \in L_\gamma^- \\ F^+(u) & \text{if } u \in L_\gamma^+ \end{cases} \quad (36)$$

(i.e. S has no discontinuity of the k^{th} derivative along the lines L_β , $|\beta| = k$, except for $\beta = \gamma$).

Proof : If F^- and F^+ have a contact of order $k - 1$ along L_γ , the function S defined by (36) obviously belongs to $\mathcal{S}_\mathcal{P}$ and necessarily coincides with the unique function defined in Lemma 7. ■

Remark : It can be observed that if F^- and F^+ have a contact of order $k - 1$ along L_γ (with $\gamma = (0, 0, k)$), the blossom f of $F = F^+ - F^-$ takes the value 0 for all k -uplets containing at least one point on L_γ , that means :

$$f(t_\beta) = 0, \text{ for all } \beta, |\beta| = k, \text{ such that } \beta_0 > 0 \text{ or } \beta_1 > 0.$$

This implies, using (35) that :

$$\mu_\gamma = 0, \text{ for all } \gamma, |\gamma| = k, \text{ such that } \gamma_0 > 0 \text{ or } \gamma_1 > 0,$$

and for $\gamma = (0, 0, k)$:

$$\mu_\gamma = \frac{f(t_\gamma)}{\varphi_\gamma(t_\gamma)}$$

For $\beta = (\beta_0, \beta_1, \beta_2)$ such that $|\beta| = k$, let us denote by V_β the following set of $k + 3$ points :

$$V_\beta = \{t_{0,0}, \dots, t_{0,\beta_0}, t_{1,0}, \dots, t_{1,\beta_1}, t_{2,0}, \dots, t_{2,\beta_2}\}$$

From now we suppose that for all β , $|\beta| = k$, the points of the set V_β are in general position, i.e. that no three points of V_β are on a same straight line.

W. Dahmen and C.A. Micchelli have shown (see [4]) that the simplex spline based on such a set V_β of $k + 3$ knots in general position is the unique (up to a multiplicative factor) piecewise polynomial function of degree k on the partition defined by these knots (that is, the partition of $\mathbb{R}^2$ induced by all segments between each pair of knots) which is C^{k-1} on $\mathbb{R}^2$ and whose support is $\text{co}(V_\beta)$. The following result shows that, for a convenient choice of the multiplicative factor, this function coincides with B_β on Ω .

Theorem 9. *For all β , $|\beta| = k$, there exists a unique function $\mathcal{N}_\beta$ that is piecewise polynomial of degree k on the partition of $\mathbb{R}^2$ defined by V_β , belongs to $C^{k-1}(\mathbb{R}^2)$, has as support $\text{co } V_\beta$, and whose restriction to Ω coincides with B_β .*

Proof : Consider a fixed β , $|\beta| = k$. Let M_β be the simplex spline corresponding to the knots V_β . This function has already all the properties required in Th. 9 except that its restriction to Ω does not necessarily coincide with B_β .

Consider the straight lines L_γ , $|\gamma| = k$, defined before Lemma 7, joining the knots t_{0,γ_0} and t_{1,γ_1} , for $\gamma_0 + \gamma_1 \leq k$. Let us apply Lemma 7 for $F^-(u) \equiv 0$ and $F^+(u) \equiv F$, where F is the restriction of M_β to Ω and f is its blossom. As the support of M_β is $co(V_\beta)$, the discontinuity along L_γ is zero for $\gamma_0 > \beta_0$ or $\gamma_1 > \beta_1$. Using the expression (33) of Lemma 7, we obtain that $f(t_\gamma) = 0$ for $|\gamma| = k$ such that $\gamma_0 > \beta_0$ or $\gamma_1 > \beta_1$.

Using the same argument with the straight lines joining the knots t_{1,γ_1} and t_{2,γ_2} , for $\gamma_1 + \gamma_2 \leq k$, we obtain that $f(t_\gamma) = 0$ for all γ ($|\gamma| = k$) such that $\gamma_1 > \beta_1$ or $\gamma_2 > \beta_2$. And similarly for the lines joining the knots t_{2,γ_2} and t_{0,γ_0} , for $\gamma_0 + \gamma_2 \leq k$.

Finally, for all γ ($|\gamma| = k$) satisfying $\gamma_0 > \beta_0$ or $\gamma_1 > \beta_1$ or $\gamma_2 > \beta_2$, we have :

$$f(t_\gamma) = 0,$$

i.e. for all γ such that $\gamma \neq \beta$.

As $f(t_\beta) \neq 0$ (otherwise F would be identically zero) the polynomial $F/f(t_\beta)$ satisfies (27), i.e. is equal to B_β and consequently the function :

$$\mathcal{N}_\beta := \frac{M_\beta}{f(t_\beta)}$$

is the unique function satisfying the conditions of the Theorem. ■

Remark : Using the theory of simplex splines, in particular the recurrence relation they satisfy, it is possible to obtain that :

$$\mathcal{N}_\beta(u) = \frac{Vol_2(\Delta_\beta)}{d_k} M_\beta(u)$$

i.e. that $f(t_\beta) = d_k/Vol_2(\Delta_\beta)$, (see [5]).

Theorem 10. *Let $F \in \mathcal{P}_k(\mathbb{R}^2)$ and f be its blossom. The function :*

$$\sum_{|\beta|=k} f(t_\beta) \mathcal{N}_\beta(u)$$

is a piecewise polynomial function of degree k belonging to $C^{k-1}(\mathbb{R}^2)$ whose restriction to Ω is equal to F .

Proof : For each β , we have $\mathcal{N}_\beta(u) = B_\beta(u)$ for $u \in \Omega$ and the result follows from (28). ■

Let $\{t_0^I, t_1^I, t_2^I\}$ and $\{t_0^J, t_1^J, t_2^J\}$ be two sets of 3 affinely independent points of $\mathbb{R}^2$, such that the corresponding triangles Δ^I and Δ^J are adjacent. For instance, suppose that :

$$t_0^I = t_0^J \text{ and } t_1^I = t_1^J$$

and suppose that we introduce at each vertex a “cloud” of k additional points

$$\begin{aligned} t_{0,l}^I &= t_{0,l}^J, & l = 1, \dots, k, \\ t_{1,l}^I &= t_{1,l}^J, & l = 1, \dots, k, \\ t_{2,l}^I &, & t_{2,l}^J, & l = 1, \dots, k. \end{aligned}$$

We assume that :

$$\Omega^I = \bigcap_{|\beta| \leq k} \text{int } \Delta_\beta^I \neq \emptyset$$

and similarly for Ω^J .

Figure 2. Two adjacent triangles

Let us define :

$$V_\beta^I := \{t_{0,0}^I, \dots, t_{0,\beta_0}^I, t_{1,0}^I, \dots, t_{1,\beta_1}^I, t_{2,0}^I, \dots, t_{2,\beta_2}^I\}$$

(and likewise V_β^J) and suppose that for all β , $|\beta| = k$, the points of V_β^I (and those of V_β^J) are in general position. Let $\mathcal{N}_\beta^I$ be the normalized B-spline associated with V_β^I as defined in Theorem 9 (and similarly $\mathcal{N}_\beta^J$).

Theorem 11. *If two polynomials F^I and F^J (whose blossoms are denoted by f^I and f^J) have a contact of order $k - 1$ along L_γ with $\gamma = (0, 0, k)$, then the function :*

$$\sum_{|\beta|=k} f^I(t_\beta^I) \mathcal{N}_\beta^I(u) + \sum_{|\beta|=k} f^J(t_\beta^J) \mathcal{N}_\beta^J(u) \quad (37)$$

have no discontinuities of the k^{th} derivative along the segments $]t_{0,\beta_0}^I, t_{1,\beta_1}^I[$ for all $\beta_0, \beta_1, \beta_0 + \beta_1 \leq k$ except for $\beta_0 = 0$ and $\beta_1 = 0$.

Proof : With our assumptions, the d_k lines L_β , $|\beta| = k$, joining t_{0,β_0}^I and t_{1,β_1}^I for $\beta_0 + \beta_1 \leq k$, are distinct. It is possible to choose two points $\omega^I \in \Omega^I$ and $\omega^J \in \Omega^J$ such that the line L defined by them crosses these lines L_β in d_k

distinct points. In the same way, one can choose two lines L' and L'' parallel to L such that the open region $]L', L''[$ contains L and intersects the d_k lines in d_k disjoint intervals I_β .

The lines L_β induce a partition of $]L', L''[$. Let us denote by $\mathcal{D}^I$ the component which intersects Ω^I and by $\mathcal{D}^J$ the one intersecting Ω^J . Consider the piecewise polynomial function on $]L', L''[$ of the form (32) which coincides with F^I on $\mathcal{D}^I$ and with F^J on $\mathcal{D}^J$.

Now as

$$\Omega^I \cap \text{Support } \mathcal{N}_\beta^J = \emptyset \quad \text{and} \quad \Omega^J \cap \text{Support } \mathcal{N}_\beta^I = \emptyset$$

the expression (37) coincides with F^I on Ω^I and with F^J on Ω^J (cf. Theorem 10). But F^I and F^J have a contact of order $k-1$ along L_γ with $\gamma = (0, 0, k)$; then following Lemma 7 (and Corollary 8 restricted to $]L', L''[$), there are no discontinuities along the segments $I_\beta =]L', L''[\cap]t_{0,\beta_0}^I, t_{1,\beta_1}^I[$, for all β_0, β_1 except for $\beta_0 = 0$ and $\beta_1 = 0$.

Now it is known (see [6], lemma 1 and its corollary) that simplex splines have constant jumps of the k^{th} derivative along each segment joining two knots, in particular along $]t_{0,\beta_0}^I, t_{1,\beta_1}^I[$, which completes the proof. ■

In order to obtain the final result, expressing a piecewise polynomial function on a triangulation of $\mathbb{R}^2$ we need the following Lemma :

Lemma 12. *Let $P_0, \dots, P_m \in \mathbb{R}^2$ be in general position and let $\mathcal{P}$ be the partition of $\mathbb{R}^2$ defined by those points. Denote by $\mathcal{S}_{\mathcal{P}}^k$ the space of C^{k-1} piecewise polynomials of degree k on $\mathcal{P}$ whose support is contained in $\text{co}\{P_0, \dots, P_m\}$. Then :*

$$\mathcal{S}_{\mathcal{P}}^k = \{0\} \text{ iff } m \leq k + 1.$$

Proof : For $m = k + 2$, the simplex spline based on the points $P_0, \dots, P_m$ is a non-zero element of $\mathcal{S}_{\mathcal{P}}^k$.

Let us suppose that $m = k + 1$ and consider a straight line L which does not contain any of the points P_i but crosses $\text{co}\{P_0, \dots, P_m\}$. More generally, consider a region $]L', L''[$ with the same properties.

It is clear that the points P_i are divided by $]L', L''[$ into two groups. Let us rename those from the one side as $t_{0,0}, \dots, t_{0,\beta_0}$ and those from the other side as $t_{1,0}, \dots, t_{1,\beta_1}$ with $\beta_0 + \beta_1 = k$.

The restriction of a function $S \in \mathcal{S}_{\mathcal{P}}^k$ to the region $]L', L''[$, defines a piecewise polynomial function which is C^{k-1} . Moreover, the discontinuities of the k^{th} derivatives are located along the lines L_β . We can apply (adding points if necessary) Lemma 7 with $F^+ \equiv F^- \equiv 0$ and conclude that $S \equiv 0$. ■

Let $\mathcal{T}$ be a triangulation of $\mathbb{R}^2$

$$\mathcal{T} = \{\Delta^I \mid I \in \mathcal{I}\}$$

with $\Delta^I = \text{co}\{t_0^I, t_1^I, t_2^I\}$. With each vertex of the triangulation we associate a “cloud” of k additional knots $t_{j,1}^I, \dots, t_{j,k}^I$ ($t_{j,0}^I = t_j^I$). Observe that if $t_i^I = t_j^J$, then we have also $t_{i,l}^I = t_{j,l}^J$, $l = 1, \dots, k$.

For each triangle we assume that :

$$\Omega^I := \bigcap_{|\beta| \leq k} \text{int } \Delta_\beta^I \neq \emptyset$$

and that no triplet of points in V_β , $|\beta| = k$ are colinear.

Theorem 13. *If S is a piecewise polynomial function of degree k on the triangulation $\mathcal{T}$ which belongs to $C^{k-1}(\mathbb{R}^2)$, then*

$$S(u) = \sum_{I \in \mathcal{I}} \sum_{|\beta|=k} f^I(t_\beta^I) \mathcal{N}_\beta^I(u) \quad (38)$$

where F^I denotes the restriction of S to Δ^I and f^I is the associated blossom.

Proof : Let us denote by $\tilde{S}$ the function defined by the right side of (38).

(i) For $u \in \Omega^I$, we have using Theorem 10 :

$$\tilde{S}(u) = \sum_{|\beta|=k} f^I(t_\beta^I) \mathcal{N}_\beta^I(u) = S(u)$$

(ii) Consider two adjacent triangles Δ^I and Δ^J . Suppose that $t_0^I = t_0^J$ and $t_1^I = t_1^J$. According to Theorem 11 the function $\tilde{S}$ will have no discontinuities along the segments $]t_{0,\beta_0}^I, t_{1,\beta_1}^I[$ for all $\beta_0, \beta_1, \beta_0 + \beta_1 \leq k$ except for $\beta_0 = 0$ and $\beta_1 = 0$.

(iii) We have proved that $\tilde{S} - S$ is a C^{k-1} piecewise polynomial function which is equal to zero everywhere except on the sets $C_j^I = \text{co}\{t_{j,0}^I, \dots, t_{j,k}^I\}$. Applying the Lemma 12 with $m = k$ on a neighborhood of each of these sets we see that $\tilde{S} - S$ is identically equal to zero. ■

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