Polar Cone Splines

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Abstract

Polyhedral splines have been defined analytically as some kind of distributions, or geometrically, as measures of slices of a polyhedron. H.P. Seidel and A.H. Vermeulen propose to define the polar form of a simplex spline using a variation of the known recurrence formula which relates simplex splines of different degrees. Here we proceed on a similar idea, but we apply it to a cone spline which is a simpler object. We propose new and simple proofs for its basic properties.

1 Introduction

Simplex splines are a natural generalization of B-splines to the multivariate case. However, its definition is not well suited for computations. Given $n+1$ knots $t_0, t_1, ..., t_n \in \mathbb{R}^s$, the associated simplex spline can be defined as the distribution:

$$\varphi \longrightarrow < M, \varphi > := n! \int_{\Sigma_n} \varphi\Big(t_0 + \sum_{i=1}^{n} \lambda_i \,(t_i - t_0)\Big) d\lambda_1 \ldots d\lambda_n \ , \tag{1.1}$$

where $\Sigma_n = \{\lambda \in \mathbb{R}^n \mid \lambda_i \geq 0 \ i = 1, \ldots, n, \ \sum_{i=1}^{n} \lambda_i \leq 1\}$. From this short definition, some basic properties of simplex splines can be derived with little effort. In particular, it is easy to prove that, in the case that the knots are in general position, the distribution M is in fact an $L^2(\mathbb{R}^s)$ function characterized by:

$$M(u) := Vol_k(\Delta \cap P^{-1}(\{u\})). \tag{1.2}$$

where Δ is any simplex in $\mathbb{R}^n$ such that its vertices $x_0, x_1, ..., x_n$ verifies $P(x_i) = t_i$. Here P is the standard projection $\mathbb{R}^n \rightarrow \mathbb{R}^s$, keeping the s first components

[1]Partially supported by Fac. de Ciencias Fis. y Mat., Univ. de Chile under a 1996 research grant

of a point in R^n. In this characterization, $\Delta \cap P^{-1}(\{u\})$ corresponds to the k-dimensional slice of Δ above $u \in \mathbb{R}^s$. For a good overview of the simplex splines theory and other "polyhedral" splines see [4].

This geometric characterization, assuming the general position hypothesis, provides an equivalent definition which gives us some intuition about simplex splines. For example, we can feel that they are piecewise polynomials whose support is $[t_0, ..., t_n]$, the convex hull of the knots.

However, both formulae (1.1) and (1.2) have proved to be useless to provide practical evaluation algorithms for simplex splines. The key property for evaluating a simplex spline is the recurrence formula, proved independently by W. Dahmen [2]–[3] and C. Micchelli [6]–[7]. This formula can be stated for $u = \sum_{i=0}^{n} \lambda_i \, t_i$, with $\sum_{i=0}^{n} \lambda_i = 1$, as

$$M(u|t_0, ..., t_n) = \sum_{i=0}^{n} \lambda_i \, M(u|t_0, ..., t_{i-1}, t_{i+1}, ..., t_n). \tag{1.3}$$

From the distributional definition of M, it is clear that

$$M(u|t_0, ..., t_s) = \frac{1}{Vol_s([t_0, ..., t_s])}$$

if u is in $[t_0, ..., t_s]$, and $M(u|t_0, ..., t_s) = 0$ otherwise. With this recurrence formula, the evaluation of a simplex spline is reduced to the computation of some determinants.

The idea developed by H.P. Seidel and A.H. Vermeulen [9] was to modify slightly formula (1.3), in order to define a (piecewise) k-affine function m, the polar form of M. They merged formula (1.3) with polar form or blossoming. The polar form or blossoming of a polynomial was defined and studied by L Ramshaw [8]. Given a degree k polynomial P, its polar form is the unique k-affine symmetric function $p : \mathbb{R}^s \times ... \times \mathbb{R}^s \to \mathbb{R}$ such that $P(u) = p(u, u, ..., u)$, $\forall u \in \mathbb{R}^s$. In the piecewise polynomial case, we have a polar form for each region. Given a point v, we denote by $m_v(u_1, ..., u_k)$ the polar form of M corresponding to the region to which v belongs.

W. Dahmen introduces the cone splines [2], also called truncated power functions, and writes simplex splines as sum of cone splines. Cohen, Lyche and Riesenfeld showed [1] that there are even stronger connections between cone splines and simplex splines. For example, a simplex spline can be seen as a cone spline along hyperplanes. In this work, we will see that cone splines are homogenized version of simplex splines.

The outline of this paper is as follows. In section 2 we introduce the notation and present a technical result (Lemma 2.1). In section 3 we define the polar cone spline and derive its properties. In section 4 we study the cone spline (whose polar form is the polar cone spline) and obtain trivially all its properties. Finally, we apply it to the computation of the jump of the k^{th} derivative of a cone spline and show a relation between cone splines and simplex splines.

2 Some Preliminary Results.

Given a sequence $Y = y^0, y^1, y^2, \ldots$ in $\mathbb{R}^{s+1}$, we denote by

$$Y^n := (y^0, y^1, \ldots, y^n)$$

the corresponding *ordered list* or *matrix* of size $n+1$. Moreover,

$$Y_i^n := (y^0, \ldots, y^{i-1}, y^{i+1}, \ldots, y^n)$$

will be the matrix Y^n without the i^{th} column. This notation is extended to the case of a matrix Y^n without j of its columns as $Y^n_{i_1, i_2, \ldots, i_j}$. We assume from here on that Y is in general position, that is, all its subsets of $s+1$ vectors are linearly independent.

The convex hull of Y^n is denoted by $[Y^n]$ and its cone convex hull by $cc(Y^n)$. An s-face of Y^n is any set of the form $S = cc\{y^{i_1}, \ldots, y^{i_s}\}$ $(1 \le i_j \le n)$. The characteristic function χ_A of a set A is the function that takes the value 1 on A and vanishes outside A.

The following lemma is a technical result on which the theory rests.

Lemma 2.1. *Suppose that $0 \notin [Y^{s+1}]$ and that v is not on an s-face of $cc(Y^{s+1})$. Then, for any reals $\mu_0, \ldots, \mu_{s+1}$ such that $\mu_0\, y^0 + \cdots + \mu_{s+1}\, y^{s+1} = 0$ we have that*

$$\sum_{i=0}^{s+1} \mu_i\, c_v(Y_i^{s+1}) = 0, \tag{2.1}$$

where $c_v(Z)$ is $|det\, Z|^{-1}$ if v is in $cc(Z)$ and is 0 otherwise.

We can get the geometric meaning of this result by transforming this statement into a statement about convex cones in R^{s+1} as follows. Because of the general position assumption for Y, $\{y^0, \ldots, y^s\}$ is a basis of $\mathbb{R}^{s+1}$ and there exist $s+1$ linear functionals (the coordinates functionals) $\ell_0, \ldots, \ell_s$ such that

$$v = \sum_{i=0}^{s} \ell_i(v)\, y^i \quad \forall v \in \mathbb{R}^{s+1}.$$

With these functionals, writing the condition over the reals μ_i as $\mu_0\, y^0 + \cdots + \mu_s\, y^s = -\mu_{s+1}\, y^{s+1}$, and writing t instead of $-\mu_{s+1}$, we have that $\mu_i = t\, \ell_i(y^{s+1}),\ \ i = 0, ..., s$. The lemma is then clearly equivalent to the relation

$$\sum_{i=0}^{s} t\, \ell_i(y^{s+1})\, c_v(Y_i^{s+1}) - t\, c_v(Y^s) = 0 \quad \forall t \in \mathbb{R},$$

from where, taking $t = 1$, we obtain

$$c_v(Y^s) = \sum_{i=0}^{s} \ell_i(y^{s+1})\, c_v(Y_i^{s+1}).$$

We can obtain an explicit expression for ℓ_i applying Cramer's rule to solve the system $\ell_i(y^j) = \delta_{ij},\ j = 0, ..., s$. This leads us to

$$\frac{det\, Y^s}{|det\, Y^s|} \chi_{cc(Y^s)}(v) = \sum_{i=0}^{s} \frac{det\, [y^0, ..., y^{s+1}, ..., y^s]}{|det\, [y^0, ..., y^{s+1}, ..., y^s]|}\, \chi_{cc(Y_i^{s+1})}(v)$$

where y^{s+1} is on the i^{th} column of the right hand side determinants. Let us suppose that the basis $\{y^0, ..., y^s\}$ is positively oriented, that is, $det\, Y^s > 0$, and let us denote by ε_i the sign of the orientation of the basis $\{y^0, ..., y^{i-1}, y^{s+1}, y^{i+1}, ..., y^s\}$. From this last relation we show finally that formula (2.1) is equivalent to

$$\chi_{cc(Y^s)}(v) = \sum_{i=0}^{s} \varepsilon_i \chi_{cc(Y_i^{s+1})}(v) \tag{2.2}$$

for all v not in an s-face of Y^{s+1}. This geometric relation can be easily checked in $\mathbb{R}^2$ or even in $\mathbb{R}^3$. A general proof of (2.2) is beyond the scope of this paper and will be published elsewhere.

From Lemma 2.1, a first simple result can be obtained.

Proposition 2.2. *Suppose that* $0 \notin [Y^{s+1}]$*, then*

$$c_v(u\ |Y^{s+1}) := \sum_{i=0}^{s+1} \mu_i\, c_v(Y_i^{s+1}) \tag{2.3}$$

is a well defined linear functional with respect to $u = \sum_{i=0}^{s+1} \mu_i\, y^i$.

Proof. The functional c_v is well defined because if there were two expressions for u, for example, $u = \sum \mu_i\, y^i = \sum \mu_i'\, y^i$, then

$$\sum_{i=0}^{s+1} (\mu_i - \mu_i')\, y^i = 0$$

which implies, using lemma 2.1, that

$$\sum_{i=0}^{s+1} (\mu_i - \mu_i')\, c_v(Y_i^{s+1}) = 0.$$

The linearity follows immediately by taking a particular choice of μ_i, for example, $\mu_{s+1} = 0$ and $\mu_i = \ell_i(u)$, $i = 0, ..., s$. $\square$

The following result is a direct consequence of Proposition 2.2. It follows from the fact that two points not separated by an s-face of Y^{s+1}, belong necessarily to the same convex cones $cc(Y_i^{s+1})$.

Proposition 2.3. *Under the same assumptions, if* v^1 *and* v^2 *are not separated by an* s*-face of* Y^{s+1}*, then*

$$c_{v^1}(\ \cdot\ |Y^{s+1}) = c_{v^2}(\ \cdot\ |Y^{s+1})$$

3 The Polar Cone Spline.

Definition 3.1. *A polar cone spline is defined recursively by*

$$c_v(u^1, ..., u^k|Y^{s+k}) := \sum_{i=0}^{s+k} \mu_i\, c_v(u^1, ..., u^{k-1}|Y_i^{s+k}) \tag{3.1}$$

where $u^k = \sum_i \mu_i\, y^i$. *For* $k = 0$, *we define*

$$c_v(Y^s) := \frac{\chi_{cc(Y^s)}(v)}{|det\, Y^s|} \tag{3.2}$$

The next result justifies the use of the adjective "polar" for c_v's name.

Theorem 3.2. *The polar cone spline* $c_v(u^1, ..., u^k|Y^{s+k})$ *is a well defined* k*-linear symmetric function of its arguments* $u^1, ..., u^k$.

Proof. We proceed by induction. The case $k=1$ is given by Proposition 2.2. Suppose that the theorem is true for $1, 2, ..., k-1$ arguments.

To prove that $c_v(u^1, ..., u^k|Y^{s+k})$ is well defined (that is, it doesn't depend on the way we write u^k as a linear combination of the vectors y^i) it is enough to prove that

$$\text{if } \sum_{i=0}^{s+k} \mu_i\, y^i = 0, \quad \text{then } \sum_{i=0}^{s+k} \mu_i\, c_v(u^1, ..., u^{k-1}|Y_i^{s+k}) = 0. \tag{3.3}$$

Let us note first that by writing $y^i = 1 \cdot y^i$, and using our induction hypothesis, we have that, for any $j \neq i$,

$$c_v(u^1, ..., u^{k-2}, y^i|Y_j^{s+k}) = c_v(u^1, ..., u^{k-2}|Y_{i,j}^{s+k}),$$

where we recall that $Y_{i,j}^{s+k}$ is Y^{s+k} without its i^{th} and j^{th} columns.

The same reasoning produces also the formula

$$c_v(u^1, ..., u^{k-2}, y^j|Y_i^{s+k}) = c_v(u^1, ..., u^{k-2}|Y_{i,j}^{s+k}).$$

Combining the last two formulae we find that

$$c_v(u^1, ..., u^{k-2}, y^i|Y_j^{s+k}) = c_v(u^1, ..., u^{k-2}, y^j|Y_i^{s+k}). \tag{3.4}$$

The proof of (3.3) continues as follows: We write $u^{k-1} = \sum_j \alpha_j\, y^j$ for some reals α_j. By induction hypothesis, the c_v's functions are linear in all its $k-1$ arguments, hence

$$\begin{aligned}
&\sum_i \mu_i\, c_v(u^1, ..., u^{k-2}, u^{k-1}|Y_i^{s+k}) \\
&\quad = \sum_i \mu_i\, c_v(u^1, ..., u^{k-2}, \sum_j \alpha_j\, y^j|Y_i^{s+k}) \\
&\quad = \sum_i \sum_j \mu_i\, \alpha_j\, c_v(u^1, ..., u^{k-2}, y^j|Y_i^{s+k}) \\
&\quad = \sum_j \alpha_j \sum_i \mu_i\, c_v(u^1, ..., u^{k-2}, y^i|Y_j^{s+k}) \\
&\quad = \sum_j \alpha_j\, c_v(u^1, ..., u^{k-2}, 0|Y_j^{s+k}) \\
&\quad = 0
\end{aligned}$$

Next, consider the linearity of c_v. By induction, c_v is already linear with respect to $u^1, ..., u^{k-1}$. To prove the linearity with respect to u^k, we just choose $\mu_i = \ell_i(u)$, $i = 0, ..., s$.

The symmetry of c_v comes from the symmetry of $Y^{s+k}_{i,j}$ with respect to i and j. In fact

$$\begin{aligned} & c_v(u^1, ..., u^{k-2}, y^i, y^j | Y^{s+k}) \\ &= c_v(u^1, ..., u^{k-2} | Y^{s+k}_{i,j}) \\ &= c_v(u^1, ..., u^{k-2}, y^j, y^i | Y^{s+k}). \end{aligned}$$

□

Theorem 3.3. *For polar cone splines the following properties hold:*

i) $c_v(u^1, ..., u^k, w | Y^{s+k} w) = c_v(u^1, ..., u^k | Y^{s+k}) \quad \forall w \in \mathbb{R}^{s+1}$, *where* $Y^{s+k}w$ *means the matrix* Y^{s+k} *augmented with the column* w.

ii) $c_v(u^1, ..., u^k | Y^{s+k}) = \sum_i \mu_i c_v(u^1, ..., u^k | Y^{s+k}_i y)$, *where* $y = \sum_i \mu_i y^i$.

iii) $c_v(u^1, ..., u^k | Y^{s+k})$ *is symmetric with respect to the ordering of* Y^{s+k}.

iv) If v^1 *and* v^2 *are not separated by an s-face of* Y^{s+k}

$$c_{v^1}(\ \cdot\ | Y^{s+k}) = c_{v^2}(\ \cdot\ | Y^{s+k}).$$

Proof. Property i) comes from writing $w = 1 \cdot w$.

Next, applying it with $w = y$, we prove ii) as follows:

$$\begin{aligned} & c_v(u^1, ..., u^k | Y^{s+k}) = c_v(u^1, ..., u^k, y | Y^{s+k} y) \\ &= \sum_{i=0}^{s+k} \mu_i\, c_v(u^1, ..., u^k, y^i | Y^{s+k} y) \\ &= \sum_{i=0}^{s+k} \mu_i\, c_v(u^1, ..., u^k | Y^{s+k}_i y). \end{aligned}$$

The iii) property is obtained by choosing $y = y^i$ in ii) and observing that

$$c_v(u^1, ..., u^k | Y^{s+k}) = c_v(u^1, ..., u^k | Y^{s+k}_j y^j).$$

Finally, iv) is obtained from Proposition 2.3. □

4 The Cone Spline and its Properties.

Definition 4.1. *The cone spline corresponding to Y^n, with $n = s+k$, is given by*

$$C(u|Y^n) := c_u(\underbrace{u, ..., u}_{k \text{ times}}|Y^n).$$

Let us show how to obtain the main properties of a cone spline.

- $C(\ \cdot\ |Y^n)$ is a piecewise homogeneous polynomial of degree k, because c_u is k-linear. Theorem 3.2 tell us that c_{u_0}, for a fixed $u_0 \in \mathbb{R}^{s+1}$, is k-linear. From the other side, the s-faces of Y^{s+k} induce a partition of $\mathbb{R}^{s+k}$. Thus, as long as u belongs to the same region of the partition than u_0, Theorem 3.3 (iv) says that $c_u = c_{u_0}$, which is k-linear all over this region.

- Recurrence relation: For $u = \sum_i \mu_i y^i$, the k-linearity of c_u and property i) of Theorem 3.3 produces:

$$\begin{aligned} C(u|Y^n) &= c_u(u, ..., u, \sum_i \mu_i y^i|Y^n) \\ &= \sum_i \mu_i c_u(u, ..., u, y^i|Y^n) \\ &= \sum_i \mu_i c_u(u, ..., u|Y_i^n) \\ &= \sum_i \mu_i C(u|Y_i^n) \end{aligned}$$

- Derivative formula. This is obtained from the derivative of a k-linear symmetric functional. More precisely

$$D_h C(u|Y^n) = k\, c_u(u, ..., u, h|Y^n) \quad \forall h \in \mathbb{R}^{s+1}$$

  From here, we obtain the following particular cases:

  · $h = u$

$$D_u C(u\ |Y^n) = k\, C(u\ |Y^n)$$

  · $h = \sum_i \alpha_i y^i$

$$D_h C(u|Y^n) = k \sum_i \alpha_i C(u|Y_i^n)$$

· $h = y^j$

$$D_{y^j} C(u|Y^n) = kC(u|Y^n_j).$$

Applying the derivative formula ν times, we obtain the following result:

Proposition 4.2. *For any point v not separated from u by an s-face,*

$$D_{h^1 \dots h^\nu} C(u|Y^n) = \frac{k!}{(k-\nu)!} c_v(u, ..., u, h^1, ..., h^\nu|Y^n)$$

The following theorem is a fundamental classical result.

Theorem 4.3. *The cone spline $C(\ \cdot\ |Y^n) \in \mathcal{C}^{k-1}$ where $k = n - s$.*

Proof

Let us recall that we are always assuming that Y^n is in general position. The s-faces of Y^n produce a partition of the domain of the cone spline. Two adjacent regions of the resulting partition are separated by exactly one s-face. If a and b are points in two adjacent regions, they are separated by an s-face which we can assume is of the form $S = cc\{y^{j_1}, ..., y^{j_s}\}$. But a and b are no more separated by an s-face of $Y^n_{j_i}$, for $i = 1, ..., s$. From this simple observation, if $a \to \bar{u} \in S$, the $k-1$ derivative tends to $k!\, c_a(\bar{u}, h^1, ..., h^{k-1}|Y^n)$. From the fact that $\bar{u} \in S$, there exist $\mu_1, ..., \mu_s$ such that $\bar{u} = \sum_{i=1}^s \mu_i\, y^{j_i}$. With this, we obtain finally

$$\begin{aligned} &k! c_a(\bar{u}, h^1, ..., h^{k-1}|Y^n) \\ &= k! \sum_{i=1}^{s} \mu_i\, c_a(y^{j_i}, h^1, ..., h^{k-1}|Y^n) \\ &= k! \sum_{i=1}^{s} \mu_i\, c_a(h^1, ..., h^{k-1}|Y^n_{j_i}) \\ &= k! \sum_{i=1}^{s} \mu_i\, c_b(h^1, ..., h^{k-1}|Y^n_{j_i}) \end{aligned}$$

In the last expression, we replaced c_a by c_b. This is possible because, as we noticed, a and b are not separated by an s-face of $Y^n_{j_i}$.

We conclude that the limit of the $k-1$ derivative of $C(\ \cdot\ |Y^n)$ at $\bar{u}$ is the same from both sides of S.

□

The next and last theorem is the cone spline version of a result due to H. Hakopian giving a formula for the size of the jump of the k^{th} derivative of a simplex spline. The proof proposed here is based mainly on the algebraic properties of cone spline and is much easier to follow than the original one [5].

Theorem 4.4. *Let $\bar{u} \in S = cc\{y^{j_1}, ..., y^{j_s}\}$ and $h \in \mathbb{R}^{s+1}$, a direction not in S. The jump of $D^k_{h...h}C(u|Y^n)$ at $\bar{u} \in S$, in the direction h, is given by*

$$\Delta^k_{S,h} = k!\, sgn(det[hy^{j_1}...y^{j_s}]) \frac{(det[h\ y^{j_1}...y^{j_s}])^k}{\Pi_{i \notin \{j_1,...,j_s\}} det[y^i y^{j_1}...y^{j_s}]}$$

Proof

It is clear that the k^{th} derivative is constant on each region not separated by an s-face. More precisely, if a is a point of the region, the k^{th} derivative is $k!c_a(h,...,h|Y^n)$. Assuming that h points towards a region where b is located and a is a point on the adjacent region, the jump can be written as $\Delta^k_{S,h} = k!c_b(h,...,h|Y^n) - k!c_a(h,...,h|Y^n)$. For each i not in $\{j_1,...,j_s\}$ we can consider the basis $\{y^i, y^{j_1}, ..., y^{j_s}\}$ and h can be written as the linear combination $h = \alpha_0\, y^i + \alpha_1\, y^{j_1} + \ldots + \alpha_s\, y^{j_s}$. From Cramer's rule, α_0 can be computed explicitly

$$\alpha_0 = \frac{det[h\ y^{j_1}...y^{j_s}]}{det[y^i y^{j_1}...y^{j_s}]}.$$

Using this expression, the jump can be computed as

$$\begin{aligned} \Delta^k_{S,h} &= k!\alpha_0 \Big(c_b(h,...,h,y^i|Y^n) - c_a(h,...,h,y^i|Y^n) \Big) \\ &= k! \frac{det[h\ y^{j_1}...y^{j_s}]}{det[y^i y^{j_1}...y^{j_s}]} \Big(c_b(h,...,h|Y^n_i) - c_a(h,...,h|Y^n_i) \Big). \end{aligned}$$

After k applications of the same idea, each time with a different vector $y^i \notin \{y^{j_1},...,y^{j_s}\}$ there will remains exactly one more y^i vector not used in this process. If we denote $\bar{Y} = (y^i y^{j_1}...y^{j_s})$, the formula obtained at this stage is

$$\Delta^k_{S,h} = k! \frac{\Big(det[h\ y^{j_1}...y^{j_s}]\Big)^k}{\Pi_{\ell \notin \{i,j_1,...,j_s\}} det[y^\ell y^{j_1}...y^{j_s}]} \Big(c_b(\bar{Y}) - c_a(\bar{Y}) \Big).$$

To complete the proof, it is necessary to consider the two possibilities for y^i, that is, whether it points towards the a region or towards the b region. In each case, the difference $c_b(\bar{Y}) - c_a(\bar{Y})$ is reduced to one term and is evaluated as

$$\pm \frac{1}{det[y^i y^{j_1}...y^{j_s}]}.$$

The sign depends upon whether y^i and h point both towards the same region or not. It is clear that in all cases the sign must be $sgn(det[h\ y^{j_1}...y^{j_s}])$

□

Consider the affine hyperplane H^s of $\mathbb{R}^{s+1}$ consisting of all points whose last component is 1, that is, points of the form $t^* = (t, 1)$ for each $t \in \mathbb{R}^s$. We say that $n+1$ points $\{t^0, ..., t^n\}$ in $\mathbb{R}^s$ are in general position if no subset of size $s+1$ is included in an affine hyperplane of $\mathbb{R}^s$. This is equivalent to the assertion that the vectors $\{t^{0*}, ..., t^{n*}\}$ are in general position in $\mathbb{R}^{s+1}$ according to the definition given in section 2. We can define

$$N(u|t^0, ..., t^n) := C(u^*|t^{0*}, ..., t^{n*})$$

which correspond to the restriction of C to the hyperplane H^s. It is possible to check that the respective polar forms share the same relation, that is,

$$n_v(u^1, ..., u^k|t^0, ..., t^n) = c_v(u^{1*}, ..., u^{k*}|t^{0*}, ..., t^{n*}),$$

and definition 3.1 can be used also to compute n_v. We observe that the recurrence relation obtained in this particular case is exactly the definition given by H.P. Seidel and A.H. Vermeulen [9] of what they called the *polar B-spline* which correspond to a simplex spline up to a multiplicative factor. This relation between cone splines and simplex splines is the same existing between a degree k polynomial in s variables and its degree k homogeneous version in $s+1$ variables obtained by the addition of an extra variable to a convenient power on each monomial term. For example, the degree 3 polynomial in two variable $2 - x + xy + y^3$ had the homogenized version $2w^3 - xw^2 + xyw + y^3$.

Starting from this observation, it is possible to obtain the properties for simplex splines rewriting the corresponding results for cone splines.

Some future work correspond to the relaxation of the general position hypothesis, and then to study of the degenerate cases. In particular, it is possible to obtain formulae for the jump of lower derivatives.

As it is possible to write a box spline as a sum of cone spline, it is possible to derive formulae for the polar form of a box spline. From here we think that it is possible to present the subdivision properties of box splines in term of its polar form.

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