B-Spline Knot-Line Elimination and Bézier Continuity Conditions

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Abstract. With respect the work of Dahmen-Micchelli-Seidel [3], where a new scheme for multivariate B-spline is proposed, we present some insight of its geometrical nature. Theirs scheme provide automaticaly C^{k-1} continuity with degree k piecewise polynomials. The cost to be payed is a complex structure of the subyacent triangulation. We propose an analogy between those B-splines and triangular Bézier patches. The affine conditions between the poles of two adjacents Bezier patches for C^r continuity are replaced by similar conditions between the poles of the B-splines. This time the conditions do not produce additional continuity but some knot-lines are eliminated.

§1. Introduction

Simplex splines were introduced by C. de Boor [4] in 1976. They are now seen as a fruitfull generalization of the univariate B-splines introduced by H.B. Curry and I.J. Schoenberg [1] in 1966. A good survey on simplex splines and some of its extension is given in [6].

Many authors have contributed to prove that simplex splines considered as single functions exhibit the same behaviour as theirs univariate analogs. But if we think of B-splines as "base"-splines, the work is far from been accomplished. Among the differents schemes proposed since 1982, the one presented in [3] is the first to combine simplicity and good properties.

The basic idea can be stated as follows. Given a triangulation of $\mathbb{R}^2$, consider each vertex as a knot having multiplicity $k+1$, where k is the degree of the piecewise polynomial scheme. If a C^{k-1} continuous scheme is needed, we had to pull apart the repeated knots. In this way we get a triangulation in which every vertex is replaced by a "cloud" of knots around it. In fact, the $k+1$ knots being differents is not sufficient to assure C^{k-1} continuity. Some non-colinearity between some of those knots is also necessary (see (H')).

Once the clouds of knots are fixed, we pick some knot subsets. We reminds that $k+3$ knots determines a unique simplex spline of degree k. This enable us to associate to each knot subset a simplex spline, obtaining in this way a set of "base"-spline or B-splines. The problem is then reduced to propose a rule for selecting knot subsets in such a way that the splines space generate by the B-splines possess nice properties.

To continue on this reasoning line we need some minimal notation. Let $\Delta^I := [t_0^I, t_1^I, t_2^I]$ be the triangle "I" of our triangulation $\mathcal{T}$ of $\mathbb{R}^2$. To each vertex t_i^I we associate a list of k distinct additional knots

$$t_i^I := t_{i,0}^I, t_{i,1}^I, \dots, t_{i,k}^I.$$

The simple rule proposed in [3] consist in producing a subset V_β^I, where $\beta = (\beta_0, \beta_1, \beta_2)$ are three non negative integers, as follows :

$$V_\beta^I := \{t_{0,0}^I, t_{0,1}^I, \dots, t_{0,\beta_0}^I, t_{1,0}^I, t_{1,1}^I, \dots, t_{1,\beta_1}^I, t_{2,0}^I, t_{2,1}^I, \dots, t_{2,\beta_2}^I\}. \tag{1}$$

If we are supposed to produce degree k simplex splines, we must impose that

$$|\beta| := \beta_0 + \beta_1 + \beta_2 = k. \tag{2}$$

The number of possibles β that verifies (2) is exactly the dimension of $\Pi^k(\mathbb{R}^2)$, the space of polynomials of (total) degree less or equal than k. In fact, $\Pi^k(\mathbb{R}^2)$ is included in the spline space generated by the B-splines. Moreover, Seidel [10] proved that this spline space contains $\Pi_{k-1}^k(\mathcal{T})$, that is, all the degree k piecewise polynomials based on the triangulation $\mathcal{T}$ and which are C^{k-1} continuous.

In section 2 we recall the main results of Dahmen-Micchelli-Seidel [3], and Seidel [10]. In section 3 we present our result about the knot-line elimination and some ilustrations.

§2. B-patches and B-splines

Let us recall the concept of blossom (or polar form) of a polynomial function

Definition 1. *Let E, Q be vector (or affine) spaces and let $F : E \longrightarrow Q$ be a degree k polynomial function. The unique function $f : E^k \longrightarrow Q$ k-affine, symetric and such that $f(u, \dots, u) = F(u)$, $\forall u \in E$, is called the blossom (or polar form) of F.*

L. Ramshaw [8] proved the existence and uniqueness of such an object. He express most of the known properties of piecewise polynomial functions in a simpler way using the blossom f instead of F itself.

H.-P. Seidel ([3], [9]) extends the de Casteljau algorithm and produces a generalization of the Bézier triangular patches, the B-patches.

Following the same notation than in section 1, we add the next ones :

$$\Delta_\beta^I := [t_{0,\beta_0}^I, t_{1,\beta_1}^I, t_{2,\beta_2}^I] \quad \text{and}$$
$$X_\beta^I := (t_{0,0}^I, t_{0,1}^I, \dots, t_{0,\beta_0-1}^I, t_{1,0}^I, \dots, t_{1,\beta_1-1}^I, t_{2,0}^I, \dots, t_{2,\beta_2-1}^I) \in (\mathbb{R}^2)^{|\beta|}.$$

If Δ_β^I is non-degenerate, it is possible to define the barycentric coordinates of $u \in \mathbb{R}^2$ with respect to this triangle :

$$u = \sum_{i=0}^{2} \lambda_{\beta,i}^I(u)\, t_{i,\beta_i}^I \quad \text{and} \quad \sum_{i=0}^{2} \lambda_{\beta,i}^I(u) = 1.$$

The generalized algorithm computes $F(u)$ starting from the values $f(X_\beta^I)$, $|\beta| = k$. Those values are called the poles of F. Let define

$$X_\beta^I\, u^\nu := X_\beta^I \times (\underbrace{u, \ldots, u}_{\nu \text{ times}}) \in (\mathbb{R}^2)^{|\beta|+\nu}$$

Putting $C_\beta^\nu(u) := f(X_\beta^I\, u^\nu)$, with $|\beta| = k - \nu$, the algorithm uses the k-affinity of f stating the recurrence relation :

$$C_\beta^0(u) := f(X_\beta^I), \quad |\beta| = k$$
$$C_\beta^{\nu+1}(u) := \sum_{i=0}^{2} \lambda_{\beta,i}^I(u)\, C_{\beta+e^i}^\nu(u)$$

where e^i denotes the canonical basis vector. It is clear that $C_0^k(u) = F(u)$.

If the algorithm is applied to the values

$$f_\gamma = \begin{cases} 1 & \text{if } \gamma = \beta \\ 0 & \text{if } \gamma \neq \beta \end{cases}$$

a particular degree k polynomial is obtained, $B_\beta^I(\cdot)$. From the linearity of the algorithm, we obtain

$$F(u) = \sum_{|\beta|=k} f(X_\beta^I)\, B_\beta^I(u). \tag{3}$$

It is possible to relate all this with simplex splines.

Let assume that

$$\Omega^I := int\Big(\bigcap_{|\beta|\le k} \Delta_\beta^I\Big) \neq \emptyset. \tag{H}$$

This assumption imply in particular that all the triangles Δ_β^I are non-degenerate. The generalized de Casteljau algorithm is then meaningfull.

The following results were state in [3] and [10]. A recent and direct proof is found in [7].

Proposition 2. *The simplex spline $M(\cdot|V_\beta^I)$ based over the knot set V_β^I and restricted to Ω^I is proportional to $B_\beta^I(\cdot)$.*

Definition 3. *The normalized B-spline, $\mathcal{N}_\beta^I(\cdot)$, is the scaled simplex spline based over the knot set V_β^I such that $\mathcal{N}_\beta^I(u) = B_\beta^I(u)$, $\forall u \in \Omega^I$.*

The main result is the following

Proposition 4. *Let S be a degree k piecewise polynomial function over the triangulation $\mathcal{T}$ which is C^{k-1} continuous, then*

$$S(u) = \sum_I \sum_{|\beta|=k} \mathcal{N}_\beta^I(u)\, f^I(X_\beta^I) \qquad \forall u \in \mathbb{R}^2$$

where f^I is the blossom of F^I, the restriction of S to the triangle Δ^I.

Finally, applying the proposition 4 to $S \equiv 1$ we get

Corolary 5.

$$\sum_I \sum_{|\beta|=k} \mathcal{N}_\beta^I(u) \equiv 1.$$

That is, the normalized B-splines form a partition of unity. This property enables us to consider poles as real points.

Remark. *Seidel's result [10] is more general than proposition 4. He presents the s-dimensional case and combines multiplicity greater than 1 with lower continuity for S.*

§3. Knot-lines elimination

A simplex spline $M(\cdot|K)$, based over a knot set $K = \{x_0, \ldots, x_n\}$, with $n = k+2$, is a degree k polynomial. It is known that if K is in general position (no line contains three points of K), $M(\cdot|K)$ is C^{k-1} continuous. Moreover, the discontinuities are located exactly along the segments $[x_i, x_j]$. We call those segments, knot-lines. This implies that a spline of the form

$$S(u) = \sum_I \sum_{|\beta|=k} \mathcal{N}_\beta^I(u)\, P_\beta^I, \tag{4}$$

where P_β^I are arbitrary points in Q called poles of the spline S, may have discontinuities (of the k^{th} derivative) at most along the corresponding knot-lines of the B-splines $\mathcal{N}_\beta^I$. But for appropiate choices of the poles P_β^I, some of the discontinuities may perhaps cancel each other.

Let us concentrate on two adjacent triangles, say Δ^I and Δ^J. Suppose, for the simplicity of the notation, that the common edge is $[t_0^I = t_0^J, t_1^I = t_1^J]$. On each sets Ω^I and Ω^J the spline is a single polynomial, say F^I and F^J. They verify

$$F^I(u) = \sum_{|\beta|=k} B_\beta^I(u)\, P_\beta^I, \tag{5}$$

and similarily for F^J. It is clear from (3) that $P_\beta^I = f^I(X_\beta^I)$.

For $|\beta| = k$, let us denote by a_β^I the affine function such that

$$a_\beta^I(t_{0,\beta_0}^I) = a_\beta^I(t_{1,\beta_1}^I) = 0 \quad , \quad a_\beta^I(t_{2,\beta_2}^I) = 1 \tag{6}$$

The straight line spanned by t^I_{0,β_0} and t^I_{1,β_1} is

$$L_\beta := \{u \in \mathbb{R}^2 | a^I_\beta(u) = 0\}.$$

We assume that

$$\textit{all the lines } L_\beta \textit{ are distincts.} \tag{H'}$$

Those lines induce a partition over $\mathbb{R}^2$, $\mathcal{T}_L$. The following result (see [7], Lemma 7) is our main tool to find out the particular choices for the poles in order to eliminate some knot-lines.

Proposition 6. *There exist a unique degree k piecewise polynomial function S over the partition $\mathcal{T}_L$ such that :*

- *S is C^{k-1} continuous.*
- *$S(u) = F^I(u)\ \forall u \in \Omega^I$ and $S(u) = F^J(u)\ \forall u \in \Omega^J$.*

Moreover, the function S has the following form :

$$S(u) = F^J(u) + \sum_{|\beta|=k} \mu_\beta \, (a^I_\beta(u))^k_+, \tag{7}$$

where the real coefficients μ_β satisfy the relations :

$$\sum_{\substack{|\beta|=k \\ \beta_0 \geq \gamma_0 \ , \ \beta_1 \geq \gamma_1}} \mu_\beta \, \varphi_\beta(X^I_\gamma) = f(X^I_\gamma) \tag{8}$$

for all γ, $|\gamma| = k$, where f is the blossom of $F = F^I - F^J$ and

$$\varphi_\beta(u_1, \ldots, u_k) := \prod_{j=1}^k a^I_\beta(u_j)$$

is the blossom of

$$\Phi_\beta(u) := a_\beta(u)^k.$$

It is easily verified that (7) defines a degree k piecewise polynomial function over $\mathcal{T}_L$ which is C^{k-1} and such that

$$S(u) = F^J(u) \quad \forall u \in \Omega^J.$$

Then, the proof (see [7] for more details) is reduced to check that

$$S(u) = F^I(u) \quad \forall u \in \Omega^I$$

and that this solution is unique. But this last condition is equivalent to

$$F^I = F^J + \sum_{|\beta|=k} \mu_\beta \, \Phi_\beta$$

The relation (8) express the fact that the poles of $F = F^I - F^J$ are the same than those of $\sum \mu_\beta \, \Phi_\beta$. The scope of the sum is reduced because of the zeros of φ_β (see (6)).

From this relation, the coefficients μ_γ can be computed recursively as

$$\mu_\gamma = \frac{f(X^I_\gamma) - \sum_{\beta \in R_\gamma} \mu_\beta \, \varphi_\beta(X^I_\gamma)}{\varphi_\gamma(X^I_\gamma)} \tag{9}$$

where :

$$R_\gamma = \{\beta \in \mathbb{Z}^3_+ \mid |\beta| = k, \beta_0 \geq \gamma_0, \beta_1 \geq \gamma_1, \beta \neq \gamma\}.$$

A knot-line L_β will be a discontinuity line for the spline S given by (4) if the coefficient μ_β of the proposition is different from zero.

Proposition 7. *The spline S given by the expression (4) has no discontinuity of its k^{th} derivative along the lines*

$$[t^I_{0,\beta_0}, t^I_{1,\beta_1}], \quad \forall \beta \; |\beta| = k, \;\; \beta_2 \leq r$$

iff

$$P^I_\beta = f^J(X^I_\beta), \quad \forall \beta \; |\beta| = k, \;\; \beta_2 \leq r. \tag{10}$$

Proof: From the recurrence relation (9), the coefficients μ_β are zero iff

$$f^I(X^I_\beta) = 0 \;\; \forall \beta \; |\beta| = k, \;\; \beta_2 \leq r.$$

But $f = f^I - f^J$, then

$$f^I(X^I_\beta) = f^J(X^I_\beta) \;\; \forall \beta \; |\beta| = k, \;\; \beta_2 \leq r,$$

and the result follows from the fact that $P^I_\beta = f^I(X^I_\beta)$. ■

Remark. *The relations (10) correspond to affine relations between the poles of F^I and F^J. Moreover, if the cloud of knots corresponding to each vertex is collapsed over its original vertex, the relation (10) became exactly the affine relations for C^r continuity between the Bézier points of the two triangular Bézier patches F^I and F^J.*

We finish this presentation with 3 figures showing the effect of the relations (10). The subjacent triangulation is made of 6 triangles with a common vertex in the center point. On the left we present the poles and on the right, the resulting surface. All the 3 figures are quadratic splines, hence all are C^1.

Figure 1 shows the situation of no relation between two adjacent patches. Visually it seem to be a discontinuous situation.

Figure 2 corresponds to the relation (10) with $r = 0$. The condition is reduced in this case to the identification of the three border poles between adjacent triangles. Visually, a C^0 effect is obtained.

Figure 3 shows the effect of (10) with $r = 1$ just for one common edge between two triangles. In this case, one of the crease line has been eliminated.

Figure 1. No conditions between poles of adjacents patches.

Figure 2. Conditions between poles of adjacent patches for $r = 0$.

Figure 3. Conditions between poles of two adjacent patches for $r = 1$.

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