# A Class of Multivariate de Boor-Fix Formulae

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A family of Marsden identities and de Boor-Fix formulae in the multivariate case are proposed. A general framework to define dual bases is presented, which requires neither polynomial functions, nor univariate functions. Nevertheless, the main focus is the multivariate polynomial case. The relation between polar form and dual bases is presented in this multivariate polynomial case.

*Key words:* Blossoming, polar form, Marsden's identity, deBoor-Fix Formula, dual bases, B-patch.

## 1 Introduction

The de Boor-Fix formula appeared in 1973 [3]: it gives a way of representing the dual functionals associated with the univariate B-splines. Another way to represent the same dual functionals can be obtained by using the blossom or polar form of a piecewise polynomial function as shown by L. Ramshaw ([11], 1987). An introduction to polar forms can be found in [14]. Recall that the polar form associated with a degree $k$ polynomial $P$ is the unique $k$-affine symmetric function $p$ such that $p(x, \ldots, x) = P(x)$. For a piecewise polynomial function, the definition is applied to each piece of $P$.

The combination of polar forms and of de Boor-Fix formula was presented by P. Barry ([1], 1990). In 1994, R. Goldman proposed the idea of dual polynomial bases ([4], 1994) in the univariate case, starting from a Marsden's type formula. A multivariate Marsden formula was also proposed and briefly analyzed.

S. Lodha and R. Goldman took the first step ([8], 1994) towards a multivariate version of the de Boor-Fix formula. With their approach they developed

[1] Partially supported by FCFM, Univ. de Chile, under a 1996 research grant and by Proyecto Fondecyt 1950561.

applications to transformations between polynomial basis. They also showed ([9], 1995) that there is a duality between degree elevation formulae and differentiation.

M.-L. Mazure and P.-J. Laurent presented ([10], 1996) a general framework for dual bases which is well suited for more general cases than those treated by R. Goldman. In particular, in a Chebyshev space they generalize the de Boor-Fix formula, define the blossoming operation and rewrite the de Casteljau algorithm.

In this paper, the general framework presented in [10] is recalled in Section 2. In Section 3, the particular case of multivariate homogeneous polynomials is considered. We recall the main properties relating polar forms and B-patch bases. Some known recursions are re-stated in a more general setting. In Section 4 we propose a particular formula of the de Boor-Fix type for homogeneous polynomials. Many results from [2] fit our framework, giving an unified presentation. The duality between degree elevation and differentiation [9] is presented from a simpler point of view (polar forms) and is generalized to arbitrary degree differentiation. Homogeneous polynomials and non-homogeneous polynomials are in fact isomorphic vector spaces, by the addition of an extra variable which balances the degree of all the monomials of the polynomial. With this standard technique, we show in Section 5 how to translate the results for homogeneous polynomials to the non-homogeneous case. Finally, in Section 6, we briefly comment on some work in progress.

## 2 General Framework

Let us consider two real function spaces $U$ and $V$ defined on two sets $X$ and $Y$ respectively, such that $dim\ U = dim\ V = n+1$. Let $E : X \times Y \to \mathbb{R}$ be a fixed function.

**Theorem 1** *The following assertions are equivalent:*

*(i) The spaces $U$ and $V$ satisfy:*
  *(a) $\{E(\cdot, y)\}_{y \in Y}$ generates $U$*
  *(b) $\forall x \in X,\ E(x, \cdot) \in V$*

*(ii) For all $\mathcal{V} = \{v_0, \dots, v_n\}$ basis of $V$ there exists $\mathcal{U} = \{u_0, \dots, u_n\}$ basis of $U$ such that*

$$E(x, y) = \sum_{i=0}^{n} u_i(x) v_i(y)$$

*(iii) There exists $\mathcal{V} = \{v_0, \ldots, v_n\}$ basis of $V$ and $\mathcal{U} = \{u_0, \ldots, u_n\}$ basis of $U$ such that*

$$E(x, y) = \sum_{i=0}^{n} u_i(x) v_i(y)$$

**Proof.** See [10] .

From now on we suppose that a function $E$ satisfying the assertions of the theorem is given. This function will be called a *reproducing function* associated with $U$ and $V$.

Let us show at this stage some standard examples of reproducing functions in the polynomial case.

(i) Consider $U = V = \Pi_n$, the vector space of the degree $n$ univariate polynomials. A very familiar reproducing function [4] is

$$\begin{aligned} E(x, y) &= (x - y)^n \\ &= \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!(n-j)!} x^j y^{n-j}. \end{aligned}$$

(ii) An unusual reproducing function for the same spaces is

$$\begin{aligned} E(x, y) &= (1 + xy)^n \\ &= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^j. \end{aligned}$$

(iii) The case $U = V = H_k(\mathbb{R}^{s+1})$ of $(s+1)$-variate homogeneous polynomials of degree $k$ will be analyzed in detail in Section 4. The reproducing function we will use is

$$\begin{aligned} E(x, y) &= \langle x, y \rangle^k \\ &= \sum_{|\beta| = k} \frac{k!}{\beta!} x^\beta y^\beta. \end{aligned}$$

where $\beta = (\beta_0, \ldots, \beta_s)$ is a multi-index (every component $\beta_i$ is a non-negative integer). Here we use $|\beta| = \beta_0 + \cdots + \beta_s$, $\beta! = \beta_0! \cdots \beta_s!$ and $x^\beta = x_0^{\beta_0} \cdots x_s^{\beta_s}$.

**Definition 2** *Two bases $\{u_0, \ldots, u_n\}$ and $\{v_0, \ldots, v_n\}$ are* dual *with respect to $E$ if*

$$E(x, y) = \sum_{i=0}^{n} u_i(x) v_i(y) \qquad \forall x \in X \quad \forall y \in Y.$$

**Remark 3** *Given any pair of bases of $U$ and $V$, $\{u_0, \ldots, u_n\}$ and $\{v_0, \ldots, v_n\}$ respectively, they will be dual with respect to the reproducing function they define, that is, with respect to*

$$\sum_{i=0}^{n} u_i(x)\, v_i(y).$$

Let $\mathcal{U} = \{u_0, \ldots, u_n\}$ and $\mathcal{V} = \{v_0, \ldots, v_n\}$ be dual bases of $U$ and $V$ with respect to $E$. For $u \in U$ and $v \in V$ there are unique representations $u = \sum_i \alpha_i u_i$ and $v = \sum_i \beta_i v_i$. These coefficients, the coordinates of $u$ and $v$ with respect to the bases $\mathcal{U}$ and $\mathcal{V}$, will be used to define a kind of scalar product in $U \times V$.

**Definition 4** *Given any pair of bases $\mathcal{U} = \{u_0, \ldots, u_n\}$ and $\mathcal{V} = \{v_0, \ldots, v_n\}$ we define the following bilinear form:*

$$[u, v]_{\mathcal{U},\mathcal{V}} := \sum_{i=0}^{n} \alpha_i\, \beta_i$$

*where $u = \sum_i \alpha_i u_i$ and $v = \sum_i \beta_i v_i$.*

Clearly, from this definition, $[u_i, v_j]_{\mathcal{U},\mathcal{V}} = \delta_{ij}$.

The following Theorem is an alternate form to present Theorem 2.2 from [10].

**Theorem 5** *(Reproduction Property) Let us suppose that the bases $\mathcal{U}$ and $\mathcal{V}$ are dual with respect to $E$. Then*

*(i)* $[E(\cdot, y), v]_{\mathcal{U},\mathcal{V}} = v(y)$ *for all* $y \in Y, v \in V$.
*(ii)* $[u, E(x, \cdot)]_{\mathcal{U},\mathcal{V}} = u(x)$ *for all* $x \in X, u \in U$.

**Proof.** (i) Let $v(\cdot) = \sum \beta_i v_i(\cdot)$. On the other hand, Definition 2 implies that $E(\cdot, y) = \sum_i v_i(y) u_i(\cdot)$, hence the coefficients $v_i(y)$ are the coordinates $\alpha_i$ of $E(\cdot, y)$ with respect to the basis $\mathcal{U}$. Applying Definition 4,

$$[E(\cdot, y), v(\cdot)]_{\mathcal{U},\mathcal{V}} = \sum_{i=0}^{n} v_i(y) \cdot \beta_i = v(y).$$

(ii) Similar. □

**Corollary 6** *The bilinear form $[\cdot, \cdot]_{\mathcal{U},\mathcal{V}}$ does not depend on the particular choice of the dual bases, as long as these bases satisfy Definition 2 with the same reproducing function $E$.*

**Proof.** The reproducing function satisfies assertion (i)–a of Theorem 1, that is, $\{E(\cdot, y) \mid y \in Y\}$ generates $U$. There must exist then $y_0, \ldots, y_n$ such that $\{E(\cdot, y_0), \ldots, E(\cdot, y_n)\}$ is a basis of $U$.

With this basis, any $u \in U$ can be written as $u = \sum_i \alpha_i E(\cdot, y_i)$. Taking now any $v \in V$, we obtain

$$
\begin{aligned}
[u, v]_{\mathcal{U},\mathcal{V}} &= [\sum \alpha_i E(\cdot, y_i), v]_{\mathcal{U},\mathcal{V}} \\
&= \sum \alpha_i [E(\cdot, y_i), v]_{\mathcal{U},\mathcal{V}} \\
&= \sum \alpha_i v(y_i).
\end{aligned}
$$

This latter expression depends only on $u$ (the $\alpha_i$'s), $v$ and $y_0, \ldots, y_n$, but not on the particular choice of the bases $\mathcal{U}$ and $\mathcal{V}$. $\square$

**Remark 7** *This result allows us to denote the bilinear form by $[\cdot, \cdot]_E$. Whenever it will be clear from the context, we will simply write $[\cdot, \cdot]$.*

**Remark 8** *The linear functionals $\{[\cdot, v_i] \mid i = 0, \ldots, n\}$ are in fact the dual functionals with respect to the basis $\mathcal{U} = \{u_0, \ldots, u_n\}$. Analogous for $\mathcal{V} = \{v_0, \ldots, v_n\}$ and $\{[u_i, \cdot] \mid i = 0, \ldots, n\}$.*

**Corollary 9** *If $u_i = E(\cdot, y_i)$, $i = 0, \ldots, n$ is a basis of $U$, then its dual basis $\{v_0, \ldots, v_n\}$ is the Lagrange basis of $V$ with respect to the nodes $y_0, \ldots, y_n$.*

**Proof.** Taking $u_i = E(\cdot, y_i)$ in Theorem 5–(i)

$$
\begin{aligned}
[u_i, F]_E &= [E(\cdot, y_i), F]_E \\
&= F(y_i).
\end{aligned}
$$

Thus

$$F(y) = \sum_i F(y_i) v_i(y).$$

$\square$

The following corollary generalizes Theorem 4.1 from R. Goldman [4].

**Corollary 10** *Let $\widetilde{\mathcal{U}} = \{\tilde{u}_0, \ldots, \tilde{u}_n\}$ be another basis of $U$ and let $m_{ij}$ be such that $u_i = \sum_j m_{ij} \tilde{u}_j$. Then the dual basis of $\widetilde{\mathcal{U}}$ is $\widetilde{\mathcal{V}} = \{\tilde{v}_0, \ldots, \tilde{v}_n\}$ where $\tilde{v}_i = \sum_j m_{ji} v_j$.*

**Proof.** Direct.

## 3 The homogeneous polynomial case

B-patches are a B-spline-like representation of multivariate polynomials that have been introduced in [12]. Seidel studies B-patches in an affine setting. Here we consider B-patches in a homogeneous setting.

In $\mathbb{R}^{s+1}$, let us consider a set of the form (see [13]):

$$X = \{x_{\ell,j} \in \mathbb{R}^{s+1} \mid \ell = 0, \ldots, s, j = 0, \ldots, k\}.$$

The set $X$ can be thought of as a rectangular array of points, or as $s+1$ "clouds" of knots around each of the vertices $x_{0,0}, x_{1,0}, \ldots, x_{s,0}$ of a simplex in $\mathbb{R}^{s+1}$.

**Definition 11** *The set $X$ is a* knot net *if $\forall \beta = (\beta_0, \ldots, \beta_s)$, $|\beta| \leq k$, $\{x_{0,\beta_0}, x_{1,\beta_1}, \ldots, x_{s,\beta_s}\}$ is linearly independent.*

For $\alpha = (\alpha_0, \ldots, \alpha_s)$, ($|\alpha| \leq k$), we introduce the linear functionals $h^*_{\ell\alpha}(\cdot) : \mathbb{R}^{s+1} \to \mathbb{R}$ defined by:

$$u = \sum_{\ell=0}^{s} h^*_{\ell\alpha}(u) \cdot x_{\ell,\alpha_\ell}. \tag{1}$$

In this section we will use the star superindex "$*$" to identify homogeneous polynomials, in contrast to the non-homogeneous case treated in the next section.

The polar form of a degree $k$ homogeneous polynomial simplifies the following discussion. We recall that the polar form of $P^* \in H_k(\mathbb{R}^{s+1})$ is the unique $k$-linear symmetric function $p^*$ such that $P^*(u) = p^*(u, \ldots, u)$. We use $k$-linear functions instead of $k$-affine functions because we are dealing here with homogeneous polynomials ([11]). A standard problem in CAGD is to compute the value of a polynomial function $P^*(u)$ starting from its control points or poles. The poles of $P^*$ with respect to the knot net $X$ are defined by

$$p^*(X_\beta) \quad |\beta| = k$$

where

$$X_\beta = \left(x_{0,0}, \ldots, x_{0,\beta_0 - 1}, \ldots, x_{s,0}, \ldots, x_{s,\beta_s - 1}\right). \tag{2}$$

Let us introduce the following quantities

$$c_\beta^0 := p^*(X_\beta) \quad \text{for} \quad |\beta| = k$$

and, for $\nu = 1, \ldots, k$,

$$c_\beta^\nu(u_1, \ldots, u_\nu) := p^*(X_\beta, u_1, \ldots, u_\nu), \quad \text{for} \quad \beta, \ |\beta| = k - \nu, \tag{3}$$

so that, in particular,

$$c^k_{(0,\ldots,0)}(u_1, \ldots, u_k) = p^*(u_1, \ldots, u_k).$$

Taking $u_1 = u_2 = \cdots = u_k = u$, we obtain

$$c^k_{(0,\ldots,0)}(u^k) = P^*(u),$$

where we are using the notation $u^k := (u, \ldots, u)$ ($k$ components).

Using the $k$-linearity of $p^*$ and (1) we write,

$$\begin{aligned} c_\beta^{\nu+1}(u_1, \ldots, u_{\nu+1}) &= p^*(X_\beta, u_1, \ldots, u_\nu, \sum_{\ell=0}^{s} h^*_{\ell\beta}(u_{\nu+1}) x_{\ell,\beta_\ell}) \\ &= \sum_{\ell=0}^{s} h^*_{\ell\beta}(u_{\nu+1}) p^*(X_\beta, u_1, \ldots, u_\nu, x_{\ell,\beta_\ell}) \\ &= \sum_{\ell=0}^{s} h^*_{\ell\beta}(u_{\nu+1}) c^\nu_{\beta+e_\ell}(u_1, \ldots, u_\nu) \end{aligned}$$

where $e_\ell = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $\ell^{th}$ canonical basis vector of $\mathbb{R}^{s+1}$, numerated from $\ell = 0$ to $s$.

Hence, the starting values $\{c_\beta^\nu\}$ will produce the coefficients $\{c_\beta^{\nu+1}\}$.

This relation, known as "up recursion", provides us with a pyramidal algorithm to compute the polynomial $P^*$ and its polar form $p^*$.

If we consider the linear mapping $L : P \to (p(X_\beta) |\ |\beta| = k)$, it is easy to show that $L$ is inyective. On the other hand, the dimension $d$ of $H_k(\mathbb{R}^{s+1})$ is equal to the cardinal of $\{\beta \mid |\beta| = k\}$, so $L$ has to be an isomorphism between $H_k(\mathbb{R}^{s+1})$ and $\mathbb{R}^d$. In other words, any set of $d$ poles can be associated to a unique homogeneous polynomial.

**Definition 12** *For each $\beta$, $|\beta| = k$ we define the function $B^*_\beta(u)$ as the homogeneous polynomial produced by the recursive method starting from the data*

$$c^0_\beta = 1, \quad c^0_{\beta'} = 0, \quad \forall \beta' \neq \beta, \ |\beta'| = k.$$

These functions constitute the B-patch basis associated with the knot net $X$.

Directly from this definition, we obtain a first expression for the dual functionals in terms of the polar form.

**Theorem 13** *The dual functionals associated to the B-Patch basis are given by*

$$\phi_\beta : \ P^* \to p^*(X_\beta) \ \text{ for all } \ |\beta| = k.$$

**Proof.** The duals functionals are defined by $\phi_\beta(B^*_{\beta'}) = \delta_{\beta\,\beta'}$, which is another way to state Definition 12. $\square$

From this result, it is clear that the coefficients $c^0_\beta$ are the coordinates of $P^*$ with respect to the B-patch basis $B^*_\beta$. The following Theorem (see [12], Cor. 3.5 for an affine version) generalizes this.

**Theorem 14** *For each $\nu = 0, \ldots, k$,*

$$P^*(u) = \sum_{|\beta| = k-\nu} c^\nu_\beta(u^\nu) B^*_\beta(u).$$

**Proof.**

Given $w \in \mathbb{R}^{s+1}$ we define

$$P^*_{w^\nu}(u) := p^*(u^{k-\nu}, w^\nu).$$

It is clear that $P^*_{w^\nu}$ is a degree $k - \nu$ homogeneous polynomial with poles given by

$$P^*_{w^\nu}(X_\beta) = p^*(X_\beta, w^\nu), \quad |\beta| = k - \nu.$$

According to (3),

$$P^*_{w^\nu}(u) = \sum_{|\beta| = k-\nu} c^\nu_\beta(w^\nu) B^*_\beta(u).$$

The result follows from taking $w = u$. $\square$

We end this section with a generalized version of the "down recursion", which shows how to compute a degree $k$ B-patch from a lower degree B-patch basis.

Suppose $|\beta| = k$. We can group the arguments of the polar form of $B^*_\beta$ as $b^*_\beta(u_1, \ldots, u_\nu, v_1, \ldots, v_{k-\nu})$. This polar form is $\nu$-linear with respect to the $u$ arguments and $(k-\nu)$-linear with respect to the $v$ arguments. Hence, $b^*_\beta(u_1, \ldots, u_\nu, v, \ldots, v)$ is a homogeneous polynomial of degree $k-\nu$ in the variable $v$. It is possible to write it as follows in the corresponding B-patch basis:

$$b^*_\beta(u_1, \ldots, u_\nu, v, \ldots, v) = \sum_{|\gamma|=k-\nu} \ell^\gamma_\beta(u_1, \ldots, u_\nu)\, B^*_\gamma(v),$$

where the functions $\ell^\gamma_\beta$ are some $\nu$-linear functions to be determined. In fact, these functions satisfy some relation of the same type as Definition 12. If we consider the polar form of this latter equation, with respect to the variable $v$, we obtain:

$$b^*_\beta(u_1, \ldots, u_\nu, v_1, \ldots, v_{k-\nu}) = \sum_{|\gamma|=k-\nu} \ell^\gamma_\beta(u_1, \ldots, u_\nu)\, b^*_\gamma(v_1, \ldots, v_{k-\nu}). \quad (4)$$

To obtain the relations satisfied by $\ell^\gamma_\beta$, evaluate $b^*_\beta(X_\beta)$ (which is equal to 1), separating $X_\beta$ into two parts. For a multi-index $\gamma \leq \beta$ (that is, every component of $\gamma$ is less than or equal to the corresponding component of $\beta$) there exists a unique set $R$ such that $X_\beta = R \cup X_\gamma$. Replacing this in equation (4), we obtain

$$\ell^\gamma_\beta(R) = 1$$

The same reasoning, but starting from $b^*_\beta(X_{\beta'}) = 0$ for any $\beta' \neq \beta$ gives

$$\ell^\gamma_\beta(R') = 0$$

where $R'$ is such that $X_{\beta'} = R' \cup X_\gamma$. These relations define a B-patch function (according to Definition 12) but with respect to a special reordering of the knot set $X$.

The particular case $\nu = 1$ leads us to consider $\gamma = \beta - e_j$ and the conditions become

$$\ell^{\beta-e_j}_\beta(x_{i,\beta_i-1}) = \delta_{ij}$$

which is the complicated version of the linear functional $h^*_{j,\beta-e_j}$(see (1)). From here we obtain the known (see [2]) particular down recursion:

$$B^*_\beta(x) = \sum_{j=0}^{s} h^*_{j,\beta-e_j}(x) B^*_{\beta-e_j}(x). \tag{5}$$

## 4 Formula of de Boor-Fix type and Polar forms

Let us recall Newton's multinomial formula:

$$\langle x, y\rangle^k = \sum_{|\beta|=k} \frac{k!}{\beta!} x^\beta y^\beta, \; x, y \in \mathbb{R}^{s+1}.$$

The function $E(x, y) = \langle x, y\rangle^k$ satisfies all the equivalent assertions of Theorem 1. In this sense, $\{x^\beta\}$ and $\{\frac{k!}{\beta!} y^\beta\}$ are dual bases.

If $P^*, Q^* \in H_k(\mathbb{R}^{s+1})$, writing them in the monomial basis, we obtain

$$P^*(x) = \sum_\beta p^*_\beta \, x^\beta,$$

and for $Q^*$

$$\begin{aligned} Q^*(y) &= \sum q^*_\beta y^\beta \\ &= \sum \frac{\beta!}{k!} q^*_\beta \, \frac{k!}{\beta!} y^\beta. \end{aligned}$$

Hence, the associated bilinear form is given by

$$[P^*, Q^*] = \sum_{|\beta|=k} \frac{\beta!}{k!} \, p^*_\beta \, q^*_\beta. \tag{6}$$

From the Taylor development of $P^*$ around 0,

$$\begin{aligned} P^*(x) &= \sum_{|\beta|\le k} \frac{1}{\beta!} D^\beta P^*(0) x^\beta \\ &= \sum_{|\beta|=k} \frac{1}{\beta!} D^\beta P^*(0) x^\beta \quad (P^* \text{ is homogeneous}), \end{aligned}$$

we obtain

$$
\begin{aligned}
p_\beta^* &= \frac{1}{\beta!} D^\beta P^*(0) \\
&= \frac{1}{\beta!} D^\beta P^*(x) \quad \forall x \in \mathbb{R}^{s+1}.
\end{aligned}
$$

This formula gives us the multivariate *de Boor-Fix* formula proposed in [8].

**Theorem 15** *Multivariate deBoor-Fix formula:*

$$
[P^*, Q^*] = \frac{1}{k!} \sum_{|\beta|=k} \frac{1}{\beta!} D^\beta P^*(x)\, D^\beta Q^*(x)
$$

**Proof.** Starting from (6), we have

$$
\begin{aligned}
\sum_{|\beta|=k} \frac{\beta!}{k!} p_\beta^* q_\beta^* &= \sum_{|\beta|=k} \frac{\beta!}{k!} \cdot \frac{1}{\beta!} D^\beta P^*(x) \cdot \frac{1}{\beta!} D^\beta Q^*(x) \\
&= \frac{1}{k!} \sum_{|\beta|=k} \frac{1}{\beta!} D^\beta P^*(x) D^\beta Q^*(x). \qquad \square
\end{aligned}
$$

Now, a simple argument shows the duality between the B-patch basis and the so-called L-basis ([2], [8]).

**Theorem 16** *The dual basis of* $\{B_\beta^*\}$ *is given by* $\{A_\beta^*\}$*, where*

$$
A_\beta^*(y) = \prod_{x_{\ell,j} \in X_\beta} \langle x_{\ell,j}, y \rangle. \tag{7}
$$

**Proof.** The dual basis $A_\beta^*$ is characterized by

$$
\langle x, y \rangle^k = \sum_{|\beta|=k} A_\beta^*(y) B_\beta^*(x). \tag{8}
$$

But, for $F^*(x) := \langle x, y \rangle^k$, its polar form is the following $k$-linear symmetric functional:

$$
f^*(u_1, \ldots, u_k) = \langle u_1, y \rangle \cdots \langle u_k, y \rangle.
$$

Then $f^*(X_\beta) = A_\beta^*(y)$ and the result is a consequence of Theorem 14 (with $\nu = 0$). $\square$

Note that the definition of $A^*_\beta$ looks simpler than the definition of $B_\beta$ (Def 12). In fact, it is possible to define $A^*_\beta$ first and then characterize $B_\beta$ with (8) [2].

A simple relation between the polar form of $P^*$ and this bilinear form is now possible.

Let us denote

$$U := \{u_1, \ldots, u_k\} \text{ and } \Psi_U(x) := \prod_{j=1}^{k} \langle x, u_j \rangle.$$

**Theorem 17** *The polar form of $P^*$ is given by*

$$p^*(u_1, \ldots, u_k) = [\Psi_U, P^*].$$

**Proof.** $\langle x, u \rangle^k = \sum\limits_{|\beta|=k} B^*_\beta(u) A^*_\beta(x)$. Polarizing with respect to $u$ we obtain:

$$\Psi_U(x) = \prod_{j=1}^{k} \langle x, u_j \rangle = \sum_{|\beta|=k} b^*_\beta(u_1, \ldots, u_k) A^*_\beta(x)$$

and from here,

$$\begin{aligned}
[\Psi_U, P^*] &= [\Psi_U, \sum p^*(X_\beta) B^*_\beta] \\
&= \sum_{|\beta|=k} b^*_\beta(u_1, \ldots, u_k) p^*(X_\beta).
\end{aligned}$$

This latter expression, according to Theorem 14 (witn $\nu = 0$), is the polar form of $P^*$, $p^*(u_1, \ldots, u_k)$ . $\square$

For a homogeneous polynomial it is very simple to obtain the following formula which relates the directional derivative with the polar form:

$$\begin{aligned}
D_h Q^*(x) &:= \lim_{\lambda \to 0} \frac{Q^*(x + \lambda h) - Q^*(x)}{\lambda} \\
&= k\, q^*(x, \ldots, x, h)
\end{aligned}$$

which, for a fixed $h$, is a degree $k-1$ homogeneous polynomial. It is also possible to compute its directional derivative in some other direction. Following this argument, we obtain in general:

$$D^{\nu}_{h^1\dots h^\nu}Q^*(x) = \frac{k!}{(k-\nu)!}q^*(x,\dots,x,h^1,\dots,h^\nu) \tag{9}$$

The following result generalizes the results from [9] about the duality between degree elevation and differentiation. We will denote with $[\cdot,\cdot]_k$ the bilinear form for degree $k$ polynomials.

**Theorem 18** *Let $P^* \in H_{k-\nu}(\mathbb{R}^{s+1})$ and $Q^* \in H_k(\mathbb{R}^{s+1})$. For all directions $h^1,\dots,h^\nu \in \mathbb{R}^{s+1}$ we have*

$$[\Pi_j\langle h^j,x\rangle P^*(x), Q^*(x)]_k = \frac{(k-\nu)!}{k!}[P^*(x), D^{\nu}_{h^1\dots h^\nu}Q^*(x)]_{k-\nu}. \tag{10}$$

**Proof.** Combining (9) with Theorem 17, formula (10) is trivially proved for any polynomial $P^*$ of the form

$$P^*(x) = \langle w^1,x\rangle\cdots\langle w^{k-\nu},x\rangle.$$

But, as these polynomials generate $H_{k-\nu}(\mathbb{R}^{s+1})$, formula (10) is valid over all $H_{k-\nu}(\mathbb{R}^{s+1})$. $\square$

As a Corollary, we obtain Theorem 2 of [9].

**Corollary 19** *For any $h \in \mathbb{R}^{s+1}$*

$$[\langle h,x\rangle P^*(x), Q^*(x)]_k = \frac{1}{k}[P^*(x), D_hQ^*(x)]_{k-1}.$$

Finally, suppose that you are looking for relations of the form

$$D^{\nu}_{h^1\dots h^\nu}B^*_\beta(x) = \sum_{|\alpha|=k-\nu} d_{\alpha\beta}(h^1,\dots,h^\nu)B^*_\alpha(x) \tag{11}$$

which relates the derivative of a B-patch with a lower order B-patch basis. Or relations between degree elevation and L-basis,

$$\Pi^{\nu}_{j=1}\langle h^j,x\rangle A^*_\alpha(x) = \sum_{|\beta|=k} c_{\alpha\beta}(h^1,\dots,h^\nu)A^*_\beta(x). \tag{12}$$

The following result is a generalization of Corollary 3 from [9].

**Corollary 20** *The coefficients of the derivative formula for B-patches (11) and the coefficients of the degree elevation formula for L-basis (12) are related by:*

$$c_{\alpha\beta}(h^1, \ldots, h^\nu) = \frac{(k-\nu)!}{k!} d_{\alpha\beta}(h^1, \ldots, h^\nu). \tag{13}$$

**Proof.**

Starting from (10), we obtain,

$$\begin{aligned}
&[\Pi_j \langle h^j, x\rangle A^*_\alpha(x)\ ,\ B^*_\beta(x)]_k \\
&\qquad = \frac{(k-\nu)!}{k!}[A^*_\alpha(x), D^\nu_{h^1\ldots h^\nu} B^*_\beta(x)]_{k-\nu} \\
&\qquad = \frac{(k-\nu)!}{k!}[A^*_\alpha(x), \sum_{|\alpha'|=k-\nu} d_{\alpha'\beta}(h^1, \ldots, h^\nu) B^*_{\alpha'}(x)]_{k-\nu} \\
&\qquad = \frac{(k-\nu)!}{k!} d_{\alpha\beta}(h^1, \ldots, h^\nu)
\end{aligned}$$

and, from (12), we obtain

$$\begin{aligned}
&[\Pi_j \langle h^j, x\rangle A^*_\alpha(x)\ ,\ B^*_\beta(x)]_k \\
&\qquad = [\sum_{|\beta'|=k} c_{\alpha\beta'}(h^1, \ldots, h^\nu) A^*_{\beta'}(x), B^*_\beta(x)]_k \\
&\qquad = c_{\alpha\beta}(h^1, \ldots, h^\nu). \qquad \square
\end{aligned}$$

Similar relations can be obtained, reversing the roles of $B^*_\beta$ and $A^*_\beta$. More precisely, let

$$D^\nu_{h^1\ldots h^\nu} A^*_\beta(x) = \sum_{|\alpha|=k-\nu} e_{\alpha\beta}(h^1, \ldots, h^\nu) A^*_\alpha(x) \tag{14}$$

and

$$\Pi^\nu_{j=1} \langle h^j, x\rangle B^*_\alpha(x) = \sum_{|\beta|=k} f_{\alpha\beta}(h^1, \ldots, h^\nu) B^*_\beta(x). \tag{15}$$

Again, there is a direct relation between the coefficients of these formulae (see Cor. 4 of [9]).

**Corollary 21** *The coefficients of the derivative formula for L-patches (14) and the coefficients of the degree elevation formula for B-patches (15) are related by:*

$$f_{\alpha\beta}(h^1, \dots, h^\nu) = \frac{(k-\nu)!}{k!} e_{\alpha\beta}(h^1, \dots, h^\nu). \tag{16}$$

## 5 The non-homogeneous case

The vector space of the polynomials in $s$ variables of degree less than or equal to $k$, $\Pi_k(\mathbb{R}^s)$, is isomorphic to $H_k(\mathbb{R}^{s+1})$. One particularly simple isomorphism between these spaces is given by $P^* \in H_k(\mathbb{R}^{s+1}) \longrightarrow P \in \Pi_k(\mathbb{R}^s)$

where $P$ is defined by $P(x_1, \dots, x_s) = P^*(1, x_1, \dots, x_s)$,

that is, by fixing the extra variable $x_0 = 1$. The inverse isomorphism is easily seen to be defined by

$$P^*(x_0, x_1, \dots, x_s) = x_0^k \, P\Big(\frac{x_1}{x_0}, \dots, \frac{x_s}{x_0}\Big) \;\text{ if }\; x_0 \neq 0$$

and $P^*(0, x_1, \dots, x_s) = 0$.

**Definition 22** *In $\Pi_k(\mathbb{R}^s)$ we propose the following bilinear form:*

$$[P, Q]_k := [P^*, Q^*]_k$$

*considering in $H_k(\mathbb{R}^{s+1})$ the previous bilinear form.*

**Theorem 23** *The isomorphism introduced between $H_k(\mathbb{R}^{s+1})$ and $\Pi_k(\mathbb{R}^s)$ maps dual bases on dual bases and the reproducing function associated with this bilinear form is given by*

$$E(x, y) = (1 + \langle x, y \rangle)^k.$$

**Proof.** If $\{P_i^*\}$ and $\{Q_i^*\}$ are dual bases in $H_k(\mathbb{R}^{s+1})$, the duality between bases $\{P_i\}$ and $\{Q_i\}$ is equivalent to the condition $[P_i, Q_j] = \delta_{ij}$, and this comes trivially from Definition 22 of the bilinear form.

Applying Definition 2, with dual bases of $\Pi_k(\mathbb{R}^s)$, we obtain,

$$
\begin{aligned}
E(x,y) &= \sum_{i=0}^{n} P_i(x)\, Q_i(y) \\
&= \sum_{i=0}^{n} P_i^*(1,x)\, Q_i^*(1,y) \\
&= \langle\, (1,x), (1,y)\, \rangle^k \\
&= (1 + \langle x, y\rangle)^k. \qquad \square
\end{aligned}
$$

**Theorem 24** *The bilinear form has the following explicit expression*

$$[F,G]_k = \frac{1}{k!} \sum_{|\beta|\le k} \frac{(k-|\beta|)!}{\beta!}\, D^\beta F(0)\, D^\beta G(0).$$

**Proof.**

Let $F(x) = \sum_{|\beta|\le k} a_\beta\, x^\beta$ and $G(x) = \sum_{|\beta|\le k} b_\beta\, x^\beta$. From our definition of the bilinear form:

$$
\begin{aligned}
[F,G] &= [F^*, G^*] \\
&= \frac{1}{k!} \sum_{|\beta^*|=k} \frac{1}{\beta^*!} D^{\beta^*} F^*(x_0, x)\, D^{\beta^*} G^*(x_0, x)
\end{aligned}
$$

which, writing $\beta^* = (\beta_0, \beta)$ with $\beta_0 = k - |\beta|$, is equivalent to

$$[F,G] = \frac{1}{k!} \sum_{|\beta|\le k} \frac{1}{\beta_0!\beta!} D^{(\beta_0,0,\ldots,0)}\, D^{(0,\beta)} x_0^k\, F(\frac{x}{x_0})\, D^{(\beta_0,0,\ldots,0)}\, D^{(0,\beta)} x_0^k\, G(\frac{x}{x_0}).$$

Observe that $x_0^k\, F(\frac{x}{x_0})$ is homogeneous of degree $k$, so the derivative reduces to one monomial term, which is also true for $x_0^k\, G(\frac{x}{x_0})$. We obtain the following formula:

$$[F,G] = \frac{1}{k!} \sum_{|\beta|\le k} \beta_0!\beta! a_\beta\, b_\beta$$

which is equivalent to the expression of the theorem. $\square$

## 6 Future work

Some other reproducing functions in the multivariate polynomial case can be considered, and also, other isomorphisms between homogeneous and non-homogeneous polynomial spaces.

The main research in this topic is the piecewise polynomial case. The continuity condition between adjacent patches can be combined with results similar to Theorem 18. It is then very promising to follow the line of [15] and [6] where the polar form of simplex splines and cone splines are studied in an algebraic setting. The goal is to produce a de Boor-Fix formula (dual functionals) for multivariate splines.

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