Collision of a solid with an incompressible fluid

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Abstract

We give a predictive theory of the collisions of a viscous incompressible fluid with solids. The theory is based on interior percussions which account for the very large stresses and contact forces resulting from the cinematic incompatibilities responsible for the collision. New equation of motion and constitutive laws result from the theory. Examples dealing with a fluid colliding with its container and with a diver impacting the water of a swimming pool are studied.

1 Introduction

Consider an incompressible fluid in a closed vertical tube. Let us turn the tube rapidly upside down. Then the liquid falls and collides with the bottom of the tube: either it bounces or remains in contact with the bottom. Another interesting situation is a swimmer diving in a swimming pool. When he impacts the water, the diver can be horizontal and do a belly flop. In this situation a violent collision occurs with the water. The third example consists of a flat boat in a rough sea. According to the creation of the waves, the boat rise up and in a small time interval its bottom hits the water. In each of these problems a solid collides with a viscous incompressible fluid.

The collision theory has mainly been developed for solids. The traditional approach (see for example, [11], [2]) is based on the coefficients of restitution which is appropriated in simple situations but may be in contradiction with the basic principles of mechanics in more complex setups [8], [1], [3], [2].

A general framework which deal with the collision of rigid bodies has been developed in [3], [4], [5], [6]. The basic ideas are to consider that the collisions are instantaneous and that the system made of two rigid bodies is deformable because their relative position changes with respect to time.

In this approach, when there is a collision, interior generalized forces appear which are referred as interior percussions. These percussions are interior

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quantities, determined by constitutive laws depending on the relative velocities before and after the collision.

The principal goal of this paper is to apply the above described ideas, in order to obtain a macroscopic predictive theory of collisions of solids with viscous incompressible fluids.

The hypothesis of instantaneousness of collisions is compatible with the fluid incompressibility. Further, in our case, in addition to the percussion on the contact surface between the fluid and solids there are also percussion stresses inside the fluid when the collision occurs. At this time, the velocity field is a discontinuous function with respect to time. There exist a velocity field before and after the collision. We denote them respectively as $\overrightarrow{U}^-$ and $\overrightarrow{U}^+$.

The first and second laws of thermodynamics lead to the assumption of the percussion stress depending on the strain rate $D(\overrightarrow{U}^+ + \overrightarrow{U}^-)$ where $D_{ij}(\overrightarrow{U}) = \frac{1}{2}(U_{i,j} + U_{j,i})$ is the usual strain rate. In the same way the percussion contact force depends on the relative velocity $\overrightarrow{U} - \overrightarrow{U}_s$ of the fluid with respect to the solid ($\overrightarrow{U}_s$ denotes the solid velocity).

Choosing the simplest constitutive law, i.e., linear constitutive laws, results in a set of partial differential equations which gives the velocities $\overrightarrow{U}^+$ and $\overrightarrow{U}_s^+$ after the collision depending on the velocities $\overrightarrow{U}^-$ and $\overrightarrow{U}_s^-$ before the collision. This set of partial differential equations is coherent in term of both mechanical and mathematical point of view. Moreover the two examples which are investigated below, show that the theory takes into accounts the basic physical properties.

The paper is organized as follows: first the equations of motions are derived from the principle of virtual work; then the constitutive laws are chosen in order to fulfill the second law of thermodynamics requirements. Examples are given: collision of a fluid with the bottom of a tube is described with a closed form solution. Then the diver problem in a swimming pool is investigated: an existence theorem is proved and numerical results are presented.

2 The equation of motion

We investigate the motion of the solid and fluid in a time interval where there is only one collision at time t_0. We denote by t_1 any time before t_0 and denote by t_2 any time after t_0.

When a solid collides with water very large internal forces appear because of the incompatibility of the velocities of the solid and water: very large stresses inside and very large forces on the contact surface. In many circumstances the duration of the violent contact is very short compare to the duration of the free motion. For instance when skipping stones on the still water of a lake, the duration of the contact of the stone with the water is small compare to its time flight. We decide as a schematisation to consider that the collisions are instantaneous. Thus the velocities are discontinuous functions of the time: for the fluid, there is the velocity $\overrightarrow{U}^-(\overrightarrow{x}, t_0)$ before the collision at time t_0 and the velocity $\overrightarrow{U}^+(\overrightarrow{x}, t_0)$ after the collision; for the solid, there is the velocity $\overrightarrow{U}_s^-(\overrightarrow{x}, t_0) = \vec{X}^-(t_0) + \overrightarrow{\varpi}^-(t_0) \times (\overrightarrow{x} - \overrightarrow{x}_G(t_0))$ before the collision and the velocity $\overrightarrow{U}_s^+(\overrightarrow{x}, t_0) = \vec{X}^+(t_0) + \overrightarrow{\varpi}^+(t_0) \times (\overrightarrow{x} - \overrightarrow{x}_G(t_0))$ after the collision. The velocity of the center of mass $\overrightarrow{x}_G(t_0)$ of the rigid body is $\vec{X}(t)$ and $\overrightarrow{\varpi}(t)$ is its rotation velocity. Because the collision is instantaneous, the very large internal forces are also concentrated in time: thus they become percussions.

2.1 The principle of virtual work

It is usual to derive the equation of motion through the principle of virtual power, [7]. This principle can be used when all the quantities have densities with respect to the Lebesgue measure. When it is not the case as in the present situation, the principle of virtual power can be avantagously replaced by the principle of virtual work. The principle is:

the virtual work of the acceleration forces between the times t_1 and t_2 is equal to the sum of the virtual work of the interior forces between the times t_1 and t_2 and of the virtual work of the exterior forces between the times t_1 and t_2.

Remark 1 *The principle of virtual work we use here is not to be confused with the principle of virtual power where the velocities are understood as small displacements (This relation is often also called as the principle of virtual work).*

We will use this principle defining first the virtual velocities. As already said, the actual velocities can be discontinuous in the evolution of the fluid. Thus the virtual velocities are to be discontinuous functions of the time. In mathematical terms, it is wise to assume that the velocities are functions of bounded variation with respect to the time. With respect to the space, it is assumed that the velocities have the usual properties (for instance, they belong to convenient Sobolev spaces). The different virtual works are linear functions of the virtual velocities. They involve regular forces and stresses together with percussions and percussion stresses. The regular forces and stresses are quantities which have a density with respect to the Lebesgue measure. The percussions and percussion stresses can be thought as time concentrated quantities which define an atomic measure (for instance a Dirac measure) and intervene only at collisions times.

Let us consider the system made of the fluid and solids. For instance, the solids are the container of the fluid and a moving obstacle (Fig. 4). For the sake of simplicity, we suppose there is only one solid (for instance the container). The fluid occupies the domain $\Omega(t)$ and is in contact with the solid on the part $\Gamma_1(t)$ of its boundary $\partial\Omega(t)$. At an arbitrary time t, we consider the virtual velocities $\left(\vec{V}(\vec{x}, t), \vec{Y}(t), \overrightarrow{\pi}(t)\right)$ defined on $\Omega(t)$. The velocity $\vec{Y}(t)$ is a virtual velocity of the center of mass of the rigid body and $\overrightarrow{\pi}(t)$ is a virtual rotation velocity of the rigid body. It is denoted by $\overrightarrow{V}_s(\overrightarrow{x}, t) = \overrightarrow{Y}(t) + \overrightarrow{\pi}(t) \times (\overrightarrow{x} - \overrightarrow{x}_G(t))$, the virtual velocity at point $\overrightarrow{x}$ of the rigid body. The mass of the rigid body is M and its mass moment of inertia is I. The density of the fluid is ρ.

2.1.1 The virtual work of the acceleration forces

Because the velocities are bounded variation functions, the quantities $\overrightarrow{V}^{+}(\vec{x},t)$ and $\overrightarrow{V}^{-}(\vec{x},t)$ are defined at any time $t' \in [t_1,t_2]$.

We choose the virtual work of the acceleration forces as

$$
\begin{aligned}
W^{acc}(t_1,t',t_2,\vec{V},\vec{Y},\overrightarrow{\pi}) = \int_{\Omega(t')} \rho \left[\overrightarrow{U}(\vec{x},t')\right] \cdot \frac{\overrightarrow{V}^{+}(\vec{x},t') + \overrightarrow{V}^{-}(\vec{x},t')}{2} d\Omega \\
+ \int_{t_1}^{t_2} \int_{\Omega} \rho \frac{d\overrightarrow{U}}{dt}(\vec{x},\tau) \cdot \vec{V}(\vec{x},\tau) d\Omega d\tau \\
+ M\left[\vec{X}(t')\right] \cdot \frac{\vec{Y}^{+}(t') + \vec{Y}^{-}(t')}{2} + M \int_{t_1}^{t_2} \frac{d\vec{X}}{dt}(\tau) \cdot \vec{Y}(\tau) d\tau \\
+ I\left[\overrightarrow{\varpi}(t')\right] \cdot \frac{\overrightarrow{\pi}^{+}(t') + \overrightarrow{\pi}^{-}(t')}{2} + I \int_{t_1}^{t_2} \frac{d\overrightarrow{\varpi}}{dt}(\tau) \cdot \overrightarrow{\pi}(\tau) d\tau,
\end{aligned}
$$

where $[A] = A^{+} - A^{-}$ is the discontinuity of the quantity A. Let us note that the actual work of acceleration forces with $t' = t_0$ is the variation of the kinetic energy $K(t)$ between times t_1 and t_2

$$
K(t_2) - K(t_1) = W^{acc}(t_1, t_{0,} t_2, \overrightarrow{U}, \vec{X}, \overrightarrow{\varpi}).
$$

2.1.2 The virtual work of the interior forces

The virtual work of the interior forces we choose, is a linear function of the virtual strain rate, $D_{ij}(\vec{V}) = (1/2)(V_{i,j} + V_{j,i})$, and of the difference of the velocities on the contact surface, $\vec{V} - \overrightarrow{V}_s$. The densities with respect to the Lebesgue measure are the stress tensor σ and the interaction force $\overrightarrow{r}$ between the fluid and the solid. The densities with respect to the Dirac measure are the percussion stress tensor Σ and the interaction percussion $\overrightarrow{R}$ between the fluid and the solid. The percussion stress and the interaction percussion are generalized interior forces which appear when collisions occur. They are, as said earlier, usual interior forces concentrated in a very short period of time. The virtual work of the interior forces we choose, is

$$
\begin{aligned}
W^{int}(t_1,t',t_2,\vec{V},\vec{Y},\overrightarrow{\pi}) = -\int_{\Omega(t')} \Sigma(\overrightarrow{x},t') : \frac{D(\overrightarrow{V}^{+})(\overrightarrow{x},t') + D(\overrightarrow{V}^{-})(\overrightarrow{x},t')}{2} d\Omega \\
- \int_{\Gamma_1(t')} \overrightarrow{R}(\overrightarrow{x},t') \cdot \frac{\overrightarrow{V}^{+}(\overrightarrow{x},t') - \overrightarrow{V}_s^{+}(\overrightarrow{x},t') + \overrightarrow{V}^{-}(\overrightarrow{x},t') - \overrightarrow{V}_s^{-}(\overrightarrow{x},t')}{2} d\Gamma \\
- \int_{t_1}^{t_2} \int_{\Omega} \sigma(\overrightarrow{x},\tau) : D(\vec{V})(\overrightarrow{x},\tau) d\Omega d\tau \\
- \int_{t_1}^{t_2} \int_{\Gamma_1(\tau)} \overrightarrow{r}(\overrightarrow{x},\tau) \cdot (\vec{V}(\overrightarrow{x},\tau) - \overrightarrow{V}_s(\overrightarrow{x},\tau)) d\Gamma d\tau.
\end{aligned}
$$

The work of the interior forces has to satisfy the axiom of the interior forces, [7]:

the work of the interior forces is zero for any rigid system virtual velocities $(\vec{V}(\vec{x},t), \vec{Y}(t), \overrightarrow{\pi}(t))$.

The rigid system velocities are such that $D(\vec{V}) = 0$ and $\vec{V} - \overrightarrow{V_s} = 0$ on the boundary Γ_1. Thus it is obvious that $W^{int}(t_1, t_2, \vec{V}, \vec{Y}, \overrightarrow{\pi}) = 0$ for such a set of velocities $(\vec{V}, \overrightarrow{Y}, \overrightarrow{\pi})$.

2.1.3 The virtual work of the exterior forces

The virtual work of the exterior forces involves the possibility to apply both the classical forces and percussions. The chosen virtual work of the exterior forces is given by

$$
\begin{aligned}
W^{ext}(t_1, t', t_2, \vec{V}, \vec{Y}, \overrightarrow{\pi}) = & \int_{\partial\Omega(t')} \overrightarrow{G}^{ext}(\overrightarrow{x}, t') \cdot \frac{\overrightarrow{V}^{+}(\overrightarrow{x}, t') + \overrightarrow{V}^{-}(\overrightarrow{x}, t')}{2} d\Gamma \\
& + \overrightarrow{H}^{ext}(t') \cdot \frac{\overrightarrow{V}_s^{+}(\overrightarrow{x}_H, t') + \overrightarrow{V}_s^{-}(\overrightarrow{x}_H, t')}{2} \\
& + \int_{t_1}^{t_2} \int_{\Omega} \overrightarrow{f}^{ext}(\overrightarrow{x}, \tau) \cdot \vec{V}(\overrightarrow{x}, \tau) d\Omega d\tau \\
& + \int_{t_1}^{t_2} \int_{\partial\Omega} \overrightarrow{g}^{ext}(\overrightarrow{x}, \tau) \cdot \vec{V}(\overrightarrow{x}, \tau) d\Omega d\tau \\
& + \int_{t_1}^{t_2} \overrightarrow{h}^{ext}(\tau) \cdot (\vec{Y}(\tau) + \overrightarrow{\pi}(\tau) \times (\overrightarrow{x}_h(\tau) - \overrightarrow{x}_G(\tau))) d\tau,
\end{aligned}
$$

where $\overrightarrow{h}^{ext}(\tau)$ is a regular exterior force applied at the point $\overrightarrow{x}_h(\tau)$ of the rigid body; $\overrightarrow{H}^{ext}(t')$ is an exterior percussion applied at the point $\overrightarrow{x}_H$ of the rigid body; $\overrightarrow{f}^{ext}(\overrightarrow{x}, \tau)$ and $\overrightarrow{g}^{ext}(\overrightarrow{x}, \tau)$ are exterior regular volume and surface forces applied to the fluid; $\overrightarrow{G}^{ext}(\overrightarrow{x}, t')$ is an exterior surface percussion applied to the fluid. The percussions $\overrightarrow{H}^{ext}(t')$ and $\overrightarrow{G}^{ext}(\overrightarrow{x}, t')$ are functions of t' which are 0 if $t' \neq t_0$.

2.2 The equations of motion

They result from the principle of virtual work

$$
\begin{aligned}
& \forall (t_1, t', t_2, \vec{V}, \vec{Y}, \overrightarrow{\pi}), \\
W^{acc}(t_1, t', t_2, \vec{V}, \vec{Y}, \overrightarrow{\pi}) \quad = \quad & W^{int}(t_1, t', t_2, \vec{V}, \vec{Y}, \overrightarrow{\pi}) + W^{ext}(t_1, t', t_2, \vec{V}, \vec{Y}, \overrightarrow{\pi}).
\end{aligned}
$$

It is easy to get, *almost everywhere in time*

$$\rho\frac{d\overrightarrow{U}}{dt}=\operatorname{div}\sigma+\overrightarrow{f}^{ext},\ in\ \Omega,$$
$$\sigma\overrightarrow{N}=-\overrightarrow{r}+\overrightarrow{g}^{ext},\ in\ \Gamma_1,$$
$$\sigma\overrightarrow{N}=\overrightarrow{g}^{ext},\ in\ \partial\Omega\backslash\Gamma_1,$$
$$M\frac{d\vec{X}}{dt}=\int_{\Gamma_1}\overrightarrow{r}d\Gamma+\overrightarrow{h}^{ext},$$
$$\frac{d}{dt}(I\vec{\omega})=\int_{\Gamma_1}(\vec{x}-\overrightarrow{x}_G)\times\vec{r}d\Gamma+(\vec{x}_h-\overrightarrow{x}_G)\times\overrightarrow{h}^{ext}, \tag{1}$$

where the outwards normal vector to the fluid is $\overrightarrow{N}$ (Fig. 4). In case there are more than one solid, the boundary conditions and the last two equations are valid for each rigid body, i.e., the container and the obstacle (of course the mechanical characteristics, M, I, are different). These are the classical equations of motion when the fluid, and the solid evolve smoothly;

and at any time t, for instance at time t_0 when there is the collision,

$$\rho\left[\overrightarrow{U}(\vec{x},t_0)\right]=\operatorname{div}\Sigma(\vec{x},t_0),\ almost\ everywhere\ in\ \Omega(t_0),$$
$$\Sigma(\vec{x},t_0)\overrightarrow{N}=-\vec{R}(\vec{x},t_0)+\vec{G}^{ext}(\vec{x},t_0),\ almost\ everywhere\ in\ \Gamma_1(t_0),$$
$$\Sigma(\vec{x},t_0)\overrightarrow{N}=\vec{G}^{ext}(\vec{x},t_0),\ almost\ everywhere\ in\ \partial\Omega(t_0)\backslash\Gamma_1(t_0),$$
$$M[\vec{X}(t_0)]=\int_{\Gamma_1(t_0)}\vec{R}(\vec{x},t_0)d\Gamma+\vec{H}^{ext}(t_0),$$
$$I[\vec{\omega}(t_0)]=\int_{\Gamma_1(t_0)}(\vec{x}-\vec{x}_G(t_0))\times\vec{R}(\vec{x},t_0)d\Gamma+(\vec{x}_H-\vec{x}_G(t_0))\times\vec{H}^{ext}(t_0).$$

The boundary conditions and the last two equations are valid for any solid if more than one are in contact with the fluid.

Constitutive laws are needed for $(\sigma,\overrightarrow{r})$ and $(\Sigma,\overrightarrow{R})$ in order to solve the smooth and non smooth equations. They result from experiments and are restricted by a basic relation, namely the second principle of thermodynamics.

3 The constitutive laws

Because we intend to deal with incompressible fluids, we assume the temperature T to be constant and the internal energy and entropy to be also constant with respect to the time.

3.1 The energy balance

The energy balance is

$$K(t_2)-K(t_1)=W^{ext}(t_1,t_{0,}t_2,\overrightarrow{U},\vec{X},\overrightarrow{\varpi})+Q^{ext}(t_1,t_2), \tag{2}$$

where $K(t)$ is the kinetic energy and $Q^{ext}(t_1,t_2)$ is the heat received from the exterior between times t_1 and t_2. It is decided that, if a collision occurs at times t_2, $K(t_2)$ is the kinetic energy after the collision and that $W^{ext}(t_1,t_2,t_2,\overrightarrow{U},\vec{X},\overrightarrow{\varpi})$ and $Q^{ext}(t_1,t_2)$ involve the mechanical and thermal effects due to the collision (the other choice gives the same result). The principle of virtual work with the actual velocities gives

$$K(t_2)-K(t_1)=W^{int}(t_1,t_0,t_2,\overrightarrow{U},\vec{X},\overrightarrow{\varpi})+W^{ext}(t_1,t_0,t_2,\overrightarrow{U},\vec{X},\overrightarrow{\varpi}). \quad (3)$$

By subtracting the previous equations (2) and (3) it results

$$0=-W^{int}(t_1,t_0,t_2,\overrightarrow{U},\vec{X},\overrightarrow{\varpi})+Q^{ext}(t_1,t_2). \quad (4)$$

3.2 The second law of thermodynamics

The second law of thermodynamics in our situation is

$$0\geq\frac{Q^{ext}(t_1,t_2)}{T}. \quad (5)$$

This relation shows that a collision produces heat. The quantity of the heat produced is given by the energy balance.

The previous relations (4) and (5) give

$$W^{int}(t_1,t_0,t_2,\overrightarrow{U},\vec{X},\overrightarrow{\varpi})\leq 0,$$

or

$$\begin{aligned}&-\int_{\Omega(t_0)}\Sigma(\overrightarrow{x},t_0):\frac{D(\overrightarrow{U}^+)(\overrightarrow{x},t_0)+D(\overrightarrow{U}^-)(\overrightarrow{x},t_0)}{2}d\Omega\\&-\int_{\Gamma_1(t_0)}\overrightarrow{R}(\overrightarrow{x},t_0)\cdot\frac{\overrightarrow{U}^+(\overrightarrow{x},t_0)-\overrightarrow{U}_s^+(\overrightarrow{x},t_0)+\overrightarrow{U}^-(\overrightarrow{x},t_0)-\overrightarrow{U}_s^-(\overrightarrow{x},t_0)}{2}d\Gamma\\&-\int_{t_1}^{t_2}\int_{\Omega}\sigma(\overrightarrow{x},\tau):D(\overrightarrow{U})(\overrightarrow{x},\tau)d\Omega d\tau\\&-\int_{t_1}^{t_2}\int_{\Gamma_1(\tau)}\overrightarrow{r}(\overrightarrow{x},\tau)\cdot(\overrightarrow{U}(\overrightarrow{x},\tau)-\overrightarrow{U}_s(\overrightarrow{x},\tau))d\Gamma d\tau\leq 0,\end{aligned} \quad (6)$$

in any actual evolution.

The relation (6) is a guide to choose the constitutive laws. It is used in the following paragraph to choose the constitutive laws for Σ and $\overrightarrow{R}$.

3.3 The constitutive laws for an incompressible fluid

The constitutive laws we choose have to satisfy the relation (6). For the smooth part, the constitutive laws are that of the incompressible viscous fluid together with the boundary conditions describing the unilateral interactions of the fluid with the solid, for instance its container

$$\sigma = 2\mu D(\overrightarrow{U}) - pI, \ \mathrm{div}\overrightarrow{U} = 0, \tag{7}$$

where μ is the viscosity and p is the pressure. It gives the regular Navier-Stokes equations. The surface interaction force is divided into a dissipative force $\overrightarrow{r}^d$, taking into account all the friction phenomenons, and a reaction force $\overrightarrow{r}^{reac}$, imposing the impenetrability

$$\overrightarrow{r} = \overrightarrow{r}^d + \overrightarrow{r}^{reac}.$$

For the dissipative force we take the simplest case: a linear function

$$\overrightarrow{r}^d = k(\overrightarrow{U} - \overrightarrow{U}_s), \text{with } k \geq 0, \tag{8}$$

where the physical quantity k accounts for the friction of the liquid on the solid. The case $k = \infty$ refers the case when the fluid remains in contact with its container, i.e., $\overrightarrow{U} = \overrightarrow{U}_s$. The case $k = 0$ describes the situation where there is no interaction between the fluid and the solid except their non impenetrability.

We choose the reaction force $\overrightarrow{r}^{reac}$ normal to the contact surface

$$\overrightarrow{r}^{reac} = r^{reac}\overrightarrow{N}, \ r^{reac} \in \partial I_-(U_n - U_{ns}), \tag{9}$$

where the outwards normal vector to the fluid is $\overrightarrow{N}$ (Fig. 4) and $U_n = \overrightarrow{U} \cdot \overrightarrow{N}$, $U_{ns} = \overrightarrow{U}_s \cdot \overrightarrow{N}$ are the normal velocities. The values of the impenetrability reaction r^{reac} belong to the subdifferential set ∂I_- of the indicator function I_- of the set of negative numbers R^-, defined by (see for example [10], Fig. 1):

$$\partial I_-(U_n - U_{ns}) = \begin{cases} \{0\}, & \text{if } U_n - U_{ns} < 0, \\ R^+, & \text{if } U_n - U_{ns} = 0, \\ \emptyset, & \text{if } U_n - U_{ns} > 0. \end{cases} \tag{10}$$

For the non smooth part, the inequality (6) and the expression of W^{int} suggest to assume Σ to depend on $D(\overrightarrow{U}^+ + \overrightarrow{U}^-)$. This important assumption is very simple. It is possible to make more sophisticated assumptions, for instance that Σ depends separately on $D(\overrightarrow{U}^+)$ and $D(\overrightarrow{U}^-)$. It is to be seen with the examples that the previous simple assumption is sufficient to take account of the basic physical results. Thus we choose

$$\Sigma = 2\gamma D(\overrightarrow{U}^+ + \overrightarrow{U}^-) - PI, \ \mathrm{div}(\overrightarrow{U}^+ + \overrightarrow{U}^-) = 0, \ \text{with } \gamma \geq 0.$$

The incompressibility condition $\mathrm{div}(\overrightarrow{U}^+ + \overrightarrow{U}^-) = 0$ or $\mathrm{div}\overrightarrow{U}^+ = 0$ because $\mathrm{div}\overrightarrow{U}^- = 0$, results in the percussion pressure P.

The interaction percussion $\vec{R}$ between the solid and the fluid has to take into account their impenetrability. Thus the percussion is divided into a reaction

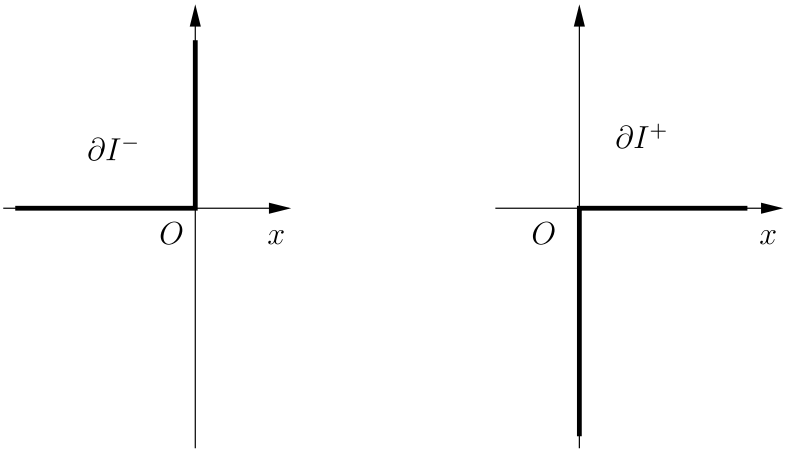

Figure 1: The subdifferential set $\partial I^-(x)$ of the indicator function $I^-(x)$ of the set of the negative numbers R^- and the subdifferential set $\partial I^+(x)$ of the indicator function $I^+(x)$ of the set of the positive numbers R^+.

$\vec{R}^{reac}$ to the impenetrability constraint and a dissipative percussion $\vec{R}^d$ which modelises all the dissipative interactions between the fluid and the solid

$$\overrightarrow{R} = \overrightarrow{R}^d + \overrightarrow{R}^{reac}.$$

The inequality (6) and the expression of W^{int} suggest to assume $\overrightarrow{R}^d$ to depend on $(\overrightarrow{U}^+(\vec{x},t_0) - \overrightarrow{U_s}^+(\vec{x},t_0) + \overrightarrow{U}^-(\vec{x},t_0) - \overrightarrow{U_s}^-(\vec{x},t_0))$. Let us repeat that more sophisticated choices may be made but we consider the relation (6) as a guide and that the basic choice of a linear function for $\overrightarrow{R}^d$ to satisfy the inequality is simple but crucial. Thus we choose

$$\overrightarrow{R}^d(\vec{x},t_0) = K(\overrightarrow{U}^+(\vec{x},t_0) - \overrightarrow{U_s}^+(\vec{x},t_0) + \overrightarrow{U}^-(\vec{x},t_0) - \overrightarrow{U_s}^-(\vec{x},t_0)),\ \text{with } K \geq 0.$$

Of course the physical parameter K depends on the nature of the colliding solid and fluid.

We choose to have the impenetrability reaction to be active only when the risk of interpenetration is present, i.e., when the contact is persistent after the collision. Thus we choose the constitutive law

$$\overrightarrow{R}^{reac} = R^{reac}\vec{N},$$
$$\text{with } R^{reac}(\vec{x},t_0) \in \partial I_-(U_n^+(\vec{x},t_0) - U_{ns}^+(\vec{x},t_0)), \tag{11}$$

where ∂I_- is the subdifferential set of the indicator function I_- of the set of the negative numbers R^- (the set ∂I_- is defined in the formulas (10), Fig. 1). To prove that the inequality (6) is satisfied, one uses the fact that the physical parameters μ, γ, k and K are positive and the property of the impenetrability reaction given in the following lemma

Lemma 2 *The impenetrability reaction (11) satisfies,*

$$\overrightarrow{R}^{reac}(\vec{x},t_0)\cdot(\overrightarrow{U}^+(\vec{x},t_0)-\overrightarrow{U_s}^+(\vec{x},t_0)+\overrightarrow{U}^-(\vec{x},t_0)-\overrightarrow{U_s}^-(\vec{x},t_0))\geq 0. \quad (12)$$

Proof. Let us recall that $U_n=\overrightarrow{U}\cdot\overrightarrow{N}$ and $U_{sn}=\overrightarrow{U_s}\cdot\overrightarrow{N}$ are the normal velocities. The relation (11) proves that the subdifferential set $\partial I(U_n^+(\vec{x},t_0)-U_{ns}^+(\vec{x},t_0))$ is not empty and thus that the impenetrability condition

$$U_n^+(\vec{x},t_0)-U_{ns}^+(\vec{x},t_0)\leq 0,$$

is satisfied. If $U_n^+(\vec{x},t_0)-U_{sn}^+(\vec{x},t_0)<0$ the properties of the subdifferential sets show that $R^{reac}=0$ and the relation (12) is satisfied. If $U_n^+(\vec{x},t_0)-U_{sn}^+(\vec{x},t_0)=0$, the properties of the subdifferential sets show that $R^{reac}\geq 0$. Because $U_n^-(\vec{x},t_0)-U_{sn}^-(\vec{x},t_0)\geq 0$, see the Fig. 4, $\overrightarrow{R}^{reac}(x,t_0)\cdot(\overrightarrow{U}^+(\vec{x},t_0)-\overrightarrow{U_s}^+(\vec{x},t_0)+\overrightarrow{U}^-(\vec{x},t_0)-\overrightarrow{U_s}^-(\vec{x},t_0))=R^{reac}(U_n^-(\vec{x},t_0)-U_{ns}^-(\vec{x},t_0))\geq 0$. Thus the relation (12) is satisfied. ■

Let us emphasizes that the impenetrability condition (11) has two meanings. The first one is to imply that $U_n^+-U_{sn}^+\leq 0$ and the second one is to give the value of the reaction R^{reac} which is 0 when contact is not maintained after the collision and which is positive when contact is maintained after the collision. Now, a direct consequence of the lemma and of the constitutive laws is that the inequality (6) is satisfied:

Theorem 3 *In any actual evolution, the inequality (6) is satisfied.*

At this point the predictive theory is completed. It is to be checked if it is consistent in terms of mathematics, numerics and if it has the ability to account for the more obvious experimental results. Let us again repeat that we have decided to choose the simplest constitutive laws which satisfy the basic requirements of mechanics. The collision constitutive laws are characterized only by two parameters γ which takes into account the volume phenomenons occurring in a collision and K which describes the surface interaction between the surface of the solid and the fluid. We are convinced that this is the minimal number of informations which are needed to predict what occur after a collision.

4 An incompressible fluid in a tube

Consider a fluid in a tube and turn the tube upside down rapidly. The fluid falls and collides with the bottom of the tube (Fig. 2). We intend to describe this collision.

We assume the velocity of the tube is zero before and after the collision and that the velocity of the fluid before the collision is vertical and homogeneous: $\overrightarrow{U}^-=(0,0,U^-)$, where U^- is constant with $U^-\leq 0$. The equations of motion of the fluid in the collision are

$$\rho\left[\overrightarrow{U}\right]=\mathrm{div}\Sigma, \quad (13)$$

$$\mathrm{div}\overrightarrow{U}^+=0, \quad (14)$$

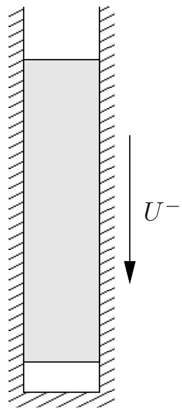

Figure 2: The bottom of the tube and the falling liquid just before the collision.

in the volume. The boundary condition on the bottom and the vertical sides are

$$\Sigma\overrightarrow{N} = -\overrightarrow{R}^d - P^{reac}\overrightarrow{N}, \tag{15}$$

The boundary condition on the top of the fluid is

$$\Sigma\overrightarrow{N} = 0. \tag{16}$$

The constitutive law are given by

$$\begin{aligned} \Sigma &= 2\gamma D(\overrightarrow{U}^+ + \overrightarrow{U}^-) - PI, \\ \overrightarrow{R}^d &= K(\overrightarrow{U}^+ + \overrightarrow{U}^-),\ P^{reac} \in \partial I_-(U_n^+). \end{aligned}$$

They are the equations which relate $\overrightarrow{U}^+$, the unknown velocity after the collision, to $\overrightarrow{U}^-$, the velocity before the collision. In order to find a closed form solution we look for the velocity $\overrightarrow{U}^+$ vertical. It results from (14) that the vertical velocity is constant

$$\overrightarrow{U}^+ = (0, 0, U^+),\ \text{where } U^+ \text{is constant.}$$

The equations (13) give

$$\rho\overrightarrow{U}^+ - 2\gamma\Delta\overrightarrow{U}^+ + \nabla P = \rho\overrightarrow{U}^- + 2\gamma\Delta\overrightarrow{U}^-, \tag{17}$$

which reduces to

$$\rho\overrightarrow{U}^+ + \nabla P = \rho\overrightarrow{U}^-.$$

This equation gives the percussion pressure

$$P(x, y, z) = P_0 - \rho(U^+ - U^-)z,$$

where P_0 is a constant. The boundary condition on the top (16) gives

$$P_0 = \rho(U^+ - U^-)L, \tag{18}$$

where L is the height of the fluid column. The boundary condition on the bottom (15) gives

$$-P_0\overrightarrow{N} = -K(\overrightarrow{U}^+ + \overrightarrow{U}^-) - P^{reac}\overrightarrow{N},$$

and replacing P_0 from (18) we obtain

$$\rho(U^+ - U^-)L = -K\left(U^+ + U^-\right) + P^{reac},$$

which is the equation giving U^+, because P^{reac} depends on U^+. This dependance is $P^{reac} \in \partial I_-(U_n^+) = \partial I_-(-U^+) = -\partial I_+(U^+)$, where ∂I_+ is the subdifferential set of the indicator function of the positive numbers R^+, (Fig. 1). It gives the following equation

$$(\rho L - K)U^- \in (K + \rho L)U^+ + \partial I_+(U^+). \tag{19}$$

The multivoque function $f(x) = (K + \rho L)x + \partial I_+(x)$, is a maximal monotone

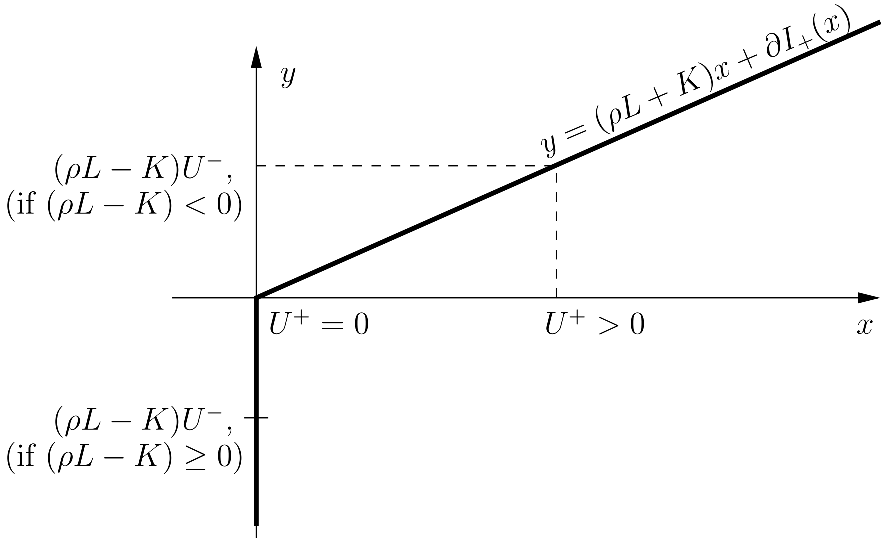

Figure 3: The graph $f(x) = (K + \rho L)x + \partial I_+(x)$ is monotone and maximal. The equation $f(x) = (\rho L - K)U^-$ has one and only one solution. When there is a large quantity of fluid, $\rho L - K > 0$, the fluid does not bounce on the bottom of the tube. When there is only a droplet, $\rho L - K < 0$, it bounces.

operator, as we can see in Fig. 3, then for any given U^- there exist a unique

U^+ solution of (19): if $K > \rho L$, that is, if there is a small amount of fluid, the solution is

$$U^+ = -\frac{K - \rho L}{K + \rho L} U^-, \text{ with } P^{reac} = 0.$$

Otherwise, if there is a large amount of fluid, $K \leq \rho L$, the solution is

$$U^+ = 0, \text{ with } P^{reac} = (K - \rho L)U^- \geq 0.$$

We conclude that a droplet of fluid bounces on the bottom of the tube but a large quantity of fluid does not bounces.

5 The diver problem

For the sake of simplicity a 2D problem is investigated. It is assumed that the swimming pool is a fixed rectangle. The horizontal swimmer Γ_1 dives at the middle of the swimming pool (Fig. 4). We assume that the diver is symmetric, flat (Fig. 4) and its thickness is zero. Its velocities when hitting the water are

$$\vec{X}^- = (0, X^-),$$
$$\vec{\varpi}^- = \vec{0}.$$

It is assumed that no external load is applied. It result from the symmetries that the solution

$$\vec{U}^+(x,y) = \big(U_1^+(x,y), U_2^+(x,y)\big)$$

is such that $\vec{U}^+(-x,y) = \big(-U_1^+(x,y), U_2^+(x,y)\big)$ and that $\vec{X}^+ = (0, X^+)$ and $\vec{\varpi}^+ = \vec{0}$. All the points of the rigid body have the same vertical velocity, which is the velocity of the middle of Γ_1.

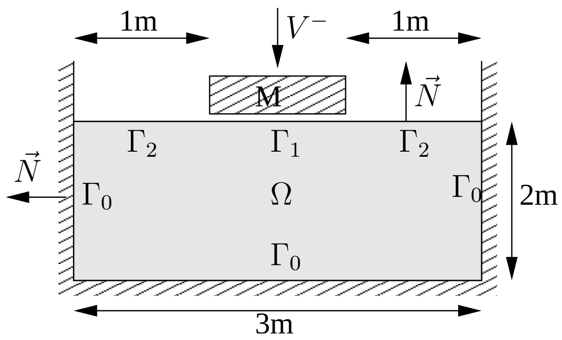

Figure 4: The diver and the swimming pool. The normal velocityy of the diver before the collision is negative, $U_n^- - U_{ns}^- = -X^- \leq 0$.

5.1 The equations

Let Ω be the swimming pool, an open subset in R^2 with piecewise $\mathcal{C}^2$ boundary with outwards normal vector $\overrightarrow{N}$, (Fig. 4). This domain contains a viscous homogeneous incompressible fluid. We suppose that at time t_0 the velocity field is known and equal to $\overrightarrow{U}^-(\vec{x}, t_0) = 0$. At this time t_0, the diver schematized by a rigid body, collides the part Γ_1 of the boundary $\partial\Omega$. We suppose that the remaining boundary $\partial\Omega \setminus \Gamma_1$ is decomposed into two additional disjoint parts: a rigid boundary Γ_0 and a free boundary Γ_2. We are interested in finding the velocities $\overrightarrow{U}^+(\vec{x}, t_0)$ and $\vec{X}^+(t_0)$ after the collision. The equations follow from the equations of motion

$$\rho\overrightarrow{U}^+ = \mathrm{div}\Sigma, \tag{20}$$

$$M[\vec{X}(t_0)] = \int_{\Gamma_1} \vec{R}_1(\vec{x}, t_0) d\Gamma, \tag{21}$$

$$\Sigma\overrightarrow{N} = -\vec{R}_0(\vec{x}, t_0), \text{ on } \Gamma_0, \tag{22}$$

$$\Sigma\overrightarrow{N} = -\vec{R}_1(\vec{x}, t_0), \text{ on } \Gamma_1, \tag{23}$$

$$\Sigma n\overrightarrow{N} = 0 \text{ on } \Gamma_2, \tag{24}$$

and the constitutive laws

$$\begin{gathered}\Sigma = 2\gamma D(\overrightarrow{U}^+) - PI,\\ \vec{R}_0 = K_0\overrightarrow{U}^+ + R_0^{reac}\overrightarrow{N},\ R_0^{reac} \in \partial I_-(U_n^+),\\ \vec{R}_1 = K_1(\overrightarrow{U}^+ - \overrightarrow{X}^+ - \overrightarrow{X}^-) + R_1^{reac}\overrightarrow{N},\ R_1^{reac} \in \partial I_-(U_n^+ - X^+),\end{gathered} \tag{25}$$

where $U_n^+ - X^+$ is the normal velocity on Γ_1 and U_n^+ is the normal velocity on Γ_0. Both normal velocities are negative due to the impenetrability condition.

5.2 The variational formulation

The set of partial differential equations (20)–(25) gives the velocities $(\overrightarrow{U}^+, \overrightarrow{X}^+)$ after the collision depending on the velocities $(\overrightarrow{U}^-, \overrightarrow{X}^-)$ before the collision.

In this subsection we show that the equations fit in the variational inequalities theory and that they have a unique solution.

We begin by defining the functional spaces framework. For the fluid velocity field we define the space

$$V = \left\{\vec{W} \in (H^1(\Omega))^2 : \mathrm{div}\vec{W} = 0\right\},$$

where $H^1(\Omega)$ is the usual Sobolev space. For the coupled system of fluid-rigid body we define the following convex cone for the cinematically admissible velocities

$$C = \left\{(\vec{W}, Y) \in V \times R : \vec{W} \cdot \overrightarrow{N} \leq 0 \text{ on } \Gamma_0, \vec{W} \cdot \overrightarrow{N} - Y\overrightarrow{N} \leq 0 \text{ on } \Gamma_1\right\}.$$

Let us consider $(\vec{W}, Y) \in C$, multiplying (20) by the function $\vec{W} - \overrightarrow{U}^+$ and (21) by the function $Y - X^+$, we prove that any classical solution must satisfy the equation

$$\begin{aligned}
&\int_\Omega \rho \overrightarrow{U}^+ \cdot (\vec{W} - \overrightarrow{U}^+) d\Omega + M(X^+ - X^-)(Y - X^+) \\
&+ \int_\Omega 2\gamma D(\overrightarrow{U}^+) : D(\vec{W} - \overrightarrow{U}^+) d\Omega + \int_{\Gamma_0} K_0 \overrightarrow{U}^+ \cdot (\vec{W} - \overrightarrow{U}^+) d\Gamma \\
&+ \int_{\Gamma_1} K_1 (\overrightarrow{U}^+ - X^+ \overrightarrow{N} - X^- \overrightarrow{N}) \cdot (\vec{W} - Y \overrightarrow{N} - (\overrightarrow{U}^+ - X^+ \overrightarrow{N})) d\Gamma \\
&= \int_{\Gamma_0} -R_0^{reac} (W_n - U_n^+) d\Gamma - \int_{\Gamma_1} R_1^{reac} \{W_n - Y - (U_n^+ - X^+)\} d\Gamma.
\end{aligned}$$

Due to the properties of the subdifferential sets, we have

$$\forall (\vec{W}, Y) \in C,\ R_0^{reac}(W_n - U_n^+) \leq 0,\ R_1^{reac}\{W_n - Y - (U_n^+ - X^+)\} \leq 0.$$

It follows

$$\begin{aligned}
&(\overrightarrow{U}^+, X^+) \in C,\ \forall (\vec{W}, Y) \in C, \\
&\int_\Omega \rho \overrightarrow{U}^+ \cdot (\vec{W} - \overrightarrow{U}^+) d\Omega + M(X^+ - X^-)(Y - X^+) \\
&+ \int_\Omega 2\gamma D(\overrightarrow{U}^+) : D(\vec{W} - \overrightarrow{U}^+) d\Omega + \int_{\Gamma_0} K_0 \overrightarrow{U}^+ \cdot (\vec{W} - \overrightarrow{U}^+) d\Gamma \\
&+ \int_{\Gamma_1} K_1 (\overrightarrow{U}^+ - X^+ \overrightarrow{N} - X^- \overrightarrow{N}) \cdot (\vec{W} - Y \overrightarrow{N} - (\overrightarrow{U}^+ - X^+ \overrightarrow{N})) d\Gamma \geq 0. \quad (26)
\end{aligned}$$

Conversely, it can be proved that a solution of the variational inequality (26), if it is smooth enough, satisfies the equation of motion (20)–(24) together with the constitutive laws (25).

5.2.1 An existence theorem

Let us define some notations

$$\begin{aligned}
a(\overrightarrow{V}, \overrightarrow{W}) &= \int_\Omega \rho \overrightarrow{V} \cdot \overrightarrow{W} d\Omega + \int_\Omega 2\gamma D(\overrightarrow{V}) : D(\overrightarrow{W}) d\Omega + \int_{\Gamma_0} K_0 \overrightarrow{V} \cdot \overrightarrow{W} d\Gamma, \\
b(Y, Z) &= MYZ, \\
c(\overrightarrow{V}, \overrightarrow{W}) &= \int_{\Gamma_1} K_1 \overrightarrow{V} \cdot \overrightarrow{W} d\Gamma, \\
L(Y) &= M(X^-)Y, \\
L_1(\overrightarrow{W}) &= \int_{\Gamma_1} K_1 (X^- \overrightarrow{N}) \cdot \overrightarrow{W} d\Gamma.
\end{aligned}$$

The equations (26) can be rewritten

$$
\begin{gathered}
(\overrightarrow{U}^+, X^+) \in C, \ \forall(\vec{W}, Y) \in C, \\
a(\overrightarrow{U}^+, \vec{W} - \overrightarrow{U}^+) + b(X^+, Y - X^+) + c(\overrightarrow{U}^+ - X^+\overrightarrow{N}, \vec{W} - Y\overrightarrow{N}) \geq \\
L(Y - X^+) + L_1(\vec{W} - \overrightarrow{U}^+) - L_1(Y\overrightarrow{N} - X^+\overrightarrow{N}).
\end{gathered} \tag{27}
$$

In this form, it is a variational inequality. It can be proved that this variational inequality (27) or (26) has an unique solution:

Theorem 4 *The variational inequality (27) has an unique solution.*

Proof. The bilinear form

$$
A((\overrightarrow{W}, Y), (\overrightarrow{V}, Z)) = a(\overrightarrow{W}, \overrightarrow{V}) + b(Y, Z) + c(\overrightarrow{W}, \overrightarrow{V}),
$$

is symmetric, continuous and coercive on $V \times R$. The linear form

$$
B(((\overrightarrow{W}, Y)) = L(Y) + L_1(\vec{W}) - L_1(Y\overrightarrow{N}),
$$

is continuous on the same space. The set C is a non-empty closed convex subset of $V \times R$. It follows from the Lions-Stampacchia [9] theorem that the variational inequality (27) equivalent to

$$
\begin{gathered}
(\overrightarrow{U}^+, X^+) \in C, \ \forall(\vec{W}, Y) \in C, \\
A((\overrightarrow{W}, Y) - (\overrightarrow{U}^+, X^+), (\overrightarrow{U}^+, X^+)) \geq B((\overrightarrow{W}, Y) - (\overrightarrow{U}^+, X^+)),
\end{gathered}
$$

has a unique solution ■

5.3 Numerical results

Let us consider the example of a diver in a swimming pool. Knowing the situation before the collision we compute the velocities of diver and water after the collision. These velocities are the initial velocities to describe the smooth evolution after the collision described by equations (1) and constitutive laws (7)–(9).

The diver has the mass $M = 100Kg$ and falls vertically from a height of $1m$ with vertical velocity $U^- = \sqrt{2g} = 4.47m/s$. The density of the water is $\rho = 1000Kg/m^3$. The parameter defining the collision constitutive laws are

$$
K_0 = K_1 = 1N/m^2, \ \gamma = 0.25N/m.
$$

The equations may be solved by the classical numerical methods for variational inequalities, for instance the Uzawa method. In the present situation it is better to take advantage of the fact that the solution is unique: the equations are solved with the assumption that the contact is maintained on the parts Γ_0 and Γ_1 and it is checked that the reaction computed with the boundary conditions are negative (in case they are positive somewhere the solution is such that the contact is not

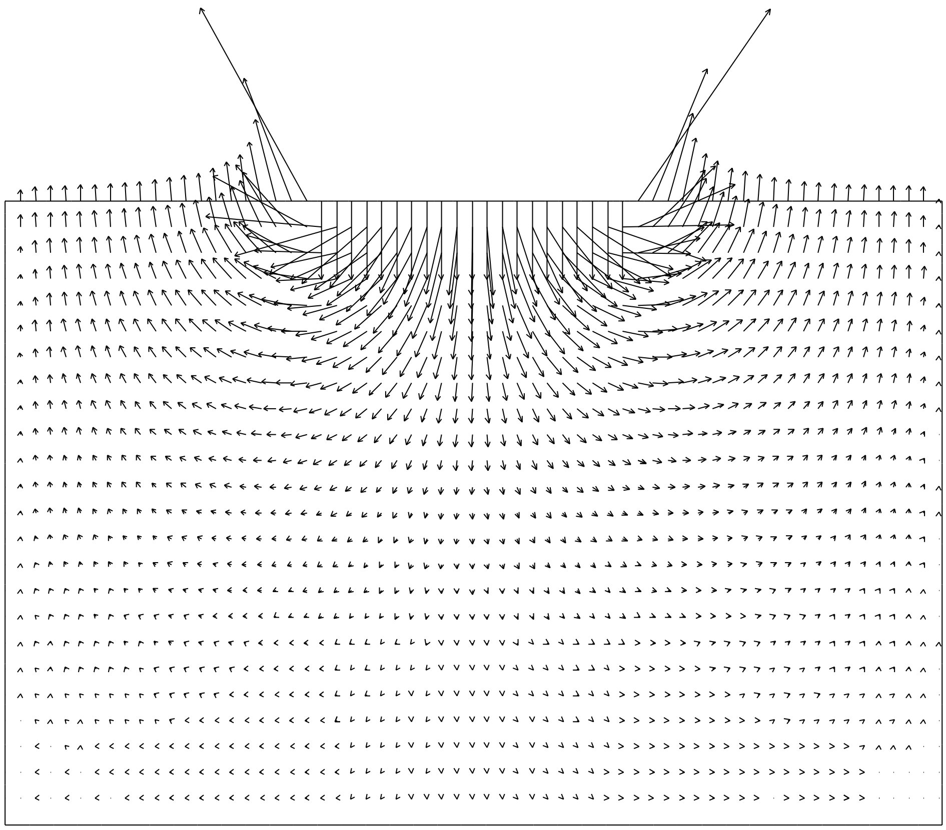

Figure 5: The velocity of the water which splashes up on the two sides of the diver.

maintained everywhere). The water splashes up on the two sides of the diver as shown on the Fig. 5. The velocity of the diver after the collision is the velocity of the water ($-0.75\,ms^{-1}$), in agreement with every day experiments!

Details are shown on the following Figs. 6 and 7: the velocity is drawn at different depth of the pool. The intensity of the vertical velocity is shown on Fig. 8. The maximum horizontal velocity is $1.96m/s$. The maximum vertical velocity is $1.95m/s$ and the minimum is $-0.75m/s$.

Remark 5 *The percussion pressure is shown on the Fig. 9. The average percussion stress* $\Sigma\overrightarrow{N}$ *under the diver is* $-373\;Nm^2s^{-1}$ *and the average percussion pressure is* $371\;Nm^2s^{-1}$.

6 Conclusions

The theory of collisions of fluids and solids outlined in this article is consistent from the mechanical point of view. It has good mathematical formulations to

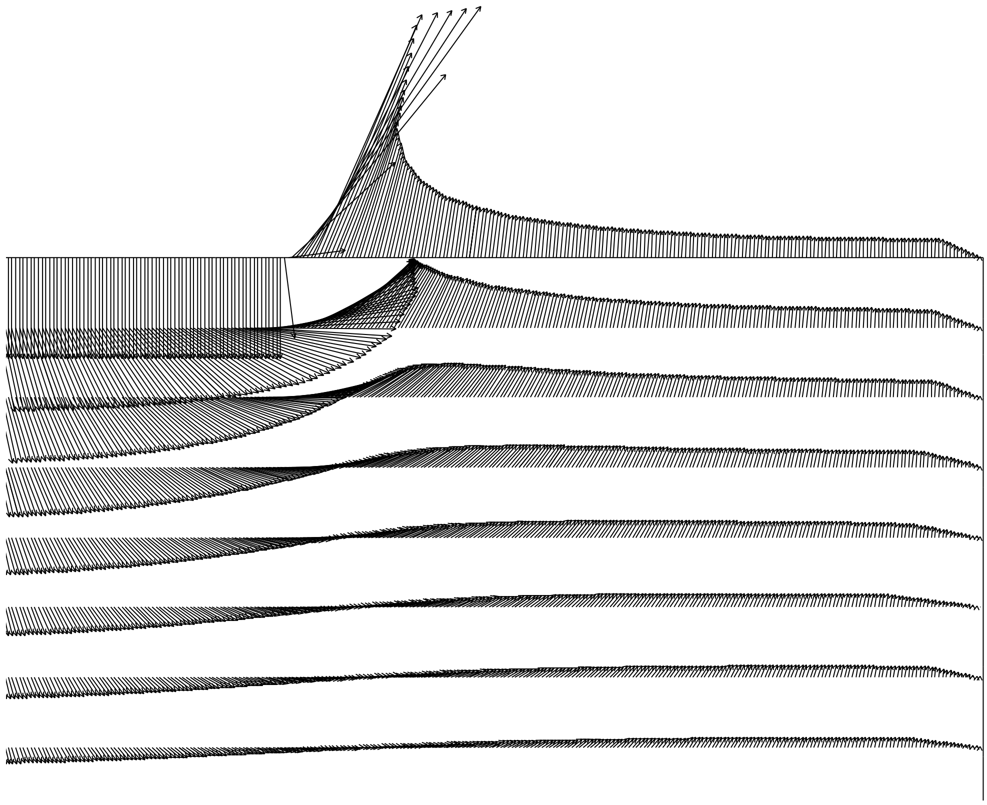

Figure 6: The velocity of the water at different depths.

which usual numerical methods apply. The examples given in Sections 4 and 5 show that the basic physical properties are taken into account.

Let us emphasize that the basic idea which is to take into account the interior forces when a collision occurs that capture and sum up the very sophisticated physical phenomenons which intervene during the short time of the collision. The interior forces are defined by duality i.e., by their work. In this setting the choice of the work of the interior forces in collisions is very crucial as well as the choice of the average strain rate $D(\overrightarrow{U}^+ + \overrightarrow{U}^-)$ to define the percussion stress.

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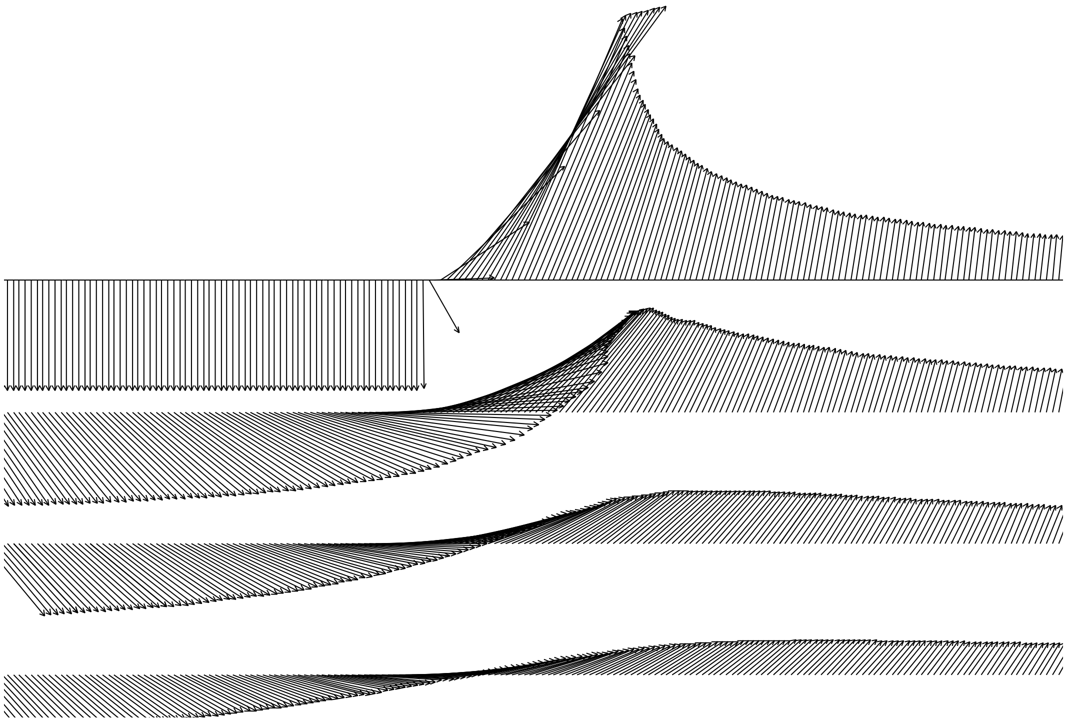

Figure 7: Details of the velocity of the water at different depths.

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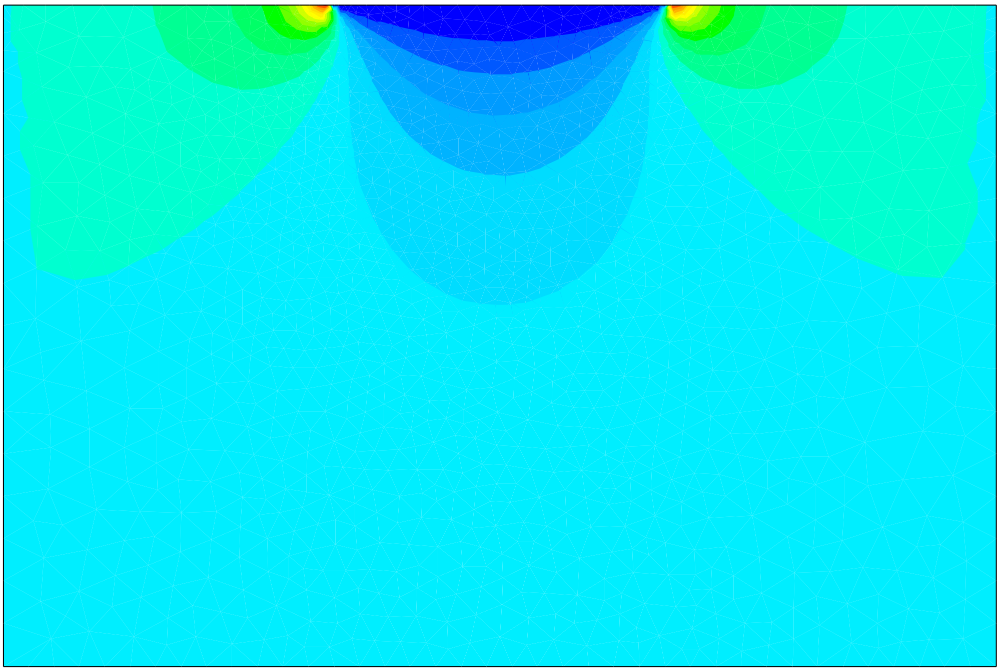

Figure 8: The intensity of the vertical velocity. Red is upwards velocity. Blue is downwards velocity. Dark blue is large downwards velocities, light blue is low downwards velocity.

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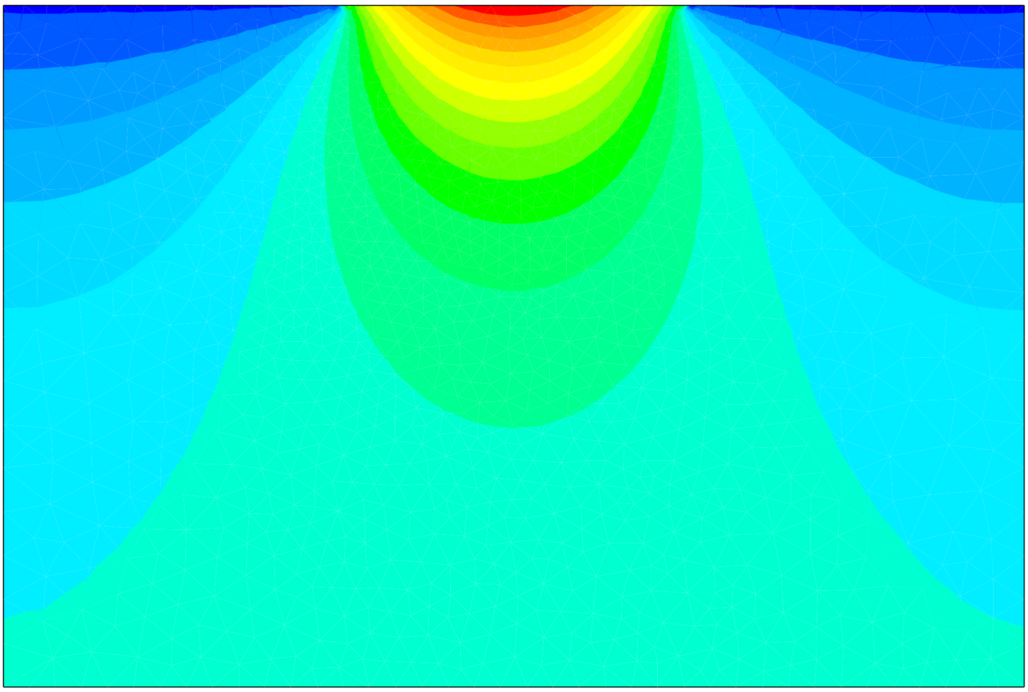

Figure 9: The pressure in the water. Red is a high pressure and blue a low pressure.