ON THE CONVERGENCE OF B-PATCH SUBDIVISION

Raúl Gormaz†
Universidad de Chile
Facultad de Ciencias Físicas y Matemáticas
Departamento de Ingeniería Matemática
Casilla 170/3 Correo 3 Santiago, Chile
rgormaz@dim.uchile.cl

Charles A. Micchelli‡
Mathematical Sciences Department
IBM Research Division
T.J. Watson Research Center
Yorktown Heights, NY 10598 U.S.A.
cam@watson.ibm.com

Abstract. We provide error bounds which establish convergence of subdivision under repeated conversion of B-patch representation.

§1. B-patch Representation

B-patches for surface representation were introduced by Seidel [7] as a generalization of Bernstein-Bézier representation. He showed that they are the only polynomial basis which provide a symmetric pyramid scheme for polynomial representation. Therefore they are intimately connected to the notion of *Blossoming* introduced into *Geometric Modeling* by Ramshaw [6]. The connection of B-patches to the multivariate B-spline was made by Dahmen, Micchelli and Seidel in [4] and using this connection a new spline space spanned by B-splines was introduced and studied. This has also led to the axiomatic study of B-splines via blossoming, by Seidel and Vermeulen in [8].

The first discussion of *subdivision* by B-patches was given in Cavaretta and Micchelli [1] where the focus was on the iterates of certain matrices constructed as permanents of associated matrices. For this reason the presentation in [1] was limited to certain types of subdivision of B-patches. The purpose of this paper is to improve upon these preliminary results of [1] on B-patch subdivision and present a complete convergence analysis of B-patch subdivision. Unlike the approach taken by Cohen and Schumaker [2] and Dahmen [3] we do not use *stability estimates* and *Schoenberg operators* for our analysis, but rather rely upon the use of blossoming. This seems to be more natural in

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the present setting and besides stability estimates for B-patches of the type needed are not readily available. Nevertheless, the Schoenberg operator based on B-patches does already appear in Cavaretta and Micchelli [1].

We begin our discussion by reviewing the definition of B-patches of degree n on $\mathbb{R}^d$. Our preference is to introduce the polynomials dual (relative to the binomial theorem) to B-patches, as was done in [1]. To this end, we recall the definition of *lineal polynomial* associated with a set X of n vectors (which may not be distinct) in $\mathbb{R}^d$.

Definition 1.1. *Let $X = \{x^1, \dots, x^n\}$ be a collection of column vectors (not necessarily distinct) in $\mathbb{R}^d$. Then the associated (affine) lineal polynomial $a(\cdot|X)$ associated with X is given by*

$$a(x|X) = \prod_{j=1}^{n}(1 + x^T x^j), \quad x \in \mathbb{R}^d. \tag{1.1}$$

Each lineal polynomial, $a(\cdot|X)$, $\#X = n$ is a polynomial of (total) degree n and hence in $\pi_n(\mathbb{R}^d)$. A fundamental question about lineal polynomials is given a collection

$$\mathcal{C} \subseteq \{X : \ X \subset \mathbb{R}^d, \ \#X = n\}$$

such that $\#\mathcal{C} = \#\pi_n(\mathbb{R}^d) = \binom{n+d}{d}$ when is $\operatorname{span}\{a(\cdot|X) : X \in \mathcal{C}\} = \pi_n(\mathbb{R}^d)$? Some information about this question is given in Cavaretta and Micchelli [1] but more understanding of this problem is needed.

For our purpose here it is important to recall the method of generating collections of "histograms" from a rectangular array in $\mathbb{R}^d$ for which the corresponding set of lineal polynomials span $\mathbb{R}^d$.

Let

$$\mathcal{X} = \{x^{i,j} : i = 0, \dots, d, \quad j = 0, 1, \dots, n\} \tag{1.2}$$

be a rectangular array in $\mathbb{R}^d$. For every lattice vector $\alpha = (\alpha_0, \dots, \alpha_d) \in \mathbb{Z}_+^{d+1}$ we associate the subset (histogram)

$$X_\alpha = \{x^{i,j} : i = 0, 1, \dots, d, \ j = 0, \dots, \alpha_i - 1\}. \tag{1.3}$$

When some coordinate of α is zero, for instance, $\alpha_r = 0$, $0 \le r \le d$, we interpret the set (1.3) as not having vectors from the rth *cloud* $\{x^{r,j} : j = 0, 1, \dots, n\}$. Since $\#X_\alpha = |\alpha| := \alpha_0 + \cdots + \alpha_d$ we see that $a_\alpha(\cdot|\mathcal{X}) := a(\cdot|X_\alpha) \in \pi_n(\mathbb{R}^d)$ when $|\alpha| = n$.

It was shown in Cavaretta and Micchelli [1] that whenever for every $|\alpha| \le n-1$ the set $\{x^{i,\alpha_i};\ i = 0, 1, \dots, d\}$ is affinely independent then

$$\operatorname{span}\{a_\alpha(\cdot|\mathcal{X}) : |\alpha| = n\} = \pi_n(\mathbb{R}^d).$$

When this property of $\mathcal{X}$ holds we call $\mathcal{X}$ a *regular configuration*. Therefore for regular configurations $\mathcal{X}$ there is a set of linearly independent polynomials $\{b_\alpha(x|\mathcal{X}) : |\alpha| = n\}$, called *B-patches*, such that

$$(1 + x^T y)^n = \sum_{|\alpha|=n} b_\alpha(x|\mathcal{X}) a_\alpha(y|\mathcal{X}), \ x, y \in \mathbb{R}^d. \tag{1.4}$$

Recall that for every $P \in \pi_n(\mathbb{R}^d)$ there is a unique symmetric function $p(x^1, \ldots, x^n)$, $x^1, \ldots, x^n \in \mathbb{R}^d$ such that in each variable $x^i, 1 \leq i \leq n$ it is affine and moreover

$$p(x, x, \ldots, x) = P(x) \quad x \in \mathbb{R}^d. \tag{1.5}$$

The function p is called the *blossom* or *polar form* of P. From (1.4) it easily follows that

$$P(x) = \sum_{|\alpha|=n} p(X_\alpha) b_\alpha(x|\mathcal{X}), \quad x \in \mathbb{R}^d \tag{1.6}$$

where $p(X_\alpha)$ is interpreted as

$$p(X_\alpha) := p(x^{0,0}, \ldots, x^{0,\alpha_0-1}, x^{1,0}, \ldots, x^{1,\alpha_1-1}, \ldots, x^{d,0}, \ldots, x^{d,\alpha_d-1}). \tag{1.7}$$

We will make use of this important identity later for our analysis of B-patch subdivision. To prepare for that we develop an inequality between P and its blossom p in the next section.

§2. Blossom Inequality

For $x^1, \ldots, x^n \in \mathbb{R}^d$ the function

$$P\left(\frac{1}{n}\left(\sum_{j=1}^{n} x^j\right)\right) \tag{2.1}$$

is symmetric and agrees with $P(x)$ when $x^1 = \cdots = x^n = x$ but *is not* affine in its variables. Therefore, it is not the blossom of p. Nonetheless, as we shall see the difference between these two functions can be conveniently bounded from above. For this purpose, we set for every compact set $K \subset \mathbb{R}^d$ and function f in $W^2_\infty(\mathbb{R}^d)$, the space of all functions which are continuously differentiable and have second derivatives which exist a.e. and are bounded,

$$\|D^2 f\|_K := \sup\left\{\frac{\partial^2 f(x)}{\partial x_i \partial x_j} : x = (x_1, \ldots, x_d) \in K, i, j = 1, \ldots, d\right\}.$$

The matrix

$$H_f(x) := \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right)_{i,j=1,\ldots,d}$$

is called the Hessian of f at x. Also, we let $\|x\|_1 := \sum_{i=1}^{d} |x_i|, x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ be the ℓ^1-norm of x.

Theorem 2.1. *Let K be any compact convex subset of $\mathbb{R}^d$ and $P \in \pi_n(\mathbb{R}^d)$ have blossom p. Then for every $x^1, \ldots, x^n \in K$ we have*

$$\left| P\left(\frac{1}{n}\sum_{j=1}^{n} x^j\right) - p(x^1, \ldots, x^n)\right| \le \frac{n^{n-2}}{n!} 2^{n-3} \|D^2 P\|_K \left(\sum_{i=1}^{n} \|x^i - \frac{1}{n}\sum_{j=1}^{n} x^j\|_1\right)^2 . \tag{2.2}$$

For any compact set $K \subset \mathbb{R}^d$ we let

$$d_1(K) := \max\{\|x - y\|_1 : \; x, y \in K\}$$

be the diameter of K (relative to the ℓ_1 norm on $\mathbb{R}^d$). Since, for every $x^1, \ldots, x^n \in \mathbb{R}^d$

$$\begin{aligned}
\sum_{i=1}^{n} \|x^i - \frac{1}{n}\sum_{j=1}^{n} x^j\|_1 &\le \frac{1}{n}\sum_{i=1}^{n} \|\sum_{j=1}^{n}(x^i - x^j)\|_1 \\
&\le \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \|x^i - x^j\|_1 \\
&\le (n-1) d_1\left(\{x^1, \ldots, x^n\}\right)
\end{aligned}$$

we get from Theorem 2.1.

Corollary 2.1. *Let K be any compact convex subset of K and $P \in \pi_n(\mathbb{R}^d)$ with blossom p. Then for every $x^1, \ldots, x^n \in K$ we have*

$$\left| P\left(\frac{1}{n}\sum_{j=1}^{n} x^j\right) - p(x^1, \ldots, x^n)\right| \le \frac{(n-1)^2 n^{n-2}}{n!} 2^{n-3} \|D^2 P\|_K (d_1(\{x^1, \ldots, x^n\})^2.$$

Proof (of Theorem 2.1). We consider the error function

$$E(x^1, \ldots, x^n) := P(\frac{1}{n}(x^1 + \cdots + x^n)) - p(x^1, \ldots, x^n)$$

on $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$, n-times, and will develop it in a Taylor series in its variables. We also sometimes write

$$E(x_1^1, \ldots, x_d^1, x_1^2, \ldots, x_d^2, \ldots, x_1^n, \ldots, x_d^n) \tag{2.3}$$

for $E(x^1, \ldots, x^n)$ where $x^i = (x_1^i, \ldots, x_d^i), i = 1, \ldots, n$.

For every $x \in \mathbb{R}^d$ it follows from (1.5) that $E(x, x, \ldots, x) = 0$. Also, E is a symmetric function of its arguments since both p and (2.1) are likewise symmetric functions. Therefore it readily follows that

$$\left(\frac{\partial E}{\partial x_s^\ell}\right)(x, x, \ldots, x) = 0, \;\; \ell = 1, \ldots, n, \;\; s = 1, \ldots, d. \tag{2.4}$$

These equations imply that for every $x \in \mathbb{R}^d$ there is a $y \in K \times \cdots \times K$, n-times such that

$$E(x^1, \dots, x^n) = \frac{1}{2} w^T H_E(y) w \tag{2.5}$$

where $w = \mathbb{R}^d \times \cdots \times \mathbb{R}^d$, n-times, is the vector

$$w = (x^1 - x, \dots, x^n - x)^T := (w^1, \dots, w^n)^T \tag{2.6}$$

and $H_E(y)$ is the Hessian of E at y.

To make use of this equation, we will in turn, separately compute the quadratic form of the Hessian of the function $P(\frac{1}{n}(x^1+\cdots+x^n))$ and $p(x^1, \dots, x^n)$. In the first instance, we observe that for

$$\bar{P}(x^1, \dots, x^n) := P(\frac{1}{n}(x^1 + \cdots + x^n)) \tag{2.7}$$

we have

$$\left(\frac{\partial \bar{P}}{\partial x_s^\ell}\right)(x^1, \dots, x^n) = \frac{1}{n} \frac{\partial P(\bar{x})}{\partial x_s}, \ \ell = 1, \dots, n, \ s = 1, \dots, d \tag{2.8}$$

where

$$\bar{x} := \frac{1}{n}(x^1 + \cdots + x^n). \tag{2.9}$$

Similarly, we have

$$\left(\frac{\partial^2 \bar{P}}{\partial x_r^\ell \partial x_s^m}\right)(x^1, \dots, x^n) = \frac{1}{n^2} \frac{\partial^2 P(\bar{x})}{\partial x_r \partial x_s}, \ r, s = 1, \ \dots, d, \ \ell, m = 1, \dots, n \tag{2.10}$$

and so we obtain

$$w^T H_{\bar{P}}(y) w = \bar{w}^T H_P(\bar{y}) \bar{w}^T$$

where

$$\bar{w} := \left(\frac{1}{n}\sum_{j=1}^n x^j - x, \dots, \frac{1}{n}\sum_{j=1}^n x^j - x\right).$$

We now choose $x = \bar{x}$ and conclude that for this choice

$$w^T H_{\bar{P}}(y) w = 0. \tag{2.11}$$

Thus we obtain from (2.5) that

$$E(x^1, \dots, x^n) = -\frac{1}{2} w^T H_p(y) w. \tag{2.12}$$

Next, we estimate the Hessian of p in terms of the Hessian of P itself. For this purpose, we use the following formula for p taken from Ramshaw [6]. That is, for $x^1, \dots, x^n \in \mathbb{R}^d$

$$p(x^1, \dots, x^n) = \sum_{\substack{S \subseteq \{1, \dots, n\} \\ S \neq \phi}} (-1)^{n - \#S} \frac{(\#S)^n}{n!} P\left(\frac{1}{\#S}\sum_{i \in S} x^i\right). \tag{2.13}$$

Hence, differentiating both sides of (2.13), for $m, \ell = 1, \dots, n; r, s = 1, \dots, \ell$ with $m \neq \ell$ and $x^1, \dots, x^n \in K$ we obtain the inequality

$$\left| \frac{\partial^2 p(x^1, \dots, x^n)}{\partial x_r^\ell \partial x_p^m} \right| \leq \frac{\|D^2 P\|_K}{n!} \sum_{\substack{S \subseteq \{1, \dots, n\} \\ \ell, m \in S}} (\#S)^{n-2}.$$

(Note that for $\ell = m$ the partial derivative corresponding second derivatives of p above are zero.) Consequently, we get

$$\|D^2 p\|_{K^n} \leq \frac{n^{n-2}}{n!} 2^{n-2} \|D^2 P\|_K$$

where $K^n := K \times \cdots \times K$, n-times. Combining this bound with (2.12) gives for $x^1, \dots, x^n \in K$ the inequality

$$\left| p(x^1, \dots, x^n) - P\left(\frac{1}{n} \sum_{j=1}^n x^j \right) \right| \leq \frac{n^{n-2}}{n!} 2^{n-3} \|w\|_1^2 \|D^2 P\|_K.$$

In view of (2.6) and our choice $x = \bar{x}$ we have

$$\|w\|_1 = \sum_{i=1}^n \|x^i - \frac{1}{n} \sum_{j=1}^n x^j \|_1$$

which prove the result. □

§3. Subdivision for B-Patches and DMS Splines

The previous result has immediate application to subdivision of B-patches and also of the spline space introduced by Dahmen, Micchelli and Seidel in [4]. We begin with B-patches.

Let $\{\mathcal{X}^k\}_{k \in \mathbb{Z}_+}$ be a collection of subsets of $\mathbb{R}^d$ such that each $\mathcal{X}^k$ is a regular configuration for $k \in \mathbb{Z}_+$. We begin with a $P \in \pi_n(\mathbb{R}^d)$ expressed in B-patch form relative to the "initial" configuration $\mathcal{X}^0$

$$P(x) = \sum_{|\alpha|=n} c_\alpha^0 b_\alpha(x|\mathcal{X}^0), \quad x \in \mathbb{R}^d. \tag{3.1}$$

Next, we successively represent P in terms of the successive configurations $\mathcal{X}^k$, $k = 1, \dots$ that is,

$$P(x) = \sum_{|\alpha|=n} c_\alpha^k b_\alpha(x|\mathcal{X}^k), \quad x \in \mathbb{R}^d. \tag{3.2}$$

We let X_α^k, $|\alpha| = n$, be a typical histogram formed from elements in $\mathcal{X}^k$. As before, we let $\bar{X}_\alpha^k$ represent the average of the elements in X_α^k and let

$$d_1(K) = \max\{\|x - y\|_1 : x, y \in K\} \tag{3.3}$$

be the diameter of a compact set $K \subset \mathbb{R}^d$ (relative to the ℓ_1-norm).

Theorem 3.1. *For every* $|\alpha| = n, \mathcal{X}^k \subseteq K$ $k = 0, 1, \dots$ *where* K *is given compact convex subset of* $\mathbb{R}^d$ *we have*

$$|P(\bar{X}_\alpha^k) - c_\alpha^k| \leq \frac{(n-1)^2 n^{n-2}}{n!} 2^{n-3} \|D^2 P\|_K (d_1(\mathcal{X}^k))^2. \tag{3.4}$$

Proof. According to (1.6) we see that

$$c_\alpha^k = p(X_\alpha^k), \ |\alpha| = n, \ k \in \mathbb{Z}_+$$

and hence Corollary 2.1 gives the estimate

$$|P(\bar{X}_\alpha^k) - c_\alpha^k| \leq (n-1)^2 \frac{n^{n-2}}{n!} 2^{n-3} \|D^2 P\|_K (d_1(X_\alpha^k))^2$$

but since $X_\alpha^k \subseteq \mathcal{X}^k, |\alpha| = n$ we get (3.4). $\square$

Theorem 3.1. establishes quadratic convergence of B-patch subdivision when $\lim_{k\to\infty} d_1(X^k) = 0$. As a special case, suppose T is a contractive affine map on $\mathbb{R}^d$ with (unique) fixed point $y \in \mathbb{R}^d$ and

$$\mathcal{X}^k := \{T^k x : \ x \in \mathcal{X}^0\}, \quad k \in \mathbb{Z}_+ \tag{3.5}$$

where $\mathcal{X}^0$ is some initial regular configuration in $\mathbb{R}^d$. Then each successive configuration $\mathcal{X}^k, \ k \in \mathbb{Z}_+$ is regular. Moreover, since T is contractive, there is a $\lambda \in (0,1)$ such that

$$\|Tx - Ty\|_1 \leq \lambda \|x - y\|_1 \ \ x, y \in \mathbb{R}^d$$

and so it follows that $d_1(\mathcal{X}^k) \leq \lambda^k d_1(\mathcal{X}^0)$, for $k \in \mathbb{Z}_+$. Moreover, since

$$\bar{X}_\alpha^k = T^k(\bar{X}_\alpha^0), \ |\alpha| = n,$$

it follows that $\lim_{k\to\infty} \bar{X}_\alpha^k = y$, for all $\alpha \in \mathbb{Z}_+^{d+1}$ with $|\alpha| = n$. Thus we obtain the following result proved in [1] by other means.

Corollary 3.1. *Let* T *be a contractive affine map with fixed point* $y \in \mathbb{R}^d$*. Then for the regular configurations (3.5) and the B-patch representation (3.2) we have*

$$\lim_{k\to\infty} c_\alpha^k = P(y)$$

for all $|\alpha| = n$*.*

To derive an error bound for subdivision by DMS splines additional work is necessary. We recall the definition of the B-splines that span the DMS spline space introduced in [4].

A d-dimensional simplex with vertices $x_0, \dots, x_d$ is denoted by $[x_0, x_1, \dots, x_d]$. We associate with each d-simplex $[x_0^I, x_1^I, \dots x_d^I]$ of a simplical decomposition $\mathcal{T} := \{\Delta^I = [x_0^I, x_1^I, ..., x_d^I] \ : I \in \mathcal{I}\}$ of $\mathbb{R}^d$ a rectangular array of vectors as in (1.2)

$$\mathcal{X}^I = \{x_{i,j}^I : i = 0, \dots, d, \quad j = 0, 1, \dots, n\}. \tag{3.6}$$

We assume that for each $I \in \mathcal{I}$ the set

$$\Omega^I := \text{interior} \left(\bigcap_{|\alpha| \le n} \Delta_\alpha^I \right)$$

is non-empty, where Δ_α^I is the convex hull of the set of vectors $\{x_{0,\alpha_0}^I, x_{1,\alpha_1}^I, \dots, x_{d,\alpha_d}^I\}$. It is easy to prove that this condition implies that each $\mathcal{X}^I$ is a regular configuration.

The B-splines associated with the rectangular array χ^I are defined for each $\alpha \in \mathbb{Z}^{d+1}$ with $|\alpha| = n$, as

$$\mathcal{N}_\alpha^I(x) := \frac{n!d!}{(n+d)!} Vol_d(\Delta_\alpha^I)\, M_\alpha^I(x)$$

where $M_\alpha^I(\cdot)$ is the multivariate B-spline $M(\cdot|W_\alpha^I)$ relative to the set of vectors

$$W_\alpha^I := \{x_{i,j}^I : i = 0, \dots, d, \quad j = 0, 1, \dots, \alpha_i\},$$

cf [5]. It was proved in [4] that this B-spline has the following property.

Theorem 3.2. *Given any* $x \in \Omega^I, \quad I \in \mathcal{I}, \ \alpha \in \mathbb{Z}^{d+1}, |\alpha| = n$ *we have* $\mathcal{N}_\alpha^I(x) = b_\alpha(x|\mathcal{X}^I)$.

A DMS-spline is any function in the span of the B-splines appearing in Theorem 3.2. That is, any function S that we can write in the following form :

$$S(x) = \sum_{I \in \mathcal{I}} \sum_{|\alpha| = n} c_\alpha^I\, \mathcal{N}_\alpha^I(x)$$

for some constants c_α^I. For each simplex $[x_0^I, x_1^I, \dots, x_d^I]$ in $\mathcal{T}$, we define F_I to be the function which coincides with S on Ω^I.

Theorem 3.2 says that, for each simplex $[x_0^I, x_1^I, \dots, x_d^I]$ the function F_I is a polynomial. Moreover, from (1.6) we obtain the relation

$$f_I(X_\alpha^I) = c_\alpha^I, \quad |\alpha| = n$$

where f_I is the blossom of the polynomial F_I.

In order to generalize Theorem 3.1 to DMS-splines, we bound the distance between the control point c_α^I and the spline S as follows:

$$|S(\bar{X}_\alpha^I) - c_\alpha^I| \le \ |S(\bar{X}_\alpha^I) - F_I(\bar{X}_\alpha^I)| + |F_I(\bar{X}_\alpha^I) - c_\alpha^I|. \tag{3.7}$$

Choose a point y_0 in Ω^I and let $K^I :=$ the convex hull of the points in the rectangular array χ^I. If S is in $W^2_\infty(\mathbb{R}^d)$, then for every $y \in K^I$ a Taylor formula about y_0 yields the inequality

$$|S(y) - S(y_0) - < \nabla S(y_0), y - y_0 > | \leq \frac{1}{2} ||D^2 S||_{K^I} \, ||y - y_0||^2, \quad y \in K^I.$$

The same bound, applied to the polynomial F_I, gives us the inequality

$$|F_I(y) - F_I(y_0) - < \nabla F_I(y_0), y - y_0 > | \leq \frac{1}{2} ||D^2 F_I||_{K^I} \, ||y - y_0||^2, \quad y \in K^I.$$

Now, let us bound the first term in the right-hand side of inequality (3.7). For this term, we choose $y = \bar{X}^I_\alpha$ above and note that $S(y_0) = F_I(y_0)$ and $\nabla S(y_0) = \nabla F_I(y_0)$. Hence

$$\begin{aligned}
&|S(\bar{X}^I_\alpha) - F_I(\bar{X}^I_\alpha)| \\
&\qquad \leq |S(\bar{X}^I_\alpha) - S(y_0) - < \nabla S(y_0), y - y_0 > | \\
&\qquad + \ |F_I(\bar{X}^I_\alpha) - F_I(y_0) - < \nabla F_I(y_0), y - y_0 > | \\
&\qquad \leq \frac{1}{2}(||D^2\, S||_{K^I} + ||D^2\, F_I||_{K^I}) \ ||\bar{X}^I_\alpha - y_0||^2 \\
&\qquad \leq \frac{1}{2}(||D^2\, S||_{K^I} + ||D^2\, F_I||_{K^I}) \ (d_1(\mathcal{X}^I))^2
\end{aligned}$$

For the second term of the right-hand side of inequality (3.7), we apply Theorem 3.1 to obtain

$$|F_I(\bar{X}^I_\alpha) - c^I_\alpha| \leq C_n \ ||D^2\, F_I||_{K^I}(d_1(\mathcal{X}^I))^2$$

where $C_n := (n-1)^2 n^{n-2} 2^{n-3}/n!$. Combining this two inequalities we get

$$|S(\bar{X}^I_\alpha) - c^I_\alpha| \leq \ \{\frac{1}{2}||D^2\, S||_{K^I} + (C_n + \frac{1}{2})||D^2\, F_I||_{K^I}\}(d_1(\mathcal{X}^I))^2. \tag{3.8}$$

Unfortunately, this bound depends on the L^∞-norm of the polynomials $D^2 F_I$ on the set K^I which generally contains the set Ω^I as a *proper* subset. To compensate for this difficulty we need to choose the rectangular array χ^T associated with the simplex $[x^I_0, x^I_1, \ldots, x^I_d]$ with some care. To explain what we have in mind, we fix some terminology and notation. A d-dimensional *rectangle* is any set of the form

$$R := [a_1, b_1] \times \cdots \times [a_d, b_d]$$

and its center is

$$c_R := \frac{1}{2}(a_1 + b_1, \dots, a_d + b_d).$$

A *parallelpiped* is any *affine* image of a rectangle and its center is the affine image of its associated rectangle. It is an elementary fact to prove that for any $p \in \pi_n(\mathbb{R}^d)$ and any two parallelpipeds $Q_i,\ i = 1, 2$ with the same centers and satisfying the condition $Q_1 \supseteq Q_2$ that

$$\|P\|_{Q_1} \leq \left(2^d \frac{\mathrm{vol}_d Q_1}{\mathrm{vol}_d Q_2}\right)^n \|P\|_{Q_2} \tag{3.9}$$

where $\|P\|_Q = \max\{|P(x)| : x \in Q\}$.

For each simplex $[x_0^I, x_1^I, \dots, x_d^I]$ and rectangular array χ^I of vectors we define

$$\rho^I := 2^d \min\left\{\frac{\mathrm{vol}_d Q_1}{\mathrm{vol}_d Q_2} :\ Q_1 \supseteq K^I,\ Q_2 \subseteq \Omega^I\right\} \tag{3.10}$$

where the minimum is taken over all parallelpipeds Q_1, Q_2 with the same center. In view of inequality (3.9) we have the inequality

$$\|P\|_{K^I} \leq (\rho^I)^n \|P\|_{\Omega^I}. \tag{3.11}$$

We say that the simplical decomposition $\mathcal{T}$ of $\mathbb{R}^d$ and its associated family of vectors

$$\chi := \{\chi^T : I \in \mathcal{I}\}$$

is a *balanced configuration* provided there is a $\rho = \rho(\mathcal{T}, \chi)$ such that $\rho^I \leq \rho$ for all $I \in \mathcal{I}$. For a balanced configuration our final result provides a quadratic convergence estimate for subdivision of DMS splines.

Theorem 3.3. *Let $(\mathcal{T}, \chi)$ be a balanced configuration and S a DMS spline of degree $n \geq 3$ in $W_\infty^2(\mathbb{R}^{d+1})$. Then for every $I \in \mathcal{I}$ and $\alpha \in \mathbb{Z}^d, |\alpha| = n$ we have*

$$|S(\bar{X}_\alpha^I) - c_\alpha^I| \leq \frac{n^{n-1} 2^{n-2}}{(n-2)!} (\rho(\mathcal{T}, \chi))^{n-2} \|D^2 S\|_{\mathbb{R}^d} (d_1(\chi^I))^2.$$

Proof. According to (3.11), applied to anyone of the polynomials $\left(\frac{\partial^2}{\partial x_i \partial x_j} F_I\right)(x)\ i,\ j = 1, \dots, d$ of total degree at most $n-2$ we have

$$\begin{aligned} \|D^2 F_I\|_{K^I} &\leq (\rho(\mathcal{T}, \chi))^{n-2} \|D^2 F_I\|_{\Omega^I} \\ &= (\rho(\mathcal{T}, \chi))^{n-2} \|D^2 S\|_{\Omega^I}. \end{aligned}$$

Combining this inequality with (3.8) proves the result. □

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