The Equations of Nonhomogeneous Asymmetric Fluids: An Iterative Approach*

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Abstract

We study the existence and uniqueness of strong solutions for the equations of nonhomogeneous asymmetric fluids. We use an iterative approach and we prove that the approximate solutions constructed by this method converge to the strong solution of these equations. We also give bounds for the rate of convergence.

Key words: asymmetric fluid, Galerkin method, strong solutions.
Short Title: Nonhomogeneous Asymmetric Fluids.

Contents

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1 Introduction

In this paper, we study the existence and uniqueness of strong solutions for the equations of a nonhomogeneous viscous incompressible asymmetric fluid. These equations are considered in a bounded domain $\Omega \subset \mathbb{R}^3$, with boundary Γ, in a time interval $[0,T]$. Let $u(x,t) \in \mathbb{R}^3$, $w(x,t) \in \mathbb{R}^3$, $\rho(x,t) \in \mathbb{R}$ and $p(x,t) \in \mathbb{R}$, denote respectively, the velocity, angular velocity of internal rotation, density and pressure at a point $x \in \Omega$ and at time $t \in [0,T]$. Then, the governing equations are given by

$$\begin{aligned} &\rho\frac{\partial u}{\partial t} + \rho(u\cdot\nabla)u - (\mu+\mu_r)\Delta u + \nabla p = 2\mu_r \text{rot}\, w + \rho f,\\ &\text{div } u = 0,\\ &\rho\frac{\partial w}{\partial t} + \rho(u\cdot\nabla)w - (c_a+c_d)\Delta w - (c_0+c_d-c_a)\nabla \text{ div } w + 4\mu_r w = 2\mu_r \text{rot}\, u + \rho g,\\ &\frac{\partial \rho}{\partial t} + (u\cdot\nabla)\rho = 0, \end{aligned} \tag{1.1}$$

in $Q_T := \Omega \times (0,T)$, with the following boundary and initial conditions

$$\begin{aligned} &u(x,t) = 0,\ w(x,t) = 0 \text{ on } \Gamma\times(0,T),\\ &u(x,0) = u_0(x),\ w(x,0) = w_0(x) \text{ in } \Omega,\\ &\rho(x,0) = \rho_0(x) \text{ in } \Omega. \end{aligned} \tag{1.2}$$

Here, $f(x,t)$ and $g(x,t)$ are respectively, the densities of the linear and angular momentum. The conditions on u_0, w_0 and ρ_0 are given in Section 2. The positive constants μ, μ_r, c_0, c_a, c_d characterize the isotropic properties of the fluid; μ is the usual Newtonian viscosity; μ_r, c_0, c_a, c_d are the new positive viscosities related to the asymmetry of the stress tensor and consequently related to the appearance of the field of internal rotation w; these constants satisfy $c_0 + c_d > c_a$. In this paper, ∇, Δ, div and rot denote, the gradient, Laplacian, divergence and rotational operators respectively (we also denote $\frac{\partial u}{\partial t}$ by u_t); the i$^{\underline{th}}$ component of $(u\cdot\nabla)v$ in the cartesian coordinates is given by $[(u\cdot\nabla)v]_i = \sum_{j=1}^3 u_j \dfrac{\partial v_i}{\partial x_j}$.

For the derivation of equations (1.1)-(1.2), and for its physical interpretations, see D.W. Condiff and J.S. Dahler [5], L.G. Petrosyan [17] and the recent book by G. Lukaszewicz [15]. We observe that this model of fluids includes the classical Navier-Stokes equations as a particular case, which has been thoroughly studied by several authors (see, for instance, the classical books of O. Ladyzhenskaya [7], J.L. Lions [10] and R. Temam [20] and the references therein).

It also includes the reduced model of the nonhomogeneous Navier-Stokes equations, which has been less studied than the previous case (see, for instance, S. Antontsev, A. Kazhikov and V. Monakhov [2], J. Simon [19], J. Kim [6], O. Ladyzhenskaya and V. Solonnikov [8], R. Salvi [18], J.L. Boldrini and M.A. Rojas-Medar [3], and P.L. Lions [11]).

Concerning the generalized model of an asymmetric fluid as considered in this paper, G. Lukaszewicz [14] established the existence of local weak solutions for (1.1)-(1.2) using linearization and a fixed point theorem. In the same paper, G. Lukaszewicz mentioned the possibility of proving the existence of strong solutions (under the hypothesis that the initial density is separated from zero) by the techniques used in G. Lukaszewicz [12] and [13] (linearization and fixed point theorems, under the assumption of constant density).

The first result on the existence and uniqueness of strong solution (local and global) for problem (1.1)-(1.2) was proved by J.L. Boldrini and M.A. Rojas-Medar [4] using the spectral semi-Galerkin method and compactness arguments. The rate of convergence of this method is also established in [4].

In this paper, we use another approach to establish the existence and uniqueness of a strong solution. We use here an iterative process, by considering a sequence of linear problems. For each one of these problems it is easy to show the existence and uniqueness of a strong solution (for instance, by using the spectral semi-Galerkin method as in J.L. Boldrini and M.A. Rojas-Medar [4]). Then, we obtain a priori estimates for the sequence generated by the iterative process. Also we show that the sequence is a Cauchy sequence in an appropriate Banach space, and consequently, we obtain the strong convergence. From these convergences, the existence of a strong solution for the original nonlinear problem (1.1)-(1.2) is easily obtained. The uniqueness of the solution is also proved. Further, we obtain bounds for the rate of convergence.

We hope that the technique developed here can be adapted to the full discretization case. This question is presently under investigation.

This paper is organized as follows: in Section 2, we state some well-known results that will be used in the rest of the paper; and also describe the approximation method and state the result of existence and uniqueness of a strong solution and the bounds for the rate of convergence. In Section 3, we derive a priori estimates for the linearized systems. In Section 4, we establish that the solutions of the sequence of linearized problems is a Cauchy sequence and we prove our main result. Section 5 provides an existence and uniqueness result of the pressure.

Finally, as it is usual in this context, in order to simplify the notation we will denote by C, C_Ω, $C_1, \dots, M$, $M_1, \dots$ generic positive constants depending only on the domain and the fixed data of the problem.

2 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary Γ, $T > 0$ be an arbitrary real number. The functions going to be cosidered in this paper are either $\mathbb{R}$ or $\mathbb{R}^3$-valued, and sometimes we will not distinguish between them in our notation. This will be clear from the context itself. We will consider the usual Sobolev spaces

$$W^{m,q}(D) = \{ \, f \in L^q(\Omega) \mid \|\partial^\alpha f\|_{L^q(D)} < \infty, \ |\alpha| \le m \, \},$$

for $m \in \mathbb{N}$, $1 \leq p < \infty$, $D = \Omega$ or $D = Q_T$, with the usual norm. When $q = 2$ we denote $H^m(D) = W^{m,2}(D)$ and $H_0^m(D) =$ closure of $\mathscr{D}(D)$ in $H^m(D)$. We put

$$\begin{aligned} \mathscr{V}(\Omega) &= \{v \in \mathscr{D}(\Omega)^3 | \text{div } v = 0 \quad \text{in} \quad \Omega\} \\ H &= \quad \text{closure of} \quad \mathscr{V}(\Omega) \quad \text{in} \quad L^2(\Omega)^3, \\ V &= \quad \text{closure of} \quad \mathscr{V}(\Omega) \quad \text{in} \quad H^1(\Omega)^3. \end{aligned}$$

It is well-known that

$$V = \{v \in H_0^1(\Omega) \mid \text{div}\, v = 0 \ \text{ in } \ \Omega\}.$$

We denote by V^* the dual space of V and by H^{-1} the dual space of $H_0^1(\Omega)$. We recall the Helmholtz decomposition of vector fields $L^2(\Omega) = H \oplus G$, where $G = \{\phi \mid \phi = \nabla p,\ p \in H^1(\Omega)\}$.

Throughout this paper P denotes the orthogonal projection from $L^2(\Omega)$ onto H. Then, the operator $A : D(A) \hookrightarrow H \longrightarrow H$ given by $A = -P\Delta$ with domain $D(A) = V \cap H^2(\Omega)$ is called the Stokes operator. It is well known that A is a positive definite, self-adjoint operator and is characterized by the relation

$$(Aw, v) = (\nabla w, \nabla v), \quad \forall\, w \in D(A), \quad v \in V.$$

If Ω is of class $\mathscr{C}^{1,1}$, then the norms $\|u\|_{H^2}$ and $\|Au\|$ are equivalent in $D(A)$ (see C. Amrouche and V. Girault [1]). We assume the other known properties of A, as given in O. Ladyzhenskaya [7], J.L. Lions [10] or R. Temam [20]. The same remark is also valid for the Laplacian operator $B = -\Delta$ with homogeneous Dirichlet boundary conditions in the domain $D(B) = H_0^1(\Omega) \cap H^2(\Omega)$.

Applying the orthogonal projection P to problem (1.1)-(1.2), we can rewrite it as follows: Find u, w, ρ in suitable spaces (which will be defined later on), satisfying

$$P(\rho u_t) + (\mu + \mu_r)Au + P(\rho u \cdot \nabla u) = 2\mu_r P(\text{rot}\, w) + P(\rho f), \tag{2.1}$$

$$\rho w_t + (c_a + c_d)Bw + \rho u \cdot \nabla w - (c_0 + c_d - c_a)\nabla \text{ div } w + 4\mu_r w = 2\mu_r \text{rot}\, u + \rho g, \tag{2.2}$$

$$\rho_t + u \cdot \nabla \rho = 0, \tag{2.3}$$

$$u(x,0) = u_0(x),\ w(x,0) = w_0(x),\ \rho(x,0) = \rho_0(x) \ \text{ in } \Omega. \tag{2.4}$$

We consider the following iterative process for the approximate solution of problem (2.1)-(2.4). Setting

$$u^1(t) = e^{-t(\mu+\mu_r)A}u_0, \quad w^1(t) = e^{-t(c_a+c_d)B}w_0, \quad \rho^1(x,t) = \rho_0(x),$$

where $e^{-t(\mu+\mu_r)A}$ and $e^{-t(c_a+c_d)B}$ are the semigroups generated by the Stokes and Laplace operators, respectively. And for given u^n, w^n and ρ^n, we define u^{n+1}, w^{n+1} and ρ^{n+1} as the unique solution of the following system of linear equations,

$$P(\rho^n u_t^{n+1}) + (\mu + \mu_r)Au^{n+1} + P(\rho^n u^n \cdot \nabla u^{n+1}) = 2\mu_r P(\text{rot}\, w^n) + P(\rho^n f), \tag{2.5}$$

$$\rho^n w_t^{n+1} + (c_a + c_d)Bw^{n+1} + \rho^n u^n \cdot \nabla w^{n+1} - (c_o + c_d - c_a)\nabla \text{ div } w^{n+1} + 4\mu_r w^{n+1} = 2\mu_r \text{rot}\, u^n + \rho^n g, \tag{2.6}$$

$$\rho_t^{n+1} + u^{n+1} \cdot \nabla \rho^{n+1} = 0, \tag{2.7}$$

$$u^{n+1}(x,0) = u_0(x), \quad w^{n+1}(x,0) = w_0(x), \quad \rho^{n+1}(x,0) = \rho_0(x) \ \text{ in } \Omega. \tag{2.8}$$

Concerning the initial density ρ_0, we assume that it is a continuously differentiable function ($\rho_0 \in \mathscr{C}^1$), and that there exist α, β such that

$$0 < \alpha \leq \rho_0(x) \leq \beta \quad \forall x \in \bar{\Omega}.$$

In this paper, the external fields f and g are assumed to be $L^2(Q_T)$ functions, small enough with respect to the viscosities coefficients of the model μ, μ_r, c_a and c_d. More precisely, f and g are assumed to satisfy

$$\left(\|f\|^2_{L^2(Q_T)} + \|g\|^2_{L^2(Q_T)}\right)\left(\frac{\beta}{\mu+\mu_r}\right)^3\left[\frac{1}{2} + \left(\frac{4}{\lambda} + \frac{1}{4}\right)\frac{\mu_r}{c_a + c_d}e^{CT}\right] \leq \frac{\lambda^{1/4}\Phi}{160 C_\Omega^2} \tag{2.9}$$

where $\Phi = \min\{\frac{\alpha}{4\beta}, \frac{\alpha^5}{\beta^5}\}$, $C = \max\{\frac{8\mu_r^2}{\alpha(\mu+\mu_r)}, \frac{4\mu_r^2}{\alpha(c_a+c_d)}\}$ and λ is the smallest eigenvalue of the Laplace operator $B = -\Delta$ in Ω with homogeneous Dirichlet boundary condition.

Notice that this hypothesis is fulfilled either if f and g are small enough with respect to the viscosities μ and μ_r, or if the viscosities are sufficiently large with respect to the data f and g.

In J.L. Boldrini and M.A. Rojas-Medar [4], the authors used the Galerkin method to solve this linear system and showed that the solutions (u^n, w^n, ρ^n) enjoy the following conditions concerning their regularity:

$$u^n \in L^\infty(0,T;V), \tag{2.10}$$
$$u_t^n \in L^2(0,T;H), \tag{2.11}$$
$$Au^n \in L^2(0,T;L^2(\Omega)), \tag{2.12}$$
$$w^n \in L^\infty(0,T;H_0^1(\Omega)), \tag{2.13}$$
$$w_t^n \in L^2(0,T;L^2(\Omega)), \tag{2.14}$$
$$Bw^n \in L^2(0,T;L^2(\Omega)). \tag{2.15}$$

We are going to prove on the one hand that these sequences are uniformly bounded in the corresponding spaces. On the other hand, applying the method of characteristics to the continuity equation (2.7), it follows immediately that whenever ρ^n exists, it satisfies $0 < \alpha \leq \rho^n \leq \beta$. In particular, we have that

$$\{\rho^n\} \text{ is uniformly bounded in } L^\infty(0,T;L^\infty(\Omega)). \tag{2.16}$$

Furthermore, the hypothesis on the density ρ^n make it possible to apply the Ladyzhenskaya-Solonnikov's results ([9], see Lemma 1.3, p. 705). In our case, we obtain that $\nabla\rho^n$ and ρ^n

$L^\infty(0,T;L^\infty(\Omega))$ as $n \to \infty$.

we consider without loss of generality $u_0(x) = 0$ and $w_0(x) = 0$ (the general introducing an appropriate lifting of the initial conditions). Let us first esults obtained for the approximate solutions. In this case, it is clear that $w^1, \rho^1) = (0, 0, \rho_0)$.

$L^2(0,T;L^2(\Omega))$ and satisfy the conditions as given in (2.9), then the unique f problem (2.5)-(2.8) are uniformly bounded in the respective spaces as given

Lemma 2.2. *If the hypotheses of Lemma 2.1 are verified and assuming that $f,\, g \in L^2(0,T;H^1(\Omega))$ and $f_t,\, g_t \in L^2(0,T;L^2(\Omega))$, then the solution (u^n, w^n, ρ^n) of problem (2.5)-(2.8) satisfies the following estimates uniformly in n:*

$$
\begin{aligned}
\sup_t(\|u_t^n(t)\|^2 + \|w_t^n(t)\|^2) &\leq C,\\
\int_0^t (\|\nabla u_t^n(\tau)\|^2 + \|\nabla w_t^n(\tau)\|^2)d\tau &\leq C,\\
\sup_t(\|Au^n(t)\|^2 + \|Bw^n(t)\|^2) &\leq C,\\
\int_0^t (\|\nabla u^n(\tau)\|_{L^\infty}^2 + \|\nabla w^n(\tau)\|_{L^\infty}^2)d\tau &\leq C,\\
\sup_t \sigma(t)(\|\nabla u_t^n(t)\|^2 + \|\nabla w_t^n(t)\|^2) &\leq C,\\
\int_0^t \sigma(\tau)(\|u_{tt}^n(\tau)\|^2 + \|w_{tt}^n(\tau)\|^2)d\tau &\leq C,\\
\int_0^t \sigma(\tau)(\|Au_t^n(\tau)\|^2 + \|Bw_t^n(\tau)\|^2)d\tau &\leq C,
\end{aligned}
$$

for all $t \in [0,T]$, where $C > 0$ is a constant independent of n and $\sigma(t) = \min\{1,t\}$.

Theorem 2.3. *Let the conditions of Lemmas 2.1 and 2.2 be satisfied. Then the approximate solutions (u^n, w^n, ρ^n) converge to the limiting element (u, w, ρ) in the following senses*

$$
\begin{aligned}
&u^n \longrightarrow u \quad \textit{strongly in} \quad L^\infty(0,T;V) \cap L^2(0,T;V \cap H^2(\Omega)),\\
&w^n \longrightarrow w \quad \textit{strongly in} \quad L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega)),\\
&u_t^n \longrightarrow u_t \quad \textit{strongly in} \quad L^2(0,T;H),\\
&w_t^n \longrightarrow w_t \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega)),\\
&u_t^n \longrightarrow u_t \quad \textit{weakly in} \quad L^2(0,T;V) \cap L^2(\varepsilon,T;V \cap H^2(\Omega)), \quad \forall\, \varepsilon > 0,\\
&u_{tt}^n \longrightarrow u_{tt} \quad \textit{weakly in} \quad L^2(\varepsilon,T;H), \quad \forall\, \varepsilon > 0,\\
&w_t^n \longrightarrow w_t \quad \textit{weakly in} \quad L^2(0,T;H_0^1(\Omega)) \cap L^2(\varepsilon,T;H_0^1(\Omega) \cap H^2(\Omega)), \quad \forall\, \varepsilon > 0,\\
&w_{tt}^n \longrightarrow w_{tt} \quad \textit{weakly in} \quad L^2(\varepsilon,T;L^2(\Omega)), \quad \forall\, \varepsilon > 0.
\end{aligned}
$$

The limiting element (u,w,ρ) is the unique solution of problem (2.1)-(2.4) and

$$\begin{aligned}
\sup_t\{\|\nabla u^n(t)-\nabla u(t)\|^2+\|\nabla w^n(t)-\nabla w(t)\|^2\} &\le M\frac{(M_1T)^{n-1}}{(n-1)!},\\
\int_0^t(\|u_t^n(\tau)-u_t(\tau)\|^2+\|w_t^n(\tau)-w_t(\tau)\|^2)d\tau &\le M\frac{(M_1T)^{n-1}}{(n-1)!},\\
\int_0^t(\|Au^n(\tau)-Au(\tau)\|^2+\|Bw^n(\tau)-Bw(\tau)\|^2)d\tau &\le M\frac{(M_1T)^{n-1}}{(n-1)!},\\
\sup_t\|\rho^n(t)-\rho(t)\|^2_{L^\infty} &\le M\frac{(M_1T)^{n-1}}{(n-1)!},\\
\sup_t\sigma(t)(\|u_t^n(t)-u_t(t)\|^2+\|w_t^n(t)-w_t(t)\|^2) &\le M\frac{(M_1T)^{n-2}}{(n-2)!},\\
\int_0^t\sigma(\tau)(\|\nabla u_t^n(\tau)-\nabla u_t(\tau)\|^2+\|\nabla w_t^n(\tau)-\nabla w_t(\tau)\|^2)d\tau &\le M\frac{(M_1T)^{n-2}}{(n-2)!},\\
\sup_t\sigma(t)(\|Au^n(t)-Au(t)\|^2+\|Bw^n(t)-Bw(t)\|^2) &\le M\frac{(M_1T)^{n-2}}{(n-2)!},\\
\sup_t\sigma(t)(\|u^n(t)-u(t)\|^2_{L^\infty}+\|w^n(t)-w(t)\|^2_{L^\infty}) &\le M\frac{(M_1T)^{n-2}}{(n-2)!},\\
\int_0^t\sigma(\tau)(\|\nabla u^n(\tau)-\nabla u(\tau)\|^2_{L^\infty}+\|\nabla w^n(\tau)-\nabla w(\tau)\|^2_{L^\infty})d\tau &\le M\frac{(M_1T)^{n-2}}{(n-2)!}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&u\in\mathscr{C}^1([0,T];H)\cap\mathscr{C}([0,T];D(A)),\\
&w\in\mathscr{C}^1([0,T];L^2(\Omega))\cap\mathscr{C}([0,T];D(B)),\\
&\rho\in\mathscr{C}^1(Q_T).
\end{aligned}$$

3 A Priori Estimates

In this section, we prove uniform a priori estimates in n for the approximate solutions.

3.1 Proof of Lemma 2.1

3.1.1 Uniform estimates for u^n and w^n in $L^2(0,T;V)$

From (2.7), we have $(\rho_t^n v,v)=-(\text{div}\,(\rho^n u^n)v,v)=2(\rho^n u^n\cdot\nabla v,v)$ and consequently

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^n}v\|^2=\frac{1}{2}(\rho_t^n v,v)+(\rho^n v_t,v)=(\rho^n u^n\cdot\nabla v,v)+(\rho^n v_t,v),\ \forall\, v\in H_0^1,\ v_t\in L^2(\Omega).$$

With this identity in mind, multiply (2.5) by u^{n+1} and (2.6) by w^{n+1}, to obtain respectively:

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^n}u^{n+1}\|^2+(\mu+\mu_r)\|\nabla u^{n+1}\|^2=2\mu_r(\text{rot}\,w^n,u^{n+1})+(\rho^n f,u^{n+1}),\tag{3.1}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^n}w^{n+1}\|^2+(c_a+c_d)\|\nabla w^{n+1}\|^2+(c_0+c_d-c_a)\|\text{div}\ w^{n+1}\|^2+4\mu_r\|w^{n+1}\|^2\\
=2\mu_r(\text{rot}\,u^n,w^{n+1})+(\rho^n g,w^{n+1}).\end{aligned}\tag{3.2}$$

We recall that for $u \in H_0^1(\Omega)$, we have

$$\|\mathrm{rot}\, u\| \le \|\nabla u\|, \quad \|u\|_{L^4} \le 2^{1/2}\|u\|^{1/4}\|\nabla u\|^{3/4} \quad \text{and} \quad \|u\|^2 \le \lambda^{-1}\|\nabla u\|^2, \tag{3.3}$$

where λ is the smallest eigenvalue of the Laplace operator $B = -\Delta$ (see, for instance, O. Ladyzhenskaya [7]).

By Hölder and Young inequalities, and (3.3), we get from (3.1) and (3.2) the following differential inequalities

$$\frac{d}{dt}\|\sqrt{\rho^n}u^{n+1}\|^2 + (\mu+\mu_r)\|\nabla u^{n+1}\|^2 \le \frac{8\mu_r^2}{\mu+\mu_r}\|w^n\|^2 + \frac{2\beta^2\,\lambda^{-1}}{\mu+\mu_r}\|f\|^2,$$

$$\frac{d}{dt}\|\sqrt{\rho^n}w^{n+1}\|^2 + (c_a+c_d)\|\nabla w^{n+1}\|^2 + 2(c_0+c_d-c_a)\|\mathrm{div}\ w^{n+1}\|^2$$
$$\le \frac{4\mu_r^2}{c_a+c_d}\|u^n\|^2 + \frac{\beta^2}{8\mu_r}\|g\|^2.$$

Adding both inequalities and integrating both sides from 0 to t, we get the following integral inequality (recall that $u_0 = w_0 = 0$):

$$\alpha(\|u^{n+1}(t)\|^2 + \|w^{n+1}(t)\|^2) + (\mu+\mu_r)\int_0^t \|\nabla u^{n+1}(\tau)\|^2 d\tau$$
$$+(c_a+c_d)\int_0^t \|\nabla w^{n+1}(\tau)\|^2 d\tau + 2(c_0+c_d-c_a)\int_0^t \|\mathrm{div}\ w^{n+1}(\tau)\|^2 d\tau$$
$$\le \frac{8\mu_r^2}{\mu+\mu_r}\int_0^t \|w^n(\tau)\|^2 d\tau + \frac{4\mu_r^2}{c_a+c_d}\int_0^t \|u^n(\tau)\|^2 d\tau + \frac{2\beta^2\,\lambda^{-1}}{\mu+\mu_r}\|f\|^2_{L^2(Q_T)} + \frac{\beta^2}{8\mu_r}\|g\|^2_{L^2(Q_T)}$$

Then, there exist constants M and C, choose for example

$$C = \max\{\frac{8\mu_r^2}{\alpha(\mu+\mu_r)}, \frac{4\mu_r^2}{\alpha(c_a+c_d)}\} \quad \text{and} \quad M = \frac{2\beta^2\,\lambda^{-1}}{\alpha(\mu+\mu_r)}\|f\|^2_{L^2(Q_T)} + \frac{\beta^2}{8\alpha\,\mu_r}\|g\|^2_{L^2(Q_T)},$$

such that

$$\|u^{n+1}(t)\|^2 + \|w^{n+1}(t)\|^2 + \frac{\mu+\mu_r}{\alpha}\int_0^t \|\nabla u^{n+1}(\tau)\|^2 d\tau + \frac{c_a+c_d}{\alpha}\int_0^t \|\nabla w^{n+1}(\tau)\|^2 d\tau$$
$$\le C\int_0^t (\|u^n(\tau)\|^2 + \|w^n(\tau)\|^2) d\tau + M. \tag{3.4}$$

Thus, setting $\varphi_n(t) = \|u^n(t)\|^2 + \|w^n(t)\|^2$, the last inequality implies

$$\varphi_{n+1}(t) \le M + C\int_0^t \varphi_n(\tau)d\tau.$$

Observing that $\varphi_1(t) = 0$, a straightforward induction argument shows that, for all n,

$$\varphi_n(t) \le M\sum_{k=0}^{n-1}\frac{(Ct)^k}{k!} \le M\ \exp(Ct).$$

Therefore, we conclude that for all n, we have

$$\sup_{t\in[0,T]}(\,\|u^n(t)\|^2 + \|w^n(t)\|^2\,) \le \sup_{t\in[0,T]} M\exp(Ct) = M\exp(CT) \equiv M_1. \tag{3.5}$$

Notice that M_1 does not depend on n. Combining (3.4) and (3.5), we get

$$\|u^{n+1}\|^2_{L^2(0,T;V)} \le \frac{\alpha M_1}{\mu+\mu_r} \quad \text{and} \quad \|w^{n+1}\|^2_{L^2(0,T;H_0^1(\Omega))} \le \frac{\alpha M_1}{c_a+c_d} \tag{3.6}$$

where the bounds are independent of n.

3.1.2 Uniform estimates for u^n and w^n in $L^\infty(0,T;V)$

Multiplying (2.5) by $\delta\, Au^{n+1}$, and then by u_t^{n+1} and integrating in Ω, we obtain respectively

$$\begin{aligned}\delta(\mu+\mu_r)\|Au^{n+1}\|^2 &= -\delta\,(\rho^n u_t^{n+1}, Au^{n+1}) + 2\mu_r\,\delta\,(\mathrm{rot}\, w^n, Au^{n+1}) \\ &\quad +\delta\,(\rho^n f, Au^{n+1}) - \delta(\rho^n u^n.\nabla u^{n+1}, Au^{n+1}) \qquad (3.7)\end{aligned}$$

and

$$\begin{aligned}\|\sqrt{\rho^n} u_t^{n+1}\|^2 + \frac{\mu+\mu_r}{2}\frac{d}{dt}\|\nabla u^{n+1}\|^2 &= 2\mu_r(\mathrm{rot}\, w^n, u_t^{n+1}) + (\rho^n f, u_t^{n+1}) \\ &\quad -(\rho^n u^n.\nabla u^{n+1}, u_t^{n+1}). \qquad (3.8)\end{aligned}$$

Then, using $\alpha \le \rho^n \le \beta$, we get

$$\begin{aligned}&\alpha\|u_t^{n+1}\|^2 + \frac{\mu+\mu_r}{2}\frac{d}{dt}\|\nabla u^{n+1}\|^2 + \delta(\mu+\mu_r)\|Au^{n+1}\|^2 \\ &\le |\delta\,(\rho^n u_t^{n+1}, Au^{n+1})| + |2\mu_r\,\delta\,(\mathrm{rot}\, w^n, Au^{n+1})| + |2\mu_r(\mathrm{rot}\, w^n, u_t^{n+1})| + |(\rho^n f, u_t^{n+1})| \\ &\quad +|\delta\,(\rho^n f, Au^{n+1})| + |\delta(\rho^n u^n.\nabla u^{n+1}, Au^{n+1})| + |(\rho^n u^n.\nabla u^{n+1}, u_t^{n+1})|. \qquad (3.9)\end{aligned}$$

Now, using Hölder and Young inequalities, and (3.3), we get

$$\begin{aligned}|\delta(\rho^n u^n.\nabla u^{n+1}, Au^{n+1})| &\le \delta\,\beta\|u^n\|_{L^4}\|\nabla u^{n+1}\|_{L^4}\|Au^{n+1}\| \\ &\le \delta\,\beta\,\sqrt{2}\,\|u^n\|^{1/4}\|\nabla u^n\|^{3/4}\|\nabla u^{n+1}\|_{L^4}\|Au^{n+1}\| \\ &\le \delta\,\beta\,\sqrt{2}\,\lambda^{-1/8}\|\nabla u^n\|\|\nabla u^{n+1}\|_{L^4}\|Au^{n+1}\|. \qquad (3.10)\end{aligned}$$

Since $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$, for $u \in D(A)$, we have

$$\|\nabla u\|_{L^4} \le \|u\|_{W^{1,4}} \le C_\Omega\|u\|_{H^2} \le C_\Omega\|Au\|, \qquad (3.11)$$

where C_Ω is a positive constant, independent of u. Thus, from (3.10) and (3.11), we obtain

$$|\delta(\rho^n u^n.\nabla u^{n+1}, Au^{n+1})| \le \delta\,\beta\,\sqrt{2}\,\lambda^{-1/8}C_\Omega\|\nabla u^n\|\|Au^{n+1}\|^2. \qquad (3.12)$$

Similarly,

$$\begin{aligned}|(\rho^n u^n.\nabla u^{n+1}, u_t^{n+1})| &\le \beta\,\|u^n\|_{L^4}\|\nabla u^{n+1}\|_{L^4}\|u_t^{n+1}\| \\ &\le \sqrt{2}\,\beta\,\lambda^{-1/8}C_\Omega\|\nabla u^n\|\left(\delta\|Au^{n+1}\|^2 + \frac{\|u_t^{n+1}\|^2}{4\delta}\right). \qquad (3.13)\end{aligned}$$

Using the above estimates for the last two terms and the classical estimates for the remaining terms in (3.9), we obtain

$$\begin{aligned}&(\mu+\mu_r)\frac{d}{dt}\|\nabla u^{n+1}\|^2 + 2\left(\alpha - 3\eta - \frac{\beta\, 2^{1/2}\,\lambda^{-1/8}C_\Omega}{4\delta}\|\nabla u^n\|\right)\|u_t^{n+1}\|^2 \\ &+2\delta\left((\mu+\mu_r) - \frac{3\delta\,\beta^2}{4\eta} - \beta\, 2^{3/2}\,\lambda^{-1/8}C_\Omega\|\nabla u^n\|\right)\|Au^{n+1}\|^2 \\ &\le \left(\frac{8\mu_r^2\,\eta}{\beta^2} + \frac{2\mu_r^2}{\eta}\right)\|\nabla w^n\|^2 + \left(2\eta + \frac{\beta^2}{2\eta}\right)\|f\|^2.\end{aligned}$$

where η is any positive real number. Integrating the above inequality in $[0,t]$ we have

$$
\begin{aligned}
&(\mu+\mu_r)\|\nabla u^{n+1}(t)\|^2 + 2\int_0^t \left(\alpha - 3\eta - \frac{\beta\, 2^{1/2}\,\lambda^{-1/8} C_\Omega}{4\delta}\|\nabla u^n(\tau)\|\right)\|u_t^{n+1}(\tau)\|^2 d\tau \\
&+2\delta \int_0^t \left((\mu+\mu_r) - \frac{3\delta\,\beta^2}{4\eta} - \beta\, 2^{3/2}\,\lambda^{-1/8} C_\Omega \|\nabla u^n(\tau)\|\right)\|Au^{n+1}(\tau)\|^2 d\tau \\
&\le \left(\frac{8\mu_r^2\,\eta}{\beta^2} + \frac{2\mu_r^2}{\eta}\right)\int_0^t \|\nabla w^n(\tau)\|^2 d\tau + (2\eta + \frac{\beta^2}{2\eta})\|f\|^2_{L^2(Q_T)} \\
&\le \left(\frac{4\eta}{\beta^2} + \frac{1}{\eta}\right)\frac{2\mu_r^2\alpha M_1}{c_a + c_d} + (2\eta + \frac{\beta^2}{2\eta})\|f\|^2_{L^2(Q_T)}.
\end{aligned}
$$

Then, by choosing $\eta = \dfrac{\alpha}{4}$ and $\delta = \dfrac{\alpha(\mu+\mu_r)}{4\,\beta^2}$, we have

$$
\begin{aligned}
&(\mu+\mu_r)\|\nabla u^{n+1}(t)\|^2 + 2\int_0^t \left(\frac{\alpha}{4} - \frac{\beta^3\, 2^{1/2}\,\lambda^{-1/8} C_\Omega}{\alpha(\mu+\mu_r)}\|\nabla u^n(\tau)\|\right)\|u_t^{n+1}(\tau)\|^2 d\tau \\
&+\frac{\alpha(\mu+\mu_r)}{2\,\beta^2}\int_0^t \left(\frac{\mu+\mu_r}{4} - \beta\, 2^{3/2}\,\lambda^{-1/8} C_\Omega \|\nabla u^n(\tau)\|\right)\|Au^{n+1}(\tau)\|^2 d\tau \\
&\le 2(\alpha^2 + 4\beta^2)\frac{\mu_r^2}{\beta^2(c_a+c_d)} M_1 + \frac{1}{2\alpha}(\alpha^2+4\beta^2)\|f\|^2_{L^2(Q_T)} = \varepsilon^2(\text{ say}).
\end{aligned}
\tag{3.14}
$$

We use the method of induction to prove that

$$
\|\nabla u^n(t)\| \le \frac{\varepsilon}{(\mu+\mu_r)^{1/2}}, \tag{3.15}
$$

Setting $n=1$ in (3.14) and using that $u^1 = 0$, we get

$$
(\mu+\mu_r)\|\nabla u^2(t)\|^2 + 2\int_0^t \frac{\alpha}{4}\|u_t^2(\tau)\|^2 d\tau + \frac{\alpha(\mu+\mu_r)}{2\,\beta^2}\int_0^t \frac{\mu+\mu_r}{4}\|Au^2(\tau)\|^2 d\tau \le \varepsilon^2,
$$

then, for all $t \in [0,T]$, we have

$$
\|\nabla u^2(t)\| \le \frac{\varepsilon}{(\mu+\mu_r)^{1/2}}. \tag{3.16}
$$

We assume the inequality (3.15) for $n=k$ and prove for $n=k+1$. From (3.14), it suffices to show that

$$
\frac{\alpha}{4} - \frac{\beta^3\, 2^{1/2}\,\lambda^{-1/8} C_\Omega\, \varepsilon}{\alpha(\mu+\mu_r)^{3/2}} > 0 \quad \text{and} \quad \frac{\mu+\mu_r}{4} - \frac{\beta\, 2^{3/2}\,\lambda^{-1/8} C_\Omega\, \varepsilon}{(\mu+\mu_r)^{1/2}} > 0.
$$

By using (2.9) one can prove the positivity of the above terms.

Therefore, for all n, we have proved that

$$
\sup_{t\in[0,T]} \|\nabla u^n(t)\| \le \frac{\varepsilon}{(\mu+\mu_r)^{1/2}}. \tag{3.17}
$$

From (3.14) and (3.17), we have

$$
\begin{aligned}
&2\int_0^t \left(\frac{\alpha}{4} - \frac{\beta^3\, 2^{1/2}\,\lambda^{-1/8} C_\Omega\, \varepsilon}{\alpha(\mu+\mu_r)^{3/2}}\right)\|u_t^{n+1}(\tau)\|^2 d\tau \\
&\quad+\frac{\alpha(\mu+\mu_r)}{2\,\beta^2}\int_0^t \left(\frac{\mu+\mu_r}{4} - \frac{\beta\, 2^{3/2}\,\lambda^{-1/8} C_\Omega\, \varepsilon}{(\mu+\mu_r)^{1/2}}\right)\|Au^{n+1}(\tau)\|^2 d\tau \le \varepsilon^2.
\end{aligned}
$$

Therefore, we conclude that there exists a constant C, independent of n, such that

$$\int_0^t \|u_t^{n+1}(\tau)\|^2 d\tau + \int_0^t \|Au^{n+1}(\tau)\|^2 d\tau \leq C. \tag{3.18}$$

Similarly, for all n, we obtain

$$(c_a + c_d)\|\nabla w^{n+1}(t)\|^2 + c_2 \int_0^t \|Bw^{n+1}(\tau)\|^2 d\tau + \alpha \int_0^t \|w_t^{n+1}(\tau)\|^2 d\tau \leq C, \tag{3.19}$$

and the proof of Lemma 2.1 is complete.

3.2 Proof of Lemma 2.2

3.2.1 Uniform estimates for u_t^n in the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V(\Omega))$ and Au^n in $L^\infty(0,T;L^2(\Omega))$

Differentiating (2.5) with respect to t, we obtain

$$\begin{aligned} P(\rho_t^n u_t^{n+1}) + P(\rho^n u_{tt}^{n+1}) + (\mu + \mu_r)Au_t^{n+1} \\ = 2\,\mu_r\, P(\text{rot}\, w_t^n) + P(\rho_t^n f) + P(\rho^n f_t) - P(\rho_t^n u^n \cdot \nabla u^{n+1}) \\ -P(\rho^n u_t^n \cdot \nabla u^{n+1}) - P(\rho^n u^n \cdot \nabla u_t^{n+1}). \end{aligned} \tag{3.20}$$

Multiplying (3.20) by u_t^{n+1} and after some simple computations, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^n}u_t^{n+1}\|^2 + (\mu + \mu_r)\|\nabla u_t^{n+1}\|^2 \\ &= -\frac{1}{2}(\rho_t^n u_t^{n+1}, u_t^{n+1}) + 2\,\mu_r\,(\text{rot}\, w_t^n, u_t^{n+1}) + (\rho_t^n f, u_t^{n+1}) + (\rho^n f_t, u_t^{n+1})) \\ &\quad -(\rho_t^n u^n \cdot \nabla u^{n+1}, u_t^{n+1}) - (\rho^n u_t^n . \nabla u^{n+1}, u_t^{n+1}) - (\rho^n u^n \cdot \nabla u_t^{n+1}, u_t^{n+1})) \\ &= \frac{1}{2}(\text{div}\,(\rho^n u^n) u_t^{n+1}, u_t^{n+1}) + 2\,\mu_r\,(w_t^n, \text{rot}\, u_t^{n+1}) - (\text{div}\,(\rho^n u^n) f, u_t^{n+1})) \\ &\quad +(\rho^n f_t, u_t^{n+1}) + (\text{div}\,(\rho^n u^n) u^n . \nabla u^{n+1}, u_t^{n+1}) - (\rho^n u_t^n \cdot \nabla u^{n+1}, u_t^{n+1})) \\ &\quad -(\rho^n u^n \cdot \nabla u_t^{n+1}, u_t^{n+1}) \end{aligned} \tag{3.21}$$

since from (2.3), $\rho_t^n = -\text{div}\,(\rho^n u^n)$.

Using classical estimates, each of the seven terms in the right hand side of (3.21) can be bounded as follows. The first one:

$$\begin{aligned} \frac{1}{2}(\text{div}\,(\rho^n u^n) u_t^{n+1}, u_t^{n+1}) &= -(\rho^n u^n \cdot \nabla u_t^{n+1}, u_t^{n+1}) \\ &\leq \|\rho^n\|_{L^\infty}\|u^n\|_{L^4}\|\nabla u_t^{n+1}\|\|u_t^{n+1}\|_{L^4} \\ &\leq \beta\|\nabla u^n\|\|\nabla u_t^{n+1}\|\|u_t^{n+1}\|^{1/4}\|\nabla u_t^{n+1}\|^{3/4} \\ &\leq C\,\|\nabla u_t^{n+1}\|^{7/4}\|u_t^{n+1}\|^{1/4} \\ &\leq C_\eta\|u_t^{n+1}\|^2 + \eta\|\nabla u_t^{n+1}\|^2. \end{aligned}$$

The second term is simply bounded by

$$2\,\mu_r\,(w_t^n, \text{rot}\, u_t^{n+1}) \leq C_\eta\|w_t^n\|^2 + \eta\|\nabla u_t^{n+1}\|^2. \tag{3.22}$$

For the third one, integration by parts gives:

$$\begin{aligned}-(\operatorname{div}(\rho^n u^n)f, u_t^{n+1}) &= (\rho^n u^n.\nabla f, u_t^{n+1}) + (\rho^n u^n\cdot\nabla u_t^{n+1}, f)\\ &\le \beta\|u^n\|_{L^4}\|\nabla f\|\|u_t^{n+1}\|_{L^4} + \beta\|u^n\|_{L^4}\|\nabla u_t^{n+1}\|\|f\|_{L^4}\\ &\le C\,\|f\|_{H^1}\|\nabla u_t^{n+1}\| \le C_\eta\|f\|_{H^1}^2 + \eta\|\nabla u_t^{n+1}\|^2.\end{aligned}$$

For the fourth term, one can easily obtain

$$(\rho^n f_t, u_t^{n+1}) \le C_\eta\|f_t\|^2 + \eta\|\nabla u_t^{n+1}\|^2. \tag{3.23}$$

Integrating by parts the fifth term gives

$$\begin{aligned}(\operatorname{div}(\rho^n u^n)u^n\cdot\nabla u^{n+1}, u_t^{n+1}) &= \sum_{i,j,k}\int_\Omega \frac{\partial}{\partial x_i}(\rho^n u_i^n)u_j^n(\frac{\partial}{\partial x_j}u_k^{n+1})u_{k,t}^{n+1}dx\\ &= -\sum_{i,j,k}\int_\Omega \rho^n u_i^n(\frac{\partial}{\partial x_i}u_j^n)(\frac{\partial}{\partial x_j}u_k^{n+1})u_{k,t}^{n+1}dx\\ &\quad -\sum_{i,j,k}\int_\Omega \rho^n u_i^n u_j^n(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u_k^{n+1})u_{k,t}^{n+1}dx\\ &\quad -\sum_{i,j,k}\int_\Omega \rho^n u_i^n u_j^n(\frac{\partial}{\partial x_j}u_k^{n+1})(\frac{\partial}{\partial x_i}u_{k,t}^{n+1})dx\\ &\le C\,\beta\,\|u^n\|_{L^6}\|\nabla u^n\|_{L^6}\|\nabla u^{n+1}\|\|u_t^{n+1}\|_{L^6}\\ &\quad +C\,\beta\|u^n\|_{L^6}^2\|Au^{n+1}\|\|u_t^{n+1}\|_{L^6}\\ &\quad +C\,\beta\|u^n\|_{L^6}^2\|\nabla u^{n+1}\|_{L^6}\|\nabla u_t^{n+1}\|\\ &\le C\,\|Au^n\|\|\nabla u_t^{n+1}\| + C\,\|Au^{n+1}\|\|\nabla u_t^{n+1}\|\\ &\le C_\eta(\|Au^n\|^2 + \|Au^{n+1}\|^2) + 2\,\eta\|\nabla u_t^{n+1}\|^2.\end{aligned}$$

The sixth and the seventh terms can be bounded respectively as:

$$\begin{aligned}(\rho^n u_t^n\cdot\nabla u^{n+1}, u_t^{n+1}) &\le \beta\|u_t^n\|\|\nabla u^{n+1}\|_{L^3}\|u_t^{n+1}\|_{L^6}\\ &\le C\,\|u_t^n\|\|Au^{n+1}\|\|u_t^{n+1}\|_{H^1}\\ &\le C_\eta\|u_t^n\|^2\|Au^{n+1}\|^2 + \eta\|\nabla u_t^{n+1}\|^2.\end{aligned}$$

and

$$\begin{aligned}(\rho^n u^n\cdot\nabla u_t^{n+1}, u_t^{n+1}) &\le \beta\|u^n\|_{L^6}\|\nabla u_t^{n+1}\|\|u_t^{n+1}\|_{L^3}\\ &\le C\|\nabla u_t^{n+1}\|\|u_t^{n+1}\|^{1/2}\|\nabla u_t^{n+1}\|^{1/2}\\ &\le C_\eta\|u_t^{n+1}\|^2 + \eta\|\nabla u_t^{n+1}\|^2.\end{aligned}$$

Using all these bounds in (3.21) we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^n}u_t^{n+1}\|^2 + (\mu+\mu_r)\|\nabla u_t^{n+1}\|^2\\ &\quad\le C_\eta\|u_t^{n+1}\|^2 + C_\eta\|w_t^n\|^2 + C_\eta\|f\|_{H^1}^2 + C_\eta\|f_t\|^2 + C_\eta\|Au^n\|^2\\ &\qquad + C_\eta\|Au^{n+1}\|^2 + C_\eta\|u_t^n\|^2\|Au^{n+1}\|^2 + C_\eta\|u_t^{n+1}\|^2 + 8\eta\,\|\nabla u_t^{n+1}\|^2.\end{aligned}$$

By choosing $\eta = \dfrac{\mu+\mu_r}{16}$, we get

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho^n}u_t^{n+1}\|^2 + (\mu+\mu_r)\|\nabla u_t^{n+1}\|^2 \\
&\qquad \le C\,\|u_t^{n+1}\|^2 + C\,\|w_t^n\|^2 + C\,\|f\|_{H^1}^2 + C\,\|f_t\|^2 + C\,\|Au^n\|^2 + C\,\|Au^{n+1}\|^2 \\
&\qquad\quad + C\,\|u_t^n\|^2\|Au^{n+1}\|^2 + C\,. \qquad (3.24)
\end{aligned}$$

In order to get a bound for $\|Au^{n+1}\|^2$, multiply (2.5) by Au^{n+1}, we obtain

$$\begin{aligned}
(\mu+\mu_r)\|Au^{n+1}\|^2 &= -(\rho^n u_t^{n+1}, Au^{n+1}) + 2\,\mu_r\,(\operatorname{rot} w^n, Au^{n+1}) + (\rho^n f, Au^{n+1}) \\
&\quad -(\rho^n u^n.\nabla u^{n+1}, Au^{n+1}). \qquad (3.25)
\end{aligned}$$

Consider the right hand side of (3.25),

$$\begin{aligned}
|(\rho^n u^n.\nabla u^{n+1}, Au^{n+1})| &\le \beta\,\|u^n\|_{L^4}\|\nabla u^{n+1}\|_{L^4}\|Au^{n+1}\| \\
&\le C\,\|\nabla u^{n+1}\|^{1/4}\|Au^{n+1}\|^{7/4} \le C_\delta\|\nabla u^{n+1}\|^2 + \delta\|Au^{n+1}\|^2,
\end{aligned}$$

Using the above result with classical estimates in (3.25), we get

$$(\mu+\mu_r)\|Au^{n+1}\|^2 \le C_\delta\|u_t^{n+1}\|^2 + C_\delta\|\nabla w^n\|^2 + C_\delta\|f\|^2 + C_\delta\|\nabla u^{n+1}\|^2 + 4\,\delta\,\|Au^{n+1}\|^2.$$

Then, taking $\delta > 0$ sufficiently small, from the previous inequality, we obtain the bound:

$$\|Au^{n+1}\|^2 \le C\,\|u_t^{n+1}\|^2 + C. \qquad (3.26)$$

Thus, rewriting (3.24), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho^n}u_t^{n+1}\|^2 + (\mu+\mu_r)\|\nabla u_t^{n+1}\|^2 \\
&\qquad \le C\,\|u_t^{n+1}\|^2 + C\,\|w_t^n\|^2 + C\,\|f\|_{H^1}^2 + C\,\|f_t\|^2 \\
&\qquad\quad + C\,\|u_t^n\|^2\|u_t^{n+1}\|^2 + C\,\|u_t^n\|^2 + C.
\end{aligned}$$

Integrating the above inequality from 0 to t, we get

$$\begin{aligned}
&\alpha\|u_t^{n+1}(t)\|^2 + (\mu+\mu_r)\int_0^t \|\nabla u_t^{n+1}(\tau)\|^2 d\tau \\
&\quad \le C\int_0^t (\|u_t^{n+1}(\tau)\|^2 + \|w_t^n(\tau)\|^2 + \|f(\tau)\|_{H^1}^2 + \|f_t(\tau)\|^2)d\tau \\
&\qquad + C\int_0^t \|u_t^n(\tau)\|^2\,\|u_t^{n+1}(\tau)\|^2 d\tau + C\int_0^t \|u_t^n(\tau)\|^2 d\tau \\
&\qquad + \beta\|u_t^{n+1}(0)\|^2 + Ct.
\end{aligned}$$

From equation (3.8), we can easily bound the rightmost term $\|u_t^{n+1}(0)\|^2$. In fact, $d/dt\|\nabla u^{n+1}(t)\|^2$ is non negative at $t=0$, since $\nabla u^{n+1}(0) = 0$. Applying (3.18), (3.19) and the hypotheses on f and f_t, we get

$$\|u_t^{n+1}(t)\|^2 + \int_0^t \|\nabla u_t^{n+1}(\tau)\|^2 d\tau \le C + C\int_0^t \|u_t^n(\tau)\|^2\|u_t^{n+1}(\tau)\|^2 d\tau.$$

If we denote $\varphi(t) = \|u_t^{n+1}(t)\|^2$, the above inequality can be written as

$$\varphi(t) \le C + C \int_0^t \|u_t^n(\tau)\|^2 \varphi(t)\, d\tau$$

By Gronwall's lemma,

$$\varphi(t) \le C \exp(C \int_0^t \|u_t^n(\tau)\|^2 \, d\tau).$$

Using (3.18) we conclude that

$$\|u_t^{n+1}(t)\|^2 + \int_0^t \|\nabla u_t^{n+1}(\tau)\|^2 d\tau \le C. \tag{3.27}$$

Moreover, from (3.26) we have for all n

$$\sup_t \|Au^{n+1}(t)\|^2 \le C. \tag{3.28}$$

Similarly, for all n, one can prove the following:

$$\|w_t^{n+1}(t)\|^2 + \int_0^t \|\nabla w_t^{n+1}(\tau)\|^2 d\tau \le C \quad \text{and} \quad \sup_t \|Bw^{n+1}(t)\|^2 \le C. \tag{3.29}$$

3.2.2 Uniform estimates for u^n in $L^2(0,T;W^{1,\infty})$

Let us write (2.5) as

$$(\mu + \mu_r) A u^{n+1} = P(F) \tag{3.30}$$

where

$$F = 2\mu_r \mathrm{rot}\, w^n + \rho^n f - \rho^n u_t^{n+1} - \rho^n u^n \cdot \nabla u^{n+1}.$$

From the estimates given in Lemma 2.1, together with the estimates (3.27) and (3.28), we can prove that $F \in L^2(0,T;L^6(\Omega))$ and consequently by the Amrouche-Girault's results (1991), we obtain uniform bounds for u^n in $L^2(0,T;W^{2,6}(\Omega))$. Also, by using the Sobolev embedding, one can show that u^n is uniformly bounded in $L^2(0,T;W^{1,\infty}(\Omega))$.

3.2.3 Three estimates on the second order derivatives

Now, multiplying (3.20) by u_{tt}^{n+1}, and using (2.16), (3.17), Lemma 2.1, the estimates for ρ^n
ölder and Young inequalities, we obtain

$$\begin{aligned} \frac{\mu_r}{2} \frac{d}{dt}\|\nabla u_t^{n+1}\|^2 \;\le\; & C_\varepsilon \|u_t^{n+1}\|^2 + C_\varepsilon \|\nabla w_t^n\|^2 + C_\varepsilon \|f\|^2 + C_\varepsilon \|f_t\|^2 \\ & + C_\varepsilon \|Au^{n+1}\|^2 + C_\varepsilon \|\nabla u_t^n\|^2 \|Au^{n+1}\|^2 \\ & + C_\varepsilon \|\nabla u_t^{n+1}\|^2 + 7\varepsilon \|u_{tt}^{n+1}\|^2. \end{aligned}$$

Choosing $\varepsilon = \dfrac{\alpha}{14}$ and observing (3.27)-(3.28), we have

$$\begin{aligned}\alpha\,\|u_{tt}^{n+1}\|^2 + (\mu+\mu_r)\frac{d}{dt}\|\nabla u_t^{n+1}\|^2 \;&\leq\; C\,\|\nabla w_t^n\|^2 + C\,\|f\|^2 + C\,\|f_t\|^2 + C\,\|\nabla u_t^n\|^2 \\ &\quad +C\,\|\nabla u_t^{n+1}\|^2 + C\end{aligned}$$

and multiplying by $\sigma(t)=\min\{1,t\}$, results

$$\begin{aligned}&\alpha\,\sigma(t)\|u_{tt}^{n+1}\|^2 + (\mu+\mu_r)\frac{d}{dt}(\sigma(t)\|\nabla u_t^{n+1}\|^2)\\ &\quad\leq (\mu+\mu_r)\sigma'(t)\|\nabla u_t^{n+1}\|^2 + C\,\sigma(t)(\|\nabla u_t^n\|^2 + \|\nabla w_t^n\|^2)\\ &\qquad +C\,\sigma(t)(\|f\|^2+\|f_t\|^2) + C\,\sigma(t)(\|\nabla u_t^{n+1}\|^2+1).\end{aligned} \tag{3.31}$$

As a consequence of (3.27), there exists a sequence $\varepsilon_k \longrightarrow 0$, such that $\varepsilon_k\,\|\nabla u_t^{n+1}(\varepsilon_k)\|^2 \leq C$. Since $\sigma(t)\leq 1$ and $\sigma'(t)\leq 1$ a.e. in $[0,T]$, applying (3.27)-(3.29) and integrating (3.31) from ε_k to t, we obtain

$$\alpha\int_{\varepsilon_k}^{t}\sigma(\tau)\|u_{tt}^{n+1}(\tau)\|^2 d\tau + (\mu+\mu_r)\sigma(t)\|\nabla u_t^{n+1}(t)\|^2 \leq C + C\,(\mu+\mu_r)\sigma(\varepsilon_k)\,\|\nabla u_t^{n+1}(\varepsilon_k)\|^2 + C.$$

Taking limit as $\varepsilon_k \longrightarrow 0$, for all n, reduces the previous inequality to

$$\int_0^t \sigma(\tau)\|u_{tt}^{n+1}(\tau)\|^2 d\tau + \sigma(t)\|\nabla u_t^{n+1}(t)\|^2 \leq C.$$

Analogously, for all n,

$$\int_0^t \sigma(\tau)\|w_{tt}^{n+1}(\tau)\|^2 d\tau + \sigma(t)\|\nabla w_t^{n+1}(t)\|^2 \leq C.$$

To prove the last estimate given in Lemma 2.2, we observe from (3.20) that

$$(\mu+\mu_r)\int_0^t \sigma(\tau)\|Au_t^{n+1}(\tau)\|^2 d\tau \leq \int_0^t \sigma(\tau)\|G^n(\tau)\|^2 d\tau$$

where

$$\begin{aligned}G^n \;&=\; 2\mu_r \mathrm{rot}\, w_t^n + \rho_t^n f + \rho^n f_t - \rho_t^n u_t^{n+1} - \rho^n u_{tt}^{n+1} - \rho_t^n u^n\cdot\nabla u^{n+1}\\ &\quad -\rho^n u_t^n . \nabla u^{n+1} - \rho^n u^n\cdot\nabla u_t^{n+1}.\end{aligned}$$

All the above estimates imply that $\sigma^{1/2}(t)G^n$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$. Analogously, one can prove the estimates for w^n.

Remark. Using arguments of compactness and the estimates given in Lemmas 2.1 and 2.2, it is possible to prove that the approximate solutions (u^n, w^n, ρ^n) converge to a strong solution of the problem (1.1)-(1.2). This can be done in exactly the same way as in J.L. Boldrini and M.A. Rojas-Medar [4].

4 Proof of Theorem 2.3

4.1 Convergence analysis

On this subsection we show that u^n, w^n and ρ^n are Cauchy sequences. Let us introduce the following notation for the difference of two terms of a sequence. For $n,\ s \geq 1$,

$$u^{n,s}(t) = u^{n+s}(t) - u^n(t),\ \ w^{n,s}(t) = w^{n+s}(t) - w^n(t) \text{ and } \rho^{n,s}(t) = \rho^{n+s}(t) - \rho^n(t).$$

With these notations, we observe that $u^{n,s}$, $w^{n,s}$ and $\rho^{n,s}$ satisfy the following equations

$$\begin{aligned} P(\rho^{n-1+s}u_t^{n,s}) + (\mu+\mu_r)Au^{n,s} &= 2\mu_r P(\operatorname{rot} w^{n-1,s}) + P(\rho^{n-1,s}f) - P(\rho^{n-1,s}u_t^n) \\ &\quad -P(\rho^{n-1+s}u^{n-1+s}\cdot\nabla u^{n,s}) - P(\rho^{n-1+s}u^{n-1,s}\cdot\nabla u^n) \\ &\quad -P(\rho^{n-1,s}u^{n-1}\cdot\nabla u^n) \end{aligned} \tag{4.1}$$

$$\begin{aligned} \rho^{n-1+s}w_t^{n,s} &+ (c_a+c_d)Bw^{n,s} - (c_0+c_d-c_a)\nabla \operatorname{div} w^{n,s} + 4\mu_r w^{n,s} \\ &= 2\mu_r(\operatorname{rot} u^{n-1,s}) + \rho^{n-1,s}g - \rho^{n-1,s}w_t^n - \rho^{n-1+s}u^{n-1+s}\cdot\nabla w^{n,s} \\ &\quad -\rho^{n-1+s}u^{n-1,s}\cdot\nabla w^n - \rho^{n-1,s}u^{n-1}\cdot\nabla w^n \end{aligned} \tag{4.2}$$

$$\rho_t^{n,s} + u^{n,s}\cdot\nabla\rho^{n+s} + u^n\cdot\nabla\rho^{n,s} = 0. \tag{4.3}$$

The following lemma, which can be easily proven, is fundamental in order to obtain error estimates.

Lemma 4.1. *Let $0 \leq \phi_1(t) \leq M$ for all $t \in [0,T]$ and assume that for all $n \geq 2$, $n \in N$, we have the following inequality*

$$0 \leq \phi_n(t) \leq C\int_0^t \phi_{n-1}(\tau)d\tau$$

where $C > 0$ is a constant independent of n. Then,

$$\phi_n(t) \leq M\frac{(Ct)^{n-1}}{(n-1)!} \leq M\frac{(CT)^{n-1}}{(n-1)!}$$

for all $t \in [0,T]$ and $n \geq 2$. Therefore, $\phi_n(t) \longrightarrow 0$ as $n \longrightarrow \infty$, $\forall\, t \in [0,T]$.

4.1.1 Bounding the error of the density sequence

The density sequence can be bounded in terms of the velocity sequence, as stated in the following lemma.

Lemma 4.2. *Under the hypotheses of Lemma 2.2, we have*

$$\|\rho^{n,s}(t)\|^2_{L^6} \leq C\int_0^t \|\nabla u^{n,s}(\tau)\|^2 d\tau.$$

Proof. Multiplying (4.3) by $(\rho^{n,s})^5$ and integrating over Ω, we obtain

$$\begin{aligned} \frac{1}{6}\frac{d}{dt}\int_\Omega |\rho^{n,s}|^6 dx &= -\int_\Omega u^{n,s}\cdot\nabla\rho^{n+s}(\rho^{n,s})^5 dx - \frac{1}{6}\int_\Omega u^n\cdot\nabla(\rho^{n,s})^6 dx \\ &\leq \int_\Omega |u^{n,s}||\nabla\rho^{n+s}||\rho^{n,s}|^5 dx + \frac{1}{6}\int_\Omega \operatorname{div}\, u^n(\rho^{n,s})^6 dx \\ &\leq \|\nabla\rho^{n+s}\|_{L^\infty(0,T;L^\infty(\Omega))}\int_\Omega |u^{n,s}||\rho^{n,s}|^5 dx \\ &\leq C\left(\int_\Omega |u^{n,s}|^6 dx\right)^{1/6}\left(\int_\Omega |\rho^{n,s}|^6 dx\right)^{5/6}. \end{aligned}$$

This implies

$$\frac{1}{6}\frac{d}{dt}\|\rho^{n,s}\|_{L^6}^6 \leq C\,\|u^{n,s}\|_{L^6}\|\rho^{n,s}\|_{L^6}^5,$$

but,

$$\frac{1}{6}\frac{d}{dt}\|\rho^{n,s}\|_{L^6}^6 = \|\rho^{n,s}\|_{L^6}^5\frac{d}{dt}\|\rho^{n,s}\|_{L^6},$$

then, since $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$\frac{d}{dt}\|\rho^{n,s}\|_{L^6} \leq C\,\|\nabla u^{n,s}\|.$$

Integrating the last inequality from 0 to t and applying the Cauchy-Schwartz inequality, we conclude that

$$\|\rho^{n,s}(t)\|_{L^6} \leq C\int_0^t \|\nabla u^{n,s}(\tau)\|d\tau \leq C\bigg(\int_0^t \|\nabla u^{n,s}(\tau)\|^2 d\tau\bigg)^{1/2}. \tag{4.4}$$

4.1.2 Convergence of u^n and w^n in $L^\infty(0,T;H_0^1(\Omega))$

Multiplying (4.1) by $\delta A u^{n,s}$, integrating over Ω and estimating as usual, we obtain,

$$\begin{aligned}\delta(\mu+\mu_r)\|Au^{n,s}\|^2 \quad \leq \quad & \eta\|u_t^{n,s}\|^2 + \eta\|\nabla w^{n-1,s}\|^2 + \eta\|\rho^{n-1,s}\|_{L^6}^2\|f\|_{L^3}^2 \\ & +\eta\|\rho^{n-1,s}\|_{L^6}^2\|\nabla u_t^n\|^2 + \eta\|\nabla u^{n,s}\|^2 \\ & +\eta\|\nabla u^{n-1,s}\|^2 + \eta\|\rho^{n-1,s}\|_{L^6}^2 + \frac{1}{4\eta}\delta^2 C\|Au^{n,s}\|^2 \end{aligned} \tag{4.5}$$

where η is a positive parameter and C is a constant independent of n. Similarly, multiplying (4.1) by $u_t^{n,s}$, we get

$$\begin{aligned}&\alpha\|u_t^{n,s}\|^2 + \frac{\mu+\mu_r}{2}\frac{d}{dt}\|\nabla u^{n,s}\|^2 \\ &\leq C_\eta\|\nabla w^{n-1,s}\|^2 + C_\eta\|\rho^{n-1,s}\|_{L^6}^2\|f\|_{L^3}^2 + C_\eta\|\rho^{n-1,s}\|_{L^6}^2\|\nabla u_t^n\|^2 \\ &\quad +C_\eta\|\nabla u^{n,s}\|^2 + C_\eta\|\nabla u^{n-1,s}\|^2 + C_\eta\|\rho^{n-1,s}\|_{L^6}^2 + 6\eta\|u_t^{n,s}\|^2\end{aligned} \tag{4.6}$$

where C_η is a constant independent of n.

Adding (4.5) and (4.6), we get

$$\begin{aligned}&\alpha\|u_t^{n,s}\|^2 + \frac{\mu+\mu_r}{2}\frac{d}{dt}\|\nabla u^{n,s}\|^2 + \delta(\mu+\mu_r)\|Au^{n,s}\|^2 \\ &\leq 2C_\eta\|\nabla w^{n-1,s}\|^2 + 2C_\eta\|\rho^{n-1,s}\|_{L^6}^2\|f\|_{L^3}^2 + 2C_\eta\|\rho^{n-1,s}\|_{L^6}^2\|\nabla u_t^n\|^2 \\ &\quad +2C_\eta\|\nabla u^{n,s}\|^2 + 2C_\eta\|\nabla u^{n-1,s}\|^2 + 2C_\eta\|\rho^{n-1,s}\|_{L^6}^2 + 6\eta\|u_t^{n,s}\|^2 \\ &\quad +\frac{C\delta^2}{4\eta}\|Au^{n,s}\|^2.\end{aligned}$$

By choosing $\eta = \dfrac{\alpha}{12}$ and $\delta > 0$ such that $(\mu+\mu_r)\delta - \dfrac{C\delta^2}{4\eta} > 0$, we reduce the previous inequality to

$$\begin{aligned}
&\alpha\|u_t^{n,s}\|^2 + (\mu+\mu_r)\frac{d}{dt}\|\nabla u^{n,s}\|^2 + C_1\|Au^{n,s}\|^2 \\
&\qquad \le C\|\nabla w^{n-1,s}\|^2 + C\|\rho^{n-1,s}\|_{L^6}^2\|f\|_{L^3}^2 + C\|\rho^{n-1,s}\|_{L^6}^2\|\nabla u_t^n\|^2 \\
&\qquad\quad + C\|\nabla u^{n,s}\|^2 + C\|\nabla u^{n-1,s}\|^2 + C\|\rho^{n-1,s}\|_{L^6}^2 \qquad (4.7)
\end{aligned}$$

with positive constants $C_1,\ C$ independent of n

From (4.2), we have

$$\begin{aligned}
&\rho^{n-1+s}w_t^{n,s} + Lw^{n,s} + 4\mu_r w^{n,s} \\
&\qquad = 2\mu_r(\operatorname{rot} u^{n-1,s}) + \rho^{n-1,s}g - \rho^{n-1,s}w_t^n - \rho^{n-1+s}u^{n-1+s}\cdot\nabla w^{n,s} \\
&\qquad\quad - \rho^{n-1+s}u^{n-1,s}\cdot\nabla w^n - \rho^{n-1,s}u^{n-1}\cdot\nabla w^n \qquad (4.8)
\end{aligned}$$

where $Lw^{n,s} = (c_a+c_d)Bw^{n,s} - (c_0+c_d-c_a)\nabla \text{ div } w^{n,s}$. Since L is a strongly elliptic operator (see O. Ladyzhenskaya, V. Solonnikov, and N. Uralceva, [9] p. 70), there exists a positive constant N depending exclusively on $c_a+c_d,\ c_0+c_d-c_a$ and Γ such that

$$(Lw^{n,s}, Bw^{n,s}) \ge (c_a+c_d)\|Bw^{n,s}\|^2 - N_0\|\nabla w^{n,s}\|^2. \qquad (4.9)$$

Multiplying (4.8) by $\theta Bw^{n,s}$, using (4.9) and estimating as usual, we have

$$\begin{aligned}
\theta(c_a+c_d)\|Bw^{n,s}\|^2 \ \le\ & \theta C_\zeta\|\nabla w^{n,s}\|^2 + \zeta\|w_t^{n,s}\|^2 + \zeta\|\nabla u^{n-1,s}\|^2 + \zeta\|\rho^{n-1,s}\|_{L^6}^2\|g\|_{L^3}^2 \\
& + \zeta\|\rho^{n-1,s}\|_{L^6}^2\|\nabla w_t^n\|^2 + \zeta\|\nabla w^{n,s}\| + \zeta\|\nabla u^{n-1,s}\| \\
& + \zeta\|\rho^{n-1,s}\|_{L^6}^2 + \frac{\theta^2 C}{4\zeta}\|Bw^{n,s}\|^2. \qquad (4.10)
\end{aligned}$$

Multiplying (4.2) by $w_t^{n,s}$, standard estimates yield

$$\begin{aligned}
&\alpha\|w_t^{n,s}\|^2 + \frac{c_a+c_d}{2}\frac{d}{dt}\|\nabla w^{n,s}\|^2 + \frac{c_0+c_d-c_a}{2}\frac{d}{dt}\|\text{div } w^{n,s}\|^2 + 2\mu_r\frac{d}{dt}\|w^{n,s}\|^2 \\
&\quad \le C_\zeta\|\nabla u^{n-1,s}\|^2 + C_\zeta\|\rho^{n-1,s}\|_{L^6}^2\|g\|_{L^3}^2 + C_\zeta\|\rho^{n-1,s}\|_{L^6}^2\|\nabla w_t^n\|^2 \\
&\quad\quad + C_\zeta\|\nabla w^{n,s}\|^2 + C_\zeta\|\nabla u^{n-1,s}\|^2 + C_\zeta\|\rho^{n-1,s}\|_{L^6}^2 + 6\zeta\|w_t^{n,s}\|^2. \qquad (4.11)
\end{aligned}$$

Adding (4.10) and (4.11), we get

$$\begin{aligned}
&\alpha\|w_t^{n,s}\|^2 + \frac{c_a+c_d}{2}\frac{d}{dt}\|\nabla w^{n,s}\|^2 + \theta(c_a+c_d)\|Bw^{n,s}\|^2 \\
&\quad + \frac{c_0+c_d-c_a}{2}\frac{d}{dt}\|\text{div } w^{n,s}\|^2 + 2\mu_r\frac{d}{dt}\|w^{n,s}\|^2 \\
&\quad\quad \le 4C_\zeta\|\nabla u^{n-1,s}\|^2 + 2C_\zeta\|\rho^{n-1,s}\|_{L^6}^2\|g\|_{L^3}^2 + 2C_\zeta\|\rho^{n-1,s}\|_{L^6}^2\|\nabla w_t^n\|^2 \\
&\quad\quad\quad + (\theta+1)C_\zeta\|\nabla w^{n,s}\|^2 + 2C_\zeta\|\rho^{n-1,s}\|_{L^6}^2 + 6\zeta\|w_t^{n,s}\|^2 + \frac{C\theta^2}{4\zeta}\|Bw^{n,s}\|^2.
\end{aligned}$$

Now, choosing $\zeta = \dfrac{\alpha}{12}$ and $\theta = \dfrac{(c_a+c_d)\alpha}{25\beta^2}$, and setting $C_2 = \frac{(c_a+c_d)^2\alpha}{(25\beta)^2}$, we get

$$\begin{aligned}
&\alpha\|w_t^{n,s}\|^2 + (c_a+c_d)\frac{d}{dt}\|\nabla w^{n,s}\|^2 + C_2\|Bw^{n,s}\|^2 + (c_0+c_d-c_a)\frac{d}{dt}\|\text{div } w^{n,s}\|^2 \\
&\quad + 4\mu_r\frac{d}{dt}\|w^{n,s}\|^2 \le C\|\nabla u^{n-1,s}\|^2 + C\|\rho^{n-1,s}\|_{L^6}^2\|g\|_{L^3}^2 + C\|\rho^{n-1,s}\|_{L^6}^2\|\nabla w_t^n\|^2 \\
&\quad\quad + C\|\nabla w^{n,s}\|^2 + C\|\rho^{n-1,s}\|_{L^6}^2. \qquad (4.12)
\end{aligned}$$

Adding (4.7) and (4.12), and integrating the result from 0 to t, we obtain

$$
\begin{aligned}
&(\mu+\mu_r)\|\nabla u^{n,s}(t)\|^2+(c_a+c_d)\|\nabla w^{n,s}(t)\|^2+\alpha\int_0^t(\|u_t^{n,s}(\tau)\|^2+\|w_t^{n,s}(\tau)\|^2)d\tau\\
&\quad+C_1\int_0^t\|Au^{n,s}(\tau)\|^2d\tau+C_2\int_0^t\|Bw^{n,s}(\tau)\|^2d\tau+(c_0+c_d-c_a)\|\text{div }w^{n,s}(t)\|^2\\
&\quad\le C\int_0^t(\|\nabla u^{n-1,s}(\tau)\|^2+\|\nabla w^{n-1,s}(\tau)\|^2)d\tau\\
&\quad\quad+C\int_0^t\|\rho^{n-1,s}(\tau)\|_{L^6}^2(\|f(\tau)\|_{L^3}^2+\|g(\tau)\|_{L^3}^2)d\tau\\
&\quad\quad+C\int_0^t\|\rho^{n-1,s}(\tau)\|_{L^6}^2(\|\nabla u_t^n(\tau)\|^2+\|\nabla w_t^n(\tau)\|^2)d\tau\\
&\quad\quad+C\int_0^t(\|\nabla u^{n,s}(\tau)\|^2+\|\nabla w^{n,s}(\tau)\|^2)d\tau+C\int_0^t\|\rho^{n-1,s}(\tau)\|_{L^6}^2d\tau.
\end{aligned}\tag{4.13}
$$

From (4.4), $\forall\ \tau\in(0,t),\ 0<t<T$, we have

$$
\|\rho^{n-1,s}(t)\|_{L^6}^2\le C\int_0^\tau\|\nabla u^{n-1,s}(\tau)\|^2d\tau\le C\int_0^t\|\nabla u^{n-1,s}(\tau)\|^2d\tau
$$

and replacing this last inequality in (4.13), we obtain

$$
\begin{aligned}
&C_3(\|\nabla u^{n,s}(t)\|^2+\|\nabla w^{n,s}(t)\|^2)+C_3\int_0^t(\|u_t^{n,s}(\tau)\|^2+\|w_t^{n,s}(\tau)\|^2)d\tau\\
&\quad+C_3\int_0^t(\|Au^{n,s}(\tau)\|^2+\|Bw^{n,s}(\tau)\|^2)d\tau+C_3\|\text{div }w^{n,s}(t)\|^2\\
&\quad\le C\int_0^t(\|\nabla u^{n-1,s}(\tau)\|^2+\|\nabla w^{n-1,s}(\tau)\|^2)d\tau\\
&\quad\quad+C\int_0^t\|\nabla u^{n-1,s}(t_1)\|^2dt_1\int_0^t(\|f(\tau)\|_{L^3}^2+\|g(\tau)\|_{L^3}^2)d\tau\\
&\quad\quad+C\int_0^t\|\nabla u^{n-1,s}(t_1)\|^2dt_1\int_0^t(\|\nabla u_t^n(\tau)\|^2+\|\nabla w_t^n(\tau)\|^2)d\tau\\
&\quad\quad+C\int_0^t(\|\nabla u^{n,s}(\tau)\|^2+\|\nabla w^{n,s}(\tau)\|^2)d\tau+CT\int_0^t\|\nabla u^{n-1,s}(\tau)\|^2d\tau.
\end{aligned}
$$

where $C_3=\min\{\ \mu+\mu_r,\ c_a+c_d,\ \alpha,\ c_1,\ c_2\ \}$. Then,

$$
\begin{aligned}
&\|\nabla u^{n,s}(t)\|^2+\|\nabla w^{n,s}(t)\|^2+\int_0^t(\|u_t^{n,s}(\tau)\|^2+\|w_t^{n,s}(\tau)\|^2)d\tau+\int_0^t\|Au^{n,s}(\tau)\|^2d\tau\\
&\quad+\int_0^t\|Bw^{n,s}(\tau)\|^2d\tau\le C\int_0^t(\|\nabla u^{n-1,s}(\tau)\|^2+\|\nabla w^{n-1,s}(\tau)\|^2)d\tau\\
&\quad\quad+C\int_0^t(\|\nabla u^{n,s}(\tau)\|^2+\|\nabla w^{n,s}(\tau)\|^2)d\tau.
\end{aligned}
$$

Applying Gronwall's inequality (see W. Varhorn [21], Lemma 3.10 p. 122), we get

$$
\begin{aligned}
&\|\nabla u^{n,s}(t)\|^2+\|\nabla w^{n,s}(t)\|^2+\int_0^t(\|u_t^{n,s}(\tau)\|^2+\|w_t^{n,s}(\tau)\|^2)d\tau+\int_0^t\|Au^{n,s}(\tau)\|^2d\tau\\
&\quad+\int_0^t\|Bw^{n,s}(\tau)\|^2d\tau\le M_1\int_0^t(\|\nabla u^{n-1,s}(\tau)\|^2+\|\nabla w^{n-1,s}(\tau)\|^2)d\tau
\end{aligned}\tag{4.14}
$$

Thus, we have

$$\|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2 \leq M_1 \int_0^t (\|\nabla u^{n-1,s}(\tau)\|^2 + \|\nabla w^{n-1,s}(\tau)\|^2) d\tau.$$

Since $\|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2 \leq M, \ \forall \, n, s$ and $t \in [0,T]$, using Lemma 4.1, we obtain

$$\|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2 \leq M \frac{(M_1 t)^{n-1}}{(n-1)!} \leq M \frac{(M_1 T)^{n-1}}{(n-1)!}. \tag{4.15}$$

We observe that

$$M_1 \int_0^t (\|\nabla u^{n-1,s}(\tau)\|^2 + \|\nabla w^{n-1,s}(\tau)\|^2) d\tau \leq M_1 \int_0^t M \frac{(M_1 \tau)^{n-2}}{(n-2)!} d\tau \leq M \frac{(M_1 t)^{n-1}}{(n-1)!}. \tag{4.16}$$

Therefore, from (4.14) and (4.16), we have

$$\int_0^t (\|A u^{n,s}(\tau)\|^2 + \|B w^{n,s}(\tau)\|^2) d\tau \leq M \frac{(M_1 t)^{n-1}}{(n-1)!} \leq M \frac{(M_1 T)^{n-1}}{(n-1)!}, \tag{4.17}$$

from which we obtain the convergence in $L^2(0,T;H^2(\Omega))$, and

$$\int_0^t (\|u_t^{n,s}(\tau)\|^2 + \|w_t^{n,s}(\tau)\|^2) d\tau \leq M \frac{(M_1 t)^{n-1}}{(n-1)!} \leq M \frac{(M_1 T)^{n-1}}{(n-1)!} \tag{4.18}$$

which gives the convergence of u_t^n and w_t^n in $L^2(0,T;L^2(\Omega))$.

4.1.3 Convergence of the density sequence in $L^\infty(0,T;L^\infty(\Omega))$

Now, from (4.3), we have

$$\begin{aligned} \rho_t^{n,s} + u^n . \nabla \rho^{n,s} &= -u^{n,s} \cdot \nabla \rho^{n+s} \\ \rho^{n,s}(0) &= 0. \end{aligned}$$

Let $z^n(x,t,\tau)$ be the solution of the Cauchy problem

$$\begin{aligned} z_t^n &= u^n(z^n, \tau) \\ z^n &= x \ \text{ for } \ \tau = t. \end{aligned}$$

Then, using the characteristic method, we have

$$\rho^{n,s}(x,t) = -\int_0^t u^{n,s}(z^n(\tau),\tau) . \nabla \rho^{n+s}(z^n(\tau),\tau) d\tau.$$

Bearing in mind the properties of z^n (see O. Ladyzhenskaya and V. Solonnikov [8], pp. 93–96), we get

$$\|\rho^{n,s}(t)\|_{L^\infty} \leq \|\nabla \rho^{n+s}\|_{L^\infty(0,T;L^\infty(\Omega))} \int_0^t \|u^{n,s}(\tau)\|_{L^\infty} d\tau \leq C \int_0^t \|A u^{n,s}(\tau)\| d\tau.$$

Hence, applying the Cauchy-Schwartz's inequality and observing (4.17), we have

$$\|\rho^{n,s}(t)\|_{L^\infty}^2 \leq c \int_0^t \|A u^{n,s}(\tau)\|^2 d\tau \leq M \frac{(M_1 t)^{n-1}}{(n-1)!} \leq M \frac{(M_1 T)^{n-1}}{(n-1)!}. \tag{4.19}$$

4.1.4 Convergence of u^n and w^n in $L^2(\varepsilon, T; W^{1,\infty}(\Omega))$

The following bounds in the lemma require some technical manipulation. Let us differentiate (4.1) with respect to t, and multiply the result by $u_t^{n,s}$ and integrate the resultant on Ω. We get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^{n-1+s}}u_t^{n,s}\|^2 + (\mu+\mu_r)\|\nabla u_t^{n,s}\|^2 \\
&\quad = -\frac{1}{2}(\rho_t^{n-1+s}u_t^{n,s}, u_t^{n,s}) + 2\mu_r(\operatorname{rot} w_t^{n-1,s}, u_t^{n,s}) + (\rho^{n-1,s}f_t, u_t^{n,s}) \\
&\quad -(\rho^{n-1,s}u_{tt}^n, u_t^{n,s}) - (\rho_t^{n-1+s}u^{n-1+s}\cdot\nabla u^{n,s}, u_t^{n,s}) \\
&\quad -(\rho^{n-1+s}u_t^{n-1+s}.\nabla u^{n,s}, u_t^{n,s}) - (\rho^{n-1+s}u^{n-1+s}\cdot\nabla u_t^{n,s}, u_t^{n,s}) \\
&\quad -(\rho_t^{n-1+s}u^{n-1,s}\cdot\nabla u^n, u_t^{n,s}) - (\rho^{n-1+s}u_t^{n-1,s}\cdot\nabla u^n, u_t^{n,s}) \\
&\quad -(\rho^{n-1+s}u^{n-1,s}\cdot\nabla u_t^n, u_t^{n,s}) - (\rho^{n-1,s}u_t^{n-1}.\nabla u^n, u_t^{n,s}) \\
&\quad -(\rho^{n-1,s}u^{n-1}\cdot\nabla u_t^n, u_t^{n,s}) + (\rho_t^{n-1,s}f, u_t^{n,s}) \\
&\quad -(\rho_t^{n-1,s}u_t^n, u_t^{n,s}) - (\rho_t^{n-1,s}u^{n-1}\cdot\nabla u^n, u_t^{n,s}).
\end{aligned}$$

Let us group the terms containing $\rho_t^{n-1,s}$, namely

$$h_2 = (\rho_t^{n-1,s}f, u_t^{n,s}) - (\rho_t^{n-1,s}u_t^n, u_t^{n,s}) - (\rho_t^{n-1,s}u^{n-1}\cdot\nabla u^n, u_t^{n,s}),$$

and denote the remaining terms by h_1. Then, we have

$$\frac{d}{dt}\|\sqrt{\rho^{n-1+s}}u_t^{n,s}\|^2 + 2\,(\mu+\mu_r)\|\nabla u_t^{n,s}\|^2 = 2\,h_1 + 2\,h_2.$$

Multiplying this equation by $\sigma(t) = \min\{1, t\}$ and integrating the result from 0 to t, we get

$$\begin{aligned}
&\sigma(t)\|\sqrt{\rho^{n-1+s}(t)}u_t^{n,s}(t)\|^2 + 2\,(\mu+\mu_r)\int_0^t \sigma(\tau)\|\nabla u_t^{n,s}(\tau)\|^2 d\tau \\
&\quad = \int_0^t \sigma'(t)\|\sqrt{\rho^{n-1+s}(\tau)}u_t^{n,s}(\tau)\|^2 d\tau + 2\,H_1(t) + 2\,H_2(t)
\end{aligned} \tag{4.20}$$

where $H_1(t) = \displaystyle\int_0^t \sigma(\tau)\,h_1(\tau)\,d\tau$ and $H_2(t) = \displaystyle\int_0^t \sigma(\tau)\,h_2(\tau)\,d\tau$.

Now, we estimate the right-hand side of the above equation. From the fact that $0 \le \sigma'(t) \le 1$ a. e. in $t \in [0, T]$, we have

$$\int_0^t \sigma'(t)\|\sqrt{\rho^{n-1+s}}(\tau)u_t^{n,s}(\tau)\|^2 d\tau \le \beta\int_0^t \|u_t^{n,s}(\tau)\|^2 d\tau \le \beta\, M\frac{(M_1T)^{n-1}}{(n-1)!} \tag{4.21}$$

as a consequence of (4.18). It is easy to show that

$$H_1(t) \le C\,\frac{(M_1T)^{n-2}}{(n-2)!} + \frac{\mu+\mu_r}{4}\int_0^t \sigma(\tau)\|\nabla u_t^{n,s}(\tau)\|^2 d\tau. \tag{4.22}$$

For each term in $h_2(t)$, using (4.3) and integrating by parts, we can obtain the same kind of bound. In fact,

$$
\begin{aligned}
(\rho_t^{n-1,s} f, u_t^{n,s}) &= -((u^{n-1,s}\cdot\nabla\rho^{n-1+s})\psi, u_t^{n,s}) - ((u^{n-1}\cdot\nabla\rho^{n-1,s})\psi, u_t^{n,s}) \\
&= -((u^{n-1,s}\cdot\nabla\rho^{n-1+s})\psi, u_t^{n,s}) + (\rho^{n-1,s}u^{n-1}\cdot\nabla\psi, u_t^{n,s}) \\
&\quad +(\rho^{n-1,s}u^{n-1}\cdot\nabla u_t^{n,s}, \psi) \\
&\le \|u^{n-1,s}\|_{L^4}\|\nabla\rho^{n-1+s}\|_{L^\infty}\|\psi\|\|u_t^{n,s}\|_{L^4} \\
&\quad +\|\rho^{n-1,s}\|_{L^6}\|u^{n-1}\|_{L^\infty}\|\nabla\psi\|\|u_t^{n,s}\|_{L^3} \\
&\quad +\|\rho^{n-1,s}\|_{L^6}\|u^{n-1}\|_{L^\infty}\|\nabla u_t^{n,s}\|\|\psi\|_{L^3} \\
&\le C\|\nabla u^{n-1,s}\|\|\psi\|\|\nabla u_t^{n,s}\| + C\|\rho^{n-1,s}\|_{L^6}\|\psi\|_{H^1}\|\nabla u_t^{n,s}\| \\
&\le C_\eta\|\nabla u^{n-1,s}\|^2\|\psi\|^2 + C_\eta\|\rho^{n-1,s}\|_{L^6}^2\|\psi\|_{H^1}^2 + 2\eta\|\nabla u_t^{n,s}\|^2. \qquad (4.23)
\end{aligned}
$$

Taking respectively $\psi = f$, $\psi = u_t^n$ and $\psi = u^{n-1}\cdot\nabla u_t^n$, choosing $\eta = \dfrac{\mu+\mu_r}{24}$, we have

$$
\begin{aligned}
H_2(t) \le\; & C\int_0^t \|\nabla u^{n-1,s}(\tau)\|^2\|f(\tau)\|^2 d\tau + C\int_0^t \|\rho^{n-1,s}(\tau)\|_{L^6}^2\|f(\tau)\|_{H^1}^2 d\tau \\
& +C\int_0^t \|\nabla u^{n-1,s}(\tau)\|^2 d\tau + C\int_0^t \|\rho^{n-1,s}(\tau)\|_{L^6}^2\|\nabla u_t^n(\tau)\|^2 d\tau \\
& +C\int_0^t \|\rho^{n-1,s}(\tau)\|_{L^6}^2 d\tau + \frac{\mu+\mu_r}{4}\int_0^t \sigma(\tau)\|\nabla u_t^{n,s}(\tau)\|^2 d\tau.
\end{aligned}
$$

Now, using (4.4), (4.15) and (4.16), we obtain

$$
H_2(t) \le M\frac{(M_1T)^{n-2}}{(n-2)!} + \frac{\mu+\mu_r}{4}\int_0^t \sigma(\tau)\|\nabla u_t^{n,s}(\tau)\|^2 d\tau. \qquad (4.24)
$$

Therefore, carrying (4.20) and (4.22) in (4.24), we obtain

$$
\sigma(t)\|u_t^{n,s}\|^2 + \int_0^t \sigma(\tau)\|\nabla u_t^{n,s}(\tau)\|^2 d\tau \le M\frac{(M_1T)^{n-2}}{(n-2)!}. \qquad (4.25)
$$

which proves the convergence for $u_t^{n,s}$. For u_t^n the arguments are similar.

of convergence in the Theorem is directly obtained from (4.5). Similarly, the
ergence is consequence of the previous bound, thanks to the Sobolev embedding

rate of convergence of Theorem 2.3 is obtained by repeating the same arguments
(see equation (3.29)). That is, write (4.1) as

$$
(\mu+\mu_r)Au^{n,s} = P(F)
$$

$$
\begin{aligned}
= \;& 2\mu_r\ rot\ w^{n-1,s} + \rho^{n-1,s}f - \rho^{n-1,s}u_t^n - \rho^{n-1+s}\cdot\nabla u^{n,s} \\
& -\rho^{n-1+s}u^{n-1,s}\cdot\nabla u^n - \rho^{n-1,s}u^{n-1}\cdot\nabla u^n - \rho^{n-1+s}u_t^{n,s}.
\end{aligned}
$$

ep consists in showing that $F \in L^2(0,T;L^6(\Omega))$ and in applying the Amrouche-

4.2 Passage to the limit

Once the convergences have been established, the passage to the limit is a standard procedure. We obtain

$$\int_0^T \langle\, \rho\, u_t + \rho\, u\cdot\nabla u - \rho\, f - 2\,\mu_r \mathrm{rot}\, w - (\mu+\mu_r)\Delta u, v\,\rangle \phi(t)\, dt = 0,$$

$$\int_0^T \langle\, \rho\, w_t + \rho\, u\cdot\nabla w - \rho\, g - 2\,\mu_r \mathrm{rot}\, u + 4\,\mu_r\, w - (c_a+c_d)\Delta w$$
$$-(c_0+c_d-c_a)\nabla \mathrm{div}\, w, z\,\rangle \psi(t)\, dt = 0,$$

for all $z,\, v\in L^2(\Omega)$ and $\phi,\, \psi\in L^\infty(0,T)$.

These equalities together with the Du Bois - Reymond's Theorem imply

$$\langle\, \rho\, u_t + \rho\, u\cdot\nabla u - \rho\, f - 2\,\mu_r \mathrm{rot}\, w - (\mu+\mu_r)\Delta u, v\,\rangle = 0,$$

$$\langle\, \rho\, w_t + \rho\, u\cdot\nabla w - \rho\, g - 2\,\mu_r \mathrm{rot}\, u + 4\,\mu_r\, w - (c_a+c_d)\Delta w - (c_0+c_d-c_a)\nabla \mathrm{div}\, w, z\,\rangle = 0,$$

a. e. in $[0,T]$, for every $v\in H,\ z\in L^2(\Omega)$. These last two equalities imply

$$P(\,\rho\, u_t + \rho\, u\cdot\nabla u - \rho\, f - 2\,\mu_r \mathrm{rot}\, w - (\mu+\mu_r)\Delta u\,) = 0 \quad \text{and}$$

$$\rho\, w_t + \rho\, u\cdot\nabla w - \rho\, g - 2\,\mu_r \mathrm{rot}\, u + 4\,\mu_r\, w - (c_a+c_d)\Delta w - (c_0+c_d-c_a)\nabla \mathrm{div}\, w = 0.$$

For the density, we proved that

$$u^n \longrightarrow u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)),$$
$$\rho_t^n \longrightarrow \rho_t, \quad \text{and} \quad \nabla\rho^n \longrightarrow \nabla\rho \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)).$$

Thus, when $n\to\infty$ in the approximated continuity equation, we obtain

$$\rho_t + u\cdot\nabla\rho = 0 \quad \text{in the} \quad L^2(0,T;L^2(\Omega)) - \text{sense}.$$

Now, we prove the continuity established in Theorem 2.3 for the solution (u,w,ρ). Firstly, given that $u\in L^\infty(0,T;D(A))$ and $u_t\in L^2(\varepsilon,T;D(A))$, by interpolation (see R. Temam [20], p. 260) u is a.e. equal to a continuous function from $[\varepsilon,T]$ into $D(A)$, i.e.,

$$u\in \mathscr{C}([\varepsilon,T];D(A)) \ \ \forall\,\varepsilon>0.$$

On the other hand, since $u_t\in L^2(\varepsilon,T;D(A))$, $u_{tt}\in L^2(\varepsilon,T;H)$, by interpolation we have

$$u_t\in \mathscr{C}([\varepsilon,T];V), \quad \forall\,\varepsilon>0.$$

Therefore,

$$u\in \mathscr{C}^1([\varepsilon,T];V)\cap \mathscr{C}([\varepsilon,T];D(A)), \quad \forall\,\varepsilon>0.$$

Analogously, we prove that

$$w\in \mathscr{C}^1([\varepsilon,T];H_0^1(\Omega))\cap \mathscr{C}([\varepsilon,T];D(B)), \quad \forall\,\varepsilon>0.$$

To prove the continuity at $t = 0$, we proceed as follows. It is easy to show that

$$\lim_{t\to 0^+} \|u(t) - u(0)\| = 0, \quad \lim_{t\to 0^+} \|\nabla u(t) - \nabla u(0)\| = 0.$$

To prove that

$$\lim_{t\to 0^+} \|Au(t) - Au(0)\| = 0.$$

it is sufficient to show that

$$\lim_{t\to 0^+} \sup \|Au(t)\| \leq \|Au_0\|,$$

since we already know that $u(t) \longrightarrow u_0$ in $H^1(\Omega)$.

Multiplying (2.5) by Au_t^{n+1} and integrating in Ω, we have

$$\begin{aligned}
&\frac{\mu+\mu_r}{2}\frac{d}{dt}\|Au^{n+1}\|^2 + \|\sqrt{\rho^n}\nabla u_t^{n+1}\|^2 \\
&\quad = -(\rho^n u^n \cdot \nabla u^{n+1}, Au_t^{n+1}) + 2\mu_r(\operatorname{rot} w^n, Au_t^{n+1}) + (\rho^n f, Au_t^{n+1}) \\
&\quad\quad -(\nabla\rho^n \cdot \nabla u_t^{n+1}, u_t^{n+1}).
\end{aligned}$$

Then, integrating from 0 to t, we get

$$\begin{aligned}
\|Au^{n+1}(t)\|^2 \;\leq\; & \|Au_0\|^2 + \frac{2}{\mu+\mu_r}[(-\rho^n(t)u^n(t)\cdot\nabla u^{n+1}(t) + 2\mu_r \operatorname{rot} w^n(t) \\
& +\rho^n(t)f(t), Au^{n+1}(t)) - (-\rho_0^n u^n(0)\cdot\nabla u_0^{n+1} + 2\mu_r \operatorname{rot} w_0^n \\
& +\rho_0^n f(0), Au_0^{n+1})] + \frac{2}{\mu+\mu_r}N(t)
\end{aligned}$$

uniformly in n and where

$$\begin{aligned}
N(t) \;=\; & \int_0^t |(\rho_t^n u^n \cdot \nabla u^{n+1} + \rho^n u_t^n \cdot \nabla u^{n+1} + \rho^n u^n \cdot \nabla u_t^{n+1} - 2\mu_r \operatorname{rot} w_t^n \\
& -\rho_t^n f - \rho^n f_t, Au^{n+1})|\, d\tau + \int_0^t |(\nabla\rho^n \cdot \nabla u_t^{n+1}, u_t^{n+1})|\, d\tau \\
\leq\; & c\int_0^t (\|\nabla u^{n+1}\| + \|\nabla u_t^n\| + \|\nabla u_t^{n+1}\| + \|\nabla w_t^n\| + \|f\| + \|f_t\|)\, d\tau \leq c\, t^{1/2}
\end{aligned}$$

by virtue of Hölder inequality and the estimates as given in Lemma 2.2.

From this, we conclude that

$$\begin{aligned}
\|Au(t)\|^2 \;\leq\; & \|Au_0\|^2 + c\,[(-\rho(t)u(t)\cdot\nabla u(t) + 2\mu_r \operatorname{rot} w(t) + \rho(t)f(t), Au(t)) \\
& -(-\rho_0 u(0)\cdot\nabla u_0 + 2\mu_r \operatorname{rot} w_0 + \rho_0 f(0), Au_0)] + C\, t^{1/2}.
\end{aligned}$$

Since $\rho(t)u(t)\cdot\nabla u(t) \longrightarrow \rho_0 u_0 \cdot \nabla u_0$, $\rho(t)f(t) \longrightarrow \rho_0 f(0)$, $\operatorname{rot} w(t) \longrightarrow \operatorname{rot} w_0$ in $L^2(\Omega)$ and $Au(t) \longrightarrow Au_0$ weakly in $L^2(\Omega)$ as $t \to 0^+$, we obtain the desired result. From this, it is easy to show

$$\lim_{t\to 0^+} \|u_t(t) - u_t(0)\| = 0.$$

The results for w are proved in the same way.

4.3 Uniqueness of the strong solution

To prove uniqueness, let us assume that (u, w, ρ) and (u_1, w_1, ρ_1) be two solutions of (1.1)-(1.2) with the same regularity as stated in Theorem 2.3. Now, define:

$$U = u_1 - u, \; W = w_1 - w \text{ and } R = \rho_1 - \rho.$$

These auxiliary functions verify a set of equations similar to (4.1)–(4.3). If we multiply the first equation by U, the second by W and the third by R and repeat the argument as given in the proof of Lemma 2.1 in 3.1.1, we obtain for $\varphi(t) = \|U(t)\|^2 + \|W(t)\|^2 + \|R(t)\|^2$ an inequality of the following type:

$$\varphi(t) \le C \int_0^t \varphi(\tau)\, d\tau$$

which, by Gronwall's inequality, is equivalent to assert $U = 0$, $W = 0$ and $R = 0$. The proof of Theorem 2.3 is completed.

5 Existence and Uniqueness of the Pressure

Lemma 5.1. *With the hypotheses of Lemma 2.1, for each n, there exists $p^n \in L^2(0, T; H^1(\Omega)/\mathbb{R})$ such that (u^n, w^n, ρ^n, p^n) is an approximate solution of problem (1.1)-(1.2), where (u^n, w^n, ρ^n) are given by Lemma 2.1. In addition, with the hypotheses of Lemma 2.2, p^n is uniformly bounded in $L^\infty(0, T; H^1(\Omega)/\mathbb{R})$.*

Proof. One can prove this lemma from (3.29) and the Amrouche-Girault's results (1991).

Lemma 5.2. *Under the hypotheses of Lemma 2.2, we have*

$$\int_0^t \|p^{n+s}(\tau) - p^n(\tau)\|^2_{H^1(\Omega)/\mathbb{R}} d\tau \le M \frac{(M_1 T)^{n-1}}{(n-1)!},$$

$$\sup_t \sigma(t) \|p^{n+s}(t) - p^n(t)\|^2_{H^1(\Omega)/\mathbb{R}} \le M \frac{(M_1 T)^{n-2}}{(n-2)!}.$$

for all $t \in [0, T]$.

Proof. We denote $p^{n,s} = p^{n+s} - p^n, \;\; \forall\, n \ge 1$. Then, from (2.5) and (4.1), we have

$$-(\mu + \mu_r)\Delta u^{n,s} + \nabla p^{n,s} = J \tag{5.1}$$

$$\begin{aligned} \text{where} \quad J \;\; = \;\; & 2\mu_r \operatorname{rot} w^{n-1,s} + \rho^{n-1,s} f - \rho^{n-1,s} u_t^n - \rho^{n-1+s} u^{n-1+s} \cdot \nabla u^{n,s} \\ & - \rho^{n-1+s} u^{n-1,s} \cdot \nabla u^n - \rho^{n-1,s} u^{n-1} \cdot \nabla u^n - \rho^{n-1+s} u_t^{n,s}. \end{aligned} \tag{5.2}$$

Moreover,

$$\begin{aligned} \|J\|^2 \;\; \le \;\; & C\,\|\nabla w^{n-1,s}\|^2 + C\,\|\rho^{n-1,s}\|^2_{L^6} \|f\|^2_{L^3} + C\,\|\rho^{n-1,s}\|^2_{L^6} \|u_t^n\|^2_{L^3} + C\,\|\nabla u^{n,s}\|^2 \\ & + C\,\|\nabla u^{n-1,s}\|^2 + C\,\|\rho^{n-1,s}\|^2_{L^6} \|\nabla u^n\|^2_{L^3} + C\,\|u_t^{n,s}\|^2. \end{aligned} \tag{5.3}$$

Now, (5.1)-(5.3) and the Amrouche-Girault's results [1], imply

$$\|p^{n,s}\|^2_{H^1(\Omega)/\mathbb{R}} \le C\,\|J\|^2 \tag{5.4}$$

and integrating it from 0 to t, we get

$$\begin{aligned}\int_0^t \|p^{n,s}(\tau)\|^2_{H^1(\Omega)/\mathbb{R}} d\tau \quad &\le \quad C\,M\,\Big[\frac{(M_1T)^{n-1}}{(n-1)!} + \frac{(M_1T)^{n-1}}{(n-1)!}\int_0^t \|f(\tau)\|^2_{L^3} d\tau \\ &+ \frac{(M_1T)^{n-1}}{(n-1)!}\int_0^t \|\nabla u^n_t(\tau)\|^2 d\tau + \frac{(M_1T)^n}{n!} \\ &+ \frac{(M_1T)^{n-1}}{(n-1)!}\int_0^t \|Au^n(\tau)\|^2 d\tau + \frac{(M_1T)^{n-1}}{(n-1)!}\Big]\end{aligned}$$

by virtue of (4.4), (4.16) and (4.18). Therefore,

$$\int_0^t \|p^{n,s}(\tau)\|^2_{H^1(\Omega)/\mathbb{R}} d\tau \le M\frac{(M_1T)^{n-1}}{(n-1)!}.$$

Also, from (5.2) and (5.4), with $\sigma(t) = \min\{1, t\}$, we have

$$\begin{aligned}\sigma(t)\|p^{n,s}\|^2_{H^1(\Omega)/\mathbb{R}} \quad &\le \quad \|\nabla w^{n-1,s}\|^2 + C\,\|\rho^{n-1,s}\|^2_{L^\infty}\|f\|^2 + C\,\|\rho^{n-1,s}\|^2_{L^\infty} \\ &+ C\,\|\nabla u^{n,s}\|^2 + C\,\|\nabla u^{n-1,s}\|^2 + C\,\|\rho^{n-1,s}\|^2_{L^6} + C\,\sigma(t)\|u^{n,s}_t\|^2 \\ &\le \quad C\,M\frac{(M_1T)^{n-2}}{(n-2)!} + C\,M\frac{(M_1T)^{n-2}}{(n-2)!}\|f\|^2 + C\,M\frac{(M_1T)^{n-1}}{(n-1)!} \\ &+ C\,M\frac{(M_1T)^{n-1}}{(n-1)!} + C\,M\frac{(M_1T)^{n-1}}{(n-1)!} + C\,M\frac{(M_1T)^{n-2}}{(n-2)!}\end{aligned}$$

by virtue of (4.4), (4.15), (4.18) and (4.19). Therefore, by interpolation, $f \in C([0,T]; L^2(\Omega))$. From the last inequality, we conclude

$$\sigma(t)\|p^{n,s}(t)\|^2_{H^1(\Omega)/\mathbb{R}} \le M\frac{(M_1T)^{n-2}}{(n-2)!}. \tag{5.5}$$

Theorem 5.3. *Under the hypotheses of Lemma 2.2, the approximate pressure p^n converge to the limiting element p in $L^2(0,T; H^1(\Omega)/\mathbb{R})$ and (u, w, ρ, p) is the unique solution of (1.1)-(1.2), where (u, w, ρ) is the solution given in the Theorem 2.3. Moreover, we have the following error estimate*

$$\int_0^t \|p^n(\tau) - p(\tau)\|^2_{H^1(\Omega)/\mathbb{R}} d\tau \le M\frac{(M_1T)^{n-1}}{(n-1)!}.$$

Also, p^n converges to p in $L^\infty(\varepsilon, T; H^1(\Omega)/\mathbb{R})$, for all $\varepsilon > 0$ and is satisfy the following error estimate

$$\sup_t \sigma(t)\|p^n(t) - p(t)\|^2_{H^1(\Omega)/\mathbb{R}} \le M\frac{(M_1T)^{n-2}}{(n-2)!}.$$

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