Scaling limit of random planar maps
Lecture 2.

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Workshop on randomness and enumeration
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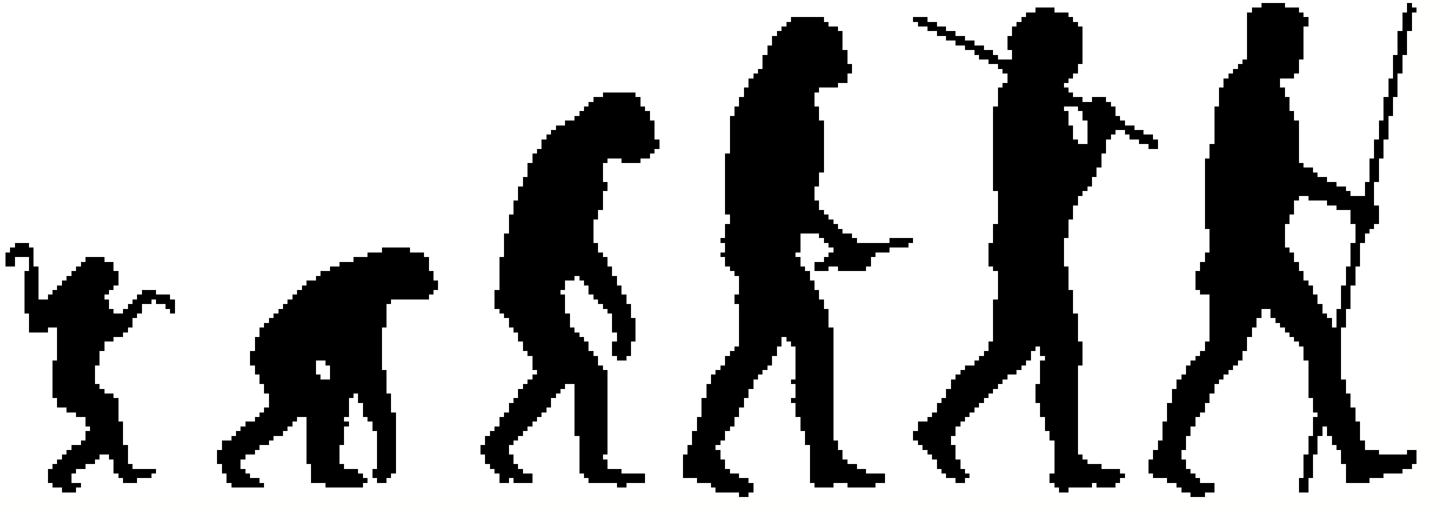
Goal

We consider quadrangulations as metric spaces: (V, d).

Question: What random metric space is the limit in distribution (for the Gromov-Hausdorff topology) of rescaled uniformly random quadrangulations of size n, when n goes to infinity ?



Outline

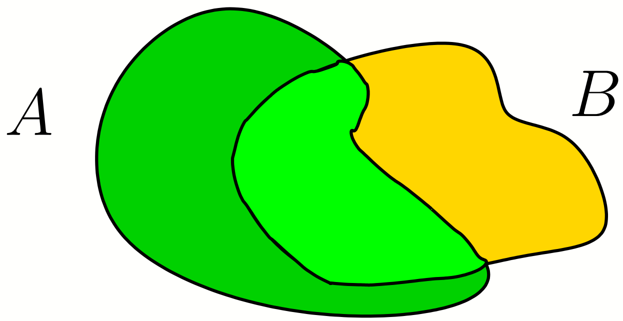
- Gromov-Hausdorff topology (on metric spaces).
- Brownian motion, convergence in distribution.
- Convergence of trees: the Continuum Random Tree.
- Convergence of maps: the Brownian map.

A metric on metric spaces:

the Gromov-Hausdorff distance

How to compare metric spaces ?

The Hausdorff distance between two sets A, B in a metric space (S, d) is the infimum of $\epsilon > 0$ such that any point of A lies at distance less than ϵ from a point of B and any point of B lies at distance less than ϵ from a point of A.



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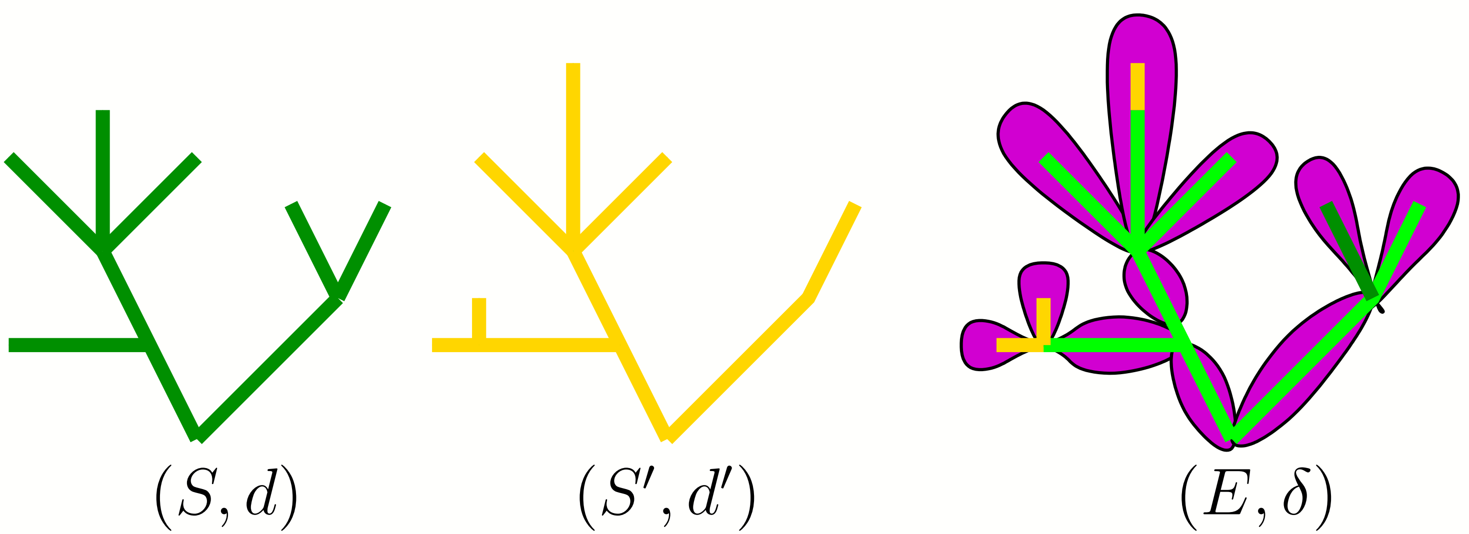
The Gromov-Hausdorff distance between two metric spaces (S, d) and (S', d') is the infimum of $d_H(A, A')$ over all isometric embeddings (A, δ), (A', δ) of (S, d) and (S', d') in an metric space (E, δ).



Hausdorff distance and distortions

Let (S, d) and (S', d') be metric spaces.

A correspondence between S and S' is a relation $R \subseteq S \times S'$ such that any point in S is in relation with some points in S' and any point in S' is in relation with some points in S.

The distortion of the correspondence R is the supremum of $|d(x,y) - d'(x',y')|$ for $x, y \in S$, $x', y' \in S'$ such that xRx' and yRy'.

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Prop: The Gromov-Hausdorff distance between (S, d) and (S', d') is half the infimum of the distortion over all correspondences.

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Corollary: The Gromov-Hausdorff distance between (T_f, d_f) and (T_g, d_g) is less than $2||f - g||_\infty$.

Proof: The distortion is $4||f - g||_\infty$ for the correspondence R between T_f and T_g defined by uRv if $\exists s \in [0, 1]$ such that $u = \tilde{s}^f$ and $v = \tilde{s}^g$. □

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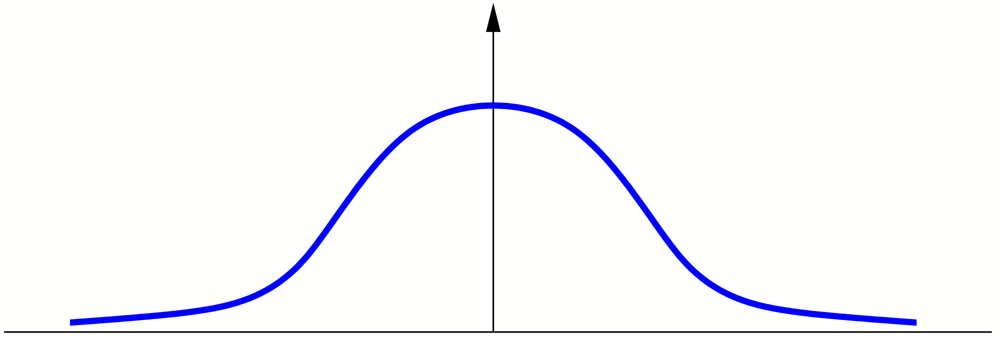
In particular, the function $f \to T_f$ is continuous from $(C([0, 1], \mathbb{R}), ||.||_\infty)$ to $(\mathbf{M}, d_{GH})$.

Brownian motion,
convergence in distribution

Gaussian variables

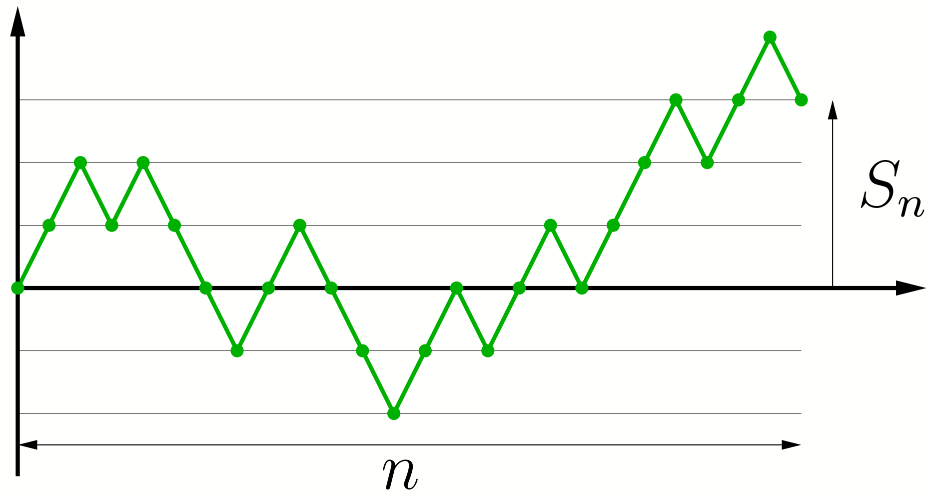
We denote by $\mathcal{N}(0, \sigma^2)$ the Gaussian probability distribution on $\mathbb{R}$ having density function $f(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{t^2}{2\sigma^2})$.



Gaussian variables

Let $(X_n)_n$ be independent random variables taking value ± 1 with probability $1/2$ and let $S_n = \sum_{i=1}^{n} X_i$.

By Central Limit Theorem, $\frac{S_n}{\sqrt{n}}$ converges in distribution toward $\mathcal{N}(0,1)$.



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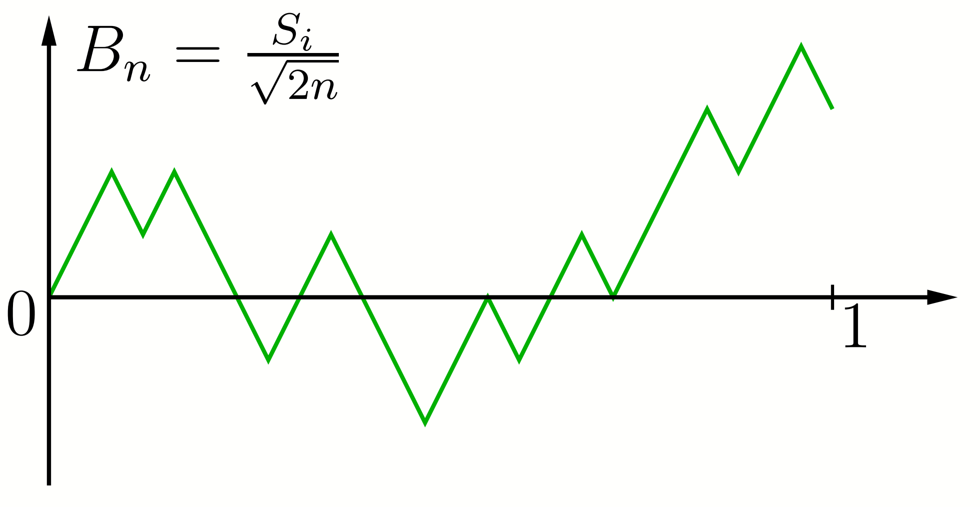
Reminder: A sequence (X_n) of real random variables converges in distribution toward X if the sequence of cumulative distribution functions (F_n) converges pointwise to F at all points of continuity.

Distribution of rescaled random paths

Let $(X_n)_n$ be independent random variables taking value ± 1 with probability $1/2$ and let $S_n = \sum_{i=1}^{n} X_i$.

By considering $\frac{1}{\sqrt{n}}(S_1, \ldots, S_n)$ as a piecewise linear function from $[0, 1]$ to $\mathbb{R}$, one obtains a random variable, denoted B_n, taking value in $C([0, 1], \mathbb{R})$.



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Proposition: For all $0 \leq t \leq 1$, the random variables $B_t^n = B_n(t)$ converge in distribution toward $\mathcal{N}(0,t)$.

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Proposition:
For all $t_0 = 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$, the random variables $B^n_{t_{i+1}} - B^n_{t_i}$ are independents and converge in distribution toward $\mathcal{N}(0, t_{i+1} - t_i)$.

Brownian motion

The Brownian motion (on $[0,1]$) is a random variable $B = (B_t)_{t\in[0,1]}$ taking value in $C([0,1],\mathbb{R})$ such that for all $t_0 = 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$ the random variables $B_{t_{i+1}} - B_{t_i}$ are independents and have distribution $\mathcal{N}(0, t_{i+1} - t_i)$.

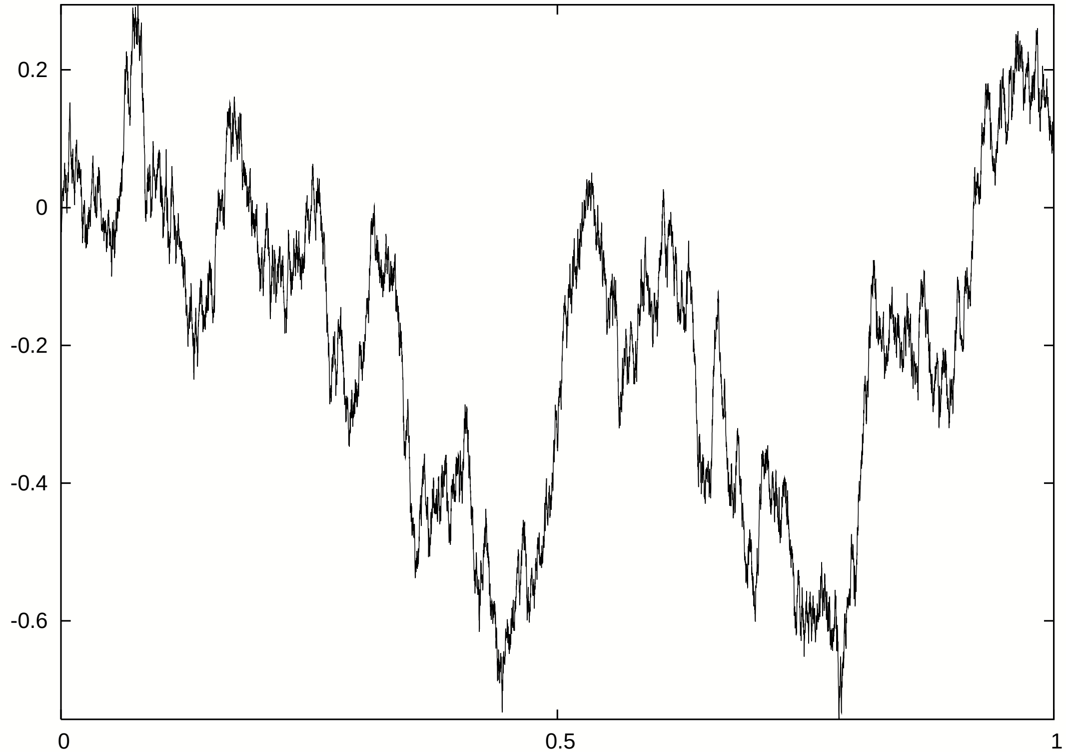

Image credit: J-F Marckert

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Remarks:

• The distribution of a process $B = (B_t)_{t\in I}$ is characterized by the finite-dimensional distributions.

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Remarks:

• The distribution of a process $B = (B_t)_{t\in I}$ is characterized by the finite-dimensional distributions.

• The existence of a process $(B_t)_{t\in[0,1]}$ with this distribution is a consequence of Kolmogorov Theorem.

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Remarks:

- The distribution of a process $B = (B_t)_{t\in I}$ is characterized by the finite-dimensional distributions.
- The existence of a process $(B_t)_{t\in[0,1]}$ with this distribution is a consequence of Kolmogorov Theorem.
- The existence of $(B_t)_{t\in[0,1]}$ with continuous trajectory can be obtained via a Lemma of Kolmogorov.

Properties of Brownian motion

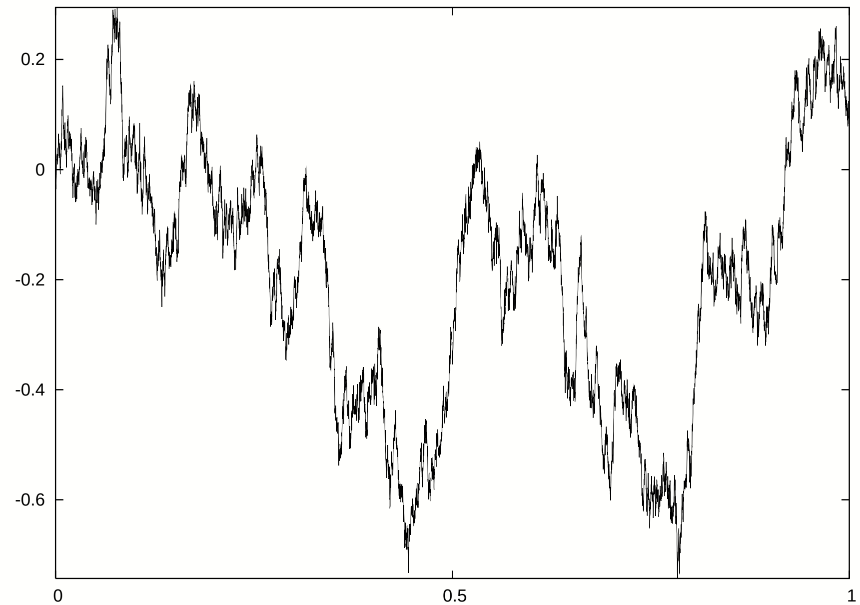
Properties almost sure of the Brownian motion:

• The Brownian motion is nowhere differentiable.

• It is Hölder continuous of exponent $1/2 - \epsilon$ for all $\epsilon > 0$.

• For fixed $t \in]0,1[$, t is not a left-minimum nor right-minimum nor …

• The value of local minima/maxima are all distinct.



Convergence in distribution

A polish space is a metric space which is complete and separable.

For instance, $(C([0,1],\mathbb{R}), ||.||_\infty)$ and $(\mathbf{M}, d_{GH})$ are Polish.

Convergence in distribution

We consider a Polish space (S, d) together with its Borel σ-algebra (generated by the open sets).

Definition: A sequence of random variables (X_n) taking value in S converges in distribution (i.e. in law, weakly) toward X if $\mathbb{E}(f(X_n)) \to \mathbb{E}(f(X_n))$ for any bounded continuous function $f : S \mapsto \mathbb{R}$.

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Remarks:

• There are other characterizations of convergence in distribution (Portmanteau Theorem).

• When $S = \mathbb{R}$, convergence in distribution is equivalent to the convergence pointwise of the cumulative distribution function at all points of continuity.

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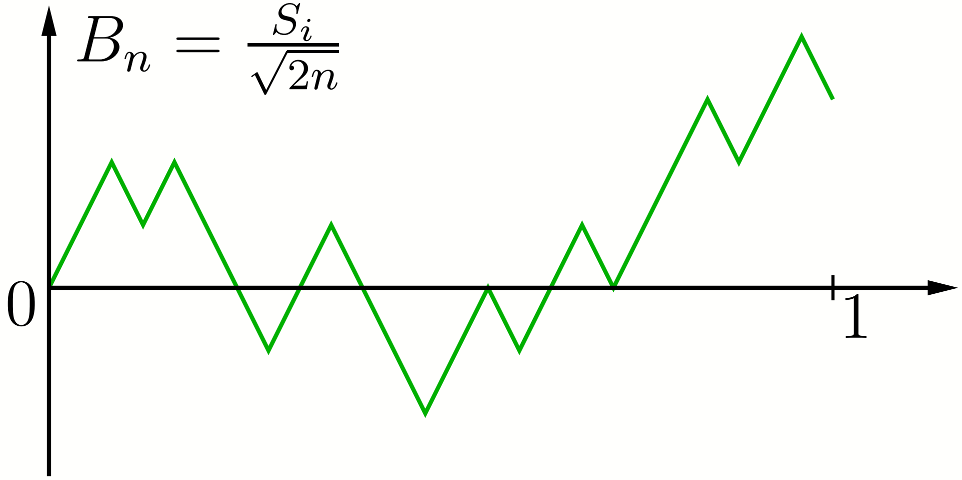
Remarks:

- Convergence almost sure implies convergence in distribution.
- Skorokhod Theorem gives a reciprocal: If $X_n \xrightarrow{\text{dist}} X$, then there are couplings $\tilde{X}_n, \tilde{X}$ such that $\tilde{X}_n \xrightarrow{\text{a.s.}} \tilde{X}$.

Brownian motion as limit of discrete paths

Let B_n be (as before) the random variables taking values in $C([0,1],\mathbb{R})$ and obtained from the uniform distribution on rescaled lattice paths on $\mathbb{N}$.



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Let B_n be (as before) the random variables taking values in $C([0,1],\mathbb{R})$ and obtained from the uniform distribution on rescaled lattice paths on $\mathbb{N}$.

We have seen that the finite-dimensionals of B_n converge (in distribution) toward the finite-dimensionals of the Brownian motion B.

This proves that if (B_n) converges in distribution in $(C([0,1],\mathbb{R}), ||.||_\infty)$ it must be toward the Brownian motion.

However, this does not prove convergence and we need to use a tightness argument.

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Let B_n be (as before) the random variables taking values in $C([0,1],\mathbb{R})$ and obtained from the uniform distribution on rescaled lattice paths on $\mathbb{N}$.

A sequence (X_n) taking value in S is tight if $\forall \epsilon > 0$ there exists a compact $K \subseteq S$ such that $\forall n,\ \mathbb{P}(X_n \in K) > 1 - \epsilon$.

Theorem (Prohorov): If (X_n) is tight, then (X_n) converges in distribution *along a subsequence*.
In particular, if the limit X is uniquely determined, then (X_n) converges to X in distribution.

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Here, to prove the tightness of (B_n), one uses the compacts $K_{(\delta_n)} \subseteq C([0,1],\mathbb{R})$ of (δ_n)-uniformly continuous functions, that is, the set of functions f such that $|s-t| \leq 2^{-n} \ \Rightarrow\ |f(s)-f(t)| \leq \delta_n$.

Brownian excursion

The Brownian excursion is a random variable $e = (e_t)_{t \in [0,1]}$ with value in $C([0,1], \mathbb{R}^+)$ satisfying $e_0 = 0$, $e_1 = 0$ and having finite-dimensional distributions (...).

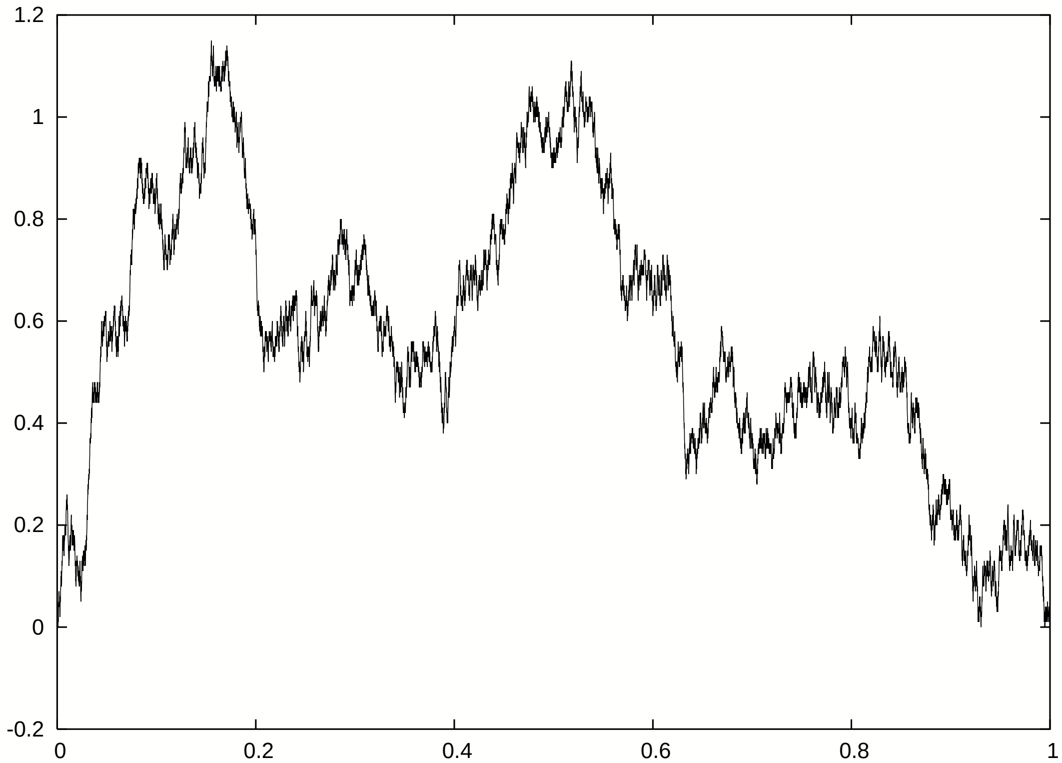

Image credit: J-F Marckert

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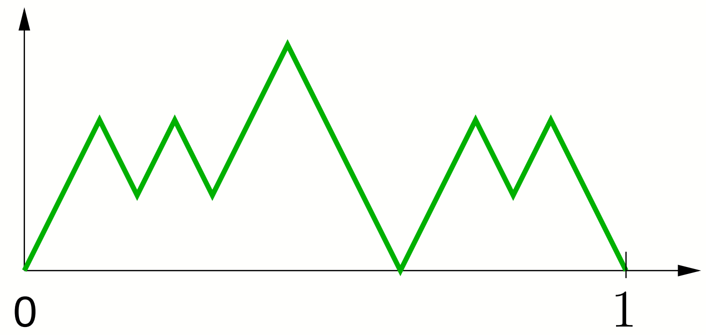
The existence of e can be obtained either
- by brute force: Kolmogorov,
- by conditioning the Brownian motion,
- by rescaling a well-chosen piece of the Brownian motion,
- as the limit of uniform Dyck paths rescaled by $\sqrt{2n}$.



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Limit of random discrete trees:
the Continuum Random Tree

The Continuum Random Tree (Aldous 91)

The Continuum Random Tree is the real tree T_e encoded by the Brownian excursion e.

This is a random variable taking value in the space $(\mathbf{M}, d_{GH})$ of compact metric spaces.

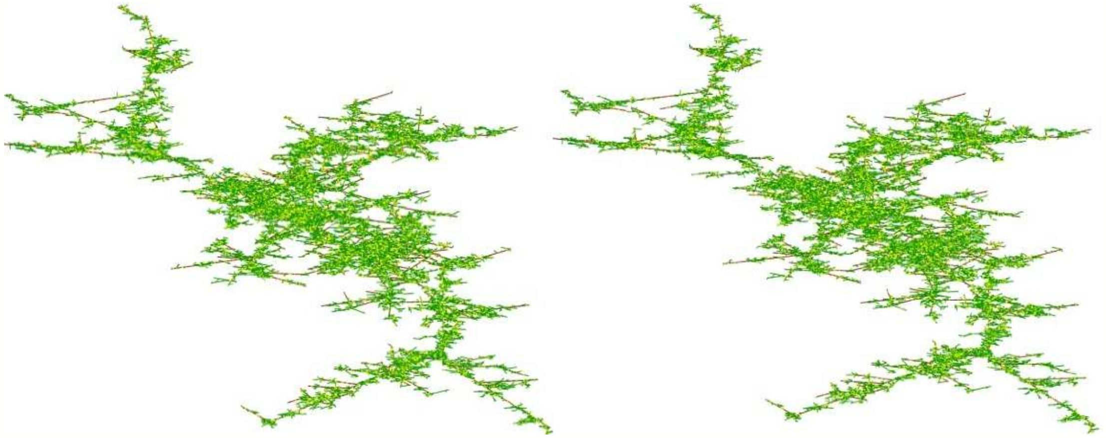

Image credit: G. Miermont

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Properties almost sure of the CRT:

- Any point has degree 1, 2 or 3.
- There are countably many points of degree 3.
- For the measure inherited from the uniform measure on $[0, 1]$, a point is a leaf with probability 1.
- The Hausdorff dimension is 2.

Hausdorff dimension

Definition: The Hausdorff dimension $\text{dim}_H(X)$ of a metric space (X, d) is defined as follows:

for $\alpha > 0$,

$C_H^\alpha = \lim_{\epsilon \to 0} \inf(\sum_i r_i{}^\alpha, \text{ where } r_i < \epsilon \text{ and } \exists x_i, \ X = \bigcup_i B(x_i, r_i))$,

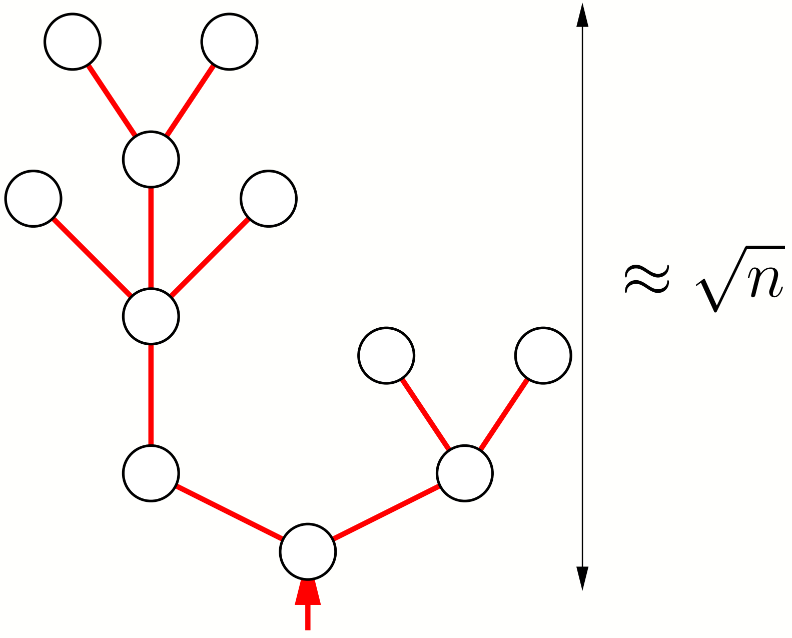
and

$\text{dim}_H(X) \ = \ \inf(\alpha : \ C_H^\alpha(X) = 0) \ = \ \sup(\alpha : \ C_H^\alpha(X) = \infty)$.

For instance, the Hausdorff dimension of a non-degenerate subset of $\mathbb{R}^d$ is d.

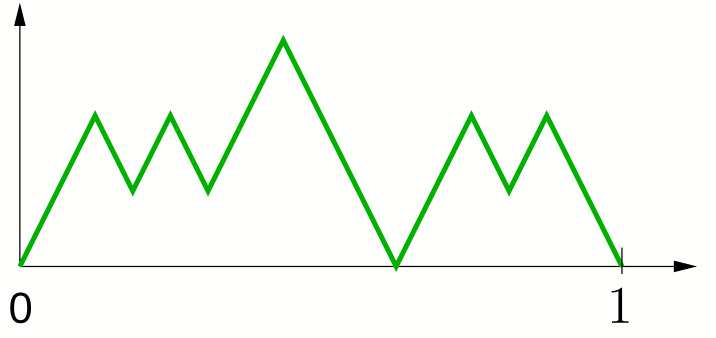
CRT as limit of discrete trees

The height of a uniformly random Dyck path of length $2n$ is of order $\sqrt{n}$.
Hence, the typical (and maximal) distance in a uniformly random tree of size n is of order $\sqrt{n}$.



CRT as limit of discrete trees

Let E_n be the random variable taking value in $C([0,1], \mathbb{R}^+)$ obtained from uniformly random Dyck paths of length $2n$ rescaled by $\sqrt{2n}$.



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We consider the random real tree T_{E_n} encoded by E_n.

In other words, T_{E_n} is the real tree corresponding to the uniformly random discrete tree of size n.

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Theorem: The sequence (T_{E_n}) converges in distribution toward the CRT T_e, in the space $(\mathbf{M}, d_{GH})$.

Proof: The random variables E_n converges toward the Brownian excursion e in distribution (in $C([0,1],\mathbb{R}^+)$). Moreover, the mapping $f \to T_f$ is continuous. $\square$

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What about the random discrete metric space $T_n = (V_n, \frac{d}{\sqrt{2n}})$?

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What about the random discrete metric space $T_n = (V_n, \frac{d}{\sqrt{2n}})$?

Theorem: The sequence (T_n) converges in distribution toward the CRT, in the space $(\mathbf{M}, d_{GH})$.

Proof: The Gromov-Hausdorff distance $d_{GH}(T_n, T_{E_n})$ is at most $\frac{1}{\sqrt{2n}}$. $\square$

CRT as limit of discrete trees

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We consider the random real tree T_{E_n} encoded by E_n.

What about the random discrete metric space $T_n = (V_n, \frac{d}{\sqrt{2n}})$?

Theorem: The sequence (T_n) converges in distribution toward the CRT, in the space $(\mathbf{M}, d_{GH})$.

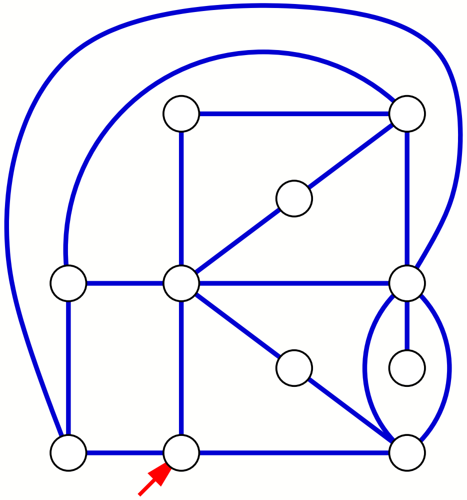
• More generally, many families of discrete trees (Galton Watson trees) converge toward the CRT.
• The theorem can be reinforced to deal with *measured metric space*: the uniform distribution on the vertices of discrete trees leads to a measure on the CRT which is the image of the uniform measure on $[0,1]$.

Limit of quadrangulations

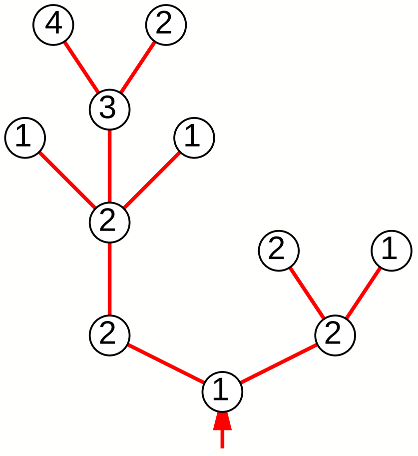
the Brownian map

From previous lecture:

Quadrangulations are in bijection with well-labelled trees.



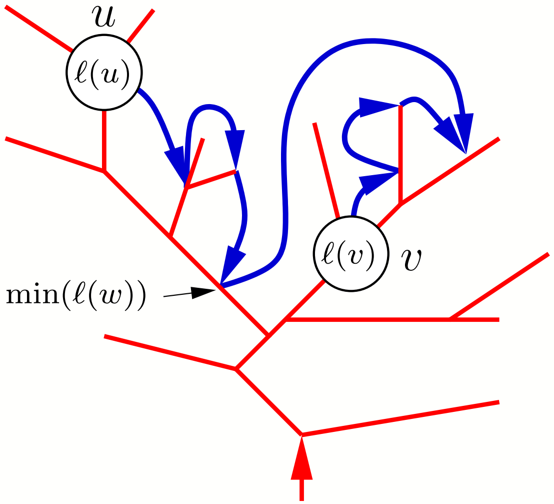
$\Longleftrightarrow$



From previous lecture:

Quadrangulations are in bijection with well-labelled trees.

The distance between vertices u, v in the quadrangulation is less than
$d^0_Q(u,v) = \ell(u) + \ell(v) + 2 - 2\min(\ell(w) : \ w \in u \leftrightsquigarrow_T v)$
hence less than
$d^*_Q(u,v) = \min_{u=u_0,u_1,\dots,u_k=v} \sum_i d^0_Q(u_i, u_{i+1})$.



Brownian map (Marckert & Mokkadem 06)

Recall that if T_f is a real tree and $g \in C(T_f, \mathbb{R}^+)$ satisfies $g(\rho) = 0$, then $T_{f,g}$ denotes the real quadrangulation obtained by quotienting T_f by the relation $D^*(u, v) = 0$, where

- $D^0(u, v) = g(u) + g(v) - 2 \inf(g(w) : \; w \in u \leftrightsquigarrow_T v)$.
- $D^*(u, v) = \inf_{u=u_0,u_1,\ldots,u_k=v} \sum_i D^0_Q(u_i, u_{i+1})$.

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- $D^*(u,v) = \inf_{u=u_0,u_1,\ldots,u_k=v} \sum_i D^0_Q(u_i, u_{i+1})$.

The Brownian map is the random real quadrangulation $(T_{e,\ell}, D^*)$, where e is the Brownian excursion and $\ell = (\ell_v)_{v \in T_e}$ is a Gaussian process such that $\ell_\rho = 0$ and $\ell_u - \ell_v$ has distribution $\mathcal{N}(0, d_T(u,v))$, *conditioned to be non-negative*.

[It is possible to make sense of this definition]

Brownian map (Marckert & Mokkadem 06)

The Brownian map is the random real quadrangulation $(T_{e,\ell}, D^*)$, where e is the Brownian excursion and $\ell = (\ell_v)_{v \in T_e}$ is a Gaussian process such that $\ell_\rho = 0$ and $\ell_u - \ell_v$ has distribution $\mathcal{N}(0, d_T(u,v))$, *conditioned to be non-negative*.

Theorem [Le Gall & Paulin 08]: Almost surely, the Brownian map is homeomorphic to the sphere and has Hausdorff dimension 4.

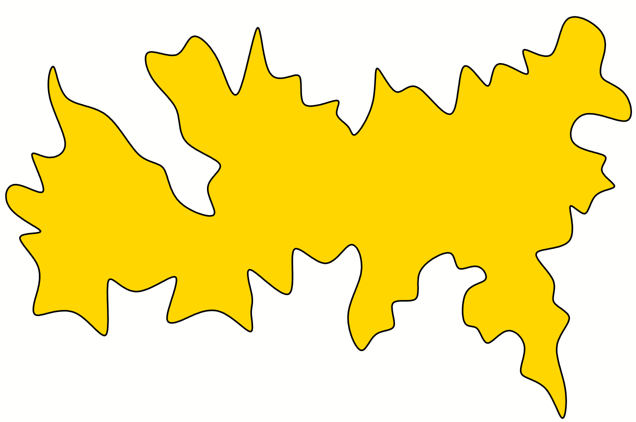
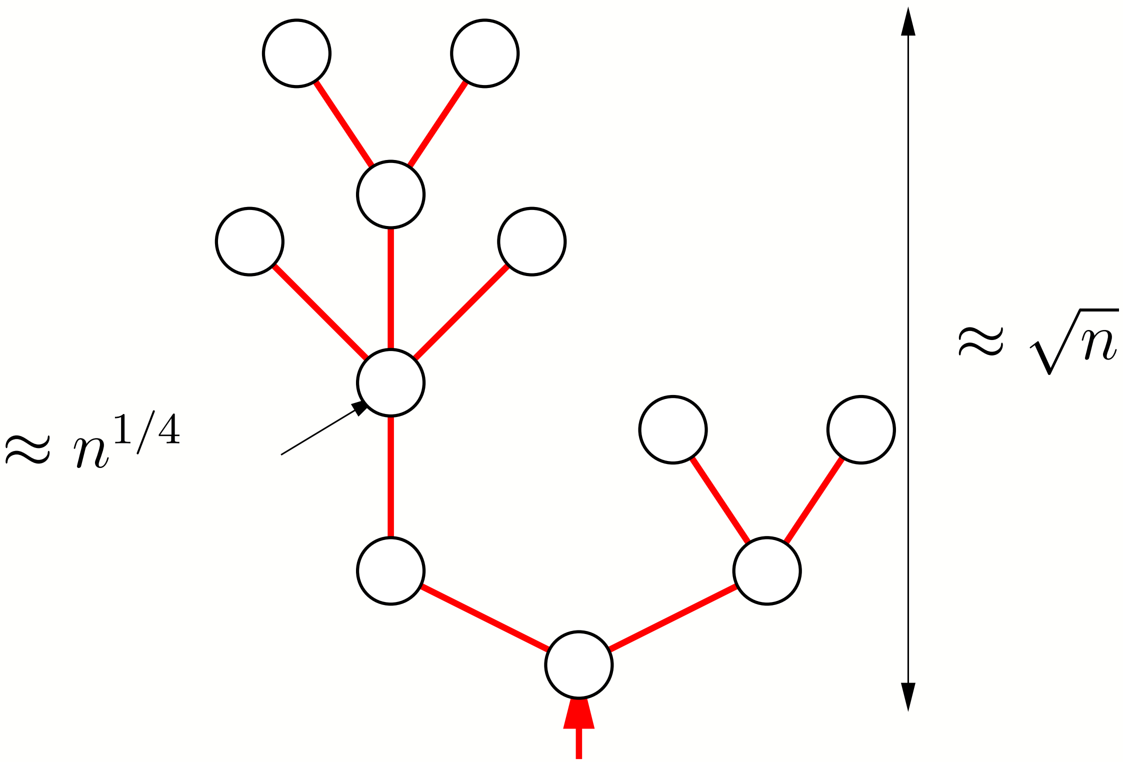


Photo non-contractuelle.

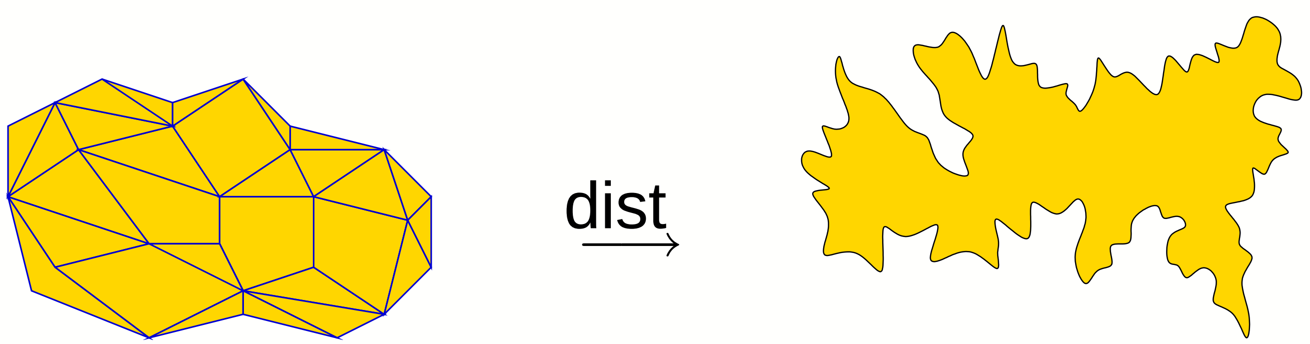
Brownian map as limit of maps

Proposition: A labelled tree of size n has height of order $\sqrt{n}$ and labels of order $n^{1/4}$.



Brownian map as limit of maps

Theorem [Le Gall 08]: The uniformly random quadrangulation Q_n considered as a metric space $(V_n, \left(\frac{9}{8n}\right)^{1/4} D_n)$ converges in distribution for the Gromov-Hausdorff topology toward $(T_{e,\ell}, D)$ along some subsequences,
where $T_{e,\ell}$ is the Brownian map and D is a distance on $T_{e,\ell}$ which is bounded by D^*.



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where $T_{e,\ell}$ is the Brownian map and D is a distance on $T_{e,\ell}$ which is bounded by D^*.

Remarks:
• Almost surely, the space $(T_{e,\ell}, D)$ is homeomorphic to the sphere (since the Brownian map is) and moreover it has Hausdorff dimension 4.
• The convergence holds for measured metric spaces.

Brownian map as limit of maps (proof)

Step 1. [Schaeffer 98] The uniformly random quadrangulation Q_n is represented by (E_n, L_n, D_n) , where

- $E_n \in C([0,1], \mathbb{R})$ is the Dyck path encoding the tree,
- $L_n \in C([0,1], \mathbb{R})$ encodes the labels,
- $D_n \in C([0,1]^2, \mathbb{R})$ encodes the distance.

Brownian map as limit of maps (proof)

Step 1. [Schaeffer 98] The uniformly random quadrangulation Q_n is represented by (E_n, L_n, D_n).

Step 2. [Chassaing & Schaeffer 04, Marckert & Mokkadem 06]
The variable $(\frac{E_n}{\sqrt{2n}}, \left(\frac{9}{8n}\right)^{1/4} L_n)$ converges in distribution toward (e, ℓ) in $C([0,1], \mathbb{R}^2)$.

Moreover, ℓ can be considered as a function from T_e to $\mathbb{R}$ and $T_{e,\ell}$ is the Brownian map.

Brownian map as limit of maps (proof)

Step 1. [Schaeffer 98] The uniformly random quadrangulation Q_n is represented by (E_n, L_n, D_n).

Step 2. [Chassaing & Schaeffer 04, Marckert & Mokkadem 06] The variable $(\frac{E_n}{\sqrt{2n}}, (\frac{9}{8n})^{1/4} L_n)$ converges in distribution toward (e, ℓ) in $C([0,1], \mathbb{R}^2)$.

Step 3. [Le Gall 08] The variable $(\frac{E_n}{\sqrt{2n}}, (\frac{9}{8n})^{1/4} L_n, (\frac{9}{8n})^{1/4} D_n)$ converges in distribution in $C([0,1], \mathbb{R}^2)$ toward (e, ℓ, D) along some subsequences.

Sketch of Proof:
- Bound the variations of D_n by those of D_n^0.
- Prove the uniform continuity of the sequence D_n^0.
- $\Rightarrow$ Tightness of D_n $\Rightarrow$ convergence along subsequences.

Brownian map as limit of maps (proof)

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Step 4. The function D defines a distance on $T_{e,\ell}/\!\approx$, where $\approx$ is the relation $D = 0$. Moreover, the convergence $(V_n, \left(\frac{9}{8n}\right)^{1/4} D_n) \xrightarrow{\text{dist}} (T_{e,\ell}/\!\approx, D)$ holds for Gromov-Hausdorff.

(Exhibit a distortion going to 0).

Brownian map as limit of maps (proof)

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Step 2. [Chassaing & Schaeffer 04, Marckert & Mokkadem 06] The variable $(\frac{E_n}{\sqrt{2n}}, (\frac{9}{8n})^{1/4} L_n)$ converges in distribution toward (e, ℓ) in $C([0,1], \mathbb{R}^2)$.

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Step 5. Almost surely, $T_{e,\ell}/\approx = T_{e,\ell}$. (Hardest part) $\square$

Properties and open questions

Theorem [Le Gall 08]: The uniformly random quadrangulation Q_n considered as a metric space $(V_n, \left(\frac{9}{8n}\right)^{1/4} D_n)$ converges in distribution for the Gromov-Hausdorff topology toward $(T_{e,\ell}, D)$ along some subsequences,
where $T_{e,\ell}$ is the Brownian map and D is a distance on $T_{e,\ell}$ which is bounded by D^*.

- Similar results hold for $2p$-angulations.
- It is an open question to know whether $D = D^*$.

In this case, the convergence in distribution would hold for the whole sequence.

Thanks