Who Wins Domineering on Rectangular Boards?

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Abstract. Using mostly elementary considerations, we find out who wins the game of Domineering on all rectangular boards of width 2, 3, 5, and 7. We obtain bounds on other boards as well, and prove the existence of polynomial-time strategies for playing on all boards of width 2, 3, 4, 5, 7, 9, and 11. We also comment briefly on toroidal and cylindrical boards.

1 Introduction

Domineering or Crosscram is a game invented by Göran Andersson and introduced to the public in [1]. Two players, say Vera and Hepzibah, have vertical and horizontal dominoes respectively. They start with a board consisting of some subset of the square lattice and take turns placing dominoes until one of them can no longer move. For instance, the 2×2 board is a win for the first player, since whoever places a domino there makes another space for herself while blocking the other player's moves.

A beautiful theory of combinatorial games of this kind, where both players have perfect information, is expounded in [2, 3]. Much of its power comes from dividing a game into smaller subgames, where a player has to choose which subgame to make a move in. Such a combination is called a *disjunctive sum*. In Domineering this happens by dividing the remaining space into several components, so that each player must choose in which component to place a domino.

Each game is either a win for Vera, regardless of who goes first, or Hepzibah regardless of who goes first, or the first player regardless of who it is, or the second regardless of who it is. These correspond to a value G which is positive, negative, fuzzy, or zero, i.e. $G > 0$, $G < 0$, $G \,\|\, 0$, or $G = 0$. (By convention a win for Hepzibah is negative.) However, we will often abbreviate these values as $G = V$, H, 1st, or 2nd. We hope this will not confuse the reader too much.

In this paper, we find who wins Domineering on all rectangles, cylinders, and tori of width 2, 3, 5, and 7. We also obtain bounds on boards of width 4, 7, 9, and 11, and partial results on many others. We also comment briefly on toroidal and cylindrical boards.

Note that this is a much coarser question than calculating the actual game-theoretical values of these boards, which determine how they act when disjunctively summed with other games. Berlekamp [4] found exact values for $2 \times n$

rectangles with n odd, and approximate values to within an infinitesimal or 'ish' (which unfortunately can change who wins in unusual situations) for other positions of width 2 and 3. In terms of who wins, the 8×8 board and many other small boards were recently solved by Breuker, Uiterwijk and van den Herik using a good system of transposition tables by [6–8]. We make use of these results below.

2 $2 \times n$ boards

On boards of width 2, it is natural to consider dividing it into two smaller boards of width 2. At first glance, Vera (the vertical player) has greater power, since she can choose where to do this. However, she can only take full advantage of this if she goes first. Hepzibah (the horizontal player) has a greater power, since whether she goes first or second, she can divide a game into two simply by *not* placing a domino across their boundary. We will see that, for sufficiently large n, this gives Hepzibah the upper hand.

We will abbreviate the value of the $2 \times n$ game as $[n]$.

Let's look at what happens when Vera goes first, and divides a board of length $m + n + 1$ into one of length m and one of length n. If she can win on both these games, i.e. if $[m] = [n] = V$, clearly she wins. If $[m] = V$ and $[n] = $ 2nd, Hepzibah will eventually lose in $[m]$ and be forced to play in $[n]$, whereupon Vera replies there and wins. Finally, if $[m] = [n] =$ 2nd, Vera replies to Hepzibah in both and wins. All this is summarized by the equation

$$\text{If } [m] \geq 0 \text{ and } [n] \geq 0, \text{ then } [m+n] \geq 0 \text{ and } [m+n+1] >\| 0. \qquad (1)$$

Since Vera can win if she goes first, $[m + n + 1]$ must be a win either for the first player or for V. This gives us the following table for combining boards of lengths m and n into boards of length $m + n + 1$:

$[m+n+1]$	2nd	V
2nd	1st or V	1st or V
V	1st or V	1st or V

(2)

Hepzibah has a similar set of tools at her disposal. By declining to ever place a domino across their boundaries, she can effectively play $[m+n]$ as a sum of $[m]$ and $[n]$ for whichever m and n are the most convenient. If Hepzibah goes first, she can win whenever $[m] =$ 1st and either $[n] =$ 2nd or $[n] = H$, by playing first in $[m]$ and replying to Vera in $[n]$. If $[m] = [n] = H$ she wins whether she goes first or second, and if $[m] =$ 2nd and $[n] = H$, the same is true since she plays in $[n]$ and replies to Vera in $[m]$. Finally, if $[m] = [n] =$ 2nd, she can win if she goes second by replying to Vera in both games. This gives the table

$[m+n]$	1st	2nd	H
1st	?	1st or H	1st or H
2nd	1st or H	2nd or H	H
H	1st or H	H	H

(3)

which can be summarized by the equation

$$[m+n] \leq [m] + [n]. \tag{4}$$

This simply states that refusing to play across a vertical boundary can only make it harder for Hepzibah.

These two tables alone, in conjunction with some search by hand and by computer, allow us to determine the following values. Values derived from smaller games using Tables 2 and 3 are shown in plain, while those found in other ways, such as David Wolfe's Gamesman's Toolkit [5], our own search program, or Berlekamp's solution for odd lengths [4] are shown in bold.

0	**2nd**	10	1st	20	H	30	H
1	**V**	11	1st	21	H	31	**H**
2	**1st**	12	H	22	**H**	32	H
3	**1st**	13	**2nd**	23	1st	33	H
4	**H**	14	1st	24	H	34	H
5	**V**	15	1st	25	H	35	H
6	1st	16	H	26	H	36	H
7	1st	17	H	27	1st	37	H
8	H	18	**1st**	28	H	38	H
9	**V**	19	1st	29	H	39	H

In fact, $[n]$ is a win for Hepzibah for all $n \geq 28$.

Some discussion is in order. Once we know that $[4] = H$, we have $[4k] = H$ for all $k \geq 1$ by Table 3. Combining Tables 2 and 3 gives $[6] = [7] = \text{1st}$, since these are both 1st or V and 1st or H. A similar argument gives $[10] = [11] = \text{1st}$ and $[14] = [15] = \text{1st}$, once we learn through search that $[13] = \text{2nd}$ (which is rather surprising, and breaks an apparent periodicity of order 4).

Combining [13] with multiples of 4 and with itself gives $[13+4k] = [26+4k] = [39+4k] = H$ for $k \geq 1$. Since $26 = 24+2 = 13+13$, we have $[26] = H$ since it is both 1st or H and 2nd or H, giving $[39] = H$ since 39=26+13.

Since $19 = 9+9+1 = 17+2$, we have $[19] = \text{1st}$ since it is both 1st or V and 1st or H. Similarly $23 = 9+13+1 = 21+2$ and $27 = 13+13+1 = 25+2$ so $[23] = [27] = \text{1st}$. A computer search gives $[22] = H$, and since $35 = 22+13$ we have $[35] = H$.

So far, we have gotten away without using the real power of game theory. However, for [31] we have found no elementary proof, and it is too large for our search program. Therefore, we turn to Berlekamp's beautiful solution for $2 \times n$ Domineering when n is odd [4], evaluate it with the Gamesman's Toolkit [5], and find the following (see [2–4] for notation):

$$[31] = \frac{1}{2} - 15 \cdot \left(\frac{1}{4} + \int^{3/4} *\right) + \int^{3/4} \int_{1/2}^{1/2*} 3\frac{7}{8} = \left\{2\,|\,0\,\|\,-\frac{1}{2}\,|\,-2\right\}\,|\,-\frac{5}{2} < 0 \tag{5}$$

Thus [31] is negative and a win for Hepzibah. This closes the last loophole, telling us who wins the $2 \times n$ game for all n.

3 Boards of width 3, 4, 5, 7, 9, 11 and others

The situation for rectangles of width 3 is much simpler. While Equation 1 no longer holds since Vera cannot divide the board in two with her first move, Equation 4 still holds, since Hepzibah can choose not to cross a vertical boundary between two games. Thus if $[m]$ and $[n]$ are both wins for Hepzibah, so is $[m+n]$. A quick search shows that $[4] = [5] = [6] = [7] = H$, so we have for width 3

$$[0] = \text{2nd},\ [1] = V,\ [2] = [3] = \text{1st, and } [n] = H \text{ for all } n \geq 4. \tag{6}$$

For width 5, we obtain

$$[0] = \text{2nd},\ [1] = [3] = V,\ [2] = [4] = H,\ [5] = \text{2nd, and } [n] = H \text{ for all } n \geq 6. \tag{7}$$

For width 7, Breuker, Uiterwijk and van den Herik found by computer search [6–8] that $[4] = [6] = [9] = [11] = H$. Then $[8] = [10] = H$ and

$$[7 \times n] = H \text{ for all } n \geq 8.$$

In all these cases, we were lucky enough that $[n] = $ 2nd or H for enough small n to generate all larger n by addition. This becomes progressively rarer for larger widths. However, we have some partial results on other widths. For width 4, Uiterwijk and van den Herik [7, 8] found by computer search that $[8] = [10] = [12] = [14] = H$, so $[n] = H$ for all even $n \geq 8$. They also found that $[15] = [17] = H$, so

$$[4 \times n] = H \text{ for all } n \geq 22.$$

This leaves $[4 \times 19]$ and $[4 \times 21]$ as the only unsolved boards of width 4.

As a general method, whenever we can find a length for which Hepzibah wins by some positive number of moves (rather than by an infinitesimal), then she wins on any board long enough to contain a sufficient number of copies of this one to overcome whatever advantage Vera might have on smaller boards. Game-theoretically, if $[n] < -r$, then $[m] < 0$ whenever $m \geq (1/r) \max_{l<n} [l]$.

For width 9, for instance, we have $[1] = 4$, $[2] = \frac{3}{2} \,|\, 0 \,\|\, -\frac{1}{2} \,|\, -\frac{5}{2}$, $[3] = 5 \,|\, 3 \,\|\, \frac{11}{4} \,|\, \frac{1}{4}$, and $[4] \leq [2] + [2] = 1 \,|\, -\frac{1}{2} \,\|\, -1 \,|\, -\frac{5}{2} < -\frac{1}{2}$. By summing these, it is easy to show that

$$[9 \times n] = H \text{ for all } n \geq 22.$$

Similarly, for width 11 we have $[1] = 5$ and $[2] = 1 \,|\, \left\{\frac{1}{2} \,|\, -1 \,\|\, -\frac{3}{2} \,|\, -\frac{7}{2}\right\}$. Then $[8] \leq [2] + [2] + [2] + [2] = 1 \,|\, -\frac{1}{2} \,\|\, -1 \,|\, -\frac{5}{2} < -\frac{1}{2}$ and $[16] \leq [8] + [8] \leq -\frac{3}{2}$, so

$$[11 \times n] = H \text{ for all } n \geq 56.$$

Unfortunately, for all other widths greater than 7, either $[2]$ or $[2]+[2]$ is positive and $[3]$ is as well, so without some way to calculate values for $[4]$ or more we can't establish this kind of bound. Nor do we know of any length for which Hepzibah wins on width 8, or a proof that she wins any board of width 6 by a positive amount.

To get results on boards of other widths, we can use a variety of tricks. First of all, just as Hepzibah can choose to cross a vertical boundary between games, Vera can choose not to cross a horizontal one. Thus Equation 4 is one of a dual pair,

$$[m \times (n_1 + n_2)] \leq [m \times n_1] + [m \times n_2] \tag{8}$$
$$[(m_1 + m_2) \times n] \geq [m_1 \times n] + [m_2 \times n] \tag{9}$$

Another useful rule is that $[n \times n] = 1\text{st}$ or 2nd, since neither player can have an advantage on a square board. In fact, in game-theoretic terms $[n \times n] + [n \times n] = 0$, so if Hepzibah goes second she can win by mimicking Vera's move, rotated 90°, in the other board. More generally we have

$$\text{If } [n \times n] = 1\text{st, then } [n \times kn] = \begin{cases} \text{2nd or } H & \text{for even } k > 1 \\ \text{1st or } H & \text{for odd } k > 1 \end{cases}$$
$$\text{If } [n \times n] = \text{2nd, then } [n \times kn] = \text{2nd or } H \text{ for all } k > 1.$$

For instance, this tells us that $[6 \times 12] = $ 2nd or H, and since $[6 \times 4] = $ 1st and $[6 \times 8] = H$ [6] we also have $[6 \times 12] = $ 1st or H. Therefore $[6 \times 12] = H$ and

$$[6 \times (4 + 4k)] = H \text{ for all } k \geq 1.$$

We can also use our addition rules backward; since no two games can sum to a square in a way that gives an advantage to either player,

$$\text{If } m < n \text{ and } [m \times n] = \text{2nd or } V\text{, then } [(n - m) \times n] \neq V \tag{10}$$

and similarly for the dual version.

Using the results of [4] and [6], some computer searches of our own, and a program that propagates these rules as much as possible gives the table shown in Figure 1. It would be very nice to deduce who wins on some large squares; the 8×8 square is the largest known so far [6]. We note that if $[9 \times 13] = $ 1st then $[13 \times 13] = $ 1st since $[4 \times 13] = V$.

4 Playing on cylinders and tori

On a torus, Hepzibah can choose not to play across a vertical boundary and Vera can choose not to play across a horizontal one. Thus cutting a torus, or pasting a rectangle along one pair of edges, to make a horizontal or vertical cylinder gives the inequalities shown in Figure 2. Note that there is no obvious relation between the value of a rectangle and that of a torus of the same size.

While it is easy to find who wins on tori and cylinders of various small widths, we do the analysis here only for tori of width 2. Since Vera's move takes both squares in the same column, these boards are equivalent to horizontal cylinders like those shown on the left of Fig. 2 (or, for that matter, Möbius strips or Klein bottles). Therefore, if Hepzibah can win on the rectangle of length n, she can win here as well.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
1	2	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
2	V	1	1	H	V	1	1	H	V	1	1	H	2	1	1	H	H	1	1	H	H	H	1	H	H	H	1	H	H	H	H	H	H
3	V	1	1	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
4	V	V	V	1	V	1	V	H	V	H	V	H	V	H	H	H	H	H	1h	H	1h	H	H	H	H	H	H	H	H	H	H	H	H
5	V	H	V	H	2	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	1	V	1	V	1	V	H	V	1	1	H	V	1h		H		1h	1h	H		1h	1h	H		1h	1h	H		1h	1h	H	
7	V	1	V	H	V	H	1	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
8	V	V	V	V	V	V	V	1	V		V		V			2h								1h								2h	
9	V	H	V	H	V	H	V	H	1	H	1h	H	1h	H	1h	H	1h	H	1h	H	1h	H	H	H	H	H	H	H	H	H	H	H	H
10	V	1	V	V	V	1	V		V	12	1v		V							2h		1h				1h				−v			
11	V	1	V	H	V	1	V	H	1v	1h	12	H	−v	1h	1h	H	1h	1h	1h	H	1h	H	1h	H	1h	H	1h	H	1h	H	1h	H	1h
12	V	V	V	V	V	V	V		V		V	12	V											2h									
13	V	2	V	H	V	H	V	H	1v	H	−h	H	12	H	−v	H	1h	H	1h	H	1h	H	1h	H	1h	H	1h	H	1h	H	1h	H	1h
14	V	1	V	V	V	1v	V		V		1v		V	12														2h		1h			
15	V	1	V	V	V		V		1v		1v		−h		12															2h		1h	
16	V	V	V	V	V	V	V	2v	V		V		V			12																2h	
17	V	V	V	V	V		V		1v		1v		1v				12																
18	V	1	V	V	V	1v	V		V		1v		V					12															
19	V	1	V	1v	V	1v	V		1v		1v		1v						12														
20	V	V	V	V	V	V	V		V	2v	V		V							12													
21	V	V	V	1v	V		V		1v		1v		1v								12												
22	V	V	V	V	V	1v	V		V	1v	V		V									12											
23	V	1	V	V	V	1v	V		V		1v		1v										12										
24	V	V	V	V	V	V	V	1v	V		V	2v	V											12									
25	V	V	V	V	V		V		V		1v		1v												12								
26	V	V	V	V	V	1v	V		V	1v	V		V													12							
27	V	1	V	V	V	1v	V		V		1v		1v														12						
28	V	V	V	V	V	V	V		V		V		V	2v														12					
29	V	V	V	V	V		V		V		1v		1v																12				
30	V	V	V	V	V	1v	V		V	−h	V		V	1v	2v															12			
31	V	V	V	V	V	1v	V		V		1v		1v																		12		
32	V	V	V	V	V	V	V	2v	V		V		V		1v	2v																12	
33	V	V	V	V	V		V		V		1v		1v																				12

Fig. 1. What we know so far about who wins Domineering on rectangular boards. 1, 2, H and V mean a win for the first player, second player, Hepzibah and Vera respectively. Things like "1h" mean either 1st or H (i.e. all we know is that Hepzibah wins if she goes first) and "-v" means that it is not a win for Vera all the time. Values outlined in black are those provided by search or other methods; all others are derived from these using our rules or by symmetry.

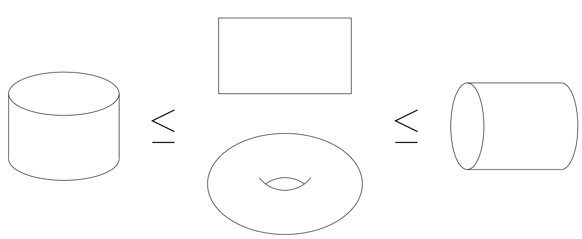

Fig. 2. Inequalities between rectangular, cylinder, and toroidal boards of the same size. Cutting vertically can only hurt Hepzibah, while cutting horizontally can only hurt Vera.

The second player has slightly more power here than she did in on the rectangle, since the first player has no control over the effect of her move. If Vera goes first, she simply converts a torus of length n into a rectangle of length $n-1$, and if Hepzibah goes first, Vera can choose where to put the rectangle's vertical boundary, in essence choosing Hepzibah's first move for her. On the other hand, in the latter case Hepzibah gets to play again, and can treat the remainder of the game as the sum of two rectangles and a horizontal space.

These observations give the following table for tori of width 2:

$[n]_{\text{rect}}$	$[n-1]_{\text{rect}}$	$[n]_{\text{torus}}$
H		H
1st	1st or H	H
1st	2nd or V	1st
2nd or V	1st or H	2nd or H

These and our table for $2 \times n$ rectangles determine $[n]_{\text{torus}}$ for all n except 5, 9, and 13. Vera loses all of these if she plays first, since she reduces the board to a rectangle which is a win for Hepzibah. For 5 and 9, Vera wins if Hepzibah plays first by playing in such a way that Hepzibah's domino is in the center of the resulting rectangle, creating a position which has zero value. Thus these boards are wins for the 2nd player. For 13, Hepzibah wins since (as the Toolkit tells us) all of Vera's replies to Hepzibah leave us in a negative position.

Our computer searches show that $n \times n$ tori are wins for the 2nd player when $n = 1$, 3, or 5, and for the 1st player when $n = 2$, 4, or 6. We conjecture that this alternation continues, and that square tori of odd and even size are wins for the 2nd and 1st players respectively. We note that a similar argument can be used to show that 9 is prime.

5 Polynomial-time strategies

While correctly playing the sum of many games is PSPACE-complete in general [9], the kinds of sums we have considered here are especially easy to play. For instance, if $[m]$ and $[n]$ are both wins for Hepzibah, she can win on $[m+n]$ by playing wherever she likes if she goes first, and replying to Vera in whichever game Vera chooses thereafter. Thus if we have strategies for both these games, we have a strategy for their sum. All our additive rules are of this kind.

Above, we showed for a number of widths that boards of any length can be reduced to sums of a finite number of lengths. Since each of these can be won with some finite strategy, and since sums of them can be played in a simple way, we have proved the following theorem:

Theorem 1. *For boards of width 2, 3, 4, 5, 7, 9, and 11, there exist polynomial-time strategies for playing on boards of any length.*

Note that we are not asking that the strategy produce optimum play, in which Hepzibah (or on small boards, 1st, 2nd or Vera) wins by as as many moves as possible, but only that it tells her how to win.

In fact, we conjecture that this theorem is true for boards of any width. This would follow if for any m there exists an n such that Hepzibah wins by some positive number of moves, which in turn implies that there is some n' such that she wins on all boards longer than n'. A similar conjecture is made in [7]. Note, however, that this is not the same as saying that there is a single polynomial-time strategy for playing on boards of any size. The size or running time of the strategy could grow exponentially in m, even if it grows polynomially when m is held constant.

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