Global fixed point attractors of circular cellular automata and periodic tilings of the plane: undecidability results [1]

Jacques Mazoyer and Ivan Rapaport

LIP - École Normale Supérieure de Lyon
46 allée d'Italie, 69364 Lyon Cedex 07, France

Abstract

A great amount of work has been devoted to the understanding of the long-time behavior of cellular automata (CA). As for any other kind of dynamical system, the long-time behavior of a CA is described by its attractors. In this context, it has been proved that it is undecidable whether every circular configuration of a given CA evolves to some fixed point (local attractor). In this paper we prove that it remains undecidable whether every circular configuration of a given CA evolves to the *same* fixed point (global attractor). Our proof is based on properties concerning NW-deterministic periodic tilings of the plane. As a corollary we conclude the (already proved) undecidability of the periodic tiling problem. Nevertheless, our approach could also be used to prove this result in a direct and very simple way.

Key words: circular cellular automata; global attractors; periodic tilings; undecidable problems.

1 Introduction

Cellular automata (CA) are discrete dynamical systems. They are defined by a *lattice of cells* and a *local rule* by which the *state* of a cell is determined as a function of the state of its neighborhood. A *configuration* of a CA is an assignment of states to the cells of the lattice. The *global transition function* is a map from the space of all configurations to itself obtained by applying the local rule simultaneously to all the cells. This global transition function corresponds to the CA *dynamics*.

[1] This work was partially supported by program ECOS-97.

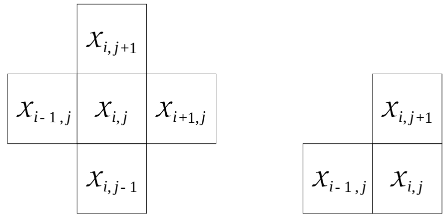
Because of the dynamical-system nature of CA, a great amount of work has been devoted to the understanding of its long-time behavior (consider, for instance, the well-known classification of Wolfram [7]). The long-time behavior of any dynamical system is described by its attractors. In this context, for the two (and higher) dimensional CA, it was proved in [3] the undecidability of the *nilpotency problem*. This problem consists, in practice, in deciding whether every configuration of a given CA evolves to a global (unique) fixed point in a finite number of steps. Later J. Kari proved in [4] the undecidability of the nilpotency problem in the one-dimensional case.

On the other hand, K. Sutner in [6] restricted this study to *circular configurations* (those spatially periodic) because of their finitary description and therefore their possibility of being handled in the framework of the ordinary computability theory. More precisely, by the use of non-standard simulations of Turing machines, it was proved that it is undecidable whether every circular configuration of a given one-dimensional CA evolves to some fixed point. A very particular property satisfied the CA of Sutner's reduction: all of them admitted an *infinite* number of circular fixed point configurations. In fact, by considering CA configurations as tapes of Turing machines, we force the existence of an infinite number of fixed points consisting of representations of the tape having no head.

In this paper we prove that it remains undecidable whether every circular configuration of a given one-dimensional CA evolves to the *same* (and therefore unique) fixed point. Our result allows us to conclude the one of Sutner in a rather direct way.

The structure of our proof is inspired on the one developed by J. Kari in [4]. In fact, our work is based on results concerning *tiling problems* and, in particular, on the useful *NW-deterministic* notion (roughly, a set of tiles is NW-deterministic if it is locally deterministic in one dimension). More precisely, here we prove that it is undecidable whether a given *NW-deterministic* set of tiles admits a *periodic* tiling of the plane. Despite the similarity with Kari's nilpotency result, our objects are different in nature: the CA configurations considered here are *circular* and the tilings of the plane are *periodic*. In this particularity lies the difficulty of our proof.

By the way, and as an obvious consequence, it can be concluded the undecidability of the *periodic tiling problem* (in which it is asked whether an *arbitrary* set of tiles admits a *periodic* tiling of the plane). This result was obtained by Y. Gurevich and I. O. Koriakov in [2]. We would like to remark that our approach could also be used to prove the Gurevich and Koriakov result in a *direct* way. In fact, when the NW-deterministic property is no required, most of the technicities of the proof are no longer needed and it becomes very simple.



(i) (ii)

Fig. 1. Local matching. (i) The general case. (ii) The NW-deterministic case.

2 Definitions

A one-dimensional cellular automaton of radius one, or simply a CA, is defined by a couple (Q, δ) where Q is a finite set of states and $\delta : Q^3 \to Q$ is a transition function. A state $q \in Q$ is said to be a spreading state if for all $x, y, z \in Q$, $q \in \{x, y, z\} \implies \delta(x, y, z) = q$. A configuration of a CA (Q, δ) is a bi-infinite sequence $\mathcal{C} \in Q^{\mathbb{Z}}$, and its global transition function $G_\delta : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ is such that $(G_\delta(\mathcal{C}))_i = \delta(\mathcal{C}_{i-1}, \mathcal{C}_i, \mathcal{C}_{i+1})$. For $t \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ it is defined recursively $G^t_\delta(\mathcal{C}) = G_\delta(G^{(t-1)}_\delta(\mathcal{C}))$ with $G^0_\delta(\mathcal{C}) = \mathcal{C}$. A family of different configurations $\{\mathcal{C}^{(0)}, \cdots, \mathcal{C}^{(T-1)}\} \subseteq Q^{\mathbb{Z}}$ is said to be a cycle of length T if for all $t \in \{0, \cdots, T-1\}$, $G^t_\delta(\mathcal{C}^{(0)}) = \mathcal{C}^{(t)}$ and $G_\delta(\mathcal{C}^{(T-1)}) = \mathcal{C}^{(0)}$. A fixed point is a cycle of length one. We say that a configuration $\mathcal{C} \in Q^{\mathbb{Z}}$ is circular if there exists $P \in \mathbb{N}^*$ such that $\mathcal{C}_i = \mathcal{C}_{i+P}$ for all $i \in \mathbb{Z}$.

In the *global fixed point attractor problem* it is asked whether every circular configuration of a given CA evolves to the *same* fixed point.

This work is mainly based on properties concerning *periodic tilings of the plane*. A tiling system is a couple $(\mathcal{T}, \varphi)$ where $\mathcal{T}$ is a finite set of tiles and $\varphi : \mathcal{T}^4 \to \mathcal{T}$ is a partial function called local matching. A tiling of the plane by $(\mathcal{T}, \varphi)$ is an assignment $\mathcal{X} \in \mathcal{T}^{\mathbb{Z}^2}$ satisfying for all $i, j \in \mathbb{Z}$, as it is represented in Figure 1-i, $\varphi(\mathcal{X}_{i-1,j}, \mathcal{X}_{i,j+1}, \mathcal{X}_{i+1,j}, \mathcal{X}_{i,j-1}) = \mathcal{X}_{i,j}$. A tiling system $(\mathcal{T}, \varphi)$ is said to be *NW-deterministic* if for every pair of tiles $x, y \in \mathcal{T}$ there exists at most one tile $z \in \mathcal{T}$ accepting x as left neighbor and y as upper neighbor. In other words, for a NW-deterministic tiling system $(\mathcal{T}, \varphi)$, the domain of the partial local matching function can be assumed to be $\mathcal{T}^2$. A tiling of the plane by a NW-deterministic set of tiles $(\mathcal{T}, \varphi)$ is an assignment $\mathcal{X} \in \mathcal{T}^{\mathbb{Z}^2}$ satisfying for all $i, j \in \mathbb{Z}$, as it is represented in Figure 1-ii, $\varphi(\mathcal{X}_{i-1,j}, \mathcal{X}_{i,j+1}) = \mathcal{X}_{i,j}$.

A tiling $\mathcal{X}$ is said to be periodic if there exist horizontal and vertical translations for which $\mathcal{X}$ remains invariant. In other words, $\mathcal{X} \in \mathcal{T}^{\mathbb{Z}^2}$ is periodic if there exists $P \in \mathbb{N}^*$ such that $\mathcal{X}_{i,j} = \mathcal{X}_{i+P,j} = \mathcal{X}_{i,j+P}$ for all $i, j \in \mathbb{Z}$.

In the *NW-deterministic periodic tiling problem* it is given a NW-deterministic tiling system and it is asked whether it admits a periodic tiling of the plane.

3 The global fixed point attractor problem

It is direct to notice that every circular configuration of a CA evolves in a finite number of steps to a finite cycle. By the use of non-standard simulations of Turing machines, K. Sutner proved in [6] that it is undecidable whether every circular configuration of a CA evolves to a fixed point (not unique). In this section we show that it remains undecidable whether every circular configuration of a CA evolves to the *same* fixed point. Our result allows us to conclude the one of Sutner directly.

The reduction to the *global fixed point attractor problem* is done from the *NW-deterministic periodic tiling problem.*

Proposition 1 *The NW-deterministic periodic tiling problem is undecidable.*

Proof. In section 4. □

Proposition 2 *The global fixed point attractor problem is undecidable.*

Proof. Let $(\mathcal{T}, \varphi)$ be a NW-deterministic tiling system. Let us consider now the CA (Q, δ) with $Q = \{\mathcal{T} \cup \{s\}\}$ such that $s \notin \mathcal{T}$, and with the transition function δ defined as follows:

$$\delta(x,y,z) = \begin{cases} \varphi(x,y) & \text{if } x,y,z \in \mathcal{T} \text{ and } \varphi(x,y) \text{ is well defined,} \\ s & \text{otherwise.} \end{cases}$$

Notice that s is a spreading state of (Q, δ). It follows that $(\mathcal{T}, \varphi)$ admits a periodic tiling of the plane if and only if there exists a circular configuration of (Q, δ) not evolving to the trivial fixed point $(\cdots sss \cdots)$.

In fact, let $\mathcal{X} \in \mathcal{T}^{\mathbb{Z}^2}$ be a periodic tiling of the plane. Let $\mathcal{C}$ be a configuration of (Q, δ) corresponding to the diagonal of $\mathcal{X}$ that intersects the origin as it appears in Figure 2. More precisely, let $\mathcal{C}_i = \mathcal{X}_{i,i}$ for all $i \in \mathbb{Z}$. The configuration $\mathcal{C}$ is a circular one because there exists $P \in \mathbb{N}^*$ such that $\mathcal{X}_{i,i} = \mathcal{X}_{i+P,i+P}$ for all $i \in \mathbb{Z}$. By construction, for all $t \in \mathbb{N}^*$, $G_\delta^t(\mathcal{C})$ corresponds to the diagonal $\{\mathcal{X}_{i+t,i}\}_{i \in \mathbb{Z}} \in \mathcal{T}^{\mathbb{Z}}$ and therefore $G_\delta^t(\mathcal{C}) \neq (\cdots sss \ldots)$.

Conversely, let $\mathcal{C}$ be a P-circular configuration not evolving to the trivial fixed point $(\cdots sss \cdots)$. Using that $s \in Q$ is a spreading state then every configuration of the cycle to which $\mathcal{C}$ evolves belongs to $\mathcal{T}^{\mathbb{Z}}$. Let $T \in \mathbb{N}^*$ be the length of this cycle and let $\mathcal{C}^*$ be an arbitrary configuration belonging to it. Let us consider the region $R = \{(i,j) \in \mathbb{Z}^2 : i \geq j\}$ that appears schematically in Figure 3. Let us tile R by the assignment $\mathcal{X} \in \mathcal{T}^R$ such that

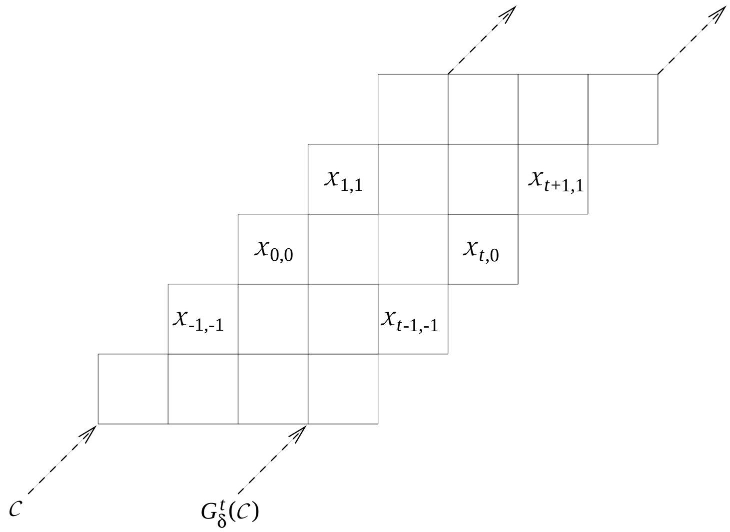


Fig. 2. The diagonals of a NW-deterministic tiling seen as CA configurations.

$\mathcal{X}_{i+t,i} = (G^t_\delta(\mathcal{C}^*))_i$, with $i \in \mathbb{Z}$ and $t \geq 0$. By the definition of (Q, δ) the previous assignment effectively corresponds to a tiling of R.

It follows that the square $\{0, 1, \cdots, PT-1\} \times \{0, -1, \cdots, -PT+1\} \subseteq R$ tiled by $\mathcal{X}$ has periodic boundary conditions and it can be repeated in order to tile the plane periodically. In fact, considering that the cycle to which $\mathcal{C}^*$ belongs has length T it holds for all $j \in \{0, -1, \cdots, -PT+1\}$:

$$\mathcal{X}_{PT,j} = \mathcal{X}_{j+(PT-j),j} = (G^{PT-j}_\delta(\mathcal{C}^*))_j = (G^{-j}_\delta(\mathcal{C}^*))_j = \mathcal{X}_{j-j,j} = \mathcal{X}_{0,j}.$$

On the other hand, considering that if $\mathcal{C}^*$ is P-periodic then also $G^t_\delta(\mathcal{C}^*)$ is P-periodic for any $t \geq 0$, then for all $i \in \{0, 1, \cdots, PT-1\}$:

$$\mathcal{X}_{i,-PT} = \mathcal{X}_{-PT+(PT+i),-PT} = (G^{PT+i}_\delta(\mathcal{C}^*))_{-PT} = (G^i_\delta(\mathcal{C}^*))_0 = \mathcal{X}_{i,0}. \qquad \square$$

In order to conclude the result of K. Sutner concerning the undecidability of the *local fixed point attractor problem* (in which it is asked whether every circular configuration of a CA evolves to a not necessarily unique fixed point), we need to use the following lemma.

Lemma 1 *Given a CA, it is decidable whether it admits a unique circular configuration as a fixed point.*

Proof. Let (Q, δ) be a CA. Let us consider the directed graph $G = (V, E)$ with $V \subseteq Q^3$ satisfying $(x, y, z) \in V$ if and only if $\delta(x, y, z) = y$ while, on the other hand, $((x_1, y_1, z_1), (x_2, y_2, z_2)) \in E$ if and only if $y_1 = x_2$ and $z_1 = y_2$.

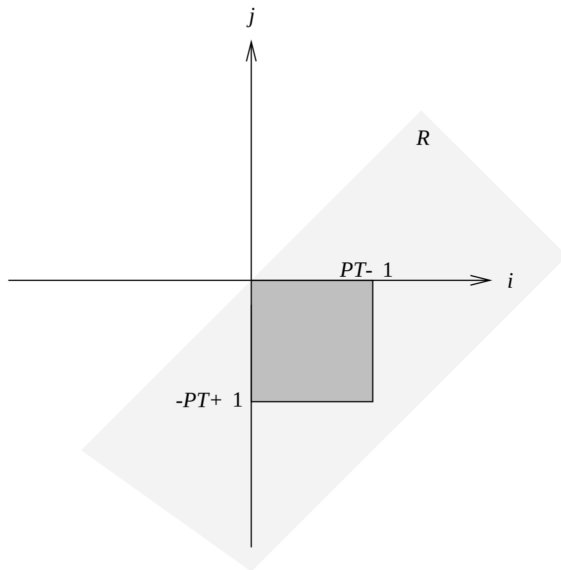


Fig. 3. A periodic square pattern of the tiled region R.

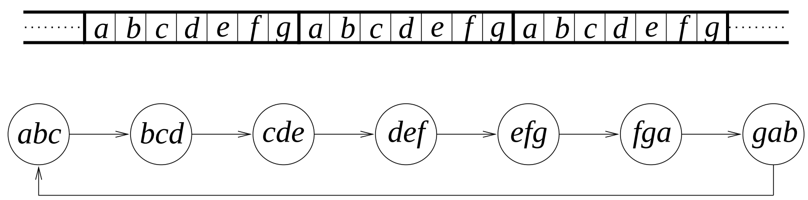


Fig. 4. A circular fixed point configuration and its associated graph cycle.

As it appears schematically in Figure 4, a cycle of the graph G corresponds to a circular fixed point of (Q, δ).

A fixed point configuration $\mathcal{C}$, in order to be unique, has to be shift invariant (i.e, $\mathcal{C}_i = \mathcal{C}_{i+j}$ for all $i, j \in \mathbb{Z}$). If not, we would obtain by a shift operation another fixed point. Considering that the only shift invariant configurations are those of the form $(\cdots qqq \cdots)$, the problem has been reduced to decide whether the graph $G = (V, E)$ admits a unique cycle that, in addition, has length one. $\square$

Corollary 1 *The local fixed point attractor problem is undecidable.*

Proof. First notice that the subproblem of the *global fixed point attractor problem* in which each instance (CA) admits a unique circular fixed point remains undecidable. In fact, let us assume that this is not true and let us denote by ALG the corresponding decision algorithm. It follows that there would exist a decision procedure for the *global fixed point attractor problem*:

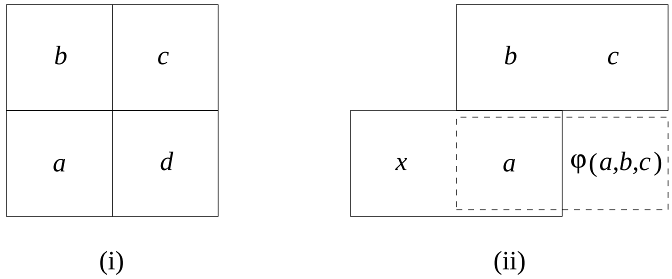


Fig. 5. (i) There exists at most one $d \in \mathcal{T}$ such that $\varphi(a,b,c) = d$. (ii) Equivalence between the two NW-deterministic notions.

given a CA decide if it admits a unique circular fixed point (see Lemma 1); if this is the case apply ALG and if this is not the case then obviously it does not hold that all the circular configurations evolve to the same fixed point. Finally, in order to conclude the undecidability of the *local fixed point attractor problem* it suffices to notice that the *global* and the *local* versions when restricted to CA having a unique circular fixed point are equivalents. □

4 The NW-deterministic periodic tiling problem

The goal of this section is to prove the undecidability of the *NW-deterministic periodic tiling problem*. As it was done in [4], in order to make the proof more readable, we are going to use an equivalent notion of NW-determinism. From now on we say that a tiling system $(\mathcal{T}, \varphi)$ is NW-deterministic if for every $a, b, c \in \mathcal{T}$ there exists at most one tile $d \in \mathcal{T}$ matching as in Figure 5-i. In this case φ can be considered as a three-arguments partial function and we note $\varphi(a,b,c) = d$.

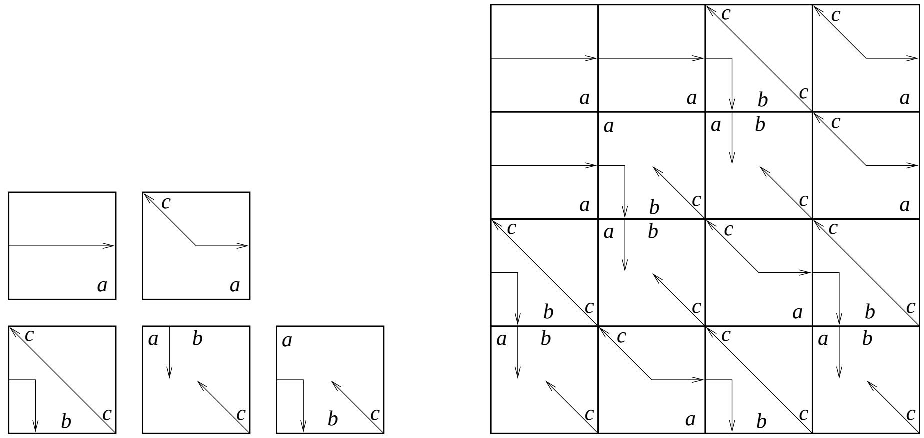
Notice that if $(\mathcal{T}, \varphi)$ is a NW-deterministic tiling system in this new sense, then there exists an "equivalent" tiling system $(\tilde{\mathcal{T}}, \tilde{\varphi})$ which is NW-deterministic in the original sense. In fact, let $\tilde{\mathcal{T}} = \mathcal{T}^2$ and let $\tilde{\varphi} : \tilde{\mathcal{T}}^2 \to \tilde{\mathcal{T}}$ be defined for all $x, a, b, c \in \mathcal{T}$ as follows (see figure 5-ii):

$$\tilde{\varphi}((x,a),(b,c)) = (a, \varphi(a,b,c))$$

It is direct to see that there exists a periodic tiling for $(\mathcal{T}, \varphi)$ if and only if there exists a periodic tiling for $(\tilde{\mathcal{T}}, \tilde{\varphi})$.

A natural superposition operation $\otimes$, which preserves the NW-deterministic property, is defined for every pair of tiling systems $(\mathcal{T}_1, \varphi_1)$ and $(\mathcal{T}_2, \varphi_2)$. More precisely, we define $(\mathcal{T}, \varphi) = (\mathcal{T}_1, \varphi_1) \otimes (\mathcal{T}_2, \varphi_2)$ with $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$ and, for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathcal{T}$:

$$\varphi((x_1,x_2),(y_1,y_2),(z_1,z_2)) = (\varphi_1(x_1,y_1,z_1), \varphi_2(x_2,y_2,z_2)).$$



(i) (ii)

Fig. 6. (i) A NW-deterministic set of tiles. (ii) A tiling of a square region.

In order to code a NW-deterministic tiling system, each tile is going to be represented as a unit-sized square having symbols and arrows on its four sides, on its upper-left corner and on its bottom-right corner. In Figure 6 appears, as an example, an arbitrary set of NW-deterministic tiles and a tiling of a square region: adjacent tiles must have the same symbol on their common edges and arrow heads must meet arrow tails. Because of the fact that the local matching function is directly encoded in the tiles representation, we are simply going to refer to "set of tiles" instead of "tiling system".

The undecidability of the *NW-deterministic periodic tiling problem* is going to be proved by a reduction from *the halting problem on Turing machines*. Before showing this reduction, we must construct a pair of NW-deterministic sets of tiles satisfying very particular conditions.

4.1 The NW-deterministic set of tiles $\mathcal{A}$

The goal here is to prove the following lemma:

Lemma 2 *There exists a NW-deterministic set of tiles $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ such that:*

- *A_1 admits only nonperiodic tilings of the plane,*
- *For any $n > 1$ there exists a square of size 2^n tiled by $\mathcal{A}$ satisfying:*
  - *It has periodic boundary conditions. In other words, this square pattern can be repeated in order to tile the plane periodically,*
  - *The tiles of $\mathcal{A}_2$ appear only on the right and bottom borders of the square as it is schematically showed in Figure 7.*

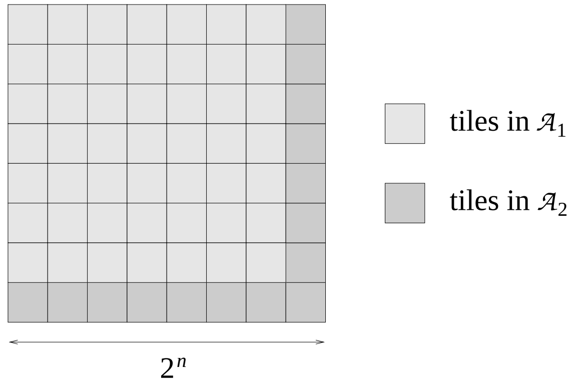


Fig. 7. A square tiled by $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ with periodic boundary conditions.

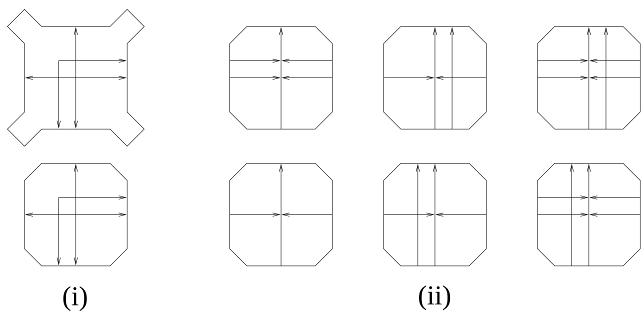


Fig. 8. The set of Robinson $\mathcal{A}_0$. (i) Crosses. (ii) Arms.

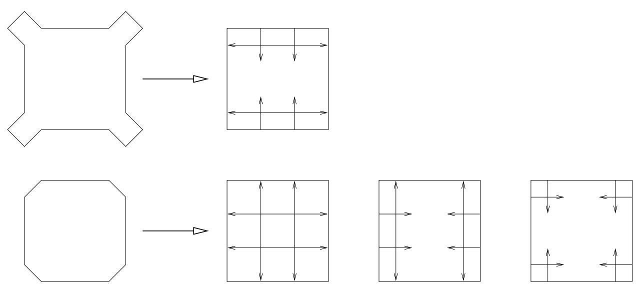


Fig. 9. Alternative representation of bumps and humps.

Proof. The set $\mathcal{A}_1$ to be considered corresponds to the one introduced in [4], which is almost identical to the well-known set of Robinson [5] denoted here by $\mathcal{A}_0$ and appearing in Figure 8. Notice that $\mathcal{A}_0$ has cardinality 32 (8 crosses and 24 arms) because all the rotations of each tile are admissible. The use of bumps and humps in the corners of the tiles is just a way to force the crosses to appear in alternate columns and in alternate rows. As it is done in [5], we show in Figure 9 how to replace bumps and humps by arrows. For clarity we prefer to keep the "bumps and humps" representation. In [5] it was proved that $\mathcal{A}_0$ admits only nonperiodic tilings of the plane.

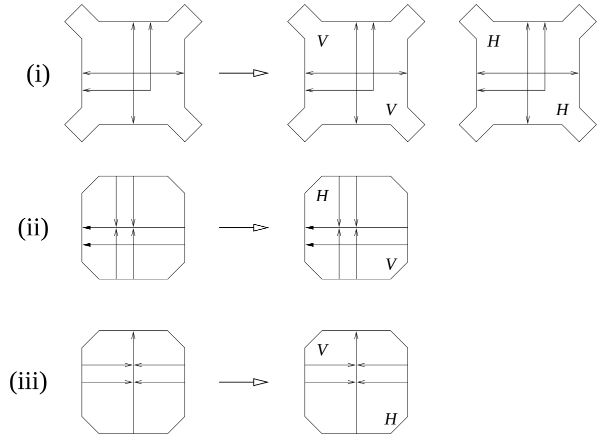


Fig. 10. Transforming $\mathcal{A}_0$ into $\mathcal{A}_1$. (i) A cross tile. (ii) An horizontally oriented arm. (iii) A vertically oriented arm.

By simply adding symbols to the upper-left and bottom-right corners, it is shown in [4] how to transform the set $\mathcal{A}_0$ into a NW-deterministic one $\mathcal{A}_1$ preserving the nonperiodicity property. More precisely, to the arms horizontally oriented (those with the principal or "one-way" arrow lying horizontally) H symbols are added on the upper-left corners and V symbols are added on the bottom-right corners. To the arms vertically oriented the V symbols are added on the upper-left corners while the H symbols are added on the bottom-right corners. Finally the crosses are duplicated by adding the same symbols (V and H) on both corners. The way the modification is done for three particular tiles corresponding to each one of previous cases appears in Figure 10.

It was proved in [4] that $\mathcal{A}_1$ is a NW-deterministic set of tiles admitting only nonperiodic tilings of the plane. The set $\mathcal{A}_1$ satisfies another key property for all $n > 1$:

> *If we denote by T_n the square of size $(2^n - 1)$ then, for each of the border conditions represented in Figures 11 and 12, there exists a corresponding tiling of T_n by $\mathcal{A}_1$ (notice that the number of border conditions is 8 since each square of Figures 11 and 12 codes in fact 2 squares because the X symbol represents either V or H).*

In fact, for $n = 2$ the 8 cases appear in Figure 13. If we suppose the property true for n we show in Figure 14 how to prove it for $n+1$ for 2 cases. The other 6 cases are similar.

Let us define the set of tiles $\mathcal{A}_2$ as the one of cardinality 5 that appears in Figure 15. The NW-determinism of $\mathcal{A} = \mathcal{A}_1 \cup A_2$ follows directly: it suffices to check that there are no two tiles in $\mathcal{A}$ with the same upper-left border. The periodic square of size 2^n with tiles of $\mathcal{A}_2$ only used on the right and bottom

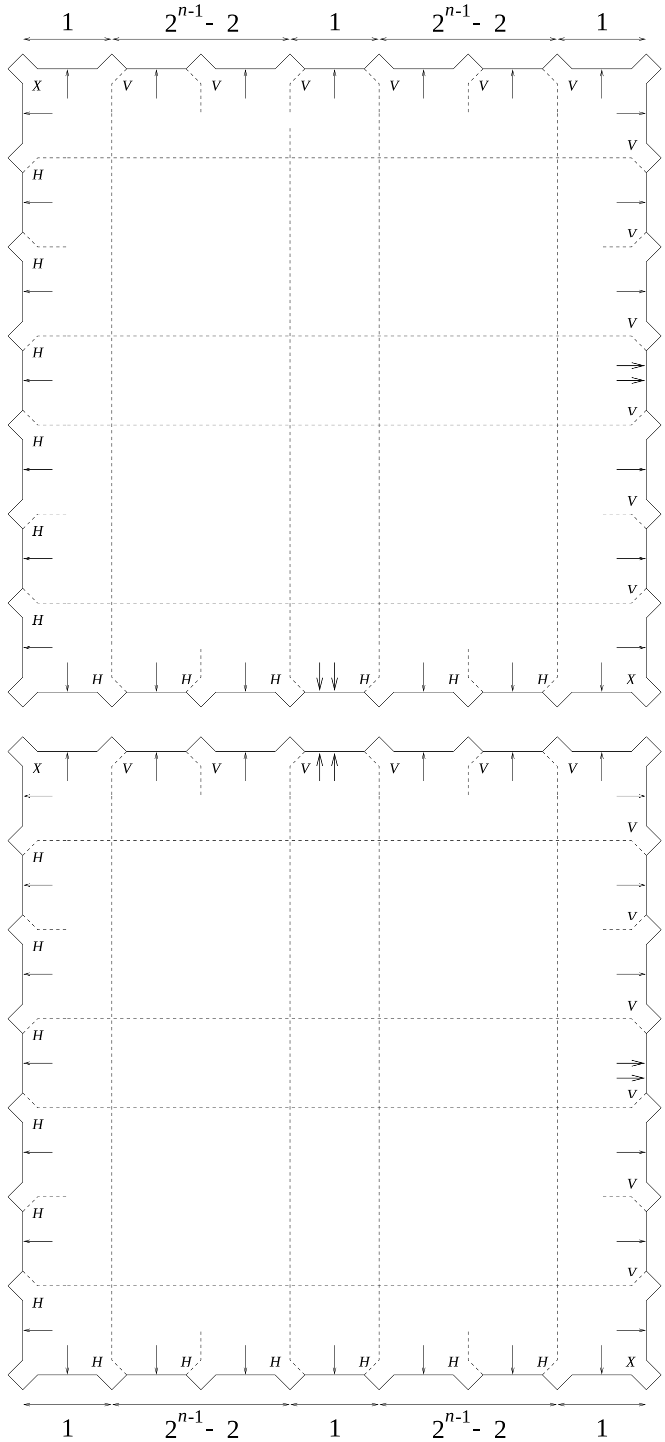


Fig. 11. The first 4 border conditions for T_n with $X \in \{V, H\}$.

borders appears schematically in Figure 16. Finally, in Figure 17 it is shown that previous pattern effectively has periodic boundary conditions. □

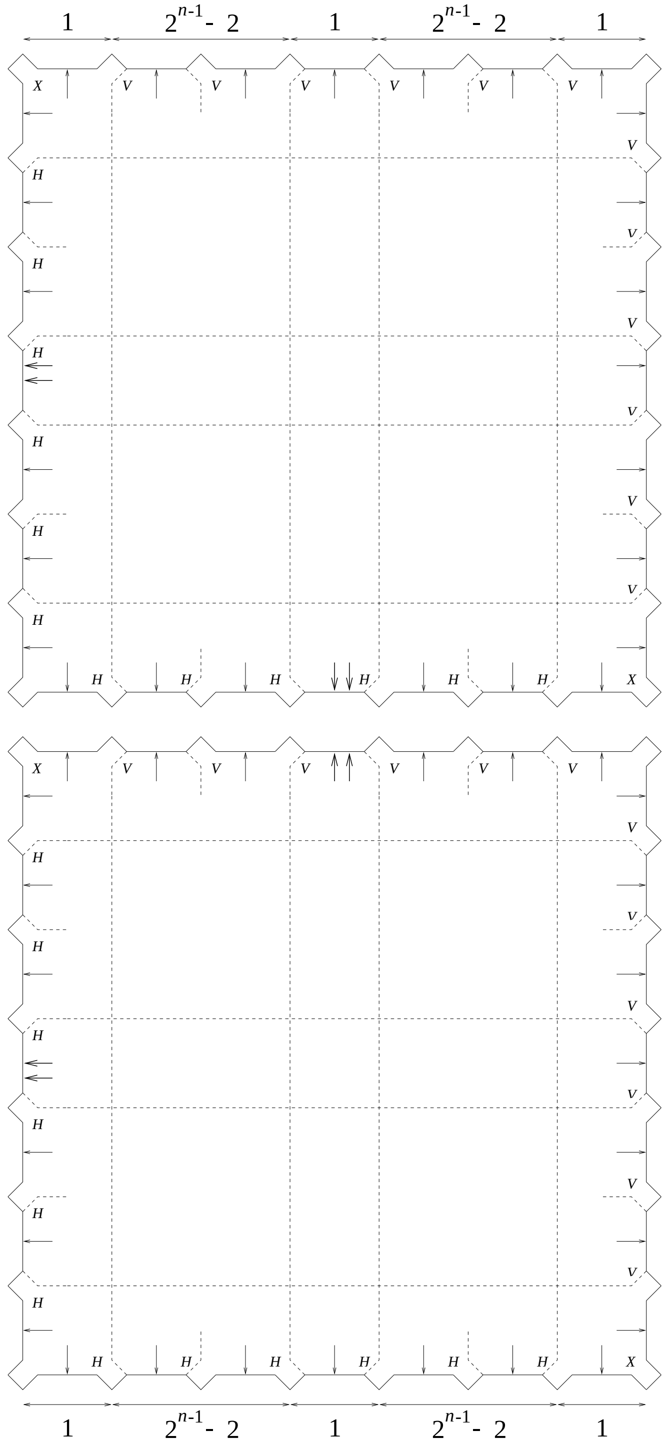


Fig. 12. The other 4 border conditions for T_n with $X \in \{V, H\}$.

4.2 The NW-deterministic set of tiles $\mathcal{AB}$

Here we generate a NW-deterministic set of tiles $\mathcal{AB}$ admitting periodic tilings of the plane and satisfying that, in any of these possible periodic tilings, some

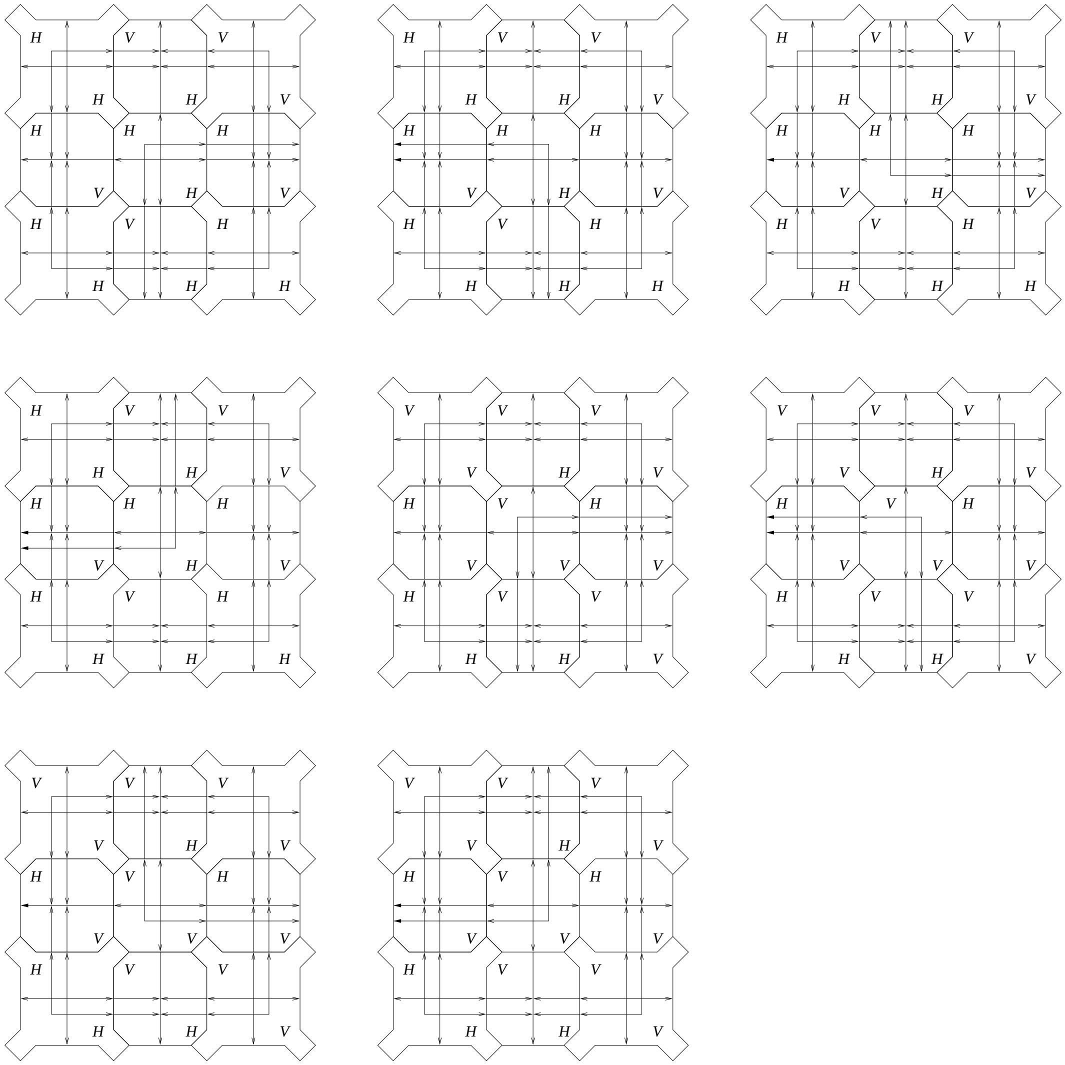


Fig. 13. The 8 cases for $n = 2$.

particular patterns called “boards” always appear. Let us start with some definitions.

Definition 1 *Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ be the NW-deterministic set of tiles of the previous section. Let $\mathcal{B} = \mathcal{B}_{int} \cup \mathcal{B}_{NW} \cup \mathcal{B}_{SE}$ be the set of Figure 18 made of internal tiles, NW-border tiles and SE-border tiles. We denote by $\mathcal{AB}$ the set obtained by the superposition that follows:*

$$\mathcal{AB} = \underbrace{\{\mathcal{A}_1 \otimes \mathcal{B}_{int}\}}_{\mathcal{AB}_{int}} \cup \underbrace{\{\mathcal{A}_1 \otimes \mathcal{B}_{NW}\} \cup \{\mathcal{A}_2 \otimes \mathcal{B}_{SE}\}}_{\mathcal{AB}_{bord}}$$

The tiles belonging to $\mathcal{AB}_{int}$ are called $\mathcal{AB}$-internal tiles while the tiles belonging to $\mathcal{AB}_{bord}$ are called $\mathcal{AB}$-border tiles.

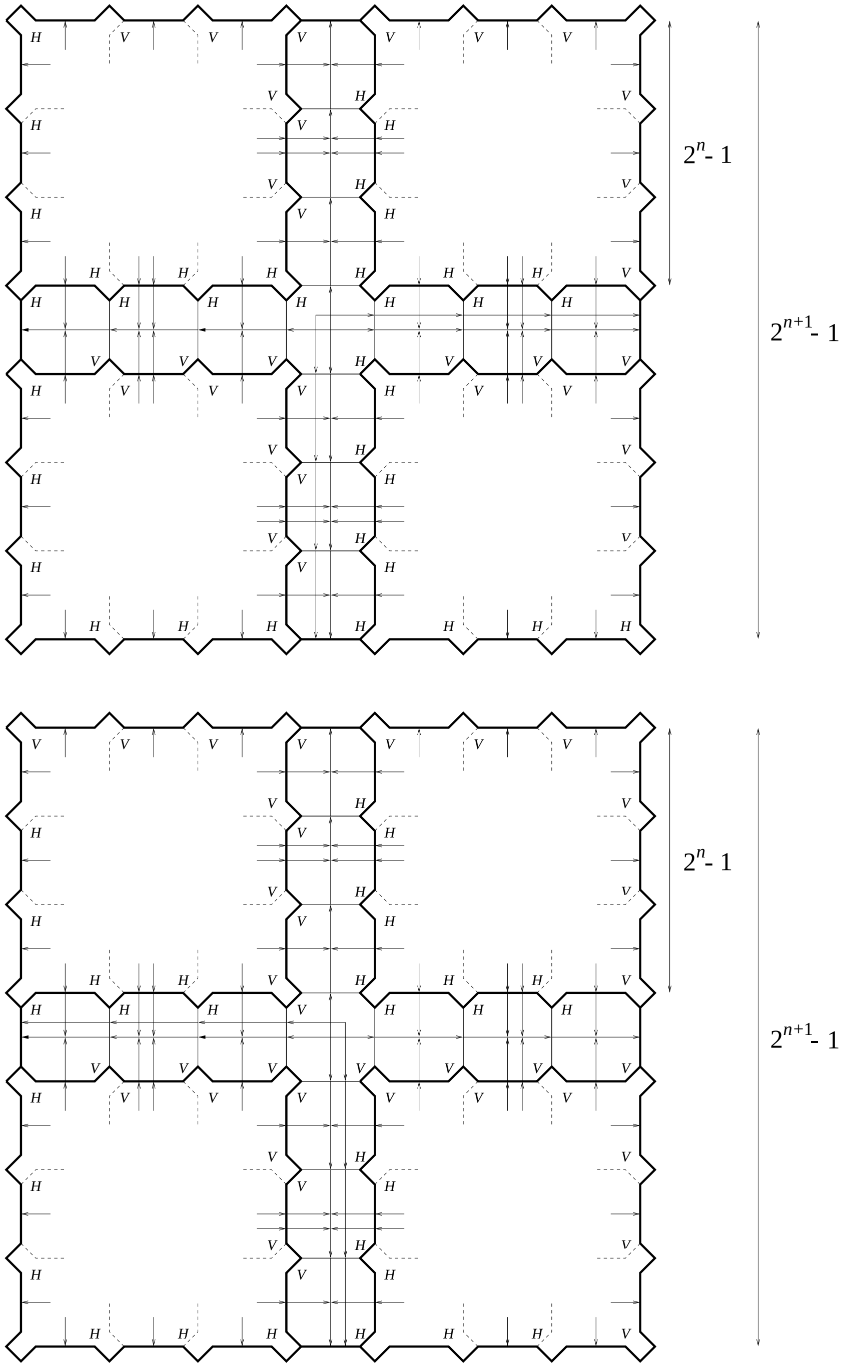


Fig. 14. The induction process in 2 cases.

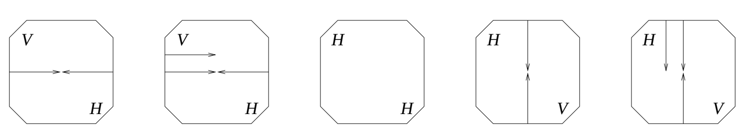


Fig. 15. The set $\mathcal{A}_2$.

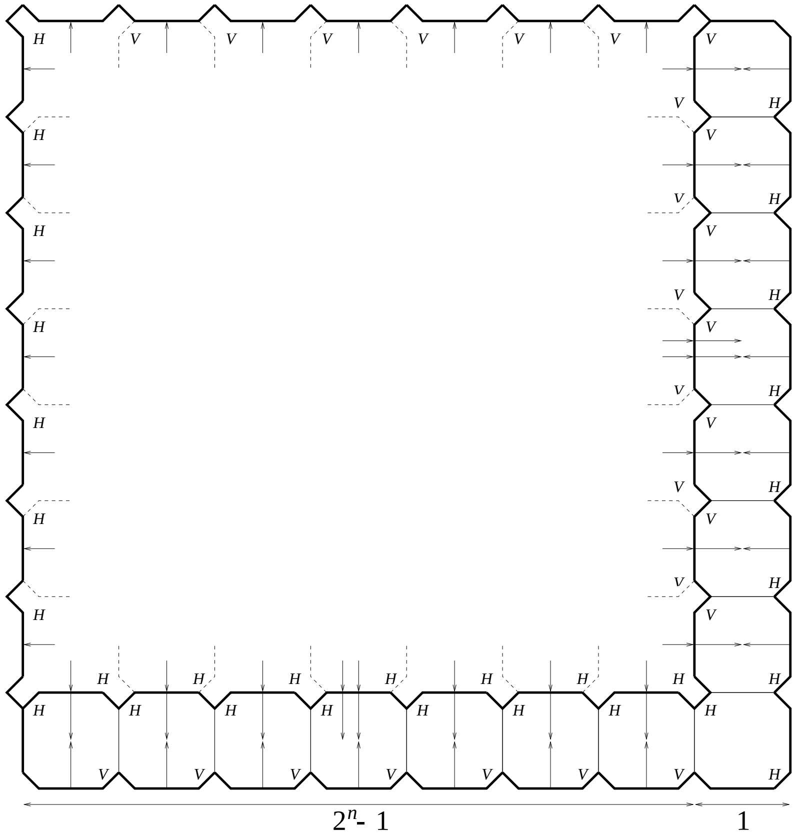


Fig. 16. The periodic pattern.

Definition 2 *An $\mathcal{AB}$-board is a square tiled by $\mathcal{AB}$ with $\mathcal{AB}$-border tiles appearing only at the border of the square as it is shown schematically in Figure 19. Notice that from now on, as it is done in Figure 19, for any tile in $\mathcal{AB}$ the presence of the $\mathcal{A}$-component will be represented by a unique shadowed background (no matter whether the $\mathcal{A}$-component belongs to $\mathcal{A}_1$ or $\mathcal{A}_2$).*

By the following two lemmas we prove that the set $\mathcal{AB}$ satisfies our requirements.

Lemma 3 *The set $\mathcal{AB}$ is NW-deterministic and for all $n > 1$ there exists an $\mathcal{AB}$-board of size 2^n with periodic boundary conditions.*

Proof. For the NW-determinism notice that $\mathcal{B}$ is NW-deterministic and that $\mathcal{AB} \subseteq \mathcal{A} \otimes \mathcal{B}$. On the other hand, for any $n > 1$, in order to obtain an $\mathcal{AB}$-board of size 2^n with periodic boundary conditions it suffices to transform a square of size 2^n tiled by $\mathcal{A}$ with periodic boundary conditions (see Figure 7) into an $\mathcal{AB}$-board by superposing in the suitable way the tiles of $\mathcal{B}$. □

Lemma 4 *In any periodic tiling of the plane by $\mathcal{AB}$ an $\mathcal{AB}$-board must appear.*

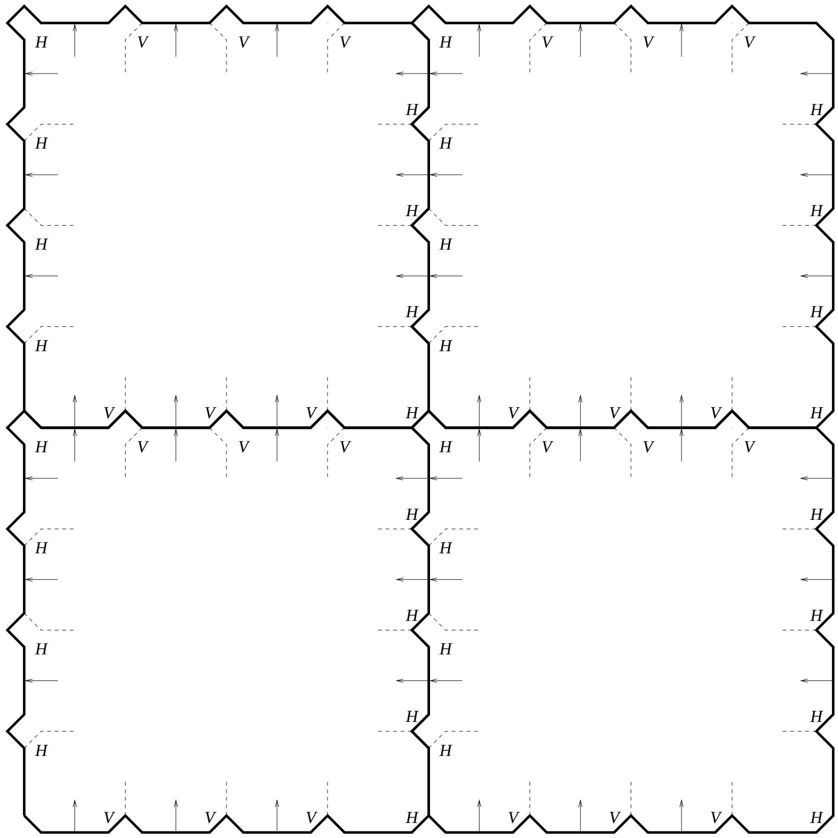


Fig. 17. The periodicity of the bounded conditions.

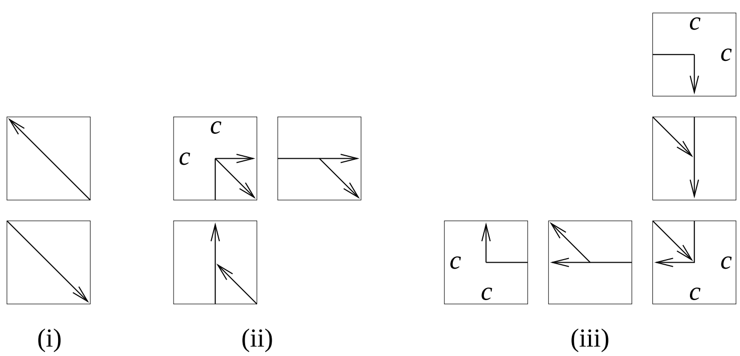


Fig. 18. The set of tiles $\mathcal{B}$. (i) Internal tiles (ii) NW-border tiles. (iii) SE-border tiles.

Proof. Let $\mathcal{P}$ be a periodic tiling of the plane by $\mathcal{AB}$. First notice that at least one $\mathcal{AB}$-border tile t_0 must appear in $\mathcal{P}$. In fact, if this is not the case then the plane would be tiled periodically by $\mathcal{A}_1 \otimes \mathcal{B}_{\text{int}}$. But this is not possible because $\mathcal{A}_1$ does not give periodic tilings of the plane. Notice also that t_0 can be assumed to be a *corner tile* (see Figure 20-i). In fact, let us suppose that there are no corner tiles in $\mathcal{P}$. If we define a *curve* to be any path in $\mathcal{P}$ determined by the (vertical and horizontal) arrows of the $\mathcal{B}$-components of

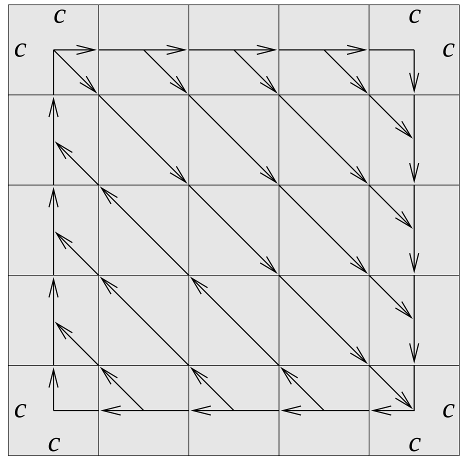


Fig. 19. An $\mathcal{AB}$-board.

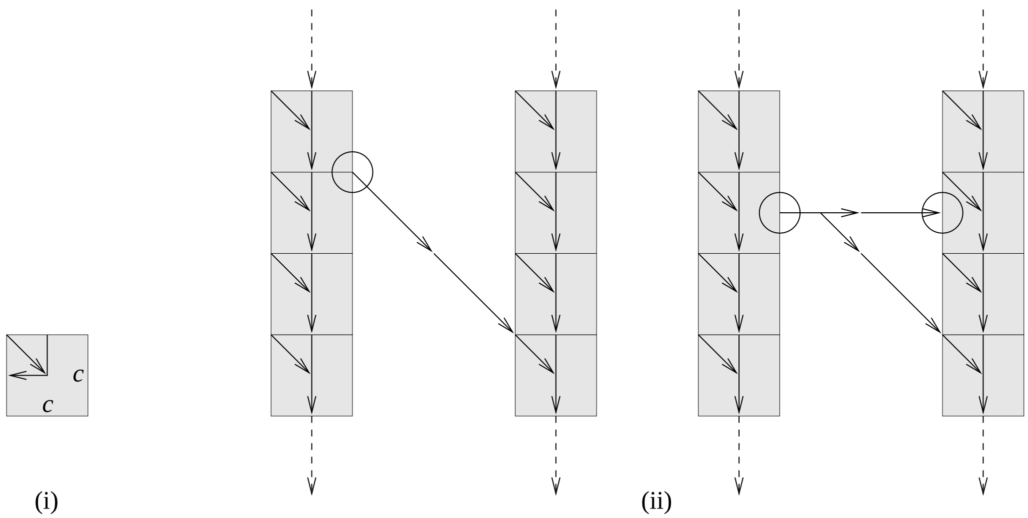


Fig. 20. (i) A corner tile. (ii) $\mathcal{P}$ does not admit infinite lines without corner tiles.

the $\mathcal{AB}$-border tiles and if we denote by C_0 the curve that passes through t_0, then C_0 has to be an infinite line. Without loss of generality this line can be assumed to be vertical and pointing down. By periodicity, there must exist a parallel line identical to C_0 and, because of the assumption that no corner tiles appear in $\mathcal{P}$, we have one of the two contradictions of Figure 20-ii.

We have just justified the fact that t_0 may be considered as a corner tile. Let us consider the curve C_0 that passes through t_0. It follows that C_0 has to be a closed curve. In fact, if this is not the case then, by the fact that the curve has to be bi-infinite, it is easy to verify that the only possibilities for C_0 are the two of Figure 21. But these two patterns cannot appear in *any* periodic tiling of the plane.

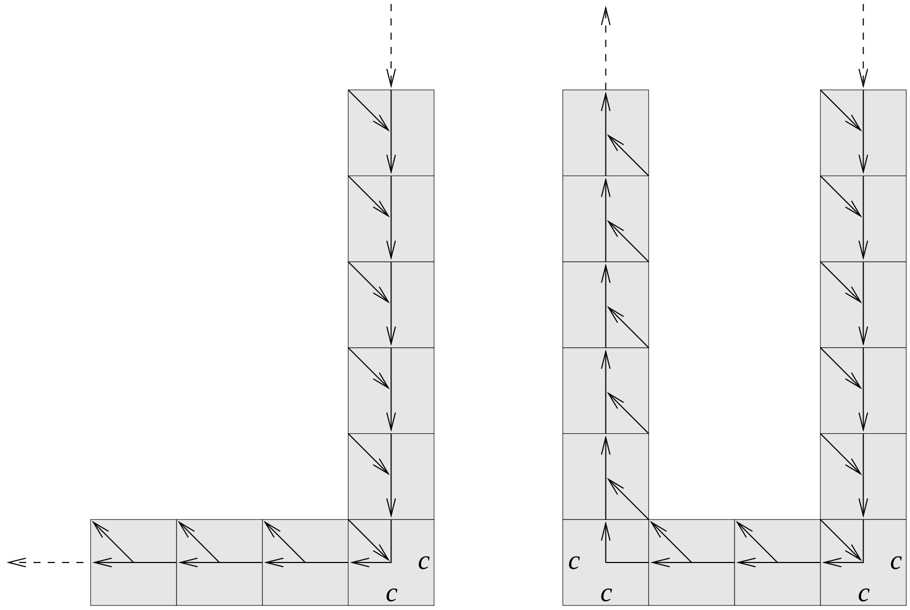
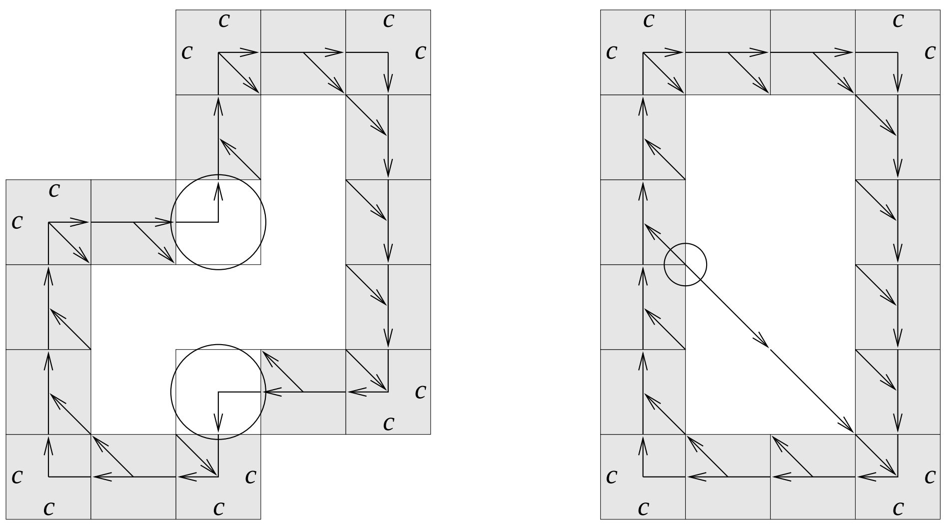


Fig. 21. Bi-infinite curves not allowing periodicity.



(i) (ii)

Fig. 22. Unfeasibility of closed curves different from squares.

Considering that C_0 is a closed curve and that different curves cannot cross each other, we can conclude that there exists a closed curve C_0^* containing in its interior no $\mathcal{AB}$-border tiles. It follows that C_0^* cannot have more than four corners. In fact, as it is shown schematically in Figure 22-i, closed curves with more than 4 corners are not feasible. Finally, it is not difficult to notice (see Figure 22-ii) that the only feasible 4 corners curve is a square. By definition, the square C_0^* delimits an $\mathcal{AB}$-board and therefore the lemma is proved. $\square$

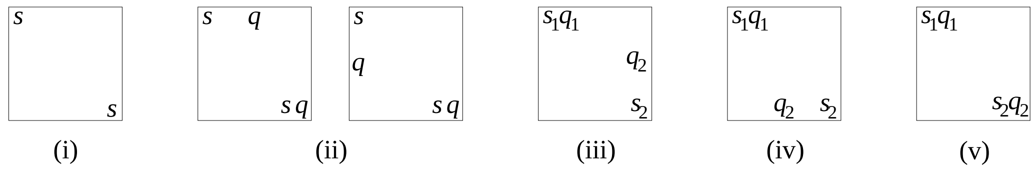


Fig. 23. The set of tiles $\mathcal{T}$. (i) Alphabet tiles. (ii) Merging tiles. (iii) Right tiles. (iv) Left tiles. (v) Stay tiles.

4.3 The reduction

Now we are able to prove the undecidability of the *NW-deterministic periodic tiling problem*. We do it by a reduction from the known undecidable *halting problem on Turing machines* in which a Turing machine $\mathcal{M} = (\Sigma, B, Q, q_0, q_h, \delta)$ is given and it is asked whether $\mathcal{M}$ reaches the halting state q_h when starting with the blank bi-infinite tape $(\cdots BB \cdots)$ and in the initial state q_0. Notice that $\delta : \Sigma \times Q \to \Sigma \times Q \times \{L, R, S\}$ represents the transition function of $\mathcal{M}$ with Σ being the alphabet, Q the set of states, and $\{L, R, S\}$ the possible moves (left, right, stay).

Proposition 3 *The NW-deterministic periodic tiling problem is undecidable.*

Proof. Let $\mathcal{M}^* = (\Sigma, B, Q, q_0, q_h, \delta)$ be an arbitrary Turing machine. Let $\mathcal{M} = (\Sigma, B, Q, q_0, q_f, \delta)$ be the same as $\mathcal{M}^*$ with the only difference that it never halts. More precisely, when it reaches the halting state q_h it erases the tape and it stays in the *final-quiescent configuration* (i.e, in the state q_f and scanning the cell located at the origin of the blank tape). By a suitable composition of a set of tiles $\mathcal{T}$ (which codes the Turing machine $\mathcal{M}$) and the set of tiles $\mathcal{AB}$ (introduced in the previous section) we are going to obtain a NW-deterministic set of tiles $\mathcal{H}$ admitting a periodic tiling of the plane if and only if $\mathcal{M}$ reaches the final-quiescent configuration.

Let $\mathcal{T}$ be the set of tiles that codes $\mathcal{M}$ of Figure 23: *alphabet tiles* are generated for each $s \in \Sigma$; *merging tiles* for every pair $(s, q) \in \Sigma \times Q$; *right, left and stay tiles* are associated to the tuples (s_1, q_1, s_2, q_2, R), (s_1, q_1, s_2, q_2, L) and (s_1, q_1, s_2, q_2, S) satisfying respectively $\delta(s_1, q_1) = (s_2, q_2, R)$, $\delta(s_1, q_1) = (s_2, q_2, L)$, and $\delta(s_1, q_1) = (s_2, q_2, S)$.

As it is shown in Figure 24, any computation of $\mathcal{M}$ can be coded as a tiling of the bottom-right quadrant of the plane ($\mathbb{N}^2$). In fact, if a t-frame is defined as a region of the form $\{(i, j) \in \mathbb{N}^2 : i = t \text{ or } j = t\}$ with $t \geq 0$, then instantaneous configurations of $\mathcal{M}$ appear coded, successively, in those t-frames where either a right, a left, or a stay tile appears. In each of these t-frames the origin of the tape is represented in the cell (t, t). The left part of the tape is represented

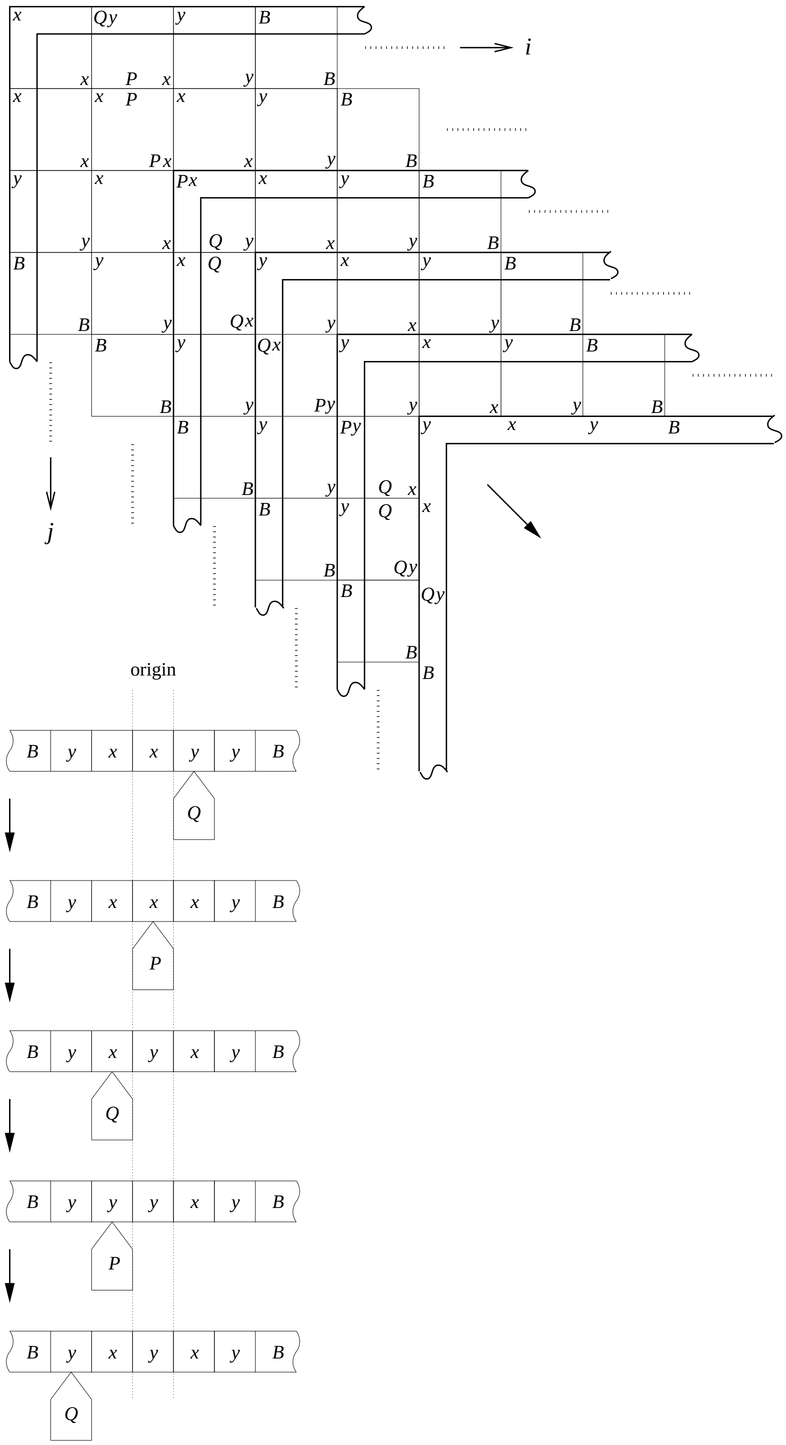


Fig. 24. Equivalence between a Turing machine computation and a tiling of the bottom-right quadrant of the plane.

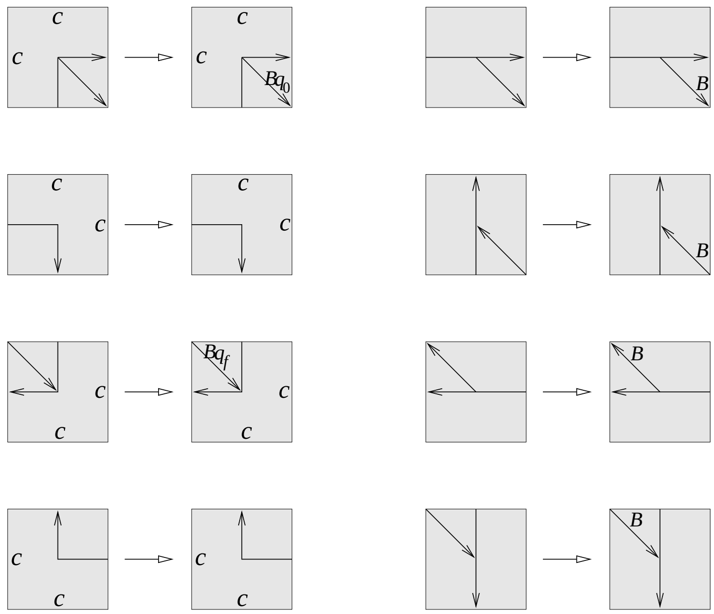


Fig. 25. Modification of $\mathcal{AB}_{\text{bord}}$ in order to obtain $\mathcal{H}_{\text{bord}}$.

in the vertical part of the frame while the right part is represented in the horizontal part of the frame. All the tiles of a frame correspond to alphabet tiles except the scanning cell and, possibly, the neighbor with which it is interacting. Notice that these tilings may be seen as alternative representations of Turing machines evolutions.

Let the set of tiles $\mathcal{H} = \mathcal{H}_{\text{int}} \cup \mathcal{H}_{\text{bord}}$ be the one with $\mathcal{H}_{\text{int}} = \mathcal{AB}_{\text{int}} \otimes \mathcal{T}$ and with $\mathcal{H}_{\text{bord}}$ being obtained by adding symbols to some tiles of $\mathcal{AB}_{\text{bord}}$ as it appears explicitly in Figure 25. The tiles belonging to $\mathcal{H}_{\text{int}}$ are called $\mathcal{H}$-internal tiles, while the tiles belonging to $\mathcal{H}_{\text{bord}}$ are called $\mathcal{H}$-border tiles.

As for the set $\mathcal{AB}$, we define an $\mathcal{H}$-board as a square tiled by $\mathcal{H}$ with the $\mathcal{H}$-border tiles appearing only at the border of the square as it is schematically shown in Figure 26. Notice that $\mathcal{H}$ is a NW-deterministic set of tiles. This fact can be easily checked by considering that $\mathcal{T}$ is NW-deterministic (because $\mathcal{M}$ is a deterministic machine) and that the set $\mathcal{AB}$ is NW-deterministic (see Lemma 3).

It remains to prove that $\mathcal{M}$ reaches the final-quiescent configuration when it starts from the blank tape *if and only if* $\mathcal{H}$ admits a periodic tiling of the plane. In fact, if $\mathcal{M}$ reaches the final-quiescent configuration then there exists a square $\mathcal{S}$ tiled by $\mathcal{T}$ with the boundary conditions that appears schematically in Figure 27-i. Without loss of generality we can assume that the size of $\mathcal{S}$ is $(2^n - 2)$ for some $n > 1$. In fact, if the size of the original square in which this computation was represented is k, then we can construct another one of size $(k+1)$ as it is explained in Figure 27-ii. Now from $\mathcal{S}$ it is direct to obtain an

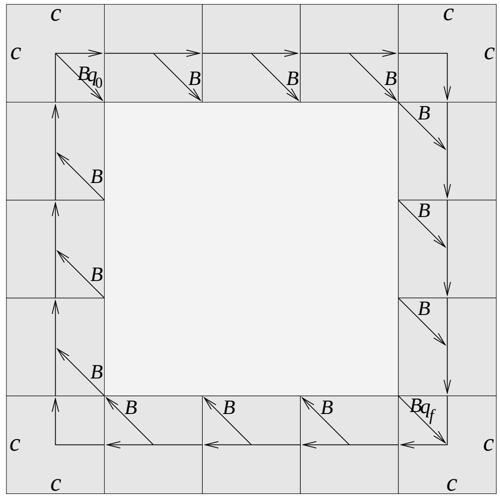


Fig. 26. An $\mathcal{H}$-board.

$\mathcal{H}$-board of size 2^n (see Figure 27-iii). Moreover, considering that there exists an $\mathcal{AB}$-board of size 2^n with periodic boundary conditions (see Lemma 3) we can assume that the $\mathcal{H}$-board has periodic boundary conditions and it can be repeated in order to tile the plane periodically.

Let us now suppose that $\mathcal{H}$ admits a periodic tiling of the plane $\mathcal{P}$. It follows that an $\mathcal{H}$-board must appear in $\mathcal{P}$. In fact, if this is not the case we would contradict Lemma 4. More precisely, if we suppose that in $\mathcal{P}$ no $\mathcal{H}$-board appears and we *extract* all the Turing machines symbols of $\mathcal{P}$ we would obtain a periodic tiling of the plane by $\mathcal{AB}$ having no $\mathcal{AB}$-boards. Finally, from an $\mathcal{H}$-board it is direct to obtain a square tiled by $\mathcal{T}$ encoding a computation of $\mathcal{M}$ evolving to the final-quiescent configuration (see again Figure 27). □

Remark 1 *The set $\mathcal{H}$ always admits a tiling of the plane. In fact, if $\mathcal{M}^*$ never halts, then it suffices to use $\mathcal{H}_{int}$ in order to tile nonperiodically the plane by representing this never-halting computation.*

Remark 2 *As an obvious consequence of Proposition 3 we can conclude the undecidability of the periodic tiling problem (in which it is asked whether an arbitrary set of tiles admits a periodic tiling of the plane). This result was obtained in [2]. Nevertheless, we would like to remark that our approach could also be used to prove the Gurevich and Koriakov result in a direct way. In fact, it suffices to notice that when the NW-deterministic property is no required, most of the technicities of the proof are no longer needed and it becomes very simple (for instance, the set $\mathcal{A}$ has just to be nonperiodic and it does not need an explicit representation).*

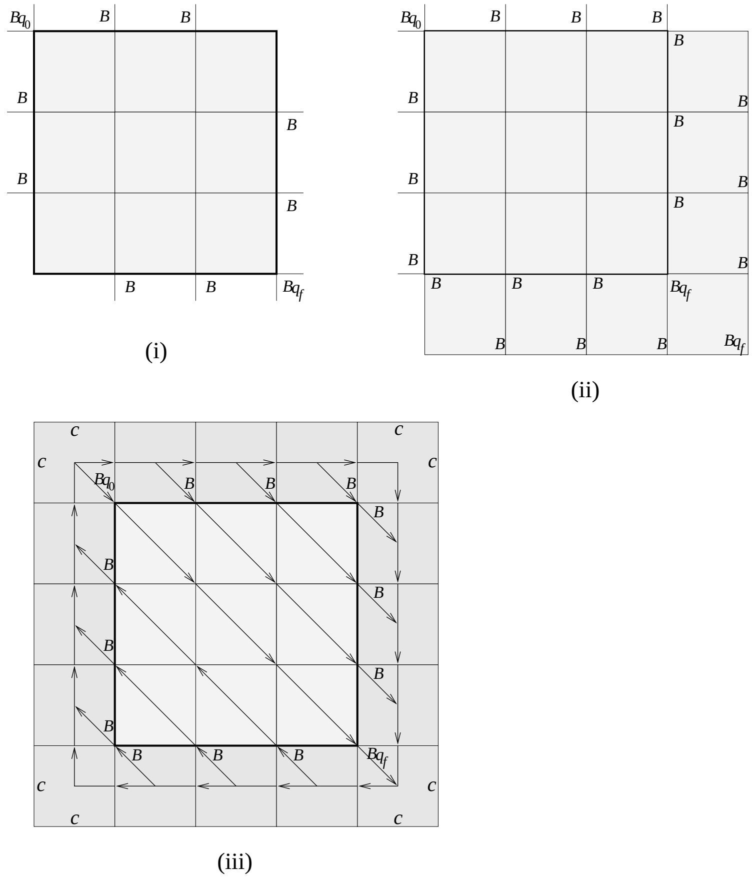


Fig. 27. (i) A square tiled by $\mathcal{T}$ representing a computation of $\mathcal{M}$ reaching the final-quiscent configuration. (ii) A bigger square. (iii) The associated $\mathcal{H}$-board.

References

[1] R. Berger. The undecidability of the domino problem. *Mem. Amer. Math. Soc.*, 66, 1966.

[2] Y. Gurevich and I. O. Koriakov. A remark on Berger's paper on the domino problem. *Siberian Mathematical Journal*, 13:319–321, 1972. (English version).

[3] K. Čulik II, J. Pachl, and S. Yu. On the limit sets of cellular automata. *SIAM Journal on Computing*, 18:831–842, 1989.

[4] J. Kari. The nilpotency problem of one-dimensional cellular automata. *SIAM Journal on Computing*, 21:571–586, 1992.

[5] R. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Inventiones Mathematicae*, 12:177–209, 1971.

[6] K. Sutner. Classifying circular cellular automata. *Physica D*, 45:386–395, 1990.

[7] S. Wolfram. Universality and complexity in cellular automata. *Physica D*, 10:1–35, 1984.